TWO-PERSON DYNAMIC EQUILIBRIUM: TRADING IN THE CAPITAL MARKET

by

Bernard Dumas

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RODNEY L. WHITE CENTER FOR FINANCIAL RESEARCH
The Wharton School
University of Pennsylvania
Philadelphia, PA 19104

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Bernard DUMAS

The author is Professor of Finance at the Wharton School of the University of Pennsylvania (U.S.A.), and at the Centre H.E.C.-I.S.A. (France). Grant #82E1177 of the French Ministry of Research is gratefully acknowledged.

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Abstract

When several investors with different risk aversions trade competitively in a capital market, the allocation of wealth fluctuates randomly between them and acts as a state variable against which each market participant will want to hedge. This hedging motive complicates the investors' portfolio choice and the equilibrium in the capital market. Although every financial economist is aware of this difficulty, to our knowledge, this issue has never been analyzed in detail. The current paper features two investors, with the same degree of impatience, one of them being logarithmic and the other having an isoelastic utility function. They face one risky constant-return-to-scale stationary production opportunity and they can borrow and lend to and from each other. The behavior of the allocation of wealth is characterized, along with the behavior of the rate of interest and that of the security market line. The two main results are: (1) investors in equilibrium do revise their portfolios over time so that some trading takes place, (2) provided some conditions are satisfied, the allocation of wealth admits a steady-state distribution at an interior point; this is in contrast to the certainty case, where one investor in the long run holds all the wealth. The existence of trading opens the way to a theory of capital flows and market trading volume.

1. Introduction

The question of dynamic asset pricing has been addressed so far, only under the assumption that all investors are alike (Lucas (1978), Cox, Ingersoll and Ross (1985)). The asset prices so obtained are then purely virtual prices, since no trading takes place in the capital market. The finance profession, that is, has no theory to offer to account for trading volume and capital flows between capital market participants, under conditions of rational symmetric information. The present paper aims to fill this gap.

Since our aim is to generate trading in the capital market, we must avoid circumstances which are known to induce constancy of the investors' ownership shares in the various assets, even though investors are not identical to each other. The work of Rubinstein (1974) has outlined these circumstances: if investors all have Hyperbolic-Absolute-Risk-Aversion (HARA) utility functions, with the same impatience parameter and the same cautiousness parameter (but are otherwise different from each other), they forever hold a fixed share of the market portfolio, and a fixed amount of a consol bond offering riskless payments. Such portfolio policies obviously require no trading: investors just live off the income generated by their constant portfolio.

There are several ways in which we can choose to deviate from the Rubinstein base case. Investors may differ in their impatience parameter. The question has been examined by Becker (1980) under conditions of certainty; the result is that the least impatient investor will hold all the wealth in the long run. It seems unlikely that new insights would be gained by the introduction of risk into Becker's analysis. We therefore retain the Rubinstein assumption of equal rates of impatience across investors.

Instead, we examine investors whose utility function is isoelastic with differing levels of relative risk aversion. Isoelastic utility functions belong to the HARA class of utility functions, and in their case, the cautiousness parameter of Rubinstein is simply equal to the relative risk aversion. In this way, the investors considered here differ in their cautiousness parameter, and we can expect that a constant portfolio policy will not be optimal for them.

The balance of this paper will show that this conjecture is correct. Along with some trading volume, our paper will produce a variable distribution of wealth across investors, but one which does not necessarily converge to 100% ownership by one of them; and it will also produce a variable rate of interest. The stochastic process for the short-term rate of interest will be shown to have several stable points.

The model is described in section 2. The equilibrium of the capital market is characterized in section 3. Section 4 provides a derivation of the dynamics of the distribution of wealth between investors. Section 5 quantifies the amount of trading in the market and derives asset prices, with special emphasis on bond prices and the term structure of interest rates.

2. The model

The capital market of our model economy is populated with but two investors, with the same rate of impatience ρ, but different risk aversions. The analysis is greatly simplified and does not lose its illustrative power if we restrict one investor to have a logarithmic utility function, while the other one exhibits any degree of risk aversion $1-\gamma$ where γ is the power of his isoelastic utility function:[1]

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[1]Note that the horizon is infinite.

(1) $$\text{Max } E \int_0^{\infty} e^{-\rho t} \frac{1}{\gamma} c^{\gamma} dt \; ; \qquad \gamma < 1 \; ,$$

and c is his finite rate of consumption of a single good.

Recall (Merton (1971)) that the logarithmic case can be obtained as the limit of the above case for $\gamma \to 0$; we shall therefore simply write the optimizing equations for the investor with the power utility function.

The two investors consume a single good and have access to two investment opportunities:

- they can buy shares in one constant-return-to-scale production activity, whose random output per unit of capital has a constant gaussian distribution with fixed parameters α and σ;
- they can borrow and lend to and from each other at the equilibrium riskless[2] rate r, which varies over time in an endogenous fashion.

Other notations are as follows:

	W:	wealth of the non-logarithmic investor;
	W*:	wealth of the logarithmic investor;
	x, x*:	share of each investor's wealth invested in the risky production opportunity;
	c, c*:	consumption rates of the two investors;
(2)	$\omega = W/(W + W^*)$:	the non-logarithmic investor's share of total wealth.

The dynamics of an investor's wealth for a given investment decision x and a given consumption decision c are well known:

(3) $$dW = \{W[r + x(\alpha - r)] - c\}dt + Wx\sigma dz \; ,$$

where dz is the random white noise affecting production. In this equation, the rate of interest r is the market rate. It is not constant over time. In

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[2]Endogenous default is left for future research.

fact, we can reasonably postulate that it is a function of the distribution of wealth ω: $r = r(\omega)$. The formulation of the two investors' optimization problem must, therefore, incorporate the behavior of the distribution of wealth.

Applying Ito's lemma to the definition (2) of ω and using the equation for the dynamics of wealth (3), as well as the analogous equation for the log investor, we obtain:

(4)
$$d\omega = \omega(1 - \omega)\Big\{\Big[(x - x^*)(\alpha - r) - \frac{c}{W} + \frac{c^*}{W^*} - (x - x^*)\big(\omega x + (1 - \omega)x\big)\sigma^2\Big]dt + (x - x^*)\sigma dz\Big\} .$$

Not surprisingly, the allocation of wealth would be constant if the two investors were to hold the same portfolio ($x = x^*$) and the allocation of wealth also admits two absorbing barriers at $\omega = 0$ and $\omega = 1$: if one investor comes to hold all the wealth, he thereafter holds all the wealth forever.

The maximization of (1) subject to (3) and (4) with respect to c and x (but taking the behavior of ω as given in a pure competitive fashion) is a standard dynamic program. The partial differential equation for its (undiscounted) value function $J(W, \omega)$ is a Hamilton-Jacobi equation which can be written easily. One can proceed in two steps, optimizing consumption first, and then the portfolio. The optimality condition with respect to consumption is:

(5)
$$c^{\gamma-1} = \frac{\partial J}{\partial W} .$$

Substituting the optimal consumption decision into the original Hamilton-Jacobi equation, one can verify that a function of the form:

(6)
$$J(W, \omega) = \frac{1}{\gamma} W^{\gamma} I(\omega)$$

is a solution, i.e., that the W variable does cancel out.[3] This being done, one is left with the second problem of optimization, with respect to the portfolio x:

(7)
$$0 = -\rho + (1 - \gamma)I^{\frac{1}{\gamma - 1}}$$
$$+ \underset{x}{\text{Max}}\ \gamma\left\{r + x(\alpha - r) + \tfrac{1}{2}(\gamma - 1)x^2\sigma^2 + \frac{I'}{I} x\sigma\hat{\omega}\right\}$$
$$+ \frac{I'}{I}\overset{\circ}{\omega} + \tfrac{1}{2}\frac{I''}{I}\hat{\omega}^2$$

where $\overset{\circ}{\omega}$ and $\hat{\omega}$ are the drift and the diffusion coefficients of ω, which appeared in equation (4) above.

The optimality condition with respect to x is evidently:

(8)
$$\alpha - r + (\gamma - 1)x\sigma^2 + \frac{I'}{I}\sigma\hat{\omega} = 0$$

so that the optimal portfolio is:

(9)
$$x = \frac{\alpha - r + \frac{I'}{I}\sigma\hat{\omega}}{(1 - \gamma)\sigma^2} .$$

The optimal portfolio is of the well-known form applying to an isoelastic investor, except for the last term of the numerator which represents hedging against shifts in the distribution of wealth, which induce shifts in the rate of interest.

3. Equilibrium

Instantaneously, the equilibrium in the capital market is characterized by:

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[3]N.B.: as a result: $c/W = I^{1/(\gamma-1)}$.

a) the non-logarithmic investor's portfolio optimality condition (8), which reads as follows, when the form of the diffusion coefficient $\overset{\circ}{\omega}$ is made explicit:

(10) $$\alpha - r = (1 - \gamma)\sigma^2 x - \frac{I'(\omega)}{I(\omega)}\,\omega(1 - \omega)\sigma^2(x - x^*) \;;$$

b) the logarithmic investor's portfolio condition: it is well known since Hakansson (1971) that such an investor exhibits no hedging motive, i.e., that his function I* is a constant; hence:

(11) $$\alpha - r = \sigma^2 x^* \;;$$

c) a 'supply equals demand' condition:

(12) $$\omega x + (1 - \omega)x^* = 1 \;.$$

Solving these three equations simultaneously, one obtains all the endogenous variables as a function of ω:

(13) $$x^* = \frac{1}{\omega\lambda + 1 - \omega}$$

(14) $$x = \frac{\lambda}{\omega\lambda + 1 - \omega}$$

(15) $$r = \alpha - \frac{\sigma^2}{\omega\lambda + 1 - \omega}$$

where:

(16) $$\lambda = \frac{1 - \frac{I'}{I}\,\omega(1 - \omega)}{1 - \gamma - \frac{I'}{I}\,\omega(1 - \omega)}$$

is best interpreted as the non-logarithmic investor's risk tolerance "adjusted for the hedging motive," since, in effect, **his decision is mean-variance optimizing, in the static sense, but at a level of risk aversion different** from $1 - \gamma$. Similarly, by analogy with the CAPM, $\omega\lambda + 1 - \omega$ can be seen as the market's risk tolerance, also "adjusted for the hedging motive."

The reason why the standard mean-variance framework survives, with a minor change,[4] despite the introduction of one more state variable and the non stationarity of the rate of interest, is that this additional state variable ω is perfectly correlated with wealth (cf. equations (3) and (4)).

Over time, the equilibrium is further characterized by the dynamics (4) of the distribution of wealth (with x, x^*, r, c/W, c^*/W^* substituted in) and by the two functions $I(\omega)$ and $I^*(\omega)$. We mentioned above that I^* is a constant; in fact, solving the logarithmic investor's partial differential equation for I^* (which is (7) with $\gamma = 0$), subject to the transversality conditions indicated in Merton (1971), produces $I^* = 1/\rho$ and therefore $c^*/W^* = \rho$. Once these elements are taken into account, the dynamics of ω, for a given I function, are:

(17) $$d\omega = \omega(1 - \omega)\{(-\frac{\omega\sigma^2(\lambda - 1)^2}{(\omega\lambda + 1 - \omega)^2} - I^{\frac{1}{\gamma-1}} + \rho)dt + \sigma dz\}$$

and, finally, substituting equations (13) to (15), and (17) above, we obtain the partial differential equation to be satisfied by the unknown function $I(\omega)$:

(18) $$0 = -\rho + (1 - \gamma)I^{\frac{1}{\gamma-1}}$$

$$+ \gamma[\alpha - \frac{\sigma^2}{\omega\lambda + 1 - \omega} - \tfrac{1}{2}(\gamma - 1)\sigma^2(\frac{\lambda}{\omega\lambda + 1 - \omega})^2]$$

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[4] As a matter of fact, if we had introduced a multiplicity of assets, we could have proved that a Tobin separation theorem applies to the present situation.

$$+ \frac{I'}{I}\,\omega(1-\omega)\Big[-\frac{\omega\sigma^2(\lambda-1)^2}{(\omega\lambda+1-\omega)^2} - I^{\frac{1}{\gamma-1}} + \rho\Big]$$

$$+ \tfrac{1}{2}\,\frac{I''}{I}\,\Big[\omega(1-\omega)\,\frac{\lambda-1}{\omega\lambda+1-\omega}\,\sigma\Big]^2 ;$$

where λ is given by (16).

The problem of the determination of equilibrium is thus reduced to that of solving the partial differential equation (18) (coupled with (16)), subject to two "natural" boundary conditions, corresponding to the two absorbing barriers $\omega = 0$ and $\omega = 1$:

(19) $$I(0) = \left|\frac{1-\gamma}{\rho - \gamma\left(\alpha - \sigma^2 + \frac{1}{2}\,\frac{\sigma^2}{1-\gamma}\right)}\right|^{1-\gamma}$$

(20) $$I(1) = \left|\frac{1-\gamma}{\rho - \gamma\left(\alpha - \frac{1}{2}(1-\gamma)\sigma^2\right)}\right|^{1-\gamma} .$$

Everything one might want to know about the equilibrium path, will follow from this $I(\omega)$ function: once it is known, equations (13) to (17), and the equation of footnote #3, give the portfolio choices, the rate of interest, the market price of risk, the dynamics of the allocation of wealth, and consumption choices.

4. Dynamics of the allocation of wealth

It is unlikely that equation (18) subject to boundary conditions (19) and (20) should have a known analytical solution. Considering, however, that the domain of variation of ω is a closed set, and that the behavior of I on the boundary is well specified, this partial differential equation lends itself nicely to numerical analysis.[5] We choose to present the results in terms of

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[5] cf. Smith (1978).

the behavior (specifically the drift) of the distribution of wealth over time (equation (17)), rather than in terms of the I function itself. The reader is referred to figures 1a and 1b, which are "expected" phase diagrams for ω.

These figures allow one to study the stability of the distribution of wealth. There are always two points which are stochastically stable, in the sense that both the drift and the volatility of ω vanish at those points: they are, of course, $\omega = 0$ and $\omega = 1$. They correspond to situations in which the entire wealth is concentrated in the hands of one investor. Depending on the value of the impatience parameter ρ, and on that of the risk aversion parameter $1 - \gamma$ (see the figures), one of these two endpoints may also be absolutely stable in the expected value sense, which means that the drift of ω would attract it towards that endpoint, when starting away from it.

Under certainty ($\sigma = 0$), one of the two endpoints would necessarily be the long-run outcome: when the two investors have the same rate of impatience ρ, the one with the higher risk aversion[6] would end up owning all the wealth, when the rate of impatience ρ is less than the earning rate α, and vice-versa.[7] There would be no stable point in-between.

Figures 1a and 1b reveal, however, that the case of uncertainty is **qualitatively different** from the case of certainty: within some range of parameter values the curve giving the drift of ω as a function of ω, cuts the ω axis with a negative slope--implying stability-- for a value of ω different from 0 or 1. For these parameter values, the distribution of wealth will forever wander around its stable interior value, tending to return to it after

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[6]Risk aversion would act then only as a measure of elasticity of substitution between periods.

[7]When $\rho = \alpha$, the long-run allocation of wealth would be determined by the initial situation.

a shock, unless a series of them causes it to hit one of the two absorbing barriers at 0 or 1.

As the allocation of wealth fluctuates, the security market line of the traditional CAPM should be viewed as pivoting around one fixed point representing the risky production opportunity, while the variable slope of the line determines the current value of the riskless rate of interest. The behavior of the allocation of wealth is thus mirrored in the stochastic behaviors of the market price of risk $1/(\lambda\omega + 1 - \omega)$ and of the equilibrium riskless rate of interest r, which are monotonic functions of ω via equations (15) and (16) above.[8] They both admit two absorbing barriers, at 1 and $1 - \gamma$ for the market price of risk, and at $\alpha - \sigma^2$ and $\alpha - (1 - \gamma)\sigma^2$ for the rate of interest. These values correspond to the endpoints $\omega = 0$ and $\omega = 1$, where one of the two investors would impose his risk aversion and his corresponding value of the rate of interest.

Whenever there exists a stable interior value of the distribution of wealth, so is there one for the rate of interest, which wanders between the two extreme values, while tending to return to the stable interior one. There are then **three** possible long-run values of the rate of interest: two which are absorbing and one which is stable in the expected-value sense (refer to figures 2a and 2b). The volatility of the rate of interest is by no means constant: it is zero at the two absorbing barriers and exhibits a maximum somewhere in-between.

Although this model is perhaps the most simple one can conceive, while still exhibiting a variable rate of interest, the process so obtained for the

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[8] r is an increasing function when $1 - \gamma > 1$, a decreasing one otherwise; the opposite is true for the market price of risk.

rate is much more complex than had been previously assumed (cf. e.g., the Ornstein-Uhlenbeck process used by Vasicek (1977)).

5. Trading, asset prices and the term structure

Asset holdings by the two investors are given by the values of x and x^* (equations (14) and (13)), for the non-logarithmic and the logarithmic investor respectively. They represent the share of each investor's wealth invested in the risky asset. However, the two investors' **shares of ownership** in the risky production opportunity are equal to $x\omega$ and $x^*(1 - \omega)$ respectively.

Trading takes place in the capital market if and only if $\omega \neq 0$ or 1 and $x\omega$ is a non constant function of a fluctuating ω; for, this implies that one investor buys shares from and sells shares to the other, as time passes. In contrast to previous theories of dynamic capital market equilibrium, the present model accounts for (some) trading volume. Indeed figures 3a and 3b display the shares of ownership as functions of the allocation of wealth and it is clear that they are no constant: when one investor owns almost all the wealth, almost all of **his** wealth is allocated to the risky asset and, by necessity, he owns almost all the shares of this asset. The other investor may or may not be a borrower, depending on his risk aversion, but his leverage always remains finite $\left(x < 1/(1 - \gamma),\ x^* < 1 - \gamma\right)$ so that he can only own a small fraction of the shares of the risky asset. As an investor's share of wealth fluctuates, so does his share of ownership of the risky asset; and, of course, his share of wealth does fluctuate because, as a result of different risk aversions, the two investors make up their portfolios differently. Our model provides scope for **capital flows** between investors; the current-account balance between them is not equal to zero.

The current model includes only one risky asset, so that it is not exactly appropriate to discuss the relative pricing of assets. Assets which are in zero net supply may nonetheless be priced. These assets would have to satisfy the following CAPM:

$$ER = r + \frac{s}{\omega\lambda + 1 - \omega} \tag{21}$$

where:

ER = expected rate of return from the asset,

s = covariance of the asset rate of return with the productive opportunity.

The price of the asset, which is a function of ω--and possibly of $W + W^*$, depending on its contractual definition--will generally have to satisfy equation (21) interpreted as a functional equation.

Since the present formulation has been able to generate an interesting behavior for the short-term rate of interest, one might think of applying it to the pricing of bonds. Rather than expressing the price of the bond as a function of the allocation of wealth which is not observable, it will be empirically more useful to express it as a function P of the rate of interest r.[9] Knowing the behavior of r, a straightforward application of Ito's lemma to the unknown function provides expressions for ER and s. Substituting them into (21) gives:

$$\frac{1}{P}\left[-\frac{\partial P}{\partial t} + \frac{\partial P}{\partial r}\mu_r + \frac{1}{2}\frac{\partial^2 P}{\partial r^2}\sigma_r^2\right] = r - \frac{1}{P}\frac{\partial P}{\partial r}\sigma_r\left[\frac{\sigma}{\omega\lambda + 1 - \omega}\right] , \tag{22}$$

where t is the time to maturity, μ_r and σ_r are the drift and the volatility of

[9]and, of course, of the time to maturity.

the rate of interest,[10] and the expression within brackets on the right-hand side is commonly known as "the market price of interest-rate risk" (cf. Vasicek (1977) or Brennan and Schwartz (1982)). This quantity is thus equal to the market price of risk, as defined above,[11] times the volatility of the output. It follows from what we said and from equation (15) that the volatility of the market price of risk and the volatility of the rate of interest are proportional to each other. It would therefore be inconsistent to assume, as is sometimes done, that the market price of interest-rate risk is constant, in a setting which allows for interest-rate uncertainty.

The boundary conditions which accompany equation (22) are easy to obtain since it is known that the rate of interest will remain constant, once it hits one of the two absorbing barriers. For these values of the rate, the price of a pure-discount unit bond is simply equal to the present value, computed at a constant rate, of one unit of consumption paid at maturity. One can then solve equation (22) numerically, working backwards from a price equal to one at maturity. The solution may be expressed in terms of the bond's yield to maturity, instead of its price. Figure 4 displays the term structure of yields for one combination of parameter values, and for various current values of the short-term rate of interest. Except when the short rate is at one of

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[10] In the present model the correlation between the rate of interest and the risky output is equal to -1.

[11] I.e., the average market risk aversion.

the absorbing barriers, the term structure is upward sloping, thus reflecting a liquidity premium.[12]

6. Conclusion

The current model, to our knowledge, is the first to present a self-contained account of dynamic capital market equilibrium, involving investors with different taste parameters. The theory is self contained in the sense that all state variables are identified and have a well-specified, endogenously determined, stochastic process. The model exhibits a stochastically variable distribution of wealth, which sometimes admits a stable interior value, and a stochastically variable short rate of interest with the same property. It also produces trading in the capital market.

The agenda for future research includes an extension to the international setting, with several productive assets, endogenous default and deviations from purchasing-power parity, and possibly also several currencies.

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[12]Even though the chosen combination of parameter values is one which produces a stable interior value for the rate of interest (see figure 2a), the tendency for the rate to return to that value, when the current value is sufficiently high, fails to generate a hump-shaped term structure. The reason for this result which is at variance with, e.g., Vasicek's structure of yields, is that the drift of the rate of interest is of small magnitude next to its volatility. Hence the risk premium almost alone (plus the uninteresting technical fact that the equation for the calculation of a yield is non linear) determines the shape of the term structure.

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Figure 1a: behavior of alloc. of wealth

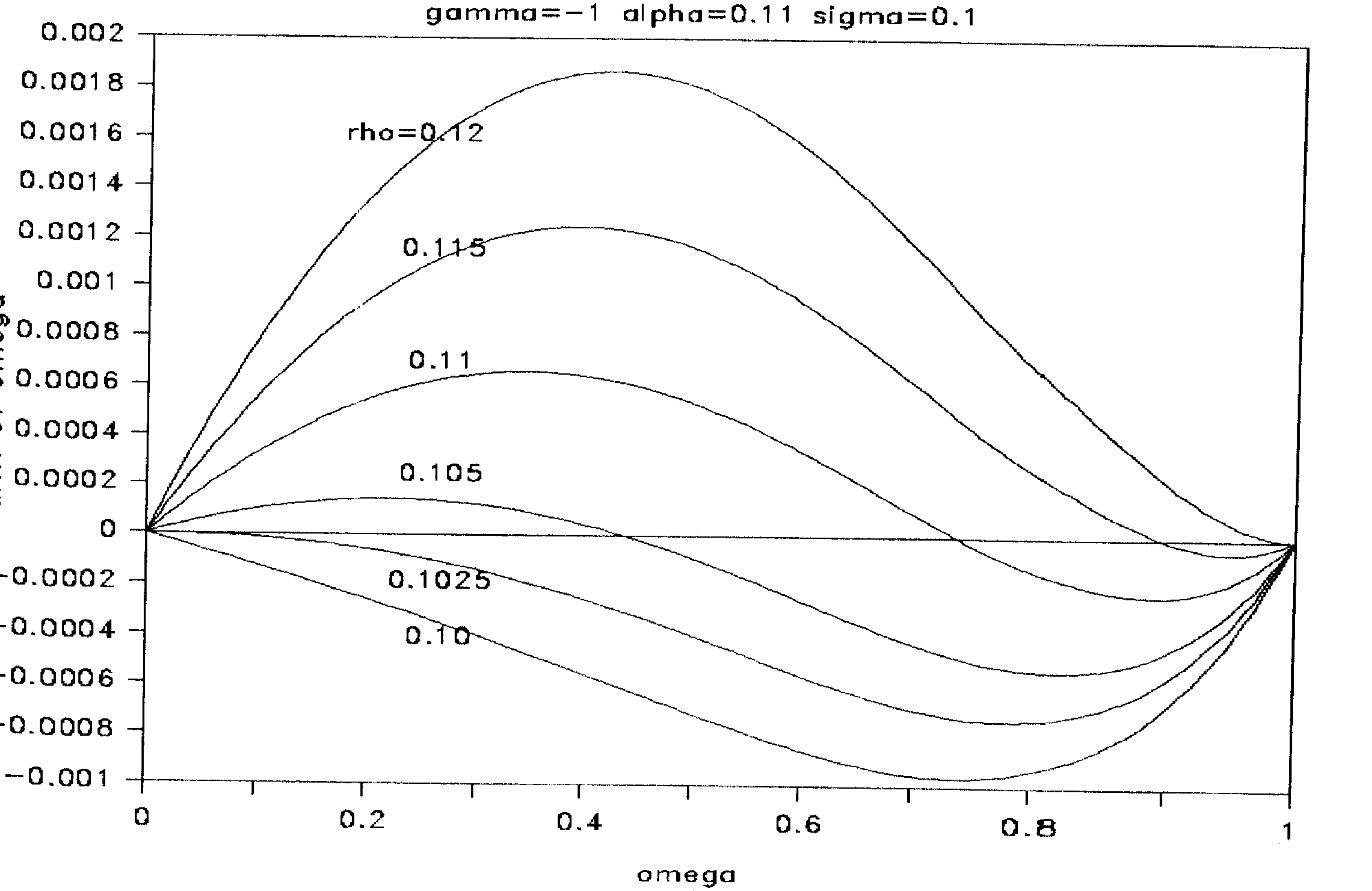

Figure 1b: behavior of alloc. of wealth

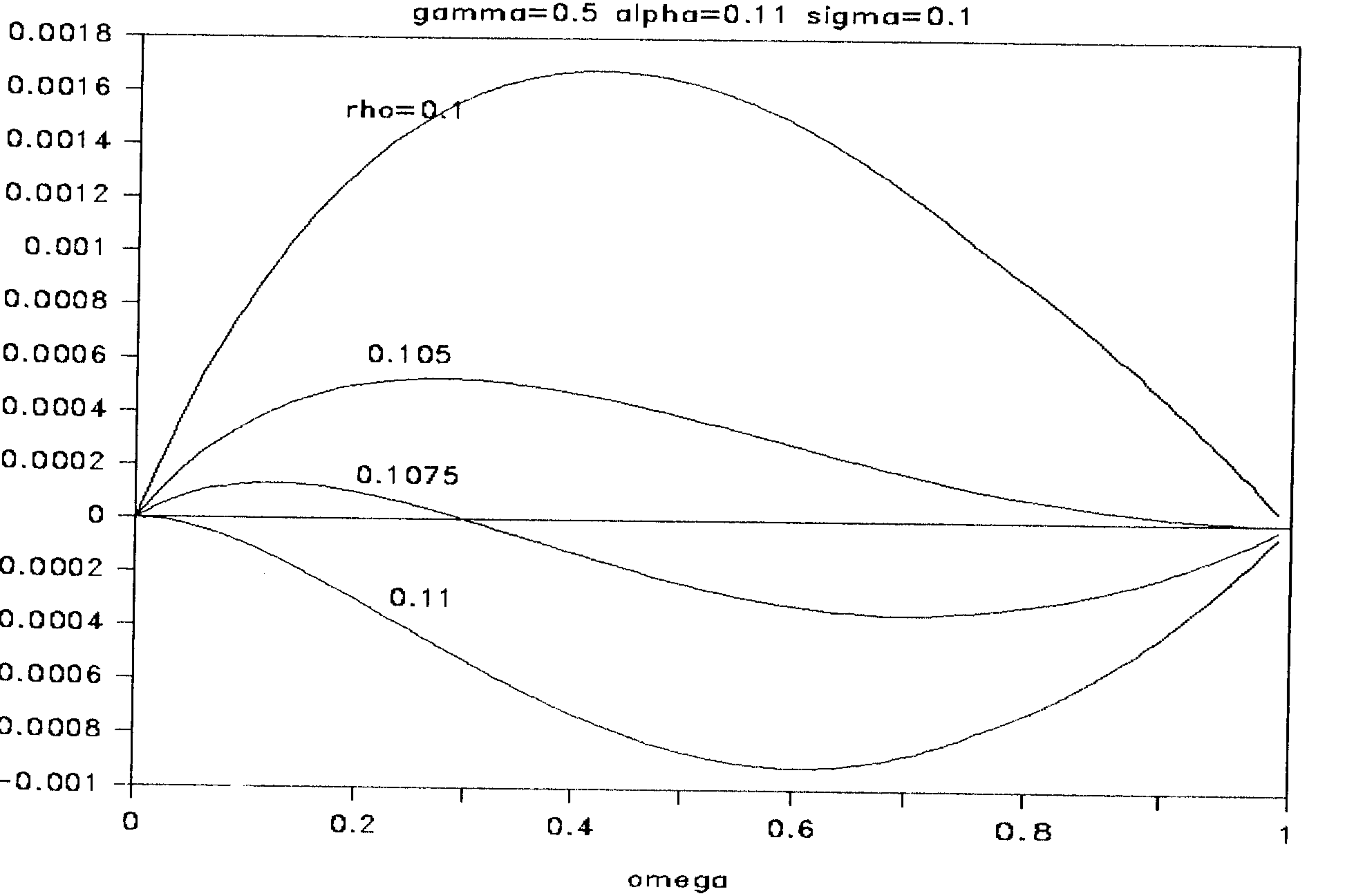

Figure 2a: behavior of rate of interest

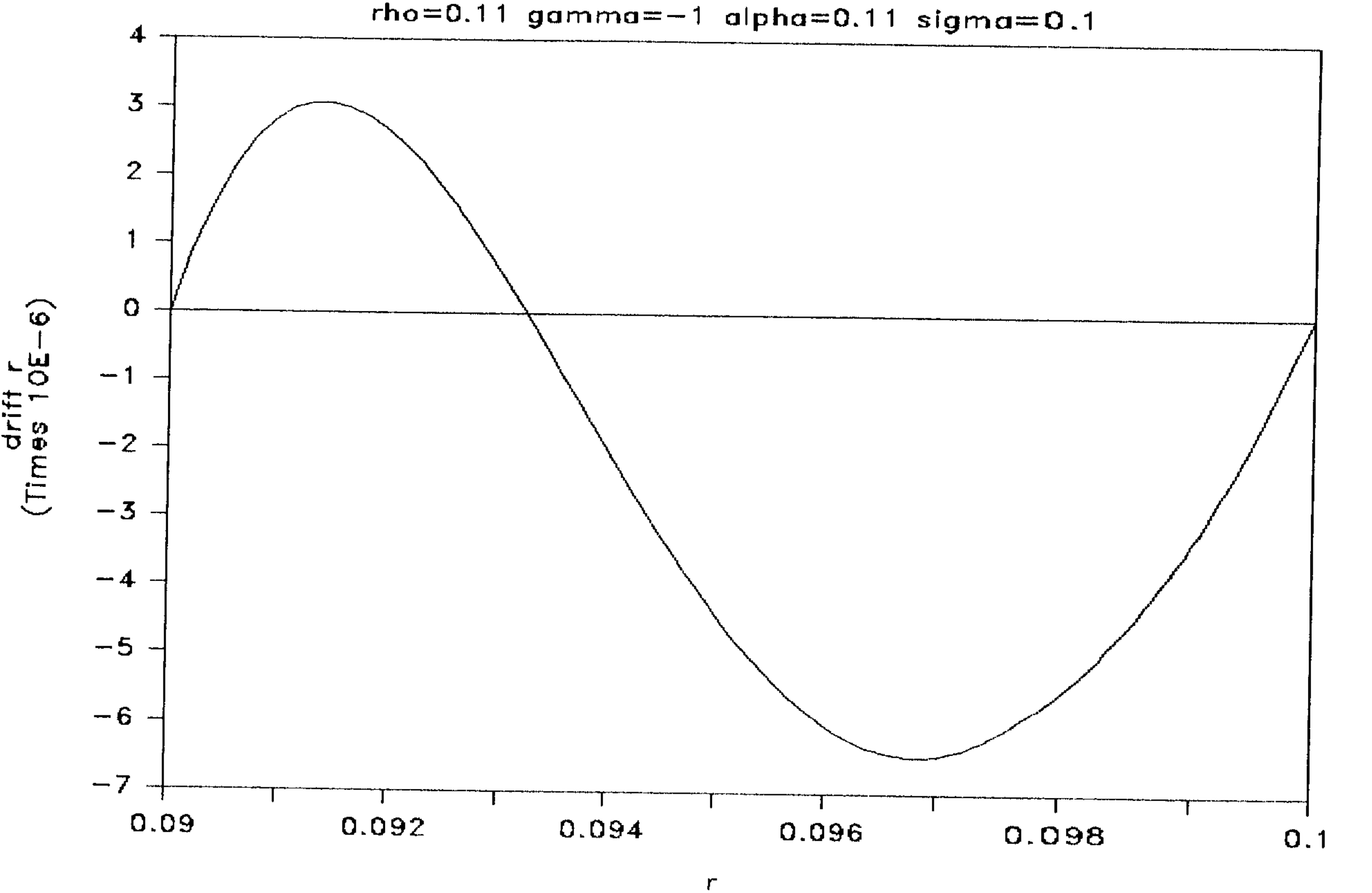

Figure 2b: behavior of rate of interest

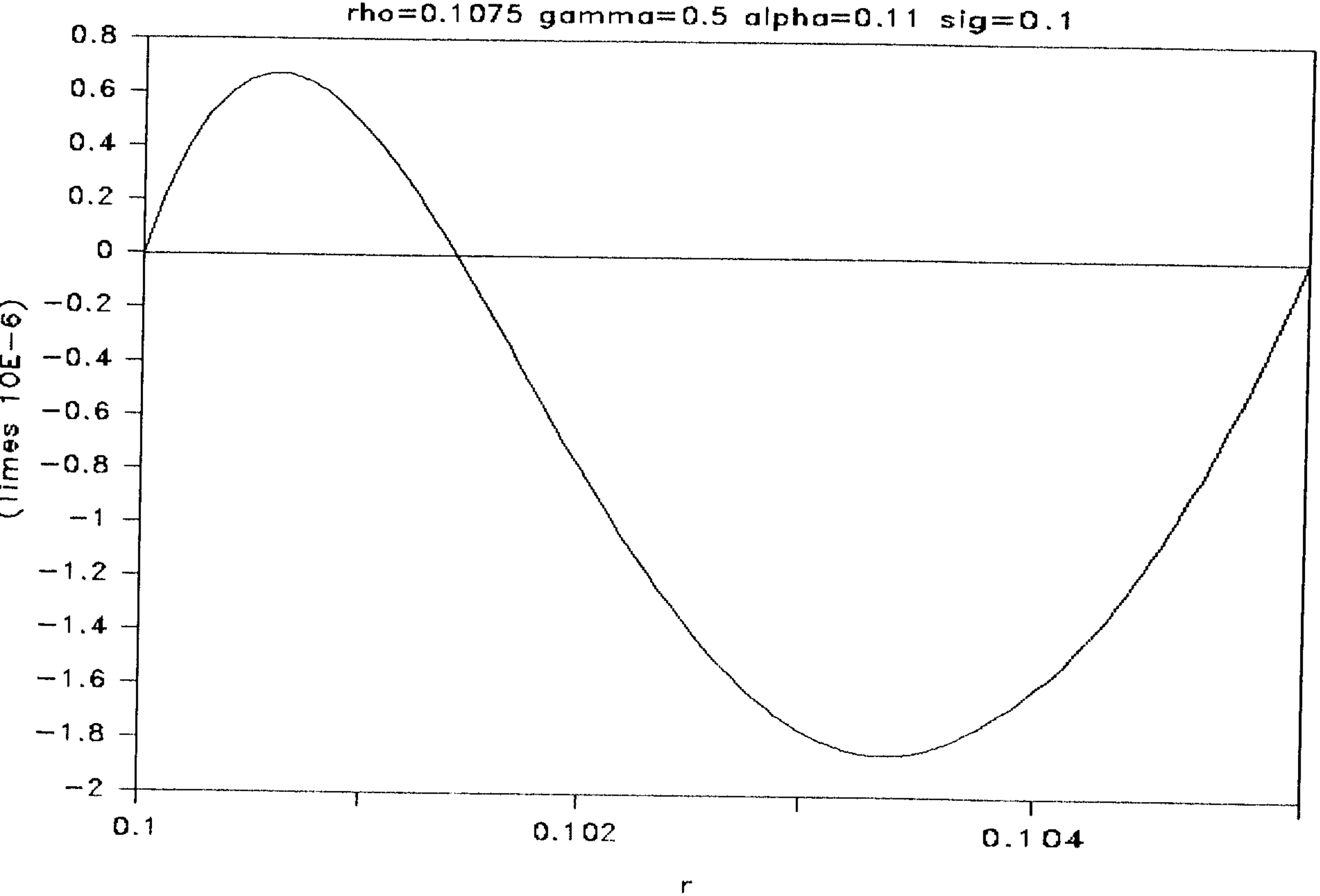

figure 3a: shares of ownership

rho=0.1075 gamma=0.5 alpha=0.11 sig=0.1

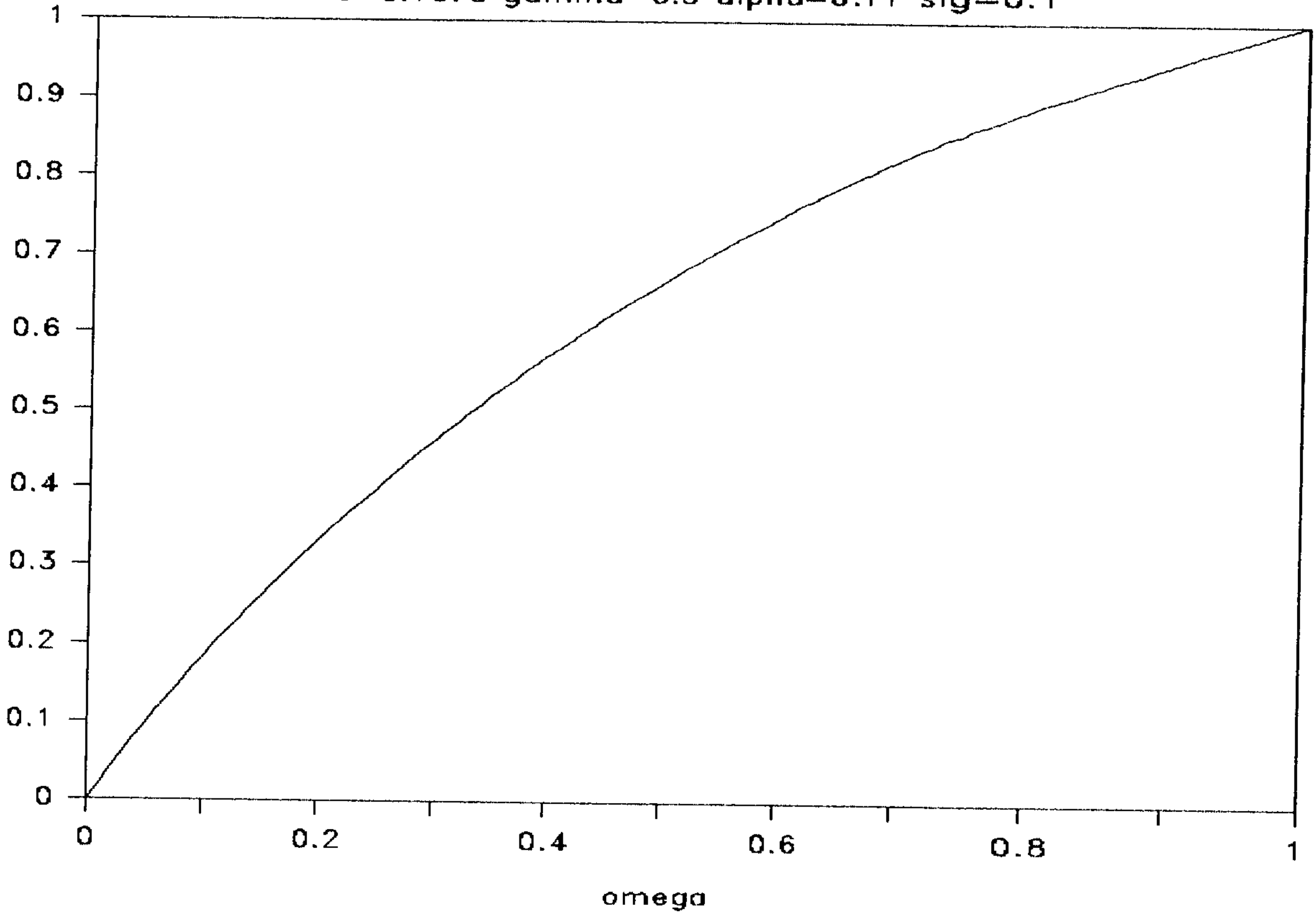

figure 3b: shares of ownership

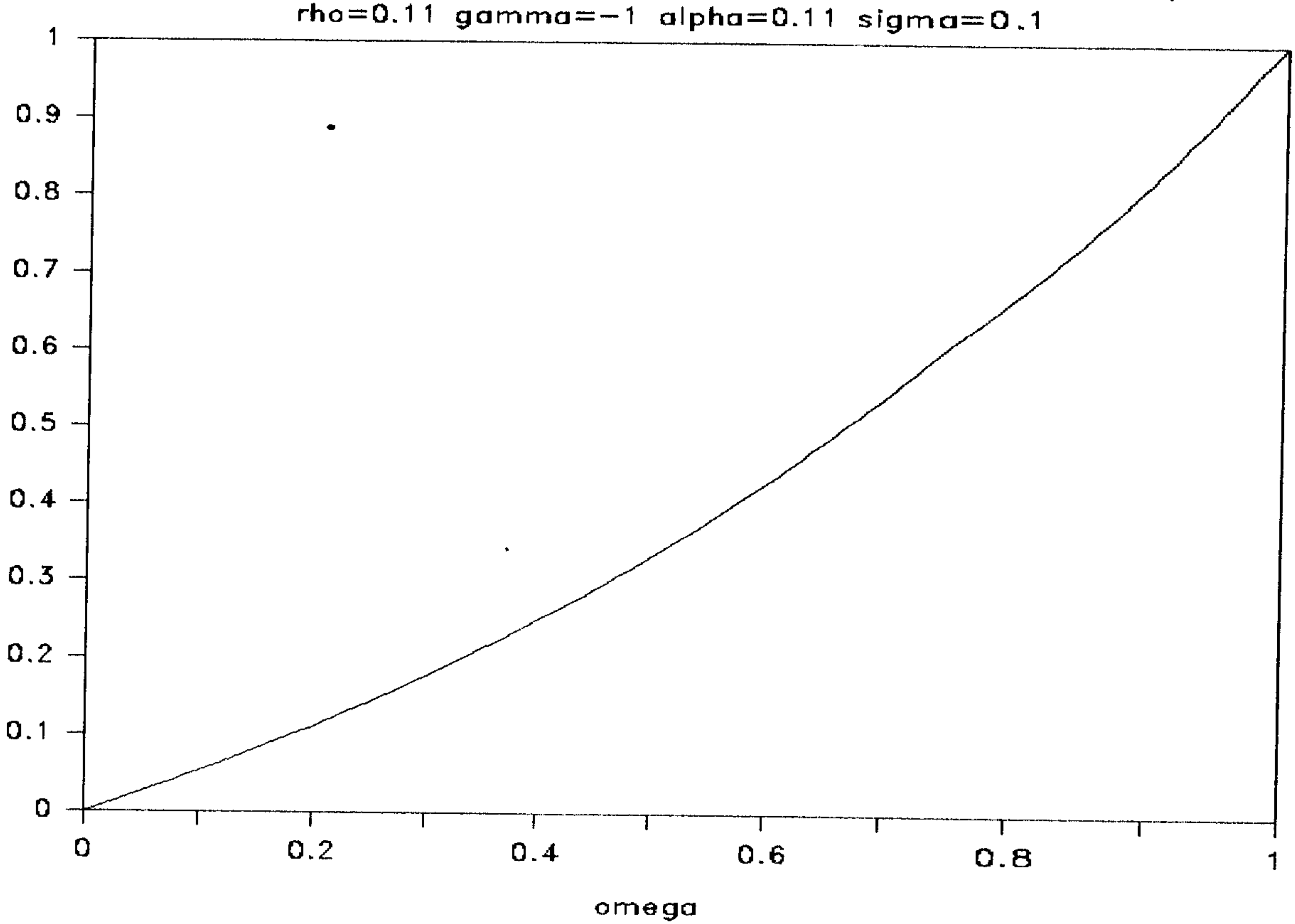

Figure 4: term structure of int rates