Screening and Adverse Selection in Frictional Markets*

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Abstract

We incorporate a search-theoretic model of imperfect competition into an otherwise standard model of asymmetric information with unrestricted contracts. We develop a methodology that allows for a sharp analytical characterization of the unique equilibrium, and then use this characterization to explore the interaction between adverse selection, screening, and imperfect competition. On the positive side, we show how the structure of equilibrium contracts—and hence the relationship between an agent's type, the quantity he trades, and the corresponding price—are jointly determined by the severity of adverse selection and the concentration of market power. This suggests that quantifying the effects of adverse selection requires controlling for the market structure. On the normative side, we show that increasing competition and reducing informational asymmetries can be detrimental to welfare. This suggests that recent attempts to increase competition and reduce opacity in markets that suffer from adverse selection could potentially have negative, unforeseen consequences.

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1 Introduction

Many large and important markets suffer from *adverse selection*, including the markets for insurance, credit, and certain financial securities. There is mounting evidence that many of these markets also feature some degree of *imperfect competition*.[1] And yet, perhaps surprisingly, the effect of imperfect competition on prices, allocations, and welfare in markets with adverse selection remains an open question.

Answering this question is important for several reasons. For one, many empirical studies attempt to quantify the effects of adverse selection in the markets mentioned above.[2] A natural question is to what extent these estimates—and the conclusions that follow—are sensitive to the assumptions being imposed on the market structure. There has also been a recent push by policymakers to make several of the markets mentioned above more competitive and less opaque.[3] Again, a crucial, but seemingly under-explored question is whether these attempts to promote competition and reduce information asymmetries are necessarily welfare-improving.

Unfortunately, the ability to answer these questions has been constrained by a shortage of appropriate theoretical frameworks.[4] A key challenge is to incorporate nonlinear pricing schedules—which are routinely used to *screen* different types of agents—into a model with asymmetric information and imperfect competition. This paper delivers such a model: we develop a novel, tractable framework of adverse selection, screening, and imperfect competition.

The key innovation is to introduce the *search-theoretic* model of imperfect competition developed by Burdett and Judd (1983) into an otherwise standard model with asymmetric information and nonlinear contracts. Within the context of this environment, we provide a full analytical characterization of the unique equilibrium, and then use this characterization to study both the positive and normative issues highlighted above.

First, we show how the structure of equilibrium contracts—and hence the relationship between an agent's type, the quantity that he trades, and the corresponding price—are *jointly* determined by the

[1] For evidence of market power in insurance markets, see Brown and Goolsbee (2002), Dafny (2010), and Cabral et al. (2014); Einav and Levin (2015) provide additional references, along with a general discussion. For evidence of market power in various credit markets, see, e.g., Ausubel (1991), Calem and Mester (1995), Petersen and Rajan (1994), Crawford et al. (2015), and Scharfstein and Sunderam (2013). In over-the-counter financial markets, a variety of data suggests that dealers extract significant rents; indeed, this finding is hard-wired into the workhorse models of this market, such as Duffie et al. (2005) and Lagos and Rocheteau (2009).

[2] See the seminal paper by Chiappori and Salanie (2000), and Einav et al. (2010a) for a comprehensive survey.

[3] Increasing competition and transparency in health insurance markets is a cornerstone of the Affordable Care Act, while the Dodd-Frank legislation addresses similar issues in over-the-counter financial markets. In credit markets, on the other hand, legislation has recently focused on *restricting* how much information lenders can demand or use from borrowers.

[4] As Chiappori et al (2006) put it, "there is a crying need for [a model] devoted to the interaction between imperfect competition and adverse selection."

severity of the adverse selection problem and the degree of imperfect competition. In particular, we show that equilibrium offers separate different types of agents in markets where competition is relatively intense or adverse selection is relatively severe, while they typically pool different types of agents in markets where principals have sufficient market power and adverse selection is sufficiently mild. Second, we explore how ex ante welfare responds to changes in the degree of competition and the severity of adverse selection. We show that increasing competition or reducing informational asymmetries is only welfare-improving in markets in which *both* market power is sufficiently concentrated and adverse selection is sufficiently severe.

Before explaining these results in greater detail, it is helpful to lay out the basic building blocks of the model. The agents in our model, whom we call "sellers," are endowed with a perfectly divisible good of either low or high quality; the quality of the good is the seller's *private information*. The principals, whom we call "buyers," offer menus containing price-quantity combinations to potentially *screen* high and low quality sellers.[5] Sellers can accept at most one contract, i.e., contracts are *exclusive.* To this otherwise canonical model of trade under asymmetric information, we introduce *imperfect competition* by endowing the buyers with some degree of market power. In particular, we assume that each seller receives offers from two buyers with probability π and from only one buyer with probability $1-\pi$. Importantly, when buyers make an offer, they are unsure whether the seller who receives it will receive an additional offer as well. This formulation allows us to capture the perfectly competitive case (a la Rothschild and Stiglitz (1976)) by setting $\pi = 1$, the monopsony case (a la Stiglitz (1977)) by setting $\pi = 0$, and everything in between.

For the general case of imperfect competition, with $\pi \in (0,1)$, the equilibrium involves buyers mixing over menus according to a non-degenerate distribution function.[6] Since each menu is comprised of two price-quantity pairs (one for each type), this implies that the main equilibrium object is a probability distribution over four-dimensional offers. A key contribution of our paper is developing a methodology that allows for a complete, yet tractable, characterization of this complicated equilibrium object.

We begin by showing that any menu can be summarized by the indirect utilities it offers to sellers of each type. This follows from two very general properties of equilibrium menus: first, the incentive

[5]The use of the labels "buyers" and "sellers" is merely for concreteness and corresponds most clearly with an asset market interpretation. These monikers can simply be switched in the context of an insurance market, so that the "buyers" of insurance are the agents with private information and the "sellers" of insurance are the principals.

[6]Mixing is to be expected for at least two reasons. First, this is a robust feature of nearly all models in which buyers are both monopsonists and Bertrand competitors with some probability, even without adverse selection or non-linear contracts. Second, even in perfectly competitive markets, it is well known that pure strategy equilibria may not exist in an environment with both adverse selection and non-linear contracts.

constraint of the low quality seller always binds; and second, the quantity traded by low quality sellers is never distorted. These properties reduce the dimensionality of the distribution from four to two. The next, and most important, step is to establish an additional property that must hold in any equilibrium: all menus offered in equilibrium are ranked in exactly the same way by both low- and high quality sellers. This property, which we call "strictly rank-preserving," simplifies the characterization considerably, as it implies that all equilibrium menus can be ranked along a single dimension. The equilibrium, then, can be described by a distribution function over a unidimensional variable—say, the indirect utility offered to low quality sellers—along with a strictly monotonic function mapping this variable to the indirect utility offered to the high quality seller.

This property allows us to provide a full analytical characterization of all equilibrium objects of interest, and to establish that the equilibrium is unique. Interestingly, our approach not only avoids the well-known problems with existence of equilibria in models of adverse selection and screening, but also requires no assumptions on off-path beliefs to get uniqueness. We then exploit this characterization to explore the implications—both positive and normative—of imperfect competition in markets suffering from adverse selection.

On the positive side, we find that the structure of menus offered in equilibrium depends on both the degree of competition, captured by π, and the severity of the adverse selection problem, which is succinctly summarized by a single statistic that is largest (i.e., adverse selection is most severe) when: (*i*) the fraction of low quality sellers is large; (*ii*) the potential surplus from trading with high quality sellers is small; and (*iii*) the information cost of separating the two types of sellers, as captured by the difference in their reservation values, is large. Given these summary statistics, we show that separating menus are more prevalent when competition is relatively strong or when adverse selection is relatively severe. Pooling menus, on the other hand, are more prevalent when competition is relatively weak and adverse selection is relatively mild. Interestingly, holding constant the severity of the adverse selection problem, the equilibrium may involve all pooling menus, all separating menus, or a mixture of the two, depending on the degree of competition. This finding suggests that attempts to infer the severity of adverse selection from the distribution of contracts that are traded should, indeed, take into account the extent to which the market is competitive.

Next we examine our model's implications for welfare, defined as the objective of a utilitarian social planner. In our context, this objective maps one-for-one to the expected quantity of high quality assets traded. We first study the relationship between welfare and the degree of competition. Our main finding

is that competition can worsen the distortions related to asymmetric information and therefore, can be detrimental to welfare. When adverse selection is mild, these negative effects are particularly stark: welfare is monotonically decreasing in π, with monopsony ($\pi = 0$) achieving the highest possible level of welfare. When adverse selection is severe, however, welfare is inverse U-shaped in π, i.e., an *interior* level of competition maximizes welfare.

To understand the hump-shape in welfare under severe adverse selection, note that an increase in competition induces buyers to allocate more of the surplus to sellers (of both types) in an attempt to retain market share. All else equal, increasing the utility offered to low quality sellers is good for welfare: by relaxing the low quality seller's incentive compatibility constraint, the buyer is able to exchange a larger quantity with high quality sellers. However, ceteris paribus, increasing the utility offered to high quality sellers is bad for welfare: it tightens the incentive constraint and forces buyers to trade less with high quality sellers. Hence, the net effect of an increase in competition on trade of high quality assets depends on whether the share of the surplus offered to high quality sellers rises faster or slower than that offered to low quality sellers.

When competition is low, buyers earn a disproportionate fraction of their profits from low quality sellers. Therefore, when buyers have lots of market power, an increase in competition leads to a faster increase in the utility offered to low quality sellers, since buyers care (relatively) more about retaining these sellers. As a result, trade with high quality sellers and welfare rise with competition. The opposite happens when competition is sufficiently high and profits come disproportionately from high quality sellers. In this case, increases in competition induce a faster increase in the offers to high quality sellers and, therefore, a decrease in expected trade and welfare. These results suggest that promoting competition—or instituting policies that have similar effects, such as price supports or minimum quantity restrictions—can lead to adverse effects on welfare in markets which are sufficiently competitive and face severe adverse selection.

Next, we study the welfare effects of providing buyers with more information—specifically, a noisy signal—about the seller's type. As in the case of increasing competition, the welfare effects of this perturbation depend on the severity of the two main frictions in the model, imperfect competition and adverse selection. When adverse selection is relatively mild or competition relatively strong, reducing informational asymmetries can actually be detrimental to welfare. The opposite is true when adverse selection and trading frictions are relatively severe. In sum, these normative results highlight how the interaction between these two frictions can have surprising implications for changes in policy (or

technological innovations), underscoring the need for a theoretical framework like ours.

Finally, we note that our baseline model was designed to be as simple as possible in order to focus on the novel interactions between adverse selection and imperfect competition. In sections 6 and 7, we analyze a number of relevant extensions and variants that demonstrate the robustness of our results, and also make our framework more amenable to applied work. First, we show how our analysis can be extended to a setting with an arbitrary number of types and contracts. Second, we relax our assumption of linear utility to analyze the canonical model of insurance under private information. Third, we allow the degree of competition to differ across sellers of different quality. Lastly, we show how to incorporate additional dimensions of heterogeneity, including horizontal and vertical differentiation.

Literature Review. Our paper contributes to a long literature on adverse selection. Our focus on contracts as screening devices puts us in the tradition of Rothschild and Stiglitz (1976), in contrast to the branch of the literature that restricts attention to single price contracts, as in the original model of Akerlof (1970). Most of the literature that studies adverse selection and screening has either assumed a monopolistic or perfectly competitive market structure.[7]

The main novelty of our analysis is to synthesize a standard model of adverse selection and unrestricted contracts with the search-theoretic model of imperfect competition developed by Butters (1977), Varian (1980), and, in particular, Burdett and Judd (1983). While this model of imperfect competition has been used extensively in both theoretical and empirical work,[8] to the best of our knowledge none of these papers address adverse selection and screening.[9] A recent paper by Garrett et al. (2014) exploits the Burdett and Judd (1983) model in an environment with screening contracts and asymmetric information, but the asymmetric information is over the agents' *private values*. This key difference implies that the role of screening—and how it interacts with imperfect competition—is ultimately very different in our paper and theirs.[10]

More closely related to our work is the literature that studies adverse selection and nonlinear contracts in an environment with *competitive search*—most notably the influential paper by Guerrieri et al.

[7] For recent contributions to this literature that assume perfectly competitive markets, see, e.g., Bisin and Gottardi (2006), Chari et al. (2014), and Azevedo and Gottlieb (2014).

[8] For recent examples, see, e.g., Sorensen (2000) and Kaplan and Menzio (2015).

[9] Carrillo-Tudela and Kaas (2015) analyze a related labor market setting with adverse selection using the on-the-job search model of Burdett and Mortensen (1998), but their focus is quite different from ours.

[10] In particular, with private values, screening is useful only for rent extraction. Competition reduces (and ultimately, eliminates) these rents and, along with them, incentives to screen. In contrast, with common values, screening plays a central role in mitigating the adverse selection problem. As a result, it disappears only when that problem is sufficiently mild - increased competition serves to strengthen incentives to separate. This interaction is also the source of non-monotonic effects on welfare from increased competition. With private values, on the other hand, welfare unambiguously increases with competition.

(2010).[11] In that paper, principals post contracts and match bilaterally with agents who direct their search efforts towards specific contracts. A matching technology determines the probability that each agent trades (or is rationed) in equilibrium, as a function of the relative measures of principals offering a specific contract and agents searching for it.

As in our paper, Guerrieri et al. (2010) present an explicit model of trade without placing any restrictions on contracts, beyond those arising from the primitive frictions. There are, however, several important differences. The first relates to the role of search frictions. In our analysis, the focus is on market power—the interaction between the resulting distortions and the underlying adverse selection problem is the central focus of this paper. Guerrieri et al. (2010) and others, on the other hand, focus on the role of search frictions in providing incentives (through the probability of trade) and not on market power *per se*. Second, depending on parameters, our equilibrium menus can be pooling, separating or a combination of both; the approach in Guerrieri et al. (2010), on the other hand, always leads to separating equilibria. In this sense, our approach has the potential to speak to a richer set of observed outcomes. Finally, we obtain a unique equilibrium without additional assumptions or refinements, whereas uniqueness in Guerrieri et al. (2010) relies on a restriction on off-equilibrium beliefs.[12]

An alternative approach to modeling imperfect competition is through product differentiation, as in Villas-Boas and Schmidt-Mohr (1999) and, more recently, Benabou and Tirole (2014), Veiga and Weyl (2012), Mahoney and Weyl (2014), and Townsend and Zhorin (2014). Identical contracts offered by various principals are valued differently by agents due to an orthogonal attribute, which is interpreted as "distance" in a Hotelling interpretation or "taste" in a random utility, discrete choice framework. This additional dimension of heterogeneity is the source of market power, and changes in competition are induced by varying the importance of this alternative attribute, i.e. by altering preferences. We take a different approach to modeling (and varying) competition, which holds constant preferences and therefore, the potential social surplus. It is also worth pointing out a few key differences in substantive results, particularly about the desirability of competition. In Benabou and Tirole (2014), a trade-off from increased competition arises not due to adverse selection *per se*, but from the need to provide incentives to allocate effort between multiple, imperfectly observable or contractible tasks. In fact, without multi-tasking, competition improves welfare even with asymmetric information. This is also the case in

[11] Other papers studying adverse selection with competitive search include Guerrieri and Shimer (2014a,b), Chang (2012), Michelacci and Suarez (2006) and Kim (2012).

[12] Still another related literature studies adverse selection in dynamic models with search frictions, where separation is achieved by having agents of different types trade at different points in time. See, e.g., Inderst (2005), Moreno and Wooders (2010), Camargo and Lester (2014), and the references therein. In all of these papers, agents are assumed to trade linear contracts.

Mahoney and Weyl (2014), where attention is restricted to single-price contracts. Veiga and Weyl (2012) also restrict attention to a single contract, but with endogenous "quality" and find that it is maximized under monopoly. In our setting, depending on parameters, competition can be beneficial or harmful. Though there are a number of differences between their setup and ours (multi-dimensional heterogeneity, contract space, equilibrium concept), which precludes a direct comparison, we interpret their results as providing a distinct but complementary insight about the interaction between competition and adverse selection.

The rest of the paper is organized as follows. Section 2 describes our model. Section 3 proves key properties of the equilibrium, followed by its construction in section 4. Section 5 contains implications for welfare and policy. Sections 6 extends the analysis to multiple types, while section 7 performs a number of robustness exercises. Section 8 concludes, and all proofs can be found in the Appendix.

2 Model

Environment. We consider an economy populated by a measure of sellers and a measure of buyers. Each seller is endowed with a single unit of a perfectly divisible good. A fraction $\mu_l \in (0,1)$ of sellers posses a low (l) quality good, while the remaining fraction $\mu_h = 1 - \mu_l$ possess a high (h) quality good. Buyers and sellers derive utility v_i and c_i, respectively, from consuming each unit of a quality $i \in \{l, h\}$ good, with $v_l < v_h$ and $c_l < c_h$. We assume that

$$v_i > c_i \quad \text{for } i \in \{l, h\}, \tag{1}$$

so that there are gains from trading both high and low quality goods.

There are two types of frictions in the market. First, there is *asymmetric information*: sellers observe the quality of the good they possess while buyers do not, though the probability μ_i that a randomly selected good is quality $i \in \{l, h\}$ is common knowledge. In order to generate the standard "lemons problem," we focus on the case in which

$$v_l < c_h. \tag{2}$$

The second type of friction is a *search* friction: the buyers in our model post offers, but sellers only sample a finite number of these offers. In particular, we assume that each seller samples one offer with probability $1 - \pi$ and two offers with probability π. Throughout the paper, we refer to sellers with one offer as "captive," while we refer to those with two offers as "non-captive" sellers. Trading is exclusive, though: a seller can accept at most one offer, even when two offers are available.

A buyer's offer is a "menu" which contains a collection of "contracts." Each contract is a pair (x, t), where x denotes the quantity of asset to be exchanged and t denotes the transfer from the buyer to the seller. In our environment, buyers offer menus with two contracts, $\{(x_l, t_l), (x_h, t_h)\} \in \left([0,1] \times \mathbb{R}_+\right)^2$, where (x_i, t_i) is the contract intended for a seller of type $i \in \{l, h\}$. In Appendix A.1, we prove that a buyer cannot gain by offering a more complicated deterministic mechanism, so that the equilibrium we construct is also an equilibrium of a game where buyers can offer arbitrary deterministic mechanisms.[13]

Payoffs. A seller who owns a quality i good and accepts a contract (x, t) receives a payoff

$$t + (1 - x)c_i$$

while a buyer who acquires a quality i good at terms (x, t) receives a payoff

$$-t + xv_i.$$

Meanwhile, a seller with a quality i good who does not trade receives a payoff c_i, while a buyer who does not trade receives zero payoff.

Strategies and Definition of Equilibrium. Let $\mathbf{z}_i = (x_i, t_i)$ denote the contract that is intended for a seller of type $i \in \{l, h\}$, and let $\mathbf{z} = (\mathbf{z}_l, \mathbf{z}_h)$. A buyer's strategy, then, is a distribution across menus, $\Phi \in \Delta\big(([0,1] \times \mathbb{R}_+)^2\big)$. A seller's strategy is much simpler: given the available menus, a seller should choose the menu with the contract that maximizes her payoffs, or mix between menus if she is indifferent. Of course, conditional on a menu, the seller chooses the contract which maximizes her payoffs. In what follows, we will take the seller's optimal behavior as given.

A symmetric equilibrium is thus a distribution $\Phi^\star(\mathbf{z})$ such that:

1. *Incentive compatibility*: for almost all $\mathbf{z} = \{(x_l, t_l), (x_h, t_h)\}$ in the support of $\Phi^\star(\mathbf{z})$,

$$t_l + c_l(1 - x_l) \geqslant t_h + c_l(1 - x_h) \tag{3}$$

$$t_h + c_h(1 - x_h) \geqslant t_l + c_h(1 - x_l). \tag{4}$$

[13] Sellers in our model have private information along two dimensions: the quality of their good and the alternative offer available to them. A natural question, then, is whether buyers can screen sellers along both dimensions, i.e., offer contracts that depend on both the asset quality and the availability (and details) of alternative menus. In Appendix A.1, we show that the answer is "no" within the set of deterministic mechanisms: since the seller's payoff conditional on accepting a contract is independent of her outside offer, it is impossible to screen along this dimension without the ability to commit to a stochastic mechansim.

2. *Buyer's optimize*: for almost all $\mathbf{z} = \{(x_l, t_l), (x_h, t_h)\}$ in the support of $\Phi^\star(\mathbf{z})$,

$$\mathbf{z} \in \arg\max_{\mathbf{z}} \sum_{i \in \{l,h\}} \mu_i(v_i x_i - t_i) \left[1 - \pi + \pi \int_{\mathbf{z}'} \chi_i(\mathbf{z}, \mathbf{z}') \Phi^\star(d\mathbf{z}')\right], \tag{5}$$

where

$$\chi_i(\mathbf{z}, \mathbf{z}') = \left\{ \begin{array}{c} 0 \\ \frac{1}{2} \\ 1 \end{array} \right\} \quad \text{if} \quad t_i + c_i(1 - x_i) \left\{ \begin{array}{c} < \\ = \\ > \end{array} \right\} t_i' + c_i(1 - x_i'). \tag{6}$$

The function χ_i reflects the seller's optimal choice. We have assumed that if the seller is indifferent between menus, then she chooses among menus with equal probability. Within a given menu, we have assumed that sellers do not randomize; for any incentive compatible contract, sellers choose the contract intended for their type, as in most of the mechanism design literature (see, e.g., Myerson (1979), Dasgupta et al. (1979)).

3 Properties of Equilibria

Characterizing the equilibrium described above requires solving for a distribution over four-dimensional menus. In this section, we establish a series of results that reduce the dimensionality of the equilibrium characterization.

First, we show that each menu offered by a buyer can be summarized by the indirect utilities that it delivers to each type of seller, so that equilibrium strategies can in fact be defined by a joint distribution over two-dimensional objects, i.e., pairs of indirect utilities. Then, we establish that the marginal distributions of offers intended for each type of seller are well-behaved, i.e., that they have fully connected support and no mass points. Finally, we establish that there is a very precise link between the two contracts offered by any buyer, which imposes even more structure on the joint distribution of offers. In particular, we show that any two menus that are offered in equilibrium are ranked in exactly the same way by both low and high type sellers; that is, one menu is strictly preferred by a low type seller if and only if it is also preferred by a high type seller. This property of equilibria, which we call "strictly rank-preserving," simplifies the characterization even more, as the marginal distribution of offers for high quality sellers can be expressed as a strictly monotonic transformation of the marginal distribution of offers for low quality sellers.

3.1 Utility Representation

As a first step, we establish two results that imply any menu can be summarized by two numbers, (u_l, u_h), where

$$u_i = t_i + c_i(1 - x_i) \tag{7}$$

denotes the utility received by a type $i \in \{l, h\}$ seller from accepting a contract $\mathbf{z}_i$.

Lemma 1. *In any equilibrium, for almost all* $\mathbf{z}$ *in the support of* $\Phi^\star$*, it must be that* $x_l = 1$ *and* $t_l = t_h + c_l(1 - x_h)$.

In words, Lemma 1 states that all equilibrium menus require that low quality sellers trade their entire endowment, and that their incentive compatibility constraint always binds. This is reminiscent of the "no-distortion-at-the-top" result in the taxation literature, or that of full-insurance for the high-risk agents in Rothschild and Stiglitz (1976).

Corollary 2. *In equilibrium, any menu of contracts* $\{(x_l, t_l), (x_h, t_h)\} \in \left([0,1] \times \mathbb{R}_+\right)^2$ *can be summarized by a pair* (u_l, u_h) *with* $x_l = 1$, $t_l = u_l$,

$$x_h = 1 - \frac{u_h - u_l}{c_h - c_l}, \text{ and} \tag{8}$$

$$t_h = \frac{u_l c_h - u_h c_l}{c_h - c_l}. \tag{9}$$

Notice that, since $0 \leqslant x_h \leqslant 1$, feasibility requires that the pair (u_l, u_h) satisfies

$$c_h - c_l \geqslant u_h - u_l \geqslant 0. \tag{10}$$

In what follows, we will often refer to the requirement $u_h \geqslant u_l$ as a "monotonicity constraint." Note that, when this constraint binds, Corollary 2 implies that $x_h = 1$ and $t_h = t_l$.

3.2 Recasting the Buyer's Problem and Equilibrium

Buyer's Problem. Lemma 1 and Corollary 2 allow us to recast the problem of a representative buyer as choosing a menu of indirect utilities, (u_l, u_h), taking as given the distribution of indirect utilities offered by other buyers. For any menu (u_l, u_h), buyers must infer the probability that the menu will be accepted by a type $i \in \{l, h\}$ seller. In order to calculate these probabilities, let us define the marginal distributions

$$F_i(u_i) = \int_{\mathbf{z}_i'} \mathbf{1}\left[t_i' + c_i\left(1 - x_i'\right) \leqslant u_i\right] \Phi\left(d\mathbf{z}_i'\right)$$

for $i \in \{l, h\}$. In words, $F_l(u_l)$ and $F_h(u_h)$ are the probability distributions of indirect utilities arising from each buyer's mixed strategy. When these distributions are continuous and have no mass points,

the probability that a contract intended for a type i seller is accepted is simply $1-\pi+\pi F_i(u_i)$, i.e., the probability that the seller is captive plus the probability that he is non-captive but receives another offer less than u_i. However, if $F_i(\cdot)$ has a mass point at u_i, then the fraction of non-captive sellers of type i attracted to a contract with value u_i is given by $\tilde{F}_i(u_i) = \frac{1}{2}F_i^-(u_i) + \frac{1}{2}F_i(u_i)$, where $F_i^-(u_i) = \lim_{u \nearrow u_i} F_i(u)$ is the left limit of F_i at u_i.Given $\tilde{F}_i(\cdot)$, each buyer solves

$$\max_{u_l \geqslant c_l,\ u_h \geqslant c_h} \quad \mu_l\left(1-\pi+\pi\tilde{F}_l(u_l)\right)\Pi_l(u_l,u_h) + \mu_h\left(1-\pi+\pi\tilde{F}_h(u_h)\right)\Pi_h(u_l,u_h) \tag{11}$$

$$\text{s. t.} \quad c_h - c_l \geqslant u_h - u_l \geqslant 0, \tag{12}$$

with

$$\Pi_l(u_l,u_h) \equiv \nu_l x_l - t_l = \nu_l - u_l \tag{13}$$

$$\Pi_h(u_l,u_h) \equiv \nu_h x_h - t_h = \nu_h - u_h\frac{\nu_h - c_l}{c_h - c_l} + u_l\frac{\nu_h - c_h}{c_h - c_l}. \tag{14}$$

In words, $\Pi_i(u_l,u_h)$ is the buyer's payoff conditional on the offer u_i being accepted by a type i seller. We refer to the objective in (11) as $\Pi(u_l,u_h)$.

Before proceeding, note that $\Pi_h(u_l,u_h)$ is increasing in u_l: by offering more utility to low quality sellers, the buyer relaxes the incentive constraint and can earn more profits when he trades with high quality sellers. As a result, one can easily show that the profit function $\Pi(u_l,u_h)$ is (at least) weakly supermodular. This property will be important in several of the results we establish below.

Equilibrium. Using the optimization problem described above, we can redefine the equilibrium in terms of the distributions of indirect utilities. In particular, for each u_l, let

$$\begin{aligned} U_h(u_l) = \arg\max_{u_h' \geqslant c_h} \quad & \Pi\left(u_l,u_h'\right) \\ \text{s.t.} \quad & c_h - c_l \geqslant u_h' - u_l \geqslant 0. \end{aligned}$$

The equilibrium can then be described by the marginal distributions $\{F_i(u_i)\}_{i\in\{l,h\}}$ together with the requirement that a joint distribution function must exist. In other words, a probability measure Φ over the set of feasible (u_l,u_h)'s must exist such that, for each $u_l > u_l'$ and $u_h > u_h'$

$$1 = \Phi\left(\{(\hat{u}_l,\hat{u}_h)\,;\hat{u}_h \in U_h(\hat{u}_l)\},\hat{u}_l \in [c_l,\nu_h]\right)$$

$$F_l^-(u_l) - F_l\left(u_l'\right) = \Phi\left(\left\{(\hat{u}_l,\hat{u}_h)\,;\hat{u}_h \in U_h(\hat{u}_l)\,,\hat{u}_l \in \left(u_l',u_l\right)\right\}\right), \tag{15}$$

$$F_h^-(u_h) - F_h\left(u_h'\right) = \Phi\left(\left\{(\hat{u}_l,\hat{u}_h)\,;\hat{u}_h \in U_h(\hat{u}_l)\,,\hat{u}_h \in \left(u_h',u_h\right)\right\}\right). \tag{16}$$

Note that this definition of equilibrium imposes two different requirements. The first is that buyers behave optimally: for each u_l, the joint probability measure puts a positive weight only on $u_h \in U_h(u_l)$. The second is aggregate consistency: the fact that F_l and F_h are marginal distributions associated with a joint measure of menus.

3.3 Basic Properties of Equilibrium Distributions

In this section, we establish that, in equilibrium, the distributions $F_l(u_l)$ and $F_h(u_h)$ are continuous and have connected support, i.e., there are neither mass points nor gaps in either distribution.

Proposition 3. *The marginal distributions F_l and F_h have connected support. They are also continuous, with the possible exception of a mass point in F_l at v_l.*

As in Burdett and Judd (1983), the proof of Proposition 3 rules out gaps and mass points in the distribution by constructing profitable deviations. A complication that arises in our model is that payoffs are interdependent, e.g., a change in the utility offered to low quality sellers changes the contract—and hence the profits—that a buyer receives from high quality sellers. We prove these properties sequentially, first for F_h and then for F_l. We sketch the proofs here, and present the formal arguments in the Appendix.

To see that F_h has connected support, suppose towards a contradiction that it is constant on some interval, and consider the equilibrium menu (u_l', u_h'), with u_h' equal to the upper bound of this interval. If either $u_l' < u_h'$ or F_l is constant over an interval containing u_l', then one can construct a profitable deviation by considering a slight decrease in u_h; as in Burdett and Judd (1983), this deviation is profitable because it attracts the same fraction of high quality sellers but makes more profit per trade.[14] The novel case to consider, then, is when $u_l' = u_h'$ and F_l has full support over the interval containing u_l'. In this case, a decrease in u_h will increase the payoff from trading with high quality sellers, but may decrease the payoff from trading with low quality sellers since the monotonicity constraint requires that u_l also declines. Using the weak supermodularity of the profit function discussed above, we show that the benefits from decreasing u_h must outweigh the costs of decreasing u_l, and hence a profitable deviation exists.

To see that F_h has no mass points, suppose towards a contradiction that it has a mass point at u_h' for some equilibrium menu (u_l', u_h'). Again, if $\Pi_h(u_l', u_h')$ is strictly positive, then the logic of finding a profitable deviation is very close to that in Burdett and Judd (1983): a small increase in u_h' will increase profits by attracting a mass of high quality sellers. The novel case that we must consider, then, is when

[14] When F_l is constant over an interval containing u_l', the profitable deviation also requires a decrease in u_l.

profits from the high quality sellers are non-positive. In this case, we show that the buyer must not be trading with high quality sellers at all—otherwise, a decrease in u_h would be both feasible and profitable.[15] From (8), if $u'_h = c_h$ and the quantity traded with high types is zero, then it must be that $u'_l = c_l$. Moreover, since we assumed a mass point at $u'_h = c_h$, then it must be that there is a mass point at $u'_l = c_l$. However, in this case, an increase in u_l is a feasible deviation and increases profits, which completes the contradiction.

Having shown that F_h is continuous and strictly increasing, we then apply an inductive argument to prove that F_l has connected support and is continuous, with a possible exception at the lower bound of the support. An important step in the induction argument, which we later use more generally, is to show that the objective function $\Pi(u_l, u_h)$ is strictly supermodular. We state this here as a lemma.

Lemma 4. *Suppose F_h has connected support and is continuous over its support. Then the profit function is strictly supermodular so that*

$$\Pi(u_{l1}, u_{h1}) + \Pi(u_{l2}, u_{h2}) \geqslant \Pi(u_{l2}, u_{h1}) + \Pi(u_{l1}, u_{h2}), \ \forall u_{i1} \geqslant u_{i2}, \ i \in \{l, h\}$$

with strict inequality when $u_{i1} > u_{i2}, \ i \in \{l, h\}$.

As noted above, the supermodularity of the buyer's profit function reflects a basic complementarity between the indirect utilities offered to low and high quality sellers. An important implication of this result is that the correspondence $U_h(u_l)$ is weakly increasing. We use this property to construct deviations similar to those described above in order to rule out gaps and mass points in the distribution F_l almost everywhere in its support; later, in Section 4, we show that these mass points only occur in a knife-edge case. Hence, generically, the marginal distribution F_l has connected support and no mass points everywhere in its support.

3.4 Strict rank-preserving

In this section, we establish that every equilibrium has the property that the menus being offered are *strictly rank-preserving*—that is, low and high quality sellers share the same ranking over the set of menus offered in equilibrium—with the possible exception of the knife-edge case discussed above. We prove this result by showing that the mapping between a buyer's optimal offer to low and high quality sellers, $U_h(u_l)$, is a well-defined, strictly increasing function. We start with the following definition.

[15] More precisely, in the candidate equilibrium under consideration, it must be the $u'_h > u'_l$, which ensures that a decline in u_h would not effect the profits earned from low quality sellers. If this were not true—i.e., if $u'_h = u'_l$—then the buyer would be making non-negative profits from both types, which cannot be true when $\pi \in (0, 1)$.

Definition 5. *For any subset* $\mathcal{U}_l$ *of* $\text{Supp}(F_l)$*, an equilibrium is* ***strictly rank-preserving over*** $\mathcal{U}_l$ *if the correspondence* $U_h(u_l)$ *is a strictly increasing function of* u_l *for all* $u_l \in \mathcal{U}_l$*. An equilibrium is* ***strictly rank-preserving*** *if it is strictly rank-preserving over* $\text{Supp}(F_l)$*.*

Equivalently, an equilibrium is strictly rank-preserving when, for any two points in the equilibrium support (u_l, u_h) and (u_l', u_h'), $u_l > u_l'$ if and only if $u_h > u_h'$. Given this terminology, we can now establish one of our key results.

Theorem 6. *All equilibria are strictly rank-preserving over the set* $\text{Supp}(F_l) \setminus \{v_l\}$*.*

As we now describe, Theorem 6 follows from the facts established above. In particular, the strict supermodularity of $\Pi(u_l, u_h)$ implies that $U_h(u_l)$ is a weakly increasing correspondence. However, since $F_l(\cdot)$ and $F_h(\cdot)$ are strictly increasing and continuous, we show that $U_h(u_l)$ can neither be multi-valued nor have flats. Intuitively, if there exists a $u_l > \underline{u}_l$ and $u_h' > u_h$ such that $u_h, u_h' \in U_h(u_l)$, then the supermodularity of $\Pi(u_l, u_h)$ implies that $[u_h, u_h'] \subset U_h(u_l)$. Since $F_h(\cdot)$ has connected support, if U_h were a correspondence for some u_l, then this would imply that $F_l(\cdot)$ must have a mass point at u_l, which contradicts Proposition 3. Similarly, if there exists u_h and $u_l' > u_l$ offered in equilibrium such that $U_h(u_l') = U_h(u_l) = u_h$, then F_h would feature a mass point, in contradiction with Proposition 3. Hence, $U_h(u_l)$ must be a strictly increasing *function* for all $u_l > \underline{u}_l$.

Notice that, if $F_l(\cdot)$ is continuous everywhere, then every menu offered in equilibrium is accepted by exactly the same fraction of low and high quality non-captive sellers. We state this result in the following Corollary to Theorem 6.

Corollary 7. *If* F_l *and* F_h *are continuous, then* $F_h(U_h(u_l)) = F_l(u_l)$*.*

Taken together, Theorem 6 and Corollary 7 simplify the construction of an equilibrium, which we undertake in the next Section. Specifically, when an equilibrium exists in which the marginal distributions F_l and F_h are continuous, then the equilibrium can be described compactly by the marginal distribution F_l and the policy function $U_h(u_l)$.

4 Construction of Equilibrium

In this section, we use the properties established above to help construct equilibria. Then we show that the equilibrium we construct is unique. In this sense, we characterize the entire set of equilibrium outcomes in our model.

4.1 Special Cases: Monopsony and Perfect Competition

To fix ideas, we first characterize equilibria in the well-known special cases of $\pi = 0$ and $\pi = 1$, i.e., when sellers face a monopsonist and when they face two buyers in Bertrand competition, respectively. As we will see, several features of the equilibrium in these two extreme cases guide our construction of equilibria for the general case of $\pi \in (0, 1)$.

Monopsony. When each seller meets with at most one buyer, the buyers solve

$$\max_{(u_l, u_h)} \quad \mu_l(v_l - u_l) + \mu_h \left[v_h - u_h \frac{v_h - c_l}{c_h - c_l} + u_l \frac{v_h - c_h}{c_h - c_l} \right],$$

subject to the monotonicity and feasibility constraints in (12). The solution to this problem, summarized in Lemma 8 below, is standard and hence we omit the proof.

Lemma 8. *Suppose $\pi = 0$, and let*

$$\phi_l \equiv 1 - \frac{\mu_h}{\mu_l} \left(\frac{v_h - c_h}{c_h - c_l} \right). \tag{17}$$

If $\phi_l > 0$, then the unique equilibrium has $u_l = c_l$ with $x_l = 1$ and $u_h = c_h$ with $x_h = 0$; if $\phi_l < 0$, then $u_l = u_h = c_h$ with $x_l = x_h = 1$; and if $\phi_l = 0$, then $u_l \in [c_l, c_h]$ with $x_l = 1$ and $u_h = c_h$ with $x_h \in [0, 1]$.

The parameter ϕ_l is a summary statistic for the adverse selection problem: it represents the net marginal cost (to the buyer) of delivering an additional unit of utility to a low quality seller. It is strictly less than 1 because the direct cost of an additional unit of transfer to a low quality seller is partially offset by the indirect benefit of relaxing this seller's incentive constraint, which allows the buyer to trade more with a high quality seller. This indirect benefit is captured by the second term on the right hand side: when this term is large, ϕ_l is small, the cost of trading with high quality sellers is low, and adverse selection is *mild*. Conversely, when this term is small, ϕ_l is large, it is costly to trade with high quality sellers, and therefore adverse selection is relatively *severe*. According to this measure, adverse selection is thus severe when the relative fraction of high quality sellers, μ_h/μ_l, is large; the gains from trading with high quality sellers, $v_h - c_h$, are relatively large; and/or the information rents associated with separating high and low quality sellers, $c_h - c_l$, are small.

When ϕ_l is positive, the net cost to a buyer of increasing u_l is positive, so she sets u_l as low as possible, i.e. $u_l = c_l$. This implies that the high quality seller is entirely shut out, i.e. $x_h = 0$. Otherwise, when $\phi_l < 0$, increasing u_l yields a net benefit to the buyer. As a result, a buyer raises u_l until the monotonicity constraint in (12) binds, i.e., she pools high and low quality sellers, offering $u_h = u_l = c_h$.

Before proceeding to the perfectly competitive case, we highlight two features of the equilibrium under monopsony. First, buyers offer *separating menus* ($u_h > u_l$) when ϕ_l is positive and *pooling menus* ($u_h = u_l$) when ϕ_l is negative. Second, they make non-negative payoffs on both types when $\phi_l > 0$, but lose money on low quality sellers when $\phi_l < 0$. In other words, the equilibrium features *cross-subsidization* when $\phi_l < 0$, but not when $\phi_l > 0$.

Bertrand Competition. When competition is perfect, i.e., when $\pi = 1$, our setup becomes the same as that in Rosenthal and Weiss (1984), and similar to that of Rothschild and Stiglitz (1976). In this case, when $\phi_l \geqslant 0$, the unique equilibrium is in pure strategies, with buyers offering the standard "least-cost separating" contract; type l sellers earn $u_l = v_l$ and type h sellers trade a fraction of their endowment at a unit price of v_h, such that the incentive constraint of the low quality seller binds. However, when $\phi_l < 0$, there is no pure strategy equilibrium.[16] In this case, an equilibrium in mixed strategies emerges, as in Dasgupta and Maskin (1986) and Rosenthal and Weiss (1984).[17] Each buyer mixes over menus, all of which involve negative profits from low quality sellers, offset exactly by positive profits from high quality sellers, leading to zero profits. The marginal distribution $F_l(\cdot)$ is such that profitable deviations are ruled out. The following lemma summarizes these results.

Lemma 9. *When $\pi = 1$, the unique equilibrium is as follows: (i) if $\phi_l \geqslant 0$, then $u_l = v_l$ with $x_l = 1$ and $u_h = \frac{v_h(c_h - c_l) + v_l(v_h - c_h)}{v_h - c_l}$ with $x_h = \frac{v_l - c_l}{v_h - c_l}$; (ii) if $\phi_l < 0$, then the symmetric equilibrium is described by the distribution*

$$F_l(u_l) = \left(\frac{u_l - v_l}{\mu_h (v_h - v_l)}\right)^{-\phi_l}, \tag{18}$$

with $\text{Supp}(F_l) = [v_l, \bar{v}]$ and $F_h(u_h) = F_l(U_h(u_l))$, where $\bar{v} = \mu_h v_h + \mu_l v_l$ and $U_h(u_l)$ satisfies

$$\mu_h \Pi_h(u_l, U_h(u_l)) + \mu_l \Pi_l(u_l, U_h(u_l)) = 0. \tag{19}$$

As with $\pi = 0$, equilibrium when $\pi = 1$ features no cross-subsidization when $\phi_l \geqslant 0$ and cross-subsidization when $\phi_l < 0$. However, unlike the case with $\pi = 0$, equilibrium with $\pi = 1$ features separating contracts for all values of ϕ_l. These properties guide our construction of equilibria in the next section, when we study the general case of $\pi \in (0, 1)$.

[16] All buyers offering the least-cost separating contract cannot be an equilibrium, as a pooling offer attracts both types and yields positive profits to the buyer. All buyers offering pooling cannot be an equilibrium either, since it is vulnerable to a cream-skimming deviation, wherein a competing buyer can draw away the high quality seller by offering a contract with $x < 1$ but at a higher price.

[17] Luz (2014) shows that the equilibrium is unique.

4.2 General Case: Imperfect Competition

We now describe how to construct equilibria when $\pi \in (0,1)$. Recall that an equilibrium is summarized by a distribution $F_l(u_l)$ and a strictly increasing function $U_h(u_l)$. A key determinant of the structure of equilibrium menus is whether the monotonicity constraint in (12) is binding. When it is slack, the local optimality (or first-order) condition for u_l, along with the strict rank-preserving condition that relates $F_h(U_h(u_l)) = F_l(u_l)$ together characterize the equilibrium distribution $F_l(u_l)$. The function $U_h(u_l)$ then follows from the requirement that all menus $(u_l, U_h(u_l))$ must yield the buyer equal profits. When the monotonicity constraint is binding, the policy function is, by definition, $U_h(u_l) = u_l$.[18]

Our analysis of $\pi = 0$ and $\pi = 1$ points to the importance of ϕ_l. Recall that when $\phi_l > 0$, the monotonicity constraint was always slack. When $\phi_l < 0$, on the other hand, the monotonicity constraint was binding only when $\pi = 0$ and slack at $\pi = 1$. Guided by these results, we discuss our construction separately for the $\phi_l > 0$ and the $\phi_l < 0$ cases.[19]

Case 1: $\phi_l > 0$**.** For this case, in line with the analysis of $\pi = 0$ and $\pi = 1$, we conjecture that, for any $\pi \in (0,1)$, the monotonicity constraint is slack, i.e., that $U_h(u_l) > u_l$ for all $u_l \in \text{Supp}(F_l)$. Proposition 10 establishes that this is indeed the case.

Proposition 10. *For any $\pi \in (0,1)$ and $\phi_l > 0$, there exists an equilibrium where F_l and U_h satisfy the following properties:*

1. *F_l solves the differential equation*

$$\frac{\pi f_l(u_l)}{1-\pi+\pi F_l(u_l)}(v_l - u_l) = \phi_l\,, \tag{20}$$

   with the boundary condition $F_l(c_l) = 0$.

2. *$U_h(u_l) > u_l$ and satisfies the equal profit condition:*

$$(1-\pi+\pi F_l(u_l))\left[\mu_h \Pi_h(u_l, U_h(u_l)) + \mu_l(v_l - u_l)\right] = \mu_l(1-\pi)(v_l - c_l)\,. \tag{21}$$

Equation (20) is derived by taking the first-order condition of (11) with respect to u_l—holding u_h fixed—and then imposing the strict rank-preserving property.[20] This necessary condition is familiar from

[18] Of course, $u_h = U_h(u_l)$ must be locally optimal as well, but this condition is implied by the joint requirements on u_l and $U_h(u_l)$ described above.

[19] The equilibrium when $\phi_l = 0$ has a slightly different structure and, for the sake of brevity, we relegate analysis of this knife-edge case to Appendix A.9.

[20] As we discuss in the proof of Proposition 10, this first order condition requires three assumptions: that $u_h > u_l$ for all menus; that there is no mass point at the lower bound of the support of $F_l(u_l)$; and that the implied quantity traded by the high quality seller is interior in all trades, i.e., $0 < x_h = (u_h - u_l)/(c_h - c_l) < 1$, except possibly at the boundary of the support of F_l. All of these assumptions are confirmed in equilibrium.

basic production theory. The left-hand side is the marginal benefit to the buyer of increasing u_l, i.e., the product of the semi-elasticity of demand and the profit per trade. The right-hand side, ϕ_l, represents the marginal cost of increasing the utility of the low quality seller, taking into account the fact that increasing u_l relaxes the incentive constraint. Note that, even though (20) ensures that local deviations by a buyer from an equilibrium menu are not profitable, completing the proof requires ensuring that there are no profitable global deviations as well; we establish that this is true in Appendix A.4.

The boundary condition requires that the lowest utility offered to the low quality seller is c_l. From (21), and using the fact that $F_l(c_l) = 0$, we find $U_h(c_l) = c_h$, so that the worst menu offered in equilibrium coincides with the monopsony outcome. Loosely speaking, if the worst menu offers more utility to low quality sellers than c_l, the buyer could profit by decreasing u_l and u_h—the gains associated with trading at better terms with the low types would exceed the losses associated from trading less quantity with high types, precisely because $\phi_l > 0$. Given that $\underline{u}_l = c_l$, if the worst equilibrium menu offers more utility to high quality sellers than c_h, then a buyer offering this menu could profit by decreasing u_h—his payoff from trading with high types would increase without changing the payoffs from trading with low types.

The final equilibrium object, $U_h(u_l)$, is characterized by the equal profit condtion: the left side of (21) defines the buyer's payoff from the menu $(u_l, U_h(u_l))$, while the right side is the profit earned from the worst contract offered in equilibrium. Figure 1 plots the two equilibrium functions in this region.

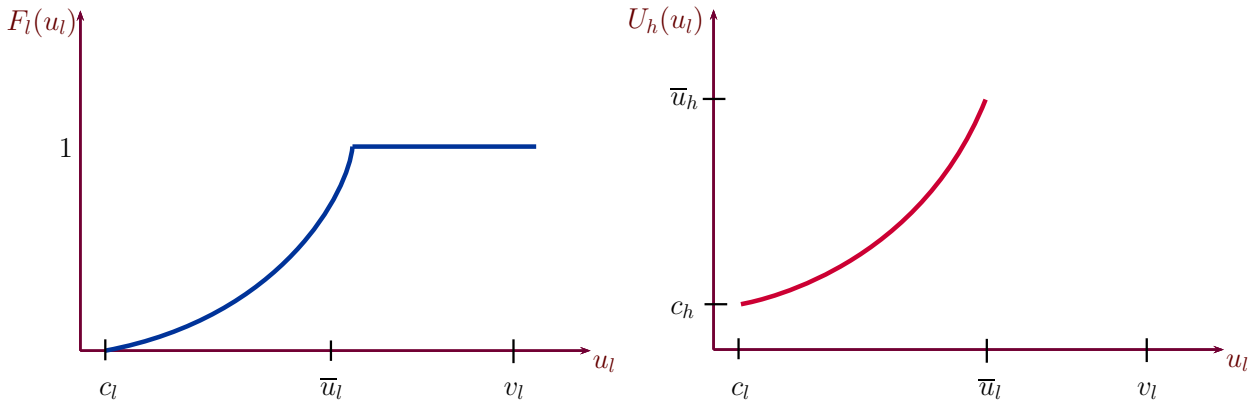

Figure 1: Equilibrium for $\pi \in (0,1)$, $\phi_l > 0$. The left panel plots the CDF $F_l(u_l)$ and the right panel plots the mapping $U_h(u_l)$.

Notice from (20) that, since $\phi_l > 0$, our equilibrium has $v_l > u_l$ for all menus in equilibrium, so that buyers earn strictly positive profits from trading with low quality sellers. It is straightforward to show that buyers also earn strictly positive profits from trading with high quality sellers. Hence, in this region,

the equilibrium features no cross-subsidization, as was the case for $\pi = 0$ and $\pi = 1$. Finally, it is also worth noting that the equilibrium distribution of offers converges to the limiting cases as π converges to both 0 and 1; in the former case, the distribution converges to a mass point at the monopsony outcome, while in the latter case, the distribution converges to a mass point at the least-cost separating outcome.

Case 2: $\phi_l < 0$**.** In this region of the parameter space, the equilibrium features a pooling menu when $\pi = 0$ and a distribution of separating menus when $\pi = 1$. This leads us to conjecture that the equilibrium for $\pi \in (0,1)$ can feature pooling, separating, or a mixture of the two, depending on the value of π. The following lemma formalizes this conjecture and shows the existence of a threshold utility for the offer to low quality sellers, such that all offers with u_l below this threshold are pooling menus while all offers above the threshold are separating menus.[21] Depending on whether this threshold lies at the lower bound, the upper bound, or in the interior of the support of $F_l(u_l)$, there are three possible cases, respectively: all equilibrium offers are separating menus, all are pooling menus, or a mixture with some pooling menus (offering relatively low utility to the seller) and some separating menus (offering higher utility). Later, in Proposition 12, we provide conditions on ϕ_l and π under which each case obtains.

Proposition 11. *For any $\pi \in (0,1)$ and $\phi_l < 0$, there exists an equilibrium where F_l and U_h satisfy the following properties:*

1. *there exists a threshold $\hat{u}_l$ such that, for any u_l in the interior of $\mathrm{Supp}(F_l)$:*

   (a) if $u_l \leqslant \hat{u}_l$, $U_h(u_l) = u_l$ and F_l satisfies

   $$\frac{\pi f_l(u_l)}{1-\pi+\pi F_l(u_l)}\left(\mu_h v_h + \mu_l v_l - u_l\right) = 1\,, \tag{22}$$

   (b) if $u_l > \hat{u}_l$, $U_h(u_l) > u_l$ and F_l satisfies (20).

2. $U_h(\underline{u}_l) = c_h$ *and* $U_h(\overline{u}_l) = \overline{u}_l$.

To understand the first set of (necessary) conditions in Proposition 11, consider the region where the buyers offer pooling menus. Here, buyers trade off profit per trade against the probability of trade, with no interaction between offers and incentive constraints—as a result, the equilibrium in this pooling region behaves as in the canonical Burdett and Judd (1983) single-quality model, with the buyer's payoff equal to the average value $\mu_h v_h + (1-\mu_h)v_l$. This yields (22). In the region where buyers offer separating

[21] At this point, it may seem arbitrary to conjecture that pooling occurs at the bottom of the distribution and separation at the top. As we discuss below, the reason this is ultimately true is that the cream-skimming deviation—which makes the pooling offer sub-optimal—becomes more attractive as the indirect utility being offered increases.

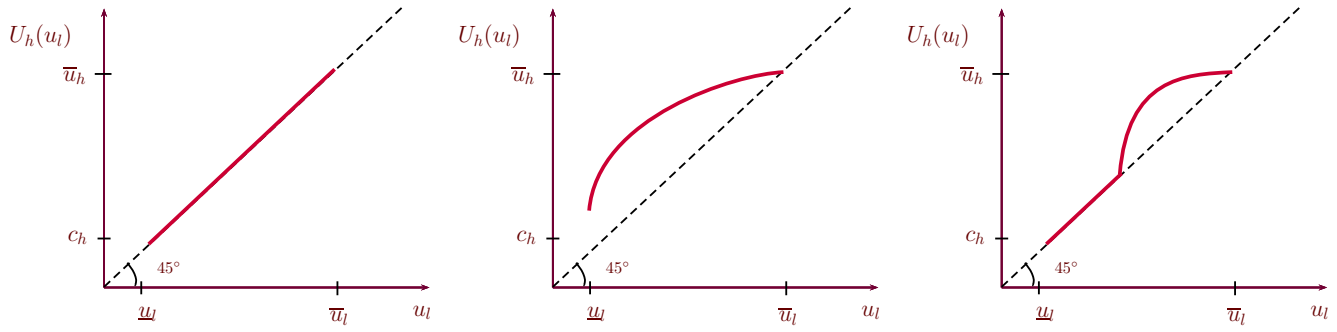

Figure 2: The mapping $U_h(u_l)$ for all pooling, all separating, and mixed equilibria when $\phi_l < 0$

menus, on the other hand, $F_l(u_l)$ is characterized by the local optimality condition (20), exactly as in the $\phi_l > 0$ case. Recall from our discussion that this differential equation accounts explicitly for the effect of an offer u_l on the seller's incentive constraint. In this region, $U_h(u_l)$ is determined by the equal profit condition.

The second part of the result describes boundary conditions for the worst and best menus offered in equilibrium. The first condition requires that the worst menu yields utility c_h to high quality sellers. To see why, suppose the worst menu is a pooling menu with $U_h(\underline{u}_l) = \underline{u}_l > c_h$. Then, lowering both u_h and u_l leads to strictly higher profits. If the worst menu is separating with $U_h(\underline{u}_l) > c_h$, then a downward deviation in only u_h is feasible and strictly increases profits. The second condition requires that the best menu offered in equilibrium is a pooling menu. Intuitively, if the best menu offered in equilibrium were a separating menu, then $x_h < 1$. This cannot be optimal when $\phi_l < 0$: the buyer can trade more with the high quality seller by increasing the utility offered to low quality sellers. Since this is already the best menu in equilibrium, this deviation has no impact on the number of sellers the buyer attracts but yields strictly higher profits.

Given these properties, we now establish two critical values—$\phi_1(\pi)$ and $\phi_2(\pi)$, with $\phi_2(\pi) < \phi_1(\pi) < 0$—which determine which of the three cases described above emerge in equilibrium. When $\phi_l < \phi_2(\pi)$, the threshold $\hat{u}_l$ is equal to the upper bound of the support $\overline{u}_l$ and there is an *all pooling* equilibrium. When $\phi_l > \phi_1(\pi)$, the monotonicity constraint is slack almost everywhere, so that $\hat{u}_l = \underline{u}_l$, and the equilibrium features *all separating* menus. Finally, if ϕ_l lies between these two critical values, we have a *mixed* equilibrium, with an intermediate threshold $\hat{u}_l \in (\underline{u}_l, \overline{u}_l)$. Figure 2 illustrates $U_h(u_l)$ for all three possibilities.

Proposition 12. *For any $\pi \in (0,1)$, there exist two cutoffs $\phi_2(\pi) < \phi_1(\pi) < 0$ such that an all pooling*

equilibrium exists for all $\phi_l \leqslant \phi_2(\pi)$*, a mixed equilibrium exists for all* $\phi_l \in (\phi_2(\pi), \phi_1(\pi))$*, and an all separating equilibrium exists for all* $\phi_l \in (\phi_1(\pi), 0)$.

Intuitively, for a pooling menu (u_l, u_l) to be offered in equilibrium, the cream-skimming deviation $(u_l - \varepsilon, u_l)$ for some $\varepsilon > 0$ cannot yield strictly higher profits. To see how incentives to cream-skim vary with ϕ_l and π, notice that there are two sources of higher profits from the menu $(u_l - \varepsilon, u_l)$, relative to the candidate pooling menu. First, it decreases the loss conditional on trading with a low quality seller. Second, it reduces the probability of trading with a non-captive low quality seller; since the buyer loses money on these sellers, this reduction in trading probability raises profits. The cost of cream-skimming is that the buyer earns lower profits on high quality sellers. Therefore, incentives to cream-skim are weak—and thus pooling is easier to sustain—when high quality sellers are relatively abundant (ϕ_l very negative) and/or there are relatively few non-captive sellers (π is small).

The higher the level of utility being offered in a pooling menu, the more vulnerable it is to cream-skimming. Therefore, if such a deviation is profitable at the lowest candidate value, c_h, then pooling cannot be sustained at all: this is the condition that determines the cutoff $\phi_1(\pi)$. Similarly, the cutoff $\phi_2(\pi)$ defines the boundary at which cream-skimming is not profitable even at the best pooling menu, $\overline{u_l}$. We derive these thresholds formally, and provide a full equilibrium characterization, in Appendix A.5.

Notice that, in all three cases, $u_l > v_l$ (since $u_l \geqslant c_h > v_l$) so that buyers always suffer losses when trading with low quality sellers. Hence, as in the extreme cases of $\pi = 0$ and $\pi = 1$, there is cross-subsidization in every equilibrium when $\phi_l < 0$. Finally, as in the case of $\phi_l > 0$, the equilibrium distribution converges to the limiting cases as π converges to both 0 and 1.

Figure 3 summarizes the various types of equilibria and the regions in which each one obtains. The x- and y-axes represent the intensity of competition and severity of adverse selection, respectively. Recall that the latter is summarized by ϕ_l, which is a function of μ_h, the fraction of high quality assets, as well as the valuations v_h, c_h, c_l. For concreteness, we use μ_h to vary ϕ_l on the y-axis—a higher fraction of high quality assets implies a lower ϕ_l, and therefore milder adverse selection.[22]

[22] The boundaries are also re-defined accordingly: $\mu_h \leqslant \mu_0$ if and only if $\phi_l \geqslant 0$ and $\mu_h \leqslant \mu_k(\pi)$ if and only if $\phi_l \geqslant \phi_k(\pi)$ for $k \in \{1, 2\}$.

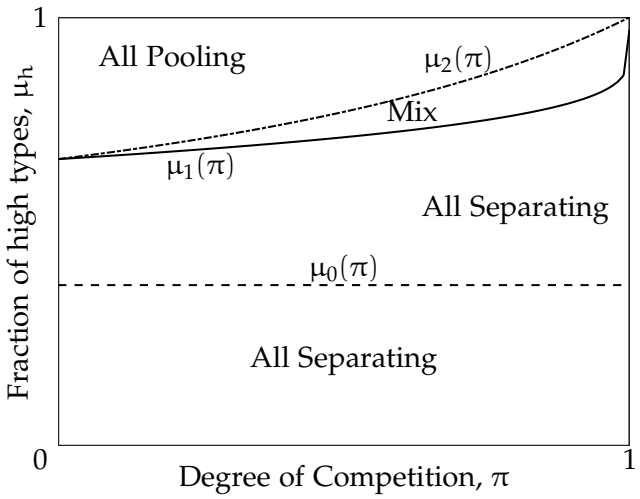

Figure 3: Equilibrium Regions

4.3 Uniqueness

In the previous section, we constructed equilibria for all $\pi \in (0,1)$ and $\phi_l \leqslant 1$. In Proposition 13, below, we establish that these equilibria are unique. For intuition, we sketch the arguments here for $\phi_l \neq 0$.[23] First, we show that for all $\phi_l \neq 0$, no equilibrium features a mass point, even at ν_l. Next, when $\phi_l > 0$, we prove that no equilibrium features pooling menus on a positive measure subset of F_l. In this case, since equilibria have no mass points and must be separating almost everywhere, the equilibrium we construct in Proposition 10 describes the unique equilibrium.

When $\phi_l < 0$, we demonstrate uniqueness of the equilibrium with a threshold $\widehat{u}_l$ in steps. First, we show that any equilibrium features pooling at the upper bound of the support of F_l. Second, we prove that any equilibrium features at most one interval of pooling menus followed by at most one interval of separating menus. Third, we prove that the equilibria characterized in Proposition 12 are mutually exclusive, so that equilibria without mass points are unique. Since no equilibrium features mass points when $\phi_l < 0$, these results establish the uniqueness of the equilibrium characterized in Proposition 12. We summarize these results in the following Theorem.

Theorem 13. *For any $\pi \in (0,1)$ and $\phi_l \in \mathbb{R}$, there exists an equilibrium and it is unique.*

4.4 Discussion

The equilibrium characterized above has a number of testable implications for transaction prices and quantities. The first set of predictions pertain to properties of equilibrium menus. We highlight three

[23] In Appendix A.9, we also prove uniqueness for the knife-edge case of $\phi_l = 0$.

robust predictions. First, the strict rank-preserving property suggests a positive correlation between the contracts that buyers offer to different types of sellers: those buyers who make attractive offers to low quality sellers will also make attractive offers to high quality sellers. Hence, in equilibrium, buyers do not specialize in trading with a particular type of seller, but rather trade with equal frequency across all types. Second, whether buyers pool different types of sellers or separate them (using a menu of options) depends crucially on the severity of the two frictions.[24] Pooling is more likely in markets where competition among buyers is relatively weak and adverse selection is relatively mild. Alternatively, separation is more likely when adverse selection is relatively severe—so that the information costs of trading with high quality sellers are large relative to the benefits—and competition is relatively strong—so that the payoffs from cream-skimming are relatively high.[25] Third, the theory also predicts that menus that are less attractive from the perspective of sellers are more likely to be pooling. In other words, those who are posting offers with relatively unattractive terms should be offering fewer options and should account for a smaller share of observed transactions.

The second set of implications pertains to dispersion. Note that, in the region with separating menus, the model predicts dispersion *within* and *across* types. This is true both for quantities traded (coverage in an insurance context or loan size in a credit market context) as well as prices (premia or interest rates, respectively). The extent of dispersion—both the support and the standard deviation of the quantity/price distributions—is determined by the interaction of competition (measured by π) and adverse selection (measured by ϕ_l).

This joint dependence calls into question the practice of identifying imperfect competition or asymmetric information in isolation using cross-sectional dispersion. For example, a common empirical strategy to identify adverse selection is to test the correlation between the quantity an agent trades and her type, as measured by ex-post outcomes.[26] In our equilibrium, there is a negative correlation between the seller's quality and the quantity she sells, but the quantitative strength of this relationship is also a function of the market structure. As a result, using the relationship between quantity and type without accounting for the imperfect nature of competition is likely to yield misleading conclusions. A similar concern applies to the strategy of identifying search frictions from price dispersion.[27] In markets where

[24] This result stands in stark contrast to, e.g., Guerrieri et al. (2010). In that model, and many like it, the quantity traded with high quality sellers is *independent* of the distribution of types in the market—trade with high quality sellers is distorted even if the fraction of low quality assets in the market is arbitrarily small.

[25] Consistent with our findings, Decarolis and Guglielmo (2015) find evidence of greater cream-skimming by health insurance providers when the market is more competitive.

[26] This technique for identifying adverse selection has been applied to a number of markets, following the seminal paper by Chiappori and Salanie (2000); recent examples include Einav et al. (2010b), Ivashina (2009), and Crawford et al. (2015).

[27] Using dispersion in prices to help identify search frictions is standard practice in the IO literature; for a recent example,

adverse selection is a concern, the extent of cross-sectional variation in terms of trade is also a function of selection-related parameters. To obtain an accurate assessment of trading frictions in such settings thus requires controlling for the underlying distribution of types.

5 Welfare and Policy

Many important markets where adverse selection is a first-order concern—such as insurance, credit, and certain financial markets—are in the midst of dramatic changes. Some of these changes are regulatory in nature; for example, there are several recent policy initiatives to make health insurance markets and over-the-counter markets for financial securities more competitive and transparent. Other changes derive from technological improvements; for example, advances in credit scoring reduce information asymmetries for lenders. In this section, we use the framework developed above to examine the effects of these types of changes. We show that increasing competition or reducing information asymmetries can worsen the distortions from adverse selection when markets are relatively competitive. As a result, initiatives to make these markets more competitive or transparent are only welfare-improving when *both* frictions are relatively severe, i.e., when buyers have a lot of market power (i.e., when π is low) *and* the adverse selection problem is relatively severe (i.e., when ϕ_l is high or, equivalently, when μ_h is small).

5.1 Welfare

Our welfare criterion is the objective of a utilitarian planner, defined as the sum of the utilities of buyers and sellers. Given this criterion, social welfare is given by

$$\begin{aligned} W(\pi,\mu_h) &= (1-\mu_h)\nu_l+\mu_h\left\{(1-\pi)\int\left[\nu_h x_h(u_l)+c_h(1-x_h(u_l))\right]dF_l(u_l)\right. \\ &\quad \left.+\pi\int\mu_h\left[\nu_h x_h(u_l)+c_h(1-x_h(u_l))\right]d\left(F_l(u_l)^2\right)\right\} \end{aligned} \tag{23}$$

where, in a slight abuse of notation, we let

$$x_h(u_l)=1-\frac{U_h(u_l)-u_l}{c_h-c_l}. \tag{24}$$

The first term represents the payoff generated by low quality assets; since $x_l=1$ with probability 1, all low quality assets are allocated to buyers. The second term captures the expected payoffs generated by trades between buyers and captive high quality sellers; such a seller trades $x_h(u_l)$ with the buyer, where u_l is drawn from $F_l(u_l)$, and consumes the remaining $1-x_h(u_l)$ herself. Finally, the last term in (23)

see Gavazza (2015).

captures the expected payoffs generated by trades between buyers and non-captive high quality sellers; since non-captive sellers choose the maximum indirect utility among the two offers they receive, they trade an amount $x_h(u_l)$ where u_l is drawn from $F_l(u_l)^2$.

5.2 Increasing Competition

We first study the effects of increasing competition, which has been a common policy response to address perceived failures in markets for insurance, credit, and certain types of financial securities. For example, a recent report by the Congressional Budget Office (2014) argues for "fostering greater competition" in health insurance plans by developing "policies that would increase the average number of sponsors per region," which would then "increase the likelihood that beneficiaries would select low-cost plans." Similarly, the The U.S. Treasury (2010) argued that the Consumer Financial Protection Bureau "will make consumer financial markets more transparent – and that's good for everyone: The agency will give Americans [...] the tools they need to comparison shop for the best prices and the best loans, which will [...] increase competition and innovations that benefit borrowers." A similar rationale underlies the Core Principles and Other Requirements for Swap Execution Facilities (Commodity Futures Trading Commission (2011)), issued under the Dodd-Frank Act, which requires that a swap facility sends a buyer's request for price quotes to a minimum number of sellers before trade can be executed.

In this section, we study the potential costs and benefits of these types of policies by examining the relationship between ex ante welfare and competition, as captured by π. We then discuss the implications of our findings for various types of policy interventions.

Welfare and Competition. In Proposition 14, we establish that welfare is decreasing in π when the adverse selection problem is relatively mild, so that more competition is unambiguously bad for welfare in this region of the parameter space. However, when the adverse selection problem is severe, we show that W is inverse U-shaped in π; i.e., there is an *interior* level of competition that maximizes welfare in this region of the parameter space.

Proposition 14. *If $\phi_l \leqslant 0$, welfare is maximized at $\pi = 0$. Otherwise, it is maximized at a $\pi \in (0,1)$.*

The first result is straightforward. Since a monopsonist offers a pooling contract in this region of the parameter space, all gains from trade are realized. Competition only serves to increase incentives to cream-skim. In order to ensure that such a deviation is not profitable, equilibrium menus offer high quality sellers a higher price but a lower quantity to trade, causing a decline in welfare.

The second result—that welfare is maximized at an interior value of π when $\phi_l > 0$—is less obvious. To see the intuition for this result, first note that, as π increases, $F_l(u_l)$ increases in the sense of first order stochastic dominance: $F_l(u_l)$ shifts to the right and $\overline{u}_l$ increases. Intuitively, in equilibrium, more competition forces buyers to allocate more surplus to sellers. Second, and crucially, $x_h(u_l)$ is hump-shaped in u_l: it increases near the monopsony offer c_l, and decreases when u_l is sufficiently close to the competitive offer, v_l. When π is close to zero, $\overline{u}_l$ is relatively small and the distribution of offers is clustered near the monopsony contract; a small increase in π causes a rightward shift in the density of offers to values of u_l associated with *higher* values of x_h, increasing the gains from trade realized between buyers and high quality sellers. In contrast, when π is close to 1, $\overline{u}_l$ is close to v_l, and the distribution of offers is clustered near the competitive contract; in this case, a small increase in π causes a shift towards values of u_l associated with *lower* values of x_h.

Therefore, understanding why welfare is hump-shaped in π ultimately requires understanding why $x_h(u_l)$ is hump-shaped in u_l. Note that, *ceteris paribus*, an increase in u_l relaxes the type l seller's incentive compatibility constraint, allowing buyers to raise x_h. In contrast, *ceteris paribus*, an increase in u_h tightens the type l seller's incentive compatibility constraint, requiring buyers to lower x_h. Thus, as offers to both types increase, the net effect on x_h depends on which one rises faster—formally, whether $U_h'(u_l)$ is greater or less than one. Figure 4 illustrates this relationship between the quantity traded with high types, x_h, and the rate at which u_h and u_l increase within the set of equilibrium menus being offered. The figure reveals that u_l rises faster than u_h for smaller values of u_l, so that x_h is increasing in this region. However, as u_l nears v_l, u_h rises faster and thus x_h is decreasing in this region.

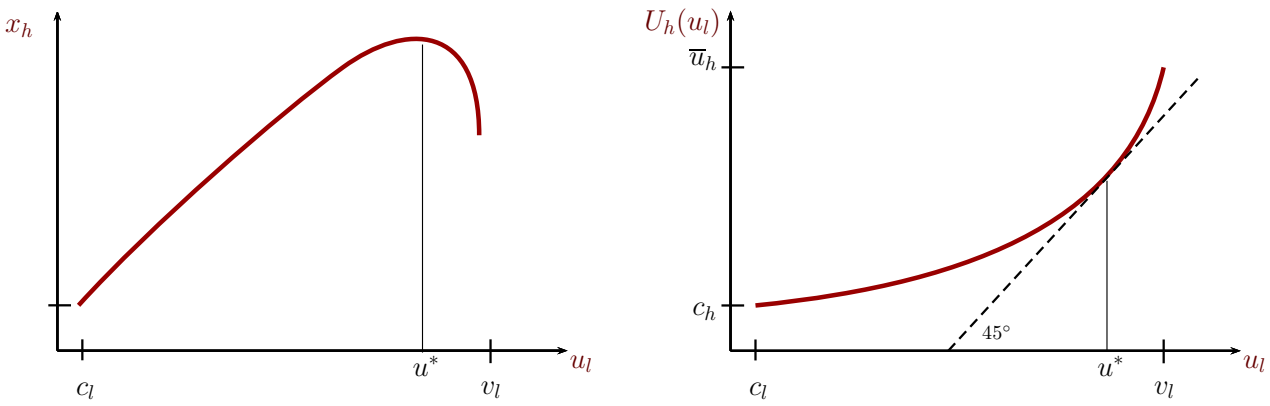

Figure 4: Trade (x_h) and Utility (U_h) of high quality seller as functions of u_l when $\pi > 0$.

To explain the hump-shape of welfare, then, we need to understand why $U_h'(u_l) < 1$ for low levels of

u_l and $U_h'(u_l) > 1$ for high levels of u_l. While this slope is a complicated equilibrium object, determined by the interaction of an individual buyer's optimal strategy and the equilibrium distribution of offers, the basic intuition can be understood through two opposing forces. First, it is cheaper for buyers to provide utility to the low type (relative to the high type) because doing so has the additional benefit of relaxing the incentive constraints; we call this the "incentive effect" and this force tends to reduce the slope, $U_h'(u_l)$. Second, as u_l rises, buyers have more incentive to attract type h sellers, relative to type l sellers; formally, one can show that $\Pi_h(u_l, U_h(u_l))/\Pi_l(u_l, u_h)$ is increasing in u_l. This effect—which we call the "composition effect"—leads them to increase u_h at faster rates at higher u_l.

To illustrate these two forces more clearly, consider the following optimality condition that any equilibrium menu $(u_l, U_h(u_l))$ must satisfy:[28]

$$U_h'(u_l) = \underbrace{\frac{\phi_l}{\phi_h}}_{\text{incentive effect}} \underbrace{\frac{\Pi_h(u_l, U_h(u_l))}{\Pi_l(u_l)}}_{\text{composition effect}} . \tag{25}$$

where $\phi_h = (v_h - c_l)/c_h - c_l)$ is the marginal cost of providing an additional unit of utility to type h sellers—i.e. $\phi_h = \frac{d\Pi_h}{du_h}$—and for notational convenience $\Pi_l(u_l) \equiv \Pi_l(u_l, u_h)$. The first term, the incentive effect, is the ratio of the marginal costs of providing utility to the two types of sellers. Since this term is strictly less than 1, all else equal, the incentive effect leads to more aggressive competition for the low type and, therefore, to u_h rising slower than u_l.

The second term, the ratio of profits, can be bigger or smaller than 1, depending on u_l. When u_l is close to the monopsony outcome, $\Pi_h \approx 0$, so the composition effect is also less than 1 and we have $U_h'(u_l) < 1$. However, as u_l approaches the upper bound v_l, this second term overwhelms the incentive effect, resulting in $U_h'(u_l) > 1$.

To see the behavior of this second term as u_l converges to v_l, we note that profits from both types go to 0 in the limit. The ratio of profits, applying L'Hospital's rule, is given by:

$$\lim_{u_l \to v_l} \frac{\Pi_h(u_l, U_h(u_l))}{\Pi_l(u_l)} = \lim_{u_l \to v_l} \frac{\frac{d\Pi_h(u_l, U_h(u_l))}{du_l}}{\frac{d\Pi_l(u_l)}{du_l}} = \lim_{u_l \to v_l} \phi_h U_h'(u_l) - \frac{v_h - c_h}{c_h - c_l} . \tag{26}$$

Now, suppose this limit is finite. Then,

$$\lim_{u_l \to v_l} \frac{\phi_l}{\phi_h} \frac{\Pi_h(u_l, U_h(u_l))}{\Pi_l(u_l)} = \phi_l U_h'(v_l) - \frac{\phi_l}{\phi_h} \frac{v_h - c_h}{c_h - c_l} \quad < \quad U_h'(v_l) , \tag{27}$$

[28] This equation combines the optimality condition (20) for u_l, the corresponding optimality condition for u_h,

$$\frac{\pi f_h}{1 - \pi + \pi F_h} \Pi_h = \phi_h ,$$

and the strict rank-preserving property $F_l(u_l) = F_h(U_h(u_l))$, which implies $f_l = f_h U_h'(u_l)$.

which implies that (25) cannot hold. In other words, if the ratio of profits is finite in the limit, buyers have incentive to offer a lower u_h. The only way to discourage such deviations is to make high types more profitable (in the limit, infinitely so), i.e. $\lim_{u_l \to v_l} \frac{\Pi_h}{\Pi_l} = \lim_{u_l \to v_l} \; U_h' = \infty$. This is why x_h has to be declining in u_l close to the Bertrand outcome.

Policy Implications. These results offer a cautionary note for policies attempting to make markets with adverse selection more competitive. They suggest that such attempts are desirable only when *both* market power and the distortions from adverse selection are relatively severe. If either friction is relatively mild, then there is actually a case for *reducing* competition. One way for policy makers to achieve this is by making entry more costly; though we have treated π as a structural parameter, it is straightforward to endogenize it with an explicit entry game and/or search costs. However, the results above are also relevant to other policies aimed at stimulating competition and trade. Perhaps the most obvious, direct form of intervention is when the government enters the market as a "buyer".[29] As we now show, studying this type of intervention in a framework that explicitly models imperfect competition and adverse selection with screening yields new, and perhaps counter-intuitive, insights.

To study this type of intervention, consider a policy in which the government announces that it stands ready to buy any quantity from sellers at a price $p \in (c_l, v_l)$. This is a natural policy to consider, as it unambiguously increases welfare when adverse selection is severe ($\phi_l > 0$) and market power is concentrated ($\pi = 0$).[30] However, as we establish in the lemma below, the effects of such a policy are ambiguous when $\pi \in (0, 1)$.

Lemma 15. *Suppose $\phi_l > 0$. If π is sufficiently small, then welfare is increasing in p for $p \in [c_l, v_l)$. If π is sufficiently close to 1, then welfare is decreasing in p for $p \in [c_l, v_l)$.*

The results in Lemma 15 mirror the findings in Proposition 14, precisely because this type of intervention mimics the effects of directly increasing competition. In particular, the government's offer raises the sellers' outside option, and this leads buyers to offer more surplus to sellers in an effort to retain market share. When π is small, as we explained above, the surplus offered to low quality sellers increases relatively fast; this helps to relax the distortions arising from adverse selection and increase the

[29] This form of intervention has been discussed and/or implemented in a variety of markets where policymakers have been concerned about the deleterious effects of imperfect competition and adverse selection. One obvious example is the market for student loans, where public and private lenders compete side by side. This type of "public option" is also available to some Americans seeking health insurance—namely, the poor and the elderly—and universal access to a government plan was a widely debated feature of the Affordable Care Act (it was ultimately left out). Finally, at the height of the recent financial crisis, policymakers considered using funds from the Troubled Asset Relief Program to enter key markets, such as those for asset-backed securities, and to buy assets directly in order to put a "floor" under prices in these markets.

[30] When $\pi = 1$, on the other hand, this policy has no effect since all prices in equilibrium are larger than v_l.

volume of trade with high quality sellers, thus increasing welfare. When π is large, the surplus offered to high quality sellers increases relatively fast; this amplifies the distortions arising from adverse selection, tightening the incentive constraints, decreasing trade with high quality sellers, and thus lowering welfare.

These results have interesting implications for the efficacy of the asset purchase programs that were proposed and debated during the recent financial crisis; see, e.g., Tirole (2012), Philippon and Skreta (2012), and Guerrieri and Shimer (2014a). In particular, since most of the models in this literature abstract from imperfect competition, they fail to capture both the positive and negative effects highlighted in Lemma 15.[31] For instance, within the context of a perfectly competitive market, several have argued that this type of intervention must necessarily lose money (for the government) in order to have beneficial effects on welfare. Lemma 15 shows that, with imperfect competition, this is not the case: when π is small, sellers never even trade with the government, and yet the availability of this outside option increases welfare. However, when π is large, our results suggest that such a program may be detrimental for welfare even if, in principle, the program makes non-negative profits. These results suggest that incorporating market structure can significantly alter—and even reverse—some of the lessons that have been drawn in the literature studying interventions in markets suffering from adverse selection.

5.3 Reducing Information Asymmetries

We now study the welfare consequences of reducing informational asymmetries. This exercise sheds light on the implications of certain policy initiatives, as well as the effects of various technological innovations. For example, an important debate in insurance, credit, and financial markets centers around information that the informed party (the seller in our context) is required to disclose and the extent to which such information can be used by the uninformed party (the buyers in our model) to discriminate.[32] Moreover, technological developments in these markets also have the potential to decrease informational asymmetries, as advanced record-keeping and more sophisticated scoring systems (e.g., credit scores)

[31] There are several papers in this literature that do not assume a competitive market structure, such as Camargo et al. (2015). See Lester et al. (2013) for a more thorough review of the literature.

[32] In insurance markets, these questions typically concern an individual's health factors, both observable (e.g. age or gender) and unobservable (e.g. pre-existing conditions) by the insurance provider. In credit markets, similar questions arise with respect to observable characteristics that can legally be used in determining a borrower's creditworthiness, as well as the amount of information about a borrower's credit history that should be available to lenders (e.g., how long a delinquency stays on an individual's credit history). In financial markets, the relevant issue is not only whether a seller discloses relevant information about an asset to a buyer, but also whether the payoff structure of the asset is sufficiently *transparent* for sellers to distinguish good from bad assets. For example, in order to support "sustainable securitisation markets," the Basel Committee on Banking Supervision and the International Organization of Securities Commissions establish a joint task force to identify criteria for "simple, transparent, and comparable" securitized assets. See http://www.bis.org/bcbs/publ/d304.pdf.

provide buyers with more and/or better information about sellers' intrinsic types.[33]

To study the effects of these changes, we introduce a noisy *public* signal $s \in \{0,1\}$ about the quality of each seller.[34] The signal is informative, so that $\Pr(s = 1|h) = \Pr(s = 0|l) > 0.5$. Since the signal is publicly observed, the buyers may condition their offers on it, i.e. they offer separate menus for sellers with $s = 0$ and $s = 1$. Thus, the economy now has two subgroups, $j \in \{0,1\}$, with the fraction of high quality sellers in subgroup j given by

$$\mu_{hj} = \frac{\mu_h \ \Pr(s = j \mid h)}{\mu_h \ \Pr(s = j \mid h) + \mu_l \ (1 - \Pr(s = j \mid l))} .$$

Note that the average across subgroups is equal to the unconditional fraction of high types, i.e. $\mathbb{E}[\mu_{hj}] = \mu_h$. The equilibrium outcome for each subgroup can be constructed using the procedure in Section 4 with the appropriate μ_{hj}. Welfare is then given by the average welfare across subgroups, i.e. $\mathbb{E}[W(\pi, \mu_{hj})]$. When buyers do not have any additional information (or, equivalently, are not permitted to condition their offers on the signal), welfare is simply $W(\pi, \mathbb{E}[\mu_{hj}])$. Hence, whether the signal increases or decreases welfare, respectively, depends on whether $W(\pi, \mu_h)$ is convex or concave in μ_h in the relevant region.

Before proceeding, two comments are in order. First, our focus is on the effect of a *small* increase in the information available to buyers; that is, we are interested in signals that induce a local mean-preserving spread around μ_h. Additional information of sufficiently high quality always improves welfare—for example, if buyers receive a perfect signal about sellers' types, all gains from trade are realized —but this is not a very interesting or realistic experiment. Second, we focus on the region with $\mu_h < \mu_0$, so that $\phi_l > 0$, which is more tractable and shows interesting interactions between competition and additional information.[35] Moreover, in this region, W is *linear* in μ_h when $\pi = 0$ or $\pi = 1$. Hence, imposing monopsony or perfect competition would lead us to the conclusion that additional information has no effect on welfare.

Welfare and Information. Proposition 16 shows that W has a strictly convex region when π is sufficiently low, implying that more information is beneficial when markets are relatively (but not perfectly) uncompetitive. Alternatively, when markets are relatively (but not perfectly) competitive, W has a strictly concave region, implying that more information actually reduces welfare.

[33] See, e.g., Chatterjee et al. (2011) and Einav et al. (2013) for a description of how the emergence of standardized scoring systems in credit markets have radically changed lenders' ability to assess a borrower's creditworthiness.

[34] The restriction to a binary signal is only for simplicity. It is easy to introduce richer information structures.

[35] Numerical simulations suggest that additional information always reduces welfare when $\mu_h > \mu_0$.

Proposition 16. *There exist* $\underline{\pi}, \overline{\pi} \in (0,1)$ *such that:* (i) *for all* $\pi \in (0, \underline{\pi})$*, there exists* $0 < \underline{\mu}_h < \overline{\mu}_h < \mu_0$ *such that* W *is strictly convex on the interval* $[\underline{\mu}_h, \overline{\mu}_h]$*; and* (ii) *for all* $\pi \in (\overline{\pi}, 1)$*, there exists* $0 < \underline{\mu}'_h < \overline{\mu}'_h < \mu_0$ *such that* W *is strictly concave on the interval* $[\underline{\mu}'_h, \overline{\mu}'_h]$.

To see the intuition behind Proposition 16, recall from the previous subsection that trade with the high quality seller (and thus welfare) is governed by the interaction of the incentive effect and the relative profit (or composition) effect. The consequences of more information can be understood in terms of these two forces, too, which depend on the severity of adverse selection. In particular, a lower ϕ_l drives down the first term in (25), which encourages more competition for low quality sellers and hence boosts trade and welfare. Now, from (17), we see that ϕ_l is a concave function of μ_h. Since the additional signal induces a mean-preserving spread of μ_h, it results in a lower ϕ_l on average, which, *ceteris paribus*, increases trade. This mechanism makes more information desirable. The effect from relative profits goes in the opposite direction. In equilibrium, milder adverse selection raises profits from high types relative to low types, which increases U'_h and hence decreases trade. Close to monopsony, since the incentive effect dominates, more information raises welfare. The opposite happens when π is close to 1 and the effect on relative profits dominates.

6 The Model with Many Types

We now extend our analysis to the case with an arbitrary, finite number of seller types. We focus our attention on equilibria where all offers are separating menus. We do so for two reasons. First, in the case of $N = 2$, this region yields some of the most interesting results—such as the non-monotonicity of welfare in π—and we want to confirm that these results are true in a more general setting. Second, in the equilibrium with all separating menus, the monotonicity constraints are slack ($x_i < x_{i+1}$), which is the most commonly studied case in the mechanism design literature.[36] We first provide a method for constructing such a separating equilibrium, and then use the constructed equilibrium to demonstrate that the welfare implications from the model with two types extend to the general case of $N > 2$.

Suppose there are $N \geqslant 2$ types, with buyers and sellers deriving utility v_i and c_i, respectively, per unit from a good of type $i \in \mathcal{N} \equiv \{1, ..., N\}$. The types are ordered so that $v_1 < v_2 < ... < v_N$ and $c_1 < c_2 < ... < c_N$, and there are gains from trading all types of goods, i.e., $v_i > c_i$ for all $i \in \mathcal{N}$. The distribution of types is summarized by the vector $(\mu_1, \ldots, \mu_N)$, with $\sum_{i \in \mathcal{N}} \mu_i = 1$. As in our benchmark model, sellers (of all types) are privately informed about the quality of their good and receive two offers

[36] See, e.g., Fudenberg and Tirole (1991).

with probability π and one offer with probability $1-\pi$.

Equilibrium Properties. The definition of strategies and a (symmetric) equilibrium are identical to those in the model with two types, and hence we omit them for brevity. We begin our analysis, in Lemma 17 below, by establishing that buyers' offers never distort the quantity traded with the lowest type of seller, and that local incentive constraints always bind "upward," i.e., equilibrium offers always leave a type i seller indifferent between his contract and the one intended for type $i+1$. As a result, a buyer's offer can again be summarized by the indirect utilities it delivers to each type $i \in \mathcal{N}$.

Lemma 17. *For almost all equilibrium menus:*

1. *There is full trade with the lowest type, so that $x_1 = 1$, and the local incentive constraints are binding upward, so that*
$$t_i + c_i(1-x_i) = t_{i+1} + c_i(1-x_{i+1}) \text{ for all } i = 1, 2.....N-1;$$

2. *Each menu can be summarized by a utility vector $\mathbf{u} = (u_1, \cdots, u_N)$ with $u_i \geqslant c_i \ \forall\, i$ and*
$$1 \geqslant \frac{u_N - u_{N-1}}{c_N - c_{N-1}} \geqslant \cdots \geqslant \frac{u_2 - u_1}{c_2 - c_1} \geqslant 0, \tag{28}$$
with the corresponding quantities and transfers given by
$$\begin{aligned} x_1 &= 1, \quad x_i = 1 - \frac{u_i - u_{i-1}}{c_i - c_{i-1}}, \quad i = 2, 3.....N \\ t_1 &= u_1, \quad t_i = u_i - \frac{c_i}{c_i - c_{i-1}}(u_i - u_{i-1}), \quad i = 2, 3.....N. \end{aligned} \tag{29}$$

Given these results, we can recast each buyer's problem in terms of the utility vector $\mathbf{u}$. In particular, given a family of marginal distributions $F_i(u_i)$ for $i \in \mathcal{N}$, each buyer chooses a vector $\mathbf{u}$ to solve
$$\max_{u_i \geqslant c_i} \sum_{i=1}^{N} \mu_i \left(1 - \pi + \pi F_i(u_i)\right) \Pi_i(u_{i-1}, u_i) \tag{30}$$
subject to the monotonicity constraints in (28), where (in a slight abuse of notation) profits per trade with a seller of quality i are given by
$$\begin{aligned} \Pi_1(u_1) &= v_1 - u_1, \\ \Pi_i(u_{i-1}, u_i) &= v_i - \frac{v_i - c_{i-1}}{c_i - c_{i-1}} u_i + \frac{v_i - c_i}{c_i - c_{i-1}} u_{i-1}, \qquad \text{for all } i = 2.....N. \end{aligned} \tag{31}$$

The program in (30) resembles a standard mechanism design problem, where the binding incentive constraints are substituted into the profit functions in (31). The monotonicity constraints in (28) are necessary to ensure that local incentive compatibility implies global incentive compatibility.

We now formally define a separating equilibrium, provide a characterization and a method for constructing such equilibria, and then use numerical examples to study their normative properties.

Definition 18. *An equilibrium is separating if the utility vector* $\mathbf{u}$ *associated with any equilibrium menu solves the relaxed problem of maximizing the objective in* (30) *ignoring the monotonicity constraints in* (28).

As a first step, in the conjectured equilibrium, one can use an induction argument to extend Proposition 3, establishing that all the distributions $F_i(u_i)$ are continuous with connected support. Since the profit function is strictly supermodular, any separating equilibrium must satisfy the strict rank-preserving property. The following proposition summarizes.

Proposition 19. *If* $\phi_1 = 1 - \frac{\mu_2}{\mu_1}\frac{\nu_2 - c_1}{c_2 - c_1} \neq 0$, *then, in any symmetric separating equilibrium,*

1. *For all* $i \in \mathcal{N}$, $F_i(\cdot)$ *has a connected support and is continuous.*

2. *There exist a sequence of strictly increasing real-valued functions* $\{U_i(u_1)\}_{i=2}^N$ *such that the utility vector associated with any equilibrium menu* $\mathbf{z}$ *satisfies:*

$$\mathbf{u}(\mathbf{z}) = \left(u_1(\mathbf{z}), U_2(u_1(\mathbf{z})), U_3(u_1(\mathbf{z})), \cdots, U_N(u_1(\mathbf{z}))\right). \tag{32}$$

As in the model with two types, Proposition 19 greatly simplifies the construction of separating equilibria: it implies that we only need to characterize the distribution of offers to the lowest type, $F_1(u_1)$, together with the sequence of functions $\{U_i(u_1)\}_{i=2}^N$.[37] The equilibrium distribution of utilities can then be derived from the fact that all types have the same ranking across equilibrium menus, i.e., $F_i(U_i(u_1)) = F_1(u_1)$ for all $i = 2, ..., N$.

Equilibrium construction. We now illustrate how to construct a separating equilibrium. Differentiability of the profit function in (30) implies that any separating equilibrium must satisfy

$$\frac{\pi f_i(U_i(u_1))}{1 - \pi + \pi F_i(U_i(u_1))}\Pi_1(u_1) = \phi_i \tag{33}$$

$$\frac{\pi f_i(U_i(u_1))}{1 - \pi + \pi F_i(U_i(u_1))}\Pi_i(U_{i-1}(u_1), U_i(u_1)) = \phi_i \quad \text{for all } i = 2, ..., N, \tag{34}$$

[37] This proposition relies on the assumption that the marginal cost of transfers to the lowest type net of any benefits arising from binding incentive constraints, ϕ_1, is non-zero. As in the two type case, this assumption is required in order to show that equilibrium distributions do not have mass points.

where ϕ_i, the marginal cost of increasing the utility of a seller of type i, is given by

$$\begin{aligned}
\phi_1 &= 1 - \frac{\mu_2}{\mu_1}\frac{v_2 - c_2}{c_2 - c_1} \\
\phi_i &= \frac{v_i - c_{i-1}}{c_i - c_{i-1}} - \frac{\mu_{i+1}}{\mu_i}\frac{v_{i+1} - c_{i+1}}{c_{i+1} - c_i}, \qquad \text{for all } i = 2, \cdots, N-1 \\
\phi_N &= \frac{v_N - c_{N-1}}{c_N - c_{N-1}}.
\end{aligned}$$

Equation (33) implies that F_1 must satisfy

$$\frac{\pi f_1(u_1)}{1 - \pi + \pi F_1(u_1)} = \frac{\phi_1}{v_1 - u_1}. \tag{35}$$

Since the strict rank-preserving property implies that each U_i must satisfy $F_i(U_i(u_1)) = F_1(u_1)$, it must be the case that $U_i'(u_1) f_1(U_i(u_1)) = f_1(u_1)$. Substituting this result into (34) implies that the equilibrium functions U_i must satisfy the set of differential equations:

$$U_i'(u_1) = \frac{\phi_1}{\phi_i}\frac{\Pi_i(U_{i-1}(u_1), U_i(u_1))}{v_1 - u_1} \qquad \text{for all } i = 2, \cdots, N. \tag{36}$$

The system of differential equations (35) and (36) are ordinary first order differential equations; to complete the characterization, we need only provide the appropriate boundary conditions. As in the two-type model, these conditions depend critically on the marginal costs, $(\phi_1, \ldots, \phi_N)$, and are closely tied to the outcome under monopsony. The following result shows that the solution to a monopsonist's problem can be represented in the form of a threshold type.

Lemma 20. *Let J denote the largest integer $i \in \{1, 2...N\}$ such that*

$$\sum_{i=1}^{J-1} \mu_i \phi_i \;<\; 0, \tag{37}$$

with $J = 1$ if $\sum_{i=1}^{k} \mu_i \phi_i > 0$ for all $k \in \{1, 2....N\}$. The solution to a monopsonist's problem is to set $u_i = c_J$ for $i \leqslant J$ and $u_i = c_i$ for $i > J$.

Intuitively, the accumulated marginal cost of trading with the first J types is negative ($\sum_{i=1}^{J-1} \mu_i \phi_i < 0$), so they are pooled. In contrast, for the remaining types, the information rents outweigh the potential gains, so the monopsonist chooses not to trade with them.[38] The next result links this threshold J to the best and worst menu when $\pi > 0$.

Lemma 21. *Let J be as defined in Lemma 20. Then, in any equilibrium, the best menu has $u_i = u_J$ for $i < J$, and the worst menu has $u_i = c_i$ for all $i \geqslant J$.*

[38] For brevity, we ignore the non-generic case where the inequality in (37) is satisfied with equality.

To see the intuition, note that the best menu trades with probability 1, i.e. attracts all captive and non-captive sellers. Therefore, it cannot be profitable for that menu to separate types that a monopsonist finds profitable to pool; if $u_i < u_J$ for some $i < J$, then increasing u_i has no effect on the probability or composition of trades but yields strictly higher profits (because the effective marginal cost of increasing u_i is negative). Similarly, it cannot be profitable for the worst menu to give any surplus to the types that the monopsonist finds optimal to shut out completely; if such a menu offers more than c_i to any type $i > J$, the buyer can raise her profits simply by lowering that utility.

The system of differential equations (35)-(36), along with the boundary conditions described in Lemma 21, describe necessary conditions for any separating equilibrium. By the Picard-Lindelof theorem, it has a unique solution. In Appendix A.10.5, we provide analytical expressions for this solution. To ensure that this solution is an equilibrium, one need only verify that the monotonicity constraints (28) are satisfied for every $u_1 \in Supp(F_1)$.

Finally, we solve two numerical examples using the method described above. The two cases both have $N = 4$, but differ in the marginal cost vector, $(\phi_1, ...\phi_N)$.[39] In the first case, $J = 1$, so the monopsonist only trades with the lowest type. In the second case, $J = 2$. We use these cases to demonstrate the robustness of the welfare results in section 5.2. In Figure 5, we plot expected trade for types 2 through 4 (recall that $x_1 = 1$ always) as a function of π. They show a non-monotonic relationship between expected trade and competition. In the first case (left panel), where the monopsonist only trades with type-1, trade by all 3 types is hump-shaped. This is analogous to the case with $\phi_l > 0$ in the two-type model. In the second case (right panel), however, trade with one of the types (type 2) is monotonically decreasing in π. This is similar to the case with $\phi_l < 0$ in the two-type model. In both cases, these patterns imply that *ex-ante* welfare is maximized at $\pi < 1$.

7 Additional Extensions and Robustness

In this section, we examine a few additional extensions of our framework, both to ensure the robustness of our results and to demonstrate that our framework is amenable to more applied work. First, we relax our assumption of linear utility to analyze the canonical model of insurance under private information. Second, we allow the degree of competition to differ across sellers of different quality. Lastly, we show how to incorporate additional dimensions of heterogeneity, including horizontal and vertical

[39] For both cases, we assume a uniform distribution $\mu_i = 0.25$ for all i, with valuations $c_i = 1, 2, 3, 4$ and $v_i = c_i\delta + 0.5$. In case 1, $\delta = 1.2$ and in case 2, $\delta = 1.3$. In each case, we solve the system (35)-(36) and verify that the monotonicity constraints are satisfied.

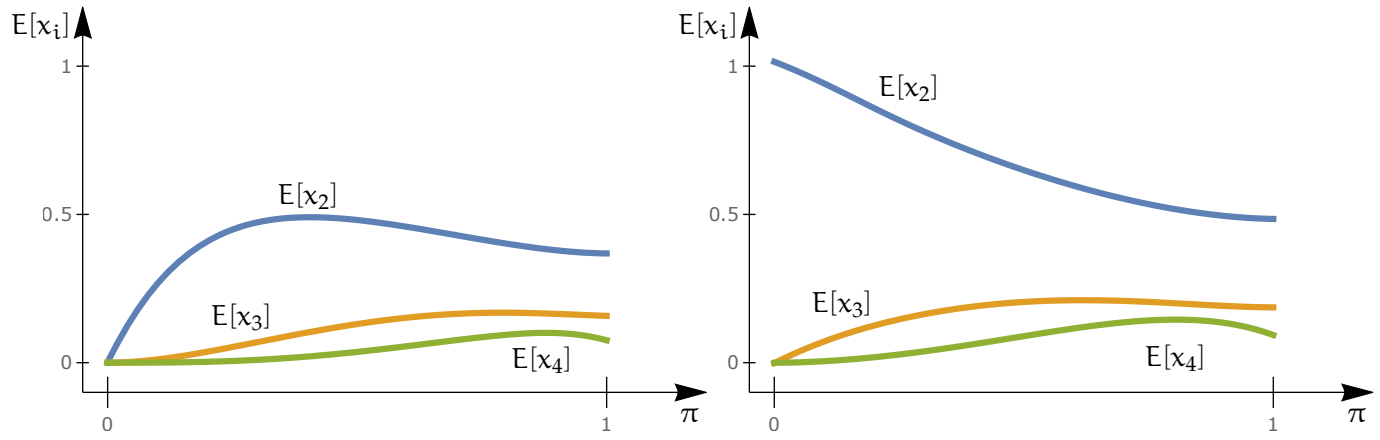

Figure 5: Expected trade and competition when $N = 4$ and $J = 1$ (left panel) or $J = 2$ (right panel).

differentiation.

7.1 A Model of Insurance

To start, we analyze a canonical model of insurance under private information, along the lines of Rothschild and Stiglitz (1976), and show that our main results—in particular, the structure of equilibrium menus and the non-monotonicity of welfare with respect to the degree of competition—extend beyond the linear, transferable utility environment.

A unit measure of agents with strictly increasing, strictly concave utility functions $w(c)$ face idiosyncratic income risk.[40] Their income in normal times is y, but they also face the risk of an "accident," which reduces their income by d. The accident itself is observable and contractible, but the probability of its occurrence, denoted θ_j, $j \in \{b, g\}$, is private information. A fraction μ_b of agents are of type b and face a higher risk of accident than type g agents, i.e. $\theta_b > \theta_g$. Principals (i.e. the insurance providers) are risk-neutral, which implies that gains from trade are strictly positive for both types. The competitive structure is exactly the same as in the baseline model: a fraction $1 - \pi$ of agents receive one offer and the remainder receive two.

A contract consists of a premium and a transfer to the agent in the event of an accident. Since trading is exclusive and the accident is observable, we can also think of the contract as directly offering a utility level in the normal and accident states. As before, we consider menus with two contracts, one for each

[40] Note that, in this application, the "buyers" of insurance are the ones with private information. To avoid confusion, we switch to a principal-agent description.

type, i.e. $\mathbf{z} = \left(u_b^n, u_b^a\right), \left(u_g^n, u_g^a\right)$ such that incentive and participation constraints are satisfied:

$$\begin{aligned}
\left(IC_j\right): \quad & \theta_j u_j^a + \left(1-\theta_j\right) u_j^n \geqslant \theta_j u_{-j}^a + \left(1-\theta_j\right) u_{-j}^n, \\
\left(PC_j\right): \quad & \theta_j u_j^a + \left(1-\theta_j\right) u_j^n \geqslant \theta_j w\left(y-d\right) + \left(1-\theta_j\right) w\left(y\right) \qquad\qquad j \in \{b, g\}.
\end{aligned}$$

To solve for the equilibrium, we follow the same steps as in Section 4. The first step is to obtain the utility representation. It is straightforward to prove that, in all equilibrium menus, the type b agent is fully insured and (IC_b) binds. This allows us to summarize equilibrium menus with a pair of expected utilities, one for each type, (u_b, u_g), with allocations given by the solution to the following system of equations:

$$u_b = u_b^a = u_b^n, \qquad u_b = \theta_b u_g^a + (1-\theta_b)\, u_g^n, \qquad u_g = \theta_g u_g^a + (1-\theta_g)\, u_g^n. \tag{38}$$

In a separating menu, the principal offers type g agents less than full insurance: $u_g^a < u_g^n$ such that (IC_b) binds. Define $C(u) \equiv w^{-1}(u)$ to be the principal's cost of providing a utility level u. Note that $C'(u), C''(u) > 0$. Then, the objective of the principal is described by (11), where the type-specific profit functions satisfy

$$\begin{aligned}
\Pi_b\left(u_b, u_g\right) &= y - \theta_b d - C\left(u_b\right), \\
\Pi_g\left(u_b, u_g\right) &= y - \theta_g d - \theta_g C\left(u_g^a\right) - \left(1-\theta_g\right) C\left(u_g^n\right).
\end{aligned}$$

Since w is strictly increasing and concave, we can show that

$$\frac{d\Pi_g\left(u_b, u_g\right)}{du_b} > 0, \qquad \text{and} \qquad \frac{d\Pi_g\left(u_b, u_g\right)}{du_g du_b} > 0.$$

The first inequality shows the effect of incentives: more surplus to type b agents relaxes their incentive constraint, allowing the principal to earn higher profits from type g agents. The second inequality shows that the marginal benefit of increasing the utility of type g agents rises with the utility offered to type b agents, implying the strict supermodularity of the profit function. In other words, the complementarity that was at the heart of the strict rank-preserving property in the linear model is present in this version as well. Using this property, we can extend the arguments in Proposition 3, implying that the marginal distributions F_j, $j \in \{b, g\}$ do not have any flat portions or mass points. Therefore, Theorem 1 applies, i.e., equilibria are strictly rank-preserving and can therefore be described by a distribution over utilities to type b agents, $F_b(u_b)$, and a strictly increasing function $U_g(u_b)$. In Appendix A.11, we use the methods from Section 4 to derive the system of differential equations that characterize these functions.

Turning now to our normative results, we consider the implications of competition for welfare. For brevity, we restrict attention to the region where all menus are separating and do not involve cross-subsidization. In this case, the consumption of type g agents necessarily varies with the state; this imperfect insurance is the analogue of distortions in the quantity traded in the baseline model. The associated resource costs are thus a natural measure of the efficiency losses (relative to a full information benchmark) in this setting. For a menu offering u_b to type b agents, this loss is given by

$$L(u_b) \quad = \quad C\left(U_g(u_b)\right) - \left[\theta_g C\left(U_g^a(u_b)\right) + (1-\theta_g)\, C\left(U_g^n(u_b)\right)\right] \ , \tag{39}$$

where U_g, U_g^a, and U_g^n are equilibrium policy functions. Average losses in the economy are then

$$\mathcal{L}(\pi) \quad \equiv \quad (1-\pi)\int L(u_b)\mathrm{d}F_b(u_b,\pi) + \pi\int L(u_b)\mathrm{d}F_b(u_b,\pi)^2 \ . \tag{40}$$

In Appendix A.11, we show, using a numerical example, that L is U-shaped in u_b, which then implies that $\mathcal{L}(\pi)$ is minimized at an interior value of π. Thus, in markets for insurance, increasing competition among providers can be detrimental for welfare.

7.2 Differential Competition Across Types

In our baseline model, we assume that the probability a seller receives one or two offers is the same for both types. If we interpret this probability as the outcome of an endogenous choice by the sellers, then this restriction is potentially important; after all, high and low quality sellers typically have different incentives to undertake costly search for more offers. In this subsection, we relax this assumption and allow π to vary across types, so that the probability a type j seller is captive is given by $1-\pi_j$. We will show that both the structure of the equilibrium and its normative properties remain largely unchanged, with the caveat that, for some parameter values, the equilibrium distribution has mass points. For brevity, we restrict attention to the $\phi_l > 0$ case, where all equilibrium menus are separating and cross-subsidization does not occur.

We first consider the case where $\pi_h > \pi_l$. In this case, the results in Proposition 3 go through unchanged and therefore, the distribution functions F_l and F_h have continuous support and no mass points. This implies that the equilibrium satisfies the strict rank-preserving property and all menus attract the same fraction of non-captive sellers. Next, we consider the case with $\pi_l > \pi_h$. In this case, both distributions still have continuous supports, but F_l has a mass point if π_l is sufficiently large. The following proposition fully characterizes the unique equilibrium and provides a condition for mass point equilibria.

Proposition 22. *If* $\frac{1-\pi_l}{1-\pi_h} < 1-\phi_l$*, then the unique equilibrium* F_l *has full mass at* v_l *and* F_h *is characterized by*

$$(1-\pi_h+\pi_h F_h(u_h))\,\Pi_h(v_l,u_h) = (1-\pi_h)\,\Pi_h(v_l,c_h)\,. \tag{41}$$

If $\frac{1-\pi_l}{1-\pi_h} \geqslant 1-\phi_l$*, then the unique equilibrium* F_l *satisfies*

$$\frac{\pi_l f_l(u_l)}{1-\pi_l+\pi_l F_l(u_l)}\Pi_l(u_l) \;=\; 1-\frac{1-\pi_h+\pi_h F_l(u_l)}{1-\pi_l+\pi_l F_l(u_l)}\left(\frac{\mu_h}{\mu_l}\right)\frac{v_h-c_h}{c_h-c_l} \tag{42}$$

and U_h *is determined by the equal profit condition.*

Equation (42) is similar in structure to (20). The key difference is that the right hand side, which again measures the (net) marginal cost of providing a unit of surplus to the low type, has an additional term that adjusts for the differential probability that an offer is *accepted* by high types relative to low types. For example, when $\pi_h > \pi_l$, this cost is large when u_l is small: the benefit from relaxing incentive constraints is weak because low offers are rarely accepted by high quality sellers when they are more likely to be captive. As u_l grows, however, these benefits grow as well, and the right hand side shrinks to reflect a smaller marginal cost of providing surplus to low types.

The construction of equilibrium follows the strategy in Section 4. The ODE in (42), with the boundary condition $F_l(c_l) = 0$, can be solved for F_l. Given F_l, the equal profit condition pins down U_h. The properties of the equilibrium—both positive and normative—are also similar to the baseline model. In particular, x_h is non-monotonic in u_l which, as before, has interesting implications for the relationship between welfare and competition.

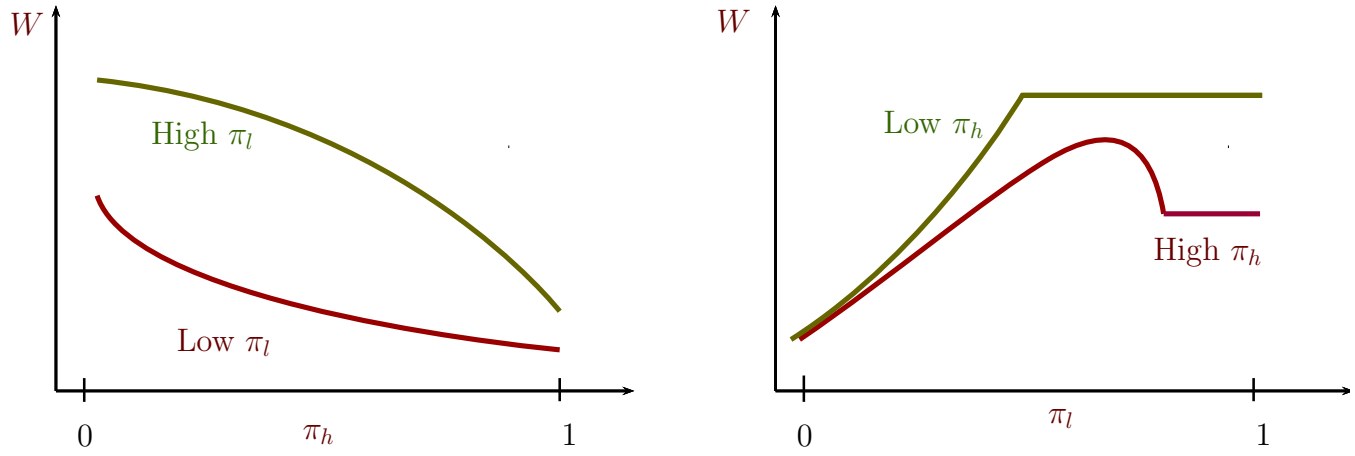

Figure 6: The effect of varying competition on welfare for high (left panel) and low (right panel) quality sellers

Figure 6 illustrates the effects of varying competition for each type separately. The left panel varies π_h, holding π_l fixed, and shows that more competition for high quality sellers always reduces welfare;

intuitively, more surplus to high quality sellers tightens the incentive constraints and reduces trade. The right panel varies π_l, holding π_h fixed, which has two effects (exactly as in section 5.2). First, it increases surplus to low quality sellers, which relaxes incentive constraints and increases trade with high quality sellers. Second, it makes low quality sellers relatively less attractive to buyers, inducing them to compete more aggressively for the high quality seller, reducing trade. These two competing forces lead to a non-monotonic relationship between π_l and welfare, provided π_h is sufficiently high.[41]

7.3 Differentiation and Multi-Dimensional Heterogeneity

In this section, in order to enhance the applicability of our framework to applied work, we introduce various types of additional heterogeneity: across buyers, across contracts, and across sellers. In various ways, these generalizations break the stark relationship between a seller's type, the offer she accepts, and the rank of that offer within the distribution of all offers. The cost of these generalizations is some degree of tractability, though we argue that, in most cases, the properties and characterization of equilibria are very similar to the baseline framework. For brevity, we restrict attention to the region of the parameter space where almost all equilibrium menus are separating and not cross-subsidizing.

Horizontal differentiation across buyers. Consider first the possibility that buyers are horizontally differentiated. Specifically, as in the discrete choice model of McFadden (1974), we assume that the payoff to a seller of type i from a contract (x, t) offered by buyer k is

$$u_{ik} \;=\; (1-x)\,c_i + t + \epsilon_k \;=\; u_i + \epsilon_k\,,$$

where ϵ_k is a buyer-specific preference shock drawn from a continuous distribution G with support $[\underline{\epsilon}, \bar{\epsilon}]$. Note that ϵ is the same for both seller types, so it has no effect on the incentive constraints. Hence, we may once again represent each equilibrium menu by a utility pair (u_l, u_h). A captive seller accepts this menu if u_{ik} is greater than her outside option, c_i, which occurs with probability

$$\tilde{F}_i^c\,(u_i) = \int_{c_i-u_i}^{\bar{\epsilon}} dG(\epsilon) = 1 - G\,(c_i - u_i)\;. \tag{43}$$

[41] When π_h is low, we enter the region with mass points before the second (negative) effect begins to dominate. Since a mass point equilibrium puts full mass at v_l, increasing π_l beyond this point has no effect on welfare.

A non-captive seller of type i accepts this menu if $u_i + \epsilon > \max(u_i' + \epsilon', c_i)$, which occurs with probability

$$\begin{aligned}\tilde{F}_i^{nc}(u_i) &= \int_{\underline{u}_i}^{\bar{u}_i} \int_{c_i - u_i}^{\bar{\epsilon}} \left(\int_{\underline{\epsilon}}^{u_i + \epsilon - u_i'} dG(\epsilon') \right) \quad dG(\epsilon) \quad dF_i(u_i') \\ &= \int_{\underline{u}_i}^{\bar{u}_i} \int_{c_i - u_i}^{\bar{\epsilon}} G(u_i + \epsilon - u_i') \quad dG(\epsilon) \quad dF_i(u_i'), \end{aligned} \tag{44}$$

where F_i is the marginal distribution of utilities offered to type i sellers in equilibrium. We can write the buyer's problem as

$$\max_{u_h', u_l'} \quad \sum_i \underbrace{\left[(1-\pi)\tilde{F}_i^c(u_i') + \pi \tilde{F}_i^{nc}(u_i')\right]}_{M_i(u_i')} \Pi_i(u_l', u_h'). \tag{45}$$

In a separating equilibrium, optimality with respect to u_l requires

$$\frac{m_l(u_l)}{M_l(u_l)}(v_l - u_l) = \phi_l. \tag{46}$$

In other words, the link between the trading probability and the utility offered to the low quality seller is exactly the same as in our baseline framework, and all of our results go through with respect to the key equilibrium objects M_l and M_h. The only caveat is that the mapping to the underlying distribution of offers F_l and F_h—which are informative about prices and allocations—typically requires numerical methods to solve.[42]

Horizontal differentiation across contracts. The extension above allows for the possibility that a seller accepts a contract from the "wrong" buyer, i.e., accepts u_i even though a contract $u_i' > u_i$ was available. In this section, we allow for the possibility that a seller accepts the "wrong" contract within a menu, i.e., accepts u_{-i} even though her type is i. In particular, suppose that a fraction δ of low quality sellers accept the contract intended for a high quality seller. It is possible to microfound this as a form of "tremble," or as arising from other unmodeled contract features that cause some low quality sellers to prefer the contract with lower quantity and higher price.[43] For example, the high price contract might carry other benefits, such as better customer service, that are valued by some low quality sellers (but not others).

Let $\tilde{v}_h \equiv \frac{\mu_h v_h + \mu_l \delta v_l}{\mu_h + \mu_l \delta}$ be the average value (to the buyer) of assets held by agents who take the contract intended for the high type. We assume that δ is sufficiently small so $\tilde{v}_h > c_h$. The expected profits of

[42] The differential equation in (46), along with the equal profit condition and the system of integral equations in (43) – (44) must be solved jointly for F_i, and this system is only analytically tractable under special assumptions on the distribution G.

[43] For simplicity, we make two additional assumptions. First, a captive low quality seller still chooses the more attractive menu, even when she takes the contract intended for the high quality seller. Second, we assume that the buyer does not (or cannot) try to use contract terms to separate out these low quality sellers.

the buyer, conditional on trade, are then given by $\tilde{\Pi}_h(u_l, u_h) = \tilde{v}_h - \left(\frac{\tilde{v}_h - c_l}{c_h - c_l}\right) u_h + \left(\frac{\tilde{v}_h - c_h}{c_h - c_l}\right) u_l$. As in our baseline model, the FOC for u_l and the equal profit condition pin down F_l and U_h:

$$\frac{\pi f_l(u_l)}{1 - \pi + \pi F_l(u_l)}(v_l - u_l) = 1 - \frac{\mu_h + \mu_l \delta}{\mu_l(1-\delta)}\left(\frac{\tilde{v}_h - c_h}{c_h - c_l}\right) \equiv \tilde{\phi}_l, \tag{47}$$

$$(1-\pi)\mu_l \delta(v_l - c_l) = (1 - \pi + \pi F_l(u_l))\left[\mu_l(1-\delta)(v_l - u_l) + (\mu_h + \mu_l \delta)\tilde{\Pi}_h(u_l, u_h)\right]. \tag{48}$$

Note that these equations are very similar to (20)–(21), with $\tilde{\Pi}_h$ and $\tilde{\phi}_l$ replacing Π_h and ϕ_l. Accordingly, the characterization and other results in the preceding sections directly extend.

Vertical differentiation across buyers. Suppose now that sellers attach a higher value to trading with certain buyers. To be more precise, suppose there are two buyers, $k \in \{1, 2\}$, and that the utility of a type $i$seller from accepting a contract (x, t) from buyer k is given by $c_i(1-x) + t + B^k$, where $B^1 \equiv B > 0$ and B^2 is normalized to zero.[44] This implies that the cost of delivering utility to sellers is lower for buyer 1 or, equivalently, his profits are higher than those of buyer 2, i.e., $\Pi_i^1(u_l, u_h) = \Pi_i^2(u_l, u_h) + B$. Not surprisingly, in this environment, the equilibrium distribution of menus is also asymmetric. Let $F_i^k(u_i)$, $k \in \{1, 2\}$ denote the marginal distribution of utilities offered by buyer k to type j sellers. In Appendix A.13, we characterize an equilibrium where these distributions satisfy the strict rank-preserving property, except at the lower bound of the support, where F_i^2 has a mass point.[45]

Multi-dimensional seller heterogeneity. Finally, our baseline framework posits a tight connection between the valuations of the seller and the buyer. While this is a natural assumption when sellers are heterogeneous along a single dimension—asset quality—it is also natural to consider the case where sellers have heterogenenous preferences as well.[46] A simple way to incorporate this additional heterogeneity into our analysis is to assume that a seller's type is a tuple $(c, \tilde{v})$, with $c \in \{c_h, c_l\}$ denoting the seller's valuation for her asset and $\tilde{v} \in \{\tilde{v}_h, \tilde{v}_l\}$ denoting the buyer's valuation. This allows for the possibility that some high (low) quality assets are held by sellers who, for idiosyncratic reasons, have a low (high) valuation for them. In an asset market interpretation, for example, this could arise from heterogeneity in discount rates or liquidity needs. Let μ_{ij} denote the proportion of sellers of type $(c_i, \tilde{v}_i)$. We can show that it is not possible for buyers to separate sellers with the same c but different $\tilde{v}'s$. Let

[44] Equivalently, and more consistent with our earlier interpretation, one could imagine a measure of buyers, with a fraction of each type $k \in \{1, 2\}$. The simplification here implies that a non-captive seller will always have one offer from a type 1 buyer and one from a type 2 buyer, though this could be relaxed.

[45] Our analysis requires one additional assumption: a seller who is indifferent between two menus chooses the one offered by buyer 1. The resulting system of differential equations can be solved numerically to obtain the equilibrium distributions.

[46] See, for example, Guerrieri and Shimer (2014b), Chang (2012), Finkelstein and McGarry (2006).

$\mu_i = \sum_j \mu_{ij}$ denote the fraction of sellers with valuation c_i, $i \in \{h, l\}$ and $v_i = \frac{\sum_j \mu_{ij} \tilde{v}_j}{\mu_i}$ denote the average value (to the buyer) of the assets held by sellers of type i. Assuming that gains from trade are positive, so that $c_i < v_i$, it is easy to see that our analysis of the baseline model goes through exactly. In other words, additional preference heterogeneity changes the interpretation of buyer values in our baseline model, but otherwise leaves the analysis unchanged.

8 Conclusion

In their survey of the literature on insurance markets, Einav et al. (2010a) note that, despite substantial progress in understanding the effects of adverse selection,

> "there has been much less progress on empirical models of insurance market competition, or on empirical models of insurance contracting that incorporate realistic market frictions. One challenge is to develop an appropriate conceptual framework. Even in stylized models of insurance markets with asymmetric information, characterizing competitive equilibrium can be challenging, and the challenge is compounded if one wants to allow for realistic consumer heterogeneity and market imperfections."

In this paper, we overcome this challenge and develop a tractable, unified framework to study adverse selection, screening, and imperfect competition. We provide a full analytical characterization of the unique equilibrium, and use it to study both positive and normative issues.

Going forward, our framework can be exploited and extended to address a variety of important issues, both applied and theoretical. On the applied side, our equilibrium provides a new structural framework that can be used to *jointly* identify the extent of adverse selection and imperfect competition in various markets, and to study how the interaction of these two frictions affects the *distribution* of contracts, prices, and quantities that are traded. On the theoretical side, there are several obvious extensions to pursue. For example, one natural extension is to endogenize the number of offers buyers receive—by allowing them to solicit offers at a fixed cost—which could help us understand how the severity of adverse selection can affect market structure. Another natural exercise is to study the analog of our model with *nonexclusive contracts*; though this would complicate the analysis considerably, it would also make our framework suitable to analyze certain markets where exclusivity is hard to enforce. We leave all of these exercises for future work.

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Appendix

A Proofs

A.1 General mechanism

Here we show that a buyer cannot gain by offering a deterministic mechanism with more than two contracts in an effort to screen sellers with different outside offers. This ensures that, within the set of deterministic mechanisms, restricting attention to menus that consist of only two contracts is without loss of generality. To show this, we first augment the type space by including competing offers. A seller's type can thus be represented by $(i, \mathbf{z})$, where i denotes the quality of her good and $\mathbf{z}$ her alternate offer. We adopt the convention that $\mathbf{z} = 0$ if a seller is captive.

A deterministic direct mechanism[47] is a mapping from the seller's reported type to an offer $(x(i, \mathbf{z}), t(i, \mathbf{z}))$. The following result shows that, in an optimum, outcomes cannot vary for sellers who differ only in their competing offers.

Claim 23. $x(i, \mathbf{z}) = x(i, \mathbf{z}') = x(i)$ *and* $t(i, \mathbf{z}) = t(i, \mathbf{z}') = t(i)$.

Proof. Consider two offers $(x(i, \mathbf{z}), t(i, \mathbf{z}))$ and $(x(i, \mathbf{z}'), t(i, \mathbf{z}'))$. If both offers are accepted with positive probability in equilibrium, incentive compatibility requires that they must deliver the same level of utility to the seller, i.e.

$$(1 - x(i, \mathbf{z}))c_i + t(i, \mathbf{z}) = (1 - x(i, \mathbf{z}'))c_i + t(i, \mathbf{z}') = u_i \,, \tag{49}$$

for some u_i. If one of them yields a lower utility than the other, then the seller who accepts the former stands to gain by changing her report of z. But, these two offers must also be equally profitable for the buyer, i.e.

$$x(i, \mathbf{z})v_i - t(i, \mathbf{z}) = x(i, \mathbf{z}')v_i - t(i, \mathbf{z}') = \Pi_i \,, \tag{50}$$

for some Π_i. Otherwise, the buyer can increase profits simply by replacing the less profitable one with the other. Since both deliver the same utility to a seller with quality i, this change has no effect on incentives to report $\mathbf{z}$ truthfully. Thus, both offers must solve the same linear system (49) − (50), and therefore must be identical. ■

A.2 Proof of Lemma 1

Proof. Both results are similar to existing results (see, for example, Dasgupta and Maskin (1986)), and thus we keep the exposition brief. To establish that $x_l = 1$ in all equilibrium menus, suppose by way of contradiction that some equilibrium menu $\mathbf{z} = (\mathbf{z}_l, \mathbf{z}_h)$ has $x_l < 1$ and $t_l \in \mathbb{R}_+$, yielding a low quality seller utility u_l. Now, consider a deviation $\mathbf{z}' = (\mathbf{z}'_l, \mathbf{z}_h)$ with $x'_l = x_l + \epsilon$ for $\epsilon \in (0, 1 - x_l]$ and $t'_l = t_l + \epsilon c_l$. Note that $u'_l = u_l$, so that $\mathbf{z}_l$ and $\mathbf{z}'_l$ are accepted with the same probability, but

$$x_l v_l - t_l < x_l v_l - t_l + \epsilon(v_l - c_l) = x'_l v_l - t'_l,$$

so that $\mathbf{z}'_l$ earns the buyer a higher payoff when it is accepted, implying existence of a profitable deviation. Therefore, no equilibrium menu features $x_l < 1$.

To establish that a low quality seller's incentive compatibility constraint binds in all equilibrium menus, suppose by way of contradiction that some equilibrium menu $\mathbf{z} = (\mathbf{z}_l, \mathbf{z}_h)$ has $t_l > t_h + c_l(1 - x_h)$. Now, consider a deviation $\mathbf{z}' = (\mathbf{z}_l, \mathbf{z}'_h)$ with $x'_h = x_h + \epsilon$ and $t'_h = t_h + \epsilon c_h$ for $\epsilon \in \left(0, \frac{t_l - t_h - c_l(1-x_h)}{c_h - c_l}\right]$, which is a non-empty interval by assumption. The upper bound on ϵ ensures that

[47] Here, we invoke a version of the revelation principle for deterministic mechanisms derived by Strausz (2003).

the incentive compatibility constraint on type l sellers is not violated. In addition, note that $u'_h = u_h$, so that $\mathbf{z}_h$ and $\mathbf{z}'_h$ are accepted with the same probability, but

$$x_h v_h - t_h < x_h v_h - t_h + \epsilon(v_h - c_h) = x'_h v_h - t'_h,$$

so that $\mathbf{z}'_h$ earns the buyer a higher payoff when it is accepted, implying existence of a profitable deviation. Therefore, in all equilibrium menus, the type l seller's incentive constraint binds. ■

A.3 Proof of Proposition 3

We prove the proposition through the following sequence of Lemmas.

Lemma 24. $F_h(\cdot)$ *has no flats.*

Proof. Suppose by way of contradiction that $F_h(\cdot)$ is flat in an interval (u_{h1}, u_{h2}). In other words, there exists $(u_{l2}, u_{h2}) \in Supp(F_l) \times Supp(F_h)$ such that, for some $\bar{\varepsilon} > 0$, the distribution F_h satisfies $F_h(u_{h2}) = F_h(u_{h2} - \varepsilon)$ for all $\epsilon \in [0, \bar{\varepsilon}]$. We prove that there must exist a profitable deviation. The particular deviation we construct depends on whether $u_{l2} < u_{h2}$ or $u_{l2} = u_{h2}$ and whether F_l is flat on an interval containing u_{l2} or not. We consider each relevant case in turn:

1. Suppose that $u_{l2} < u_{h2}$. In this case a deviation to $(u_{l2}, u_{h2} - \varepsilon')$ with $\varepsilon' < \varepsilon$ is feasible and must be profitable because such a deviation increases profits earned from trading with h types but does not change the fraction of h types attracted.

2. Suppose that $u_{l2} = u_{h2}$ and F_l is flat below u_{l2}. In this case, a deviation of the form $(u_{l2} - \varepsilon', u_{h2} - \varepsilon')$ for a small but positive ε' is profitable since it increases profits per trade (from both l and h type sellers) but does not change the fraction of either type attracted.

3. Suppose $u_{l2} = u_{h2}$ and F_l is not flat below u_{l2}. Such a situation is depicted in Figure 7. Point A represents the contract (u_{l2}, u_{h2}). Since F_h is flat by assumption, the area between the two **red** dashed lines must not contain any equilibrium menu. Since F_l is not flat below u_{l2} by assumption and there are no menus in the area between the red dashed lines, an equilibrium contract must exist in the region where the point D is located—recall, since $u_h \geqslant u_l$, the point D cannot lie below the lower red dashed line. Let point D represent such an equilibrium menu. In addition, let B represent a menu with the same offer to the low type as D but offers u_{h2} to the high type. Similarly, let C represent a menu with the same offer to the low type as A and the same offer to the high type as D.

   For any distributions, F_l and F_h, the profit function, $\Pi(u_l, u_h)$ is weakly super-modular so that

   $$\Pi_A + \Pi_D \leqslant \Pi_C + \Pi_B.$$

   Since both D and A are offered in equilibrium, we must have that $\Pi_A = \Pi_D \geqslant \Pi_C, \Pi_B$. This implies that $\Pi_A = \Pi_B$. Additionally, since F_h is flat between B and E (and these menus offer the same u_l), it must be that $\Pi_E > \Pi_B$. Therefore, this is a profitable deviation.

■

Lemma 25. $F_l(\cdot)$ *has no flats.*

Proof. Suppose by way of contradiction that F_l is flat in an interval (u_{l1}, u_{l2}). Without loss of generality, we can complete the measure Φ to include menus with first element given by u_{l1} and u_{l2}. Since the profit function is weakly supermodular, then the policy correspondence must be weakly increasing. Now consider the policy correspondences $U_h(u_{l1})$ and $U_h(u_{l2})$. Note that $Cl(U_h(u_{l1}))$ and $Cl(U_h(u_{l2}))$

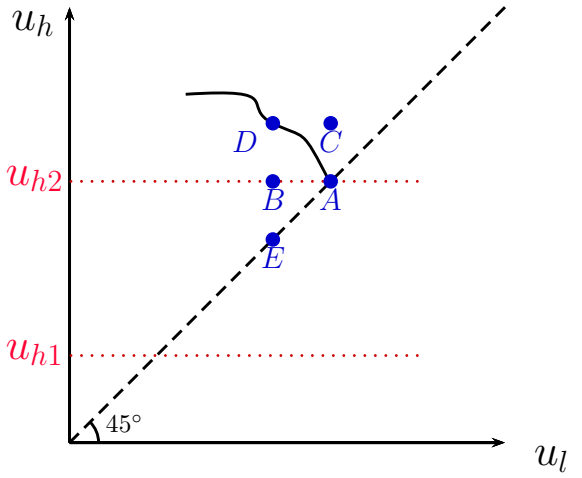


Figure 7: A graphical illustration of why F_h cannot be flat.

cannot be disjoint—if they were, then there would be a flat in the support of F_h, which contradicts Lemma 24. Let $\hat{u}_h$ be a common value in the two sets. We present a depiction of such a situation in Figure 8 below.

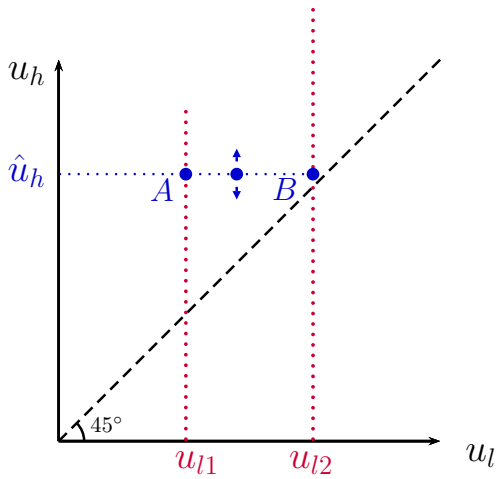


Figure 8: A graphical illustration of why F_l cannot be flat.

Holding $\hat{u}_h$ fixed, the profit function must be linear over the set (u_{l1}, u_{l2}) since $F_l(\cdot)$ is flat by assumption. Therefore, all the menus on the line AB must also deliver profits equal to equilibrium profits. However, since profits earned from trading with h types are increasing in u_l, the marginal benefit of a change in u_h is changing along the line AB. As a result, it possible to construct an upward or downward deviation along AB that increases profits, implying existence of a profitable deviation. ■

Lemma 26. *Φ has no mass point.*

Proof. Suppose by way of contradiction that Φ has a mass point at the menu (u_l, u_h). Let m denote the mass at this menu. Since for any such menu, a deviation of the form $(u_l + \varepsilon_1, u_h + \varepsilon_2)$ for small $\varepsilon_1, \varepsilon_2$ (one of which is positive or negative) must be feasible, profits earned from the mass of sellers attracted

to such deviation must be zero:

$$\mu_l \pi \frac{m}{2} \Pi_l(u_l) + \mu_h \pi \frac{m}{2} \Pi_h(u_l, u_h) = 0.$$

If the menu (u_l, u_h) is interior to the constraint set—that is, if $c_h - c_l > u_h - u_l > 0$—then a simple deviation along u_l or u_h will be feasible and profitable. However, it is possible that (u_l, u_h) is on the boundary of the set and as a result not all deviations are feasible. There are two relevant possibilities:

1. Suppose that the menu with mass, (u_l, u_h), satisfies $u_h = u_l + c_h - c_l$. In such a case, the menu features no trade with the high type. Therefore, it must be that $\Pi_h \leqslant 0$. Since equilibrium profits are strictly positive, it must be that $\Pi_l > 0$. Hence, an infinitesimal increase in u_l, which is feasible, increases profits.

2. Suppose that the menu with mass, (u_l, u_h) satisfies $u_h = u_l$. Then (u_l, u_h) is a pooling menu. Therefore, the profits from the high type must be positive. As a result, the buyer offering this contract could increase profits with an infinitesimal increase in u_h (which would attract a mass of high types), while holding u_l constant.

■

Lemma 27. $F_h(\cdot)$ *does not have a mass point.*

Proof. Suppose by way of contradiction that F_h has a mass point. From Lemma 26, we know that this mass point could not have been created from a mass point in Φ. Therefore, if F_h has a mass point at $\hat{u}_h$, it must be that a positive measure set of the form $\{(u_l, \hat{u}_h)\}$ exists. Figure 9, depicts this possibility.

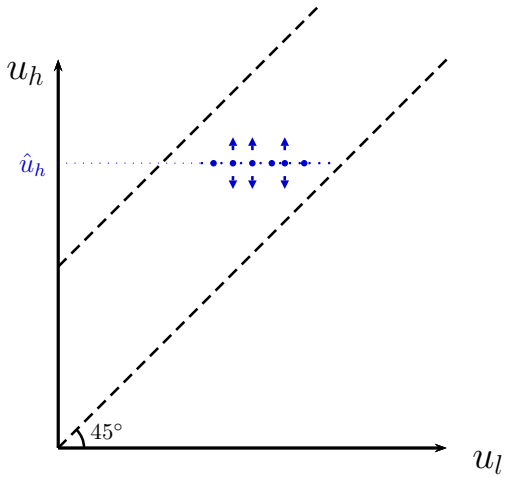


Figure 9: A graphical illustration of why F_h cannot have a mass point.

Note that at one of the points on the line, profits from the h type, $\Pi_h(u_l, \hat{u}_h)$ must be non-zero since Π_h is strictly increasing in u_l. Therefore a small deviation upward or downward increases profits; this implies existence of a profitable deviation and yields the necessary contradiction. ■

To show that F_l has no mass points, we make use of the strict super-modularity of the profit function, which only relies on the continuity of F_h. We therefore provide a proof of the strict super modularity of the profit function here.

Proof of Lemma 4. Suppose $u_{l2} > u_{l1}$ and $u_{h2} > u_{h1}$. Then, letting $\xi_1 \equiv \frac{v_h - c_h}{c_h - c_l} > 0$ and $\xi_2 \equiv \frac{v_h - c_l}{c_h - c_l} > 0$,

$$\begin{aligned}
& \Pi(u_{l1}, u_{h2}) - \Pi(u_{l1}, u_{h1}) \\
= \; & \mu_h \{[1 - \pi + \pi F_h(u_{h2})]\Pi_h(u_{l1}, u_{h2}) - [1 - \pi + \pi F_h(u_{h1})]\Pi_h(u_{l1}, u_{h1})\} \\
= \; & \mu_h \{[1 - \pi + \pi F_h(u_{h2})]\left[v_h + \xi_1 u_{l1} - \xi_2 u_{h2}\right] - [1 - \pi + \pi F_h(u_{h1})]\left[v_h + \xi_1 u_{l1} - \xi_2 u_{h1}\right]\} \\
< \; & \mu_h \{[1 - \pi + \pi F_h(u_{h2})]\left[v_h + \xi_1 u_{l2} - \xi_2 u_{h2}\right] - [1 - \pi + \pi F_h(u_{h1})]\left[v_h + \xi_1 u_{l2} - \xi_2 u_{h1}\right]\} \\
= \; & \Pi(u_{l2}, u_{h2}) - \Pi(u_{l2}, u_{h1}),
\end{aligned}$$

where the inequality follows from the fact that F_h is strictly increasing, and hence

$$\pi\xi_1(u_{l2} - u_{l1})[F_h(u_{h2}) - F_h(u_{h1})] > 0.$$

■

Lemma 28. *F_l is continuous except possibly at v_l.*

Proof. Suppose by way of contradiction that F_l is not continuous and thus has a mass point at some $\hat{u}_l$. Again, by Lemma 26, it must be that a positive measure set of the form $S = \{(\hat{u}_l, u_h)\}$ exists. It is immediate that $\Pi_l(\hat{u}_l) = 0$; otherwise, it would be profitable to increase or decrease u_l by ε if $\Pi_l(\hat{u}_l) > 0$ or $\Pi_l(\hat{u}_l) < 0$, respectively. If $\Pi_l(\hat{u}_l) = 0$, then it must be $\hat{u}_l = v_l$. ■

A.4 Proof of Proposition 10

Proof. We first show that the equilibrium allocations constructed in (20) and (21) are indeed separating and interior. Our construction ensures that local deviations are not profitable. Below we prove that the global deviations are not profitable as well.

Verifying Allocations are Separating and Interior. Note that solution to the differential equation in (21) together with boundary condition $F_l(c_l) = 0$, must satisfy

$$1 - \pi + \pi F_l(u_l) = (1 - \pi)(v_l - c_l)^{\phi_l}(v_l - u_l)^{-\phi_l}. \tag{51}$$

Therefore, from (21), $U_h(u_l)$ must satisfy

$$U_h(u_l) = \frac{1}{\mu_h \frac{v_h - c_l}{c_h - c_l}}\left[\mu_h v_h + \mu_l v_l - \mu_l \phi_l u_l - \mu_l (v_l - c_l)^{1-\phi_l}(v_l - u_l)^{\phi_l}\right].$$

For the allocation to be separating, we must verify that

$$u_l + c_h - c_l \geqslant U_h(u_l) > u_l, \forall u_l \in \text{Supp}(F_l) \tag{52}$$

where

$$\text{Supp}(F_l) = \left[c_l, v_l - (1 - \pi)^{\frac{1}{\phi_l}}(v_l - c_l)\right].$$

The second inequality in (52), $U_h(u_l) > u_l$, is satisfied if and only if

$$\mu_h v_h + \mu_l v_l > \mu_l (v_l - c_l)^{1-\phi_l}(v_l - u_l)^{\phi_l} + u_l \tag{53}$$

for all $u_l \in \text{Supp}(F_l)$. Let $H(u_l)$ denote the right hand side of (53). We argue that $H(\cdot)$ is strictly concave and attains its maximum at a value $u_l^* \in [c_l, v_l]$ with $H(u_l^*) < \mu_h v_h + \mu_l v_l$, implying that (53) is satisfied

for all $u_l \in Supp(F_l)$. To see this, note that

$$H'(u_l) = -\phi_l \mu_l (v_l - c_l)^{1-\phi_l} (v_l - u_l)^{\phi_l - 1} + 1 \tag{54}$$

$$H''(u_l) = \phi_l (\phi_l - 1) \mu_l (v_l - c_l)^{1-\phi_l} (v_l - u_l)^{\phi_l - 2} < 0, \tag{55}$$

where the inequality in (55) is implied by the fact that $0 < \phi_l < 1$. Also, since $\phi_l < 1$, $H'(v_l) = -\infty$ and $H'(c_l) = 1 - \phi_l \mu_l > 0$, so that the maximum of $H(u_l)$ is attained on the interior of $[c_l, v_l]$.

The function $H(u_l)$ is maximized at u_l^* given by

$$u_l^* = v_l - (\phi_l \mu_l)^{\frac{1}{1-\phi_l}} (v_l - c_l)$$

with

$$H(u_l^*) = v_l + (v_l - c_l) \, \mu_l^{\frac{1}{1-\phi_l}} \phi_l^{\frac{\phi_l}{1-\phi_l}} \left[1 - \phi_l\right].$$

Since $c_h \geqslant v_l$ and $\phi_l < 1$, it is immediate that

$$(\phi_l \mu_l)^{\frac{\phi_l}{1-\phi_l}} < 1 \leqslant \frac{(c_h - c_l)(v_h - v_l)}{(v_l - c_l)(v_h - c_h)}$$

which implies

$$(v_l - c_l) \, \mu_l \, (\phi_l \mu_l)^{\frac{\phi_l}{1-\phi_l}} \frac{\mu_h}{\mu_l} \frac{v_h - c_h}{c_h - c_l} < \mu_h (v_h - v_l).$$

Hence,

$$(v_l - c_l) \, \mu_l \, (\phi_l \mu_l)^{\frac{\phi_l}{1-\phi_l}} (1 - \phi_l) < \mu_h (v_h - v_l)$$

and

$$\max_{u_l \in [c_l, v_l]} H(u_l) = H(u_l^*) = v_l + (v_l - c_l) \, \mu_l \, (\phi_l \mu_l)^{\frac{\phi_l}{1-\phi_l}} (1 - \phi_l) < \mu_h (v_h - v_l) + v_l$$

as needed.

We now establish that the first inequality in (52) is true, which requires showing that

$$\frac{\mu_h v_h + \mu_l v_l - \mu_l \phi_l u_l - \mu_l (v_l - c_l)^{1-\phi_l} (v_l - u_l)^{\phi_l}}{\mu_h \frac{v_h - c_l}{c_h - c_l}} \leqslant u_l + c_h - c_l,$$

or, equivalently,

$$\mu_h c_l + \mu_l v_l \leqslant u_l + \mu_l (v_l - c_l)^{1-\phi_l} (v_l - u_l)^{\phi_l}, \forall u_l \in Supp(F_l) \subset [c_l, v_l]. \tag{56}$$

Since, the right side of (56) is a concave function, it takes its minimum values at the extremes of the interval $[v_l, c_l]$. These values are given by v_l and $\mu_l v_l + \mu_h c_l$, both of which are at least as large as the left side of (56). Hence, (56) must be satisfied for all $u_l \in [v_l, c_l]$, as needed.

Global Deviations. Note that our conditions (20) and (21) imply that local deviations with respect to u_h and u_l are not profitable. It, thus, remains to show that, for all (u_l', u_h'), $\Pi(u_l', u_h') \leqslant \mu_l (1 - \pi)(v_l - c_l)$. We consider two types of deviations:

1. Consider first deviation menus with $u_h' > \max Supp(F_h) = \bar{u}_h$. Such deviations attract all type h sellers, so that $1 - \pi + \pi F_h(u_h') = 1$. If $u_l' > \max Supp(F_l) = \bar{u}_l$, then the profits from this menu are given by

$$\mu_l (v_l - u_l') + \mu_h \Pi_h (u_l', u_h').$$

Since $\phi_l > 0$, the above function is decreasing in u'_l and u'_h, and therefore

$$\mu_l (v_l - u'_l) + \mu_h \Pi_h (u'_l, u'_h) < \mu_l (v_l - \bar{u}_l) + \mu_h \Pi_h (\bar{u}_l, \bar{u}_h) = \mu_l (1-\pi)(v_l - c_l).$$

When $u'_l \leqslant \bar{u}_l$, the partial derivative of $\Pi (u'_l, u'_h)$ with respect to u'_l is

$$-\mu_l (1 - \pi + \pi F_l (u'_l)) + \mu_l \pi f_l (u'_l)(v_l - u'_l) + \mu_h \frac{v_h - c_h}{c_h - c_l} \geqslant$$
$$-\mu_l (1 - \pi + \pi F_l (u'_l)) + \mu_l \pi f_l (u'_l)(v_l - u'_l) + \mu_h (1 - \pi + \pi F_l (u'_l)) \frac{v_h - c_h}{c_h - c_l} = 0.$$

Thus for a given value of u'_h, we must have

$$\Pi (u'_l, u'_h) \leqslant \Pi (\bar{u}_l, u'_h) < \Pi (\bar{u}_l, \bar{u}_h) = \mu_l (1-\pi)(v_l - c_l)$$

where the last inequality follows from the fact that Π_h is decreasing in u'_h. Thus, such global deviations are unprofitable.

2. Consider next deviations with $u'_h \in [c_h, \bar{u}_h]$. In this case, there must exist $\tilde{u}_l$ such that $u'_h = U_h (\tilde{u}_l)$ and thus $F_h (u'_h) = F_l (\tilde{u}_l)$. We can thus write the profits obtained from the deviation menu (u'_l, u'_h) as

$$\mu_l (1 - \pi + \pi F_l (u'_l))(v_l - u'_l) + \mu_h (1 - \pi + \pi F_l (\tilde{u}_l)) \Pi_h (u'_l, u'_h). \tag{57}$$

We show that the function defined by (57) is strictly concave in u'_l for values of $u'_l \in \text{Supp}(F_l)$ and decreasing for values of $u'_l > \bar{u}_l$ so that this function is maximized at the value of u'_l which equates its partial derivative with zero. By (20), this partial derivative is zero when evaluated at $u'_l = \tilde{u}_l$, which completes the proof.

Note that for $u'_l \in \text{Supp}(F_l)$, since Π_h is linear in u'_l, the second derivative of (57) with respect to u'_l is given by

$$\frac{\partial^2}{\partial (u'_l)^2} \mu_l (1 - \pi + \pi F_l (u'_l))(v_l - u'_l).$$

Using the form of the distribution F_l given by (51), we may rewrite this second derivative as

$$\begin{aligned} \frac{\partial^2}{\partial (u'_l)^2} \mu_l (1 - \pi + \pi F_l (u'_l))(v_l - u'_l) &= \frac{\partial^2}{\partial (u'_l)^2} \mu_l (1-\pi)(v_l - c_l)^{\phi_l} (v_l - u'_l)^{1-\phi_l} \\ &= (\phi_l - 1)\phi_l \mu_l (1-\pi)(v_l - c_l)^{\phi_l} (v_l - u'_l)^{-1-\phi_l} < 0 \end{aligned}$$

so that (57) is strictly concave in u'_l for values of $u'_l \in \text{Supp}(F_l)$. For values $u'_l > \bar{u}_l$, $1 - \pi + \pi F_l (u'_l) = 1$ and thus (57) satisfies

$$\mu_l (v_l - u'_l) + \mu_h (1 - \pi + \pi F_l (\tilde{u}_l)) \Pi_h (u'_l, u'_h).$$

The derivative of this function with respect to u'_l is given by

$$-\mu_l + \mu_h (1 - \pi + \pi F_l (\tilde{u}_l)) \frac{v_h - c_h}{c_h - c_l} < -\mu_l + \mu_h \frac{v_h - c_h}{c_h - c_l} = -\mu_l \phi_l < 0.$$

Therefore, (57) is maximized at a value of u'_l which equates the partial derivative of (57) with zero. This value must satisfy

$$-\mu_l (1 - \pi + \pi F_l (u'_l)) + \mu_l \pi f_l (u'_l)(v_l - u'_l) + \mu_h (1 - \pi + \pi F_l (\tilde{u}_l)) \frac{v_h - c_h}{c_h - c_l} = 0.$$

Note that since (57) is strictly concave, at most one u_l' exists that satisfies the above. The differential equation (20) implies that $u_l' = \tilde{u}_l$ is a solution to the above equation. This implies that (57) must be maximized at $u_l' = \tilde{u}_l$.

■

A.5 Proofs of Propositions 11 and 12

We prove these Propositions together. To begin, let ϕ_1 be the value of ϕ_l that satisfies

$$c_h \geqslant \nu_l + \frac{\pi(1-\mu_l)\,(\nu_h - \nu_l)}{(1-\pi)\left[(1-\pi)^{\frac{1-\phi_l}{\phi_l}} - 1\right]} \tag{58}$$

with equality. Similarly, let ϕ_2 be the value of ϕ_l that satisfies

$$1 - \pi \geqslant \frac{\mu_h \nu_h + \mu_l \nu_l - \nu_l}{(1-\phi_l)(\mu_h \nu_h + \mu_l \nu_l - c_h)} \tag{59}$$

with equality. We first argue that (58) represents a lower bound on ϕ_l and (59) represents an upper bound on ϕ_l which lies below the lower bound defined by (58). In other words, the inequalities (58) and (59) partition the set $(-\infty, 0]$. We then prove that the equilibrium described in Proposition 12 exists—that is, in each case, no profitable local or global deviations exist when buyers use the equilibrium strategies defined jointly by Propositions 11 and 12.

Lemma 29. *(58) is satisfied if and only if $\phi_1 \leqslant \phi_l < 0$ and (59) is satisfied if and only if $\phi_l \leqslant \phi_2$. Moreover, $\phi_2 < \phi_1 < 0$.*

Proof. First, note that equation (58), which implicitly determines the threshold ϕ_1, can be re-written as

$$(1-\pi)^{\frac{1-\phi_l}{\phi_l}} \geqslant \frac{\pi}{1-\pi}\frac{\nu_h - \nu_l}{c_h - \nu_l}\mu_h + 1, \tag{60}$$

or, after taking logs and substituting for ϕ_l, can be re-written as

$$\frac{\mu_h\,(\nu_h - c_h)}{c_h - c_l - \mu_h\,(\nu_h - c_l)} \log\,(1-\pi) - \log(\mu_h \pi\,(\nu_h - \nu_l) + (1-\pi)\,(c_h - \nu_l)) - \log\,[(1-\pi)\,(c_h - \nu_l)] \geqslant 0. \tag{61}$$

We show that the left side of (61) is a decreasing function of μ_h, that (61) is strictly satisfied when μ_h is such that $\phi_l = 0$, and that (61) is weakly violated when $\mu_h = 1$. Hence, there is a unique threshold μ_1 (and implied threshold ϕ_1) such that for all $\mu_h \leqslant \mu_1$ such that $\phi_l < 0$, the separating condition (58) is satisfied. Differentiating the left side of (61) with respect to μ_h we obtain

$$\log\,(1-\pi)\,\frac{(\nu_h - c_h)\,(c_h - c_l)}{[c_h - c_l - \mu_h\,(\nu_h - c_l)]^2} - \frac{\pi\,(\nu_h - \nu_l)}{\mu_h \pi\,(\nu_h - \nu_l) + (1-\pi)\,(c_h - \nu_l)}$$

which is negative for all $\pi \leqslant 1$. Next, as $\phi_l \to 0$ from below, it is immediate that (60) is satisfied since the left-hand side tends to infinity. As $\mu_h \to 1$, the term $(1-\phi_l)/\phi_l \to -1$ and so (60) tends to the requirement that

$$1 \geqslant \pi\frac{\nu_h - \nu_l}{c_h - \nu_l} + (1-\pi)$$

which is violated since $c_h < \nu_h$.

Next, consider equation (59), which implicitly determines the threshold ϕ_2. Substituting for ϕ_l, one can show the inequality (59) is equivalent to

$$\mu_h (v_h - v_l) \left[1 + (1-\pi)\frac{v_h - c_h}{c_h - c_l}\right] \geqslant v_h - v_l + (c_h - v_l)(1-\pi)\frac{v_h - c_h}{c_h - c_l}. \tag{62}$$

Clearly, (62) represents a lower bound on μ_h, or, equivalently, an upper bound on ϕ_l. Note that this equation is necessarily satisfied at $\mu_h = 1$. It is immediate that when μ_h is such that $\phi_l = 0$, equation (59) is violated since $c_h > v_l$.

We now establish that $\phi_2 < \phi_1$ by proving that $\phi_l \leqslant \phi_2$ implies $\phi_l < \phi_1$. Suppose $\phi_l \leqslant \phi_2$ and let $\bar{v} = \mu_h v_h + \mu_l v_l$, so that we can write (59) as

$$1 - \pi \geqslant \frac{\bar{v} - v_l}{(1-\phi_l)(\bar{v} - c_h)}. \tag{63}$$

Below, we will use the fact that (63) implies

$$1 - \phi_l \geqslant \frac{\bar{v} - v_l}{(\bar{v} - c_h)(1-\pi)} > \frac{\bar{v} - v_l}{\bar{v} - c_h} \quad \Rightarrow \quad -\phi_l > \frac{c_h - v_l}{\bar{v} - c_h}.$$

To prove that (58) is violated when $\phi_l \leqslant \phi_2$, note that (58) can be rearranged as

$$(1-\pi)\left[(1-\pi)^{\frac{1-\phi_l}{\phi_l}} - 1\right](c_h - v_l) - \pi\mu_h (v_h - v_l) \geqslant 0$$

which can be simplified to

$$(1-\pi)(\bar{v} - c_h) + (1-\pi)^{\frac{1}{\phi_l}}(c_h - v_l) \geqslant \bar{v} - v_l. \tag{64}$$

We will show that (64) is violated if (63) holds. Towards this end, define a function

$$H(\pi) = (1-\pi)(\bar{v} - c_h) + (1-\pi)^{\frac{1}{\phi_l}}(c_h - v_l)$$

so that we must show $H(\pi) < \bar{v} - v_l$. We argue that $H(\cdot)$ is a strictly convex function which is decreasing at $\pi = 0$ and that, if π satisfies (63), then $H(\pi) < H(0) = \bar{v} - v_l$, which completes the proof.

First, note that $H(\cdot)$ is strictly convex since $\phi_l < 0$ and

$$\begin{aligned} H'(\pi) &= -(\bar{v} - c_h) - \frac{1}{\phi_l}(1-\pi)^{\frac{1}{\phi_l}-1}(c_h - v_l), \\ H''(\pi) &= \frac{1}{\phi_l}\left(\frac{1}{\phi_l} - 1\right)(1-\pi)^{\frac{1}{\phi_l}-2}(c_h - v_l) > 0. \end{aligned}$$

Next, observe that $H(0) = \bar{v} - v_l$, $H'(0) \leqslant 0$ when $-\phi_l \geqslant (c_h - v_l)/(\bar{v} - c_h)$ and $\lim_{\pi \to 1} H(\pi) = \infty$. Thus, there is a unique value $\pi^s > 0$ such that for all $\pi < \pi^s$, $H(\pi) \leqslant \bar{v} - v_l$.

Next, let $\hat{\pi}$ denote the value of π such that (63) is satisfied with equality. We will prove that $H(\hat{\pi}) < \bar{v} - v_l$, so that $H(\pi) < \bar{v} - v_l$ for all $\pi \leqslant \hat{\pi}$.

Using the expression for $H(\pi)$, we have

$$H(\hat{\pi}) = \frac{\bar{v} - v_l}{(1-\phi_l)(\bar{v} - c_h)}(\bar{v} - c_h) + \left(\frac{\bar{v} - v_l}{(1-\phi_l)(\bar{v} - c_h)}\right)^{\frac{1}{\phi_l}}(c_h - v_l). \tag{65}$$

Straightforward algebra can be applied to (65) to show that $H(\hat{\pi}) < \bar{v} - v_l$ if and only if

$$\left(\frac{c_h - v_l}{\bar{v} - c_h}\right)^{\phi_l} \left(\frac{\bar{v} - v_l}{\bar{v} - c_h}\right)^{1-\phi_l} > (-\phi_l)^{\phi_l} (1-\phi_l)^{1-\phi_l}. \tag{66}$$

Since $(\bar{v} - v_l)/(\bar{v} - c_h) = 1 + (c_h - v_l)/(\bar{v} - c_h)$, if we let $B(x) = x^{\phi_l}(1+x)^{1-\phi_l}$, then (66) can be written as the requirement that

$$B\left(\frac{c_h - v_l}{\bar{v} - c_h}\right) > B(-\phi_l).$$

It is straightforward to show that $B'(x) < 0$ when $0 < x < -\phi_l$, and since (63) implies $-\phi_l > (c_h - v_l)/(\bar{v} - c_h)$, (66) must hold. Consequently, $H(\pi) < H(\hat{\pi}) < \bar{v} - v_l$, which proves that $\phi_1 > \phi_2$. ■

Definition of the Threshold, $\hat{u}_l$. To prove Propositions 11 and 12, we first define the threshold $\hat{u}_l$ for various values of $\phi_l < 0$.

Case 1: $\phi_l \leqslant \phi_2$. The threshold satisfies $\hat{u}_l = \bar{u}_l$, the upper bound of F_l, where $\bar{u}_l$ satisfies

$$\bar{v} - \bar{u}_l = (1-\pi)(\bar{v} - c_h). \tag{67}$$

Case 2: $\phi_2 < \phi_l < \phi_1$. The threshold satisfies

$$v_l + (\hat{u}_l - v_l)\left[1 - \pi + \pi F_l(\hat{u}_l)\right]^{\frac{1}{\phi_l}} = \bar{v} - (1-\pi)(\bar{v} - c_h) \tag{68}$$

where $F_l(\hat{u}_l)$ satisfies (22). As we will see below, in this case, the threshold will be such that $F_l(\hat{u}_l) \in (0,1)$ so that the equilibrium is indeed mixed.

Case 3: $\phi_1 < \phi_l < 0$. The threshold is any value such that $\hat{u}_l < \underline{u}_l$ where the lower bound of the support of F_l satisfies

$$(1-\pi)\left[\mu_l(v_l - \underline{u}_l) + \mu_h \Pi_h(\underline{u}_l, c_h)\right] = \bar{v} - \left[v_l + (1-\pi)^{\frac{1}{\phi_l}}(\underline{u}_l - v_l)\right]. \tag{69}$$

This equation determines the lower bound as the value that equates profits from the worst (separating) menu and the best (pooling) menu where the best menu is determined as the value of u_l such that $F_l(u_l) = 1$ when F_l is determined by (20).

We now prove that the conjectured equilibrium defined implicitly by the thresholds above, in each case, are indeed equilibria.

Lemma 30. *Suppose $\phi_1 \leqslant \phi_l < 0$. There exists an equilibrium with only separating menus.*

Proof. It suffices to ensure that global deviations are unprofitable for buyers since by construction, the distribution $F_l(u_l)$ ensures no local deviations are profitable. To rule out global deviations a proof similar to that of Proposition 10 can be used. We show that for a given value of u'_h, the profit function is strictly concave in u'_l and therefore it must be maximized at $u'_l = U_h^{-1}(u'_h)$ – since at this value the derivative of the profit function is equal to zero (by construction).

Profits from such a global deviation are given by

$$\mu_l\left(1 - \pi + \pi F_l(u'_l)\right)(v_l - u'_l) + \mu_h\left(1 - \pi + \pi F_h(u'_h)\right)\Pi_h(u'_l, u'_h).$$

Since Π_h is linear in u'_l, the second derivative of the above function is equal to the second derivative of profits from l type sellers. Using (20), we know that $\left(1 - \pi + \pi F_l(u'_l)\right) = \kappa(u'_l - v_l)^{-\phi_l}$ for some

non-negative constant κ. Therefore, we have

$$\begin{aligned}\frac{\partial^2}{\partial\left(u_l'\right)^2}\mu_l\left(1-\pi+\pi F_l\left(u_l'\right)\right)\left(v_l-u_l'\right) &= -\mu_l\kappa\frac{\partial^2}{\partial\left(u_l'\right)^2}\left(u_l'-v_l\right)^{1-\phi_l}\\ &= -\mu_l\kappa\left(1-\phi_l\right)\left(-\phi_l\right)\left(u_l'-v_l\right)^{-1-\phi_l}<0\end{aligned}$$

■

Lemma 31. *Suppose $\phi_l \leqslant \phi_2$. There exists an equilibrium with only pooling menus.*

Proof. We first prove that no local deviations in the pooling equilibrium strictly improve profits. Below we demonstrate global deviations are also unprofitable. Recall that in an equilibrium with only pooling menus, the distribution $F_l(u_l)$ satisfies

$$\left(1-\pi+\pi F_l(u_l)\right)\left(\bar{v}-u_l\right)=\left(1-\pi\right)\left(\bar{v}-c_h\right) \tag{70}$$

where $\bar{v}=\mu_h v_h+\mu_l v_l$, $U_h(u_l)=u_l$, $F_h(u_l)=F_l(u_l)$, and $\text{Supp}(F_l)=\left[c_h,\bar{v}-\left(1-\pi\right)\left(\bar{v}-c_h\right)\right]$. Fix any utility, u_l, interior to the support of F_l and consider a local deviation to the menu $(u_l',u_h')=(u_l,u_l+\varepsilon)$. Profits from such a local deviation satisfy

$$\begin{aligned}&\mu_l\left(1-\pi+\pi F_l(u_l)\right)\left(v_l-u_l\right)+\mu_h\left(1-\pi+\pi F_l\left(u_l+\varepsilon\right)\right)\Pi_h\left(u_l,u_l+\varepsilon\right)\\ =\;&\mu_l\left(1-\pi+\pi F_l(u_l)\right)\left(v_l-u_l\right)+\mu_h\left(1-\pi+\pi F_l\left(u_l+\varepsilon\right)\right)\left[v_h-u_l-\varepsilon\frac{v_h-c_l}{c_h-c_l}\right].\end{aligned}$$

If local deviations are unprofitable, this function must be maximized at $\varepsilon=0$, so that F_l must satisfy

$$\mu_h\pi f_l\left(u_l\right)\left[v_h-u_l\right]-\mu_h\left(1-\pi+\pi F_l\left(u_l\right)\right)\frac{v_h-c_l}{c_h-c_l}\leqslant 0.$$

Totally differentiating (70) yields the following relationship between F_l and f_l,

$$\pi f_l(u_l)\left(\bar{v}-u_l\right)=\left(1-\pi+\pi F_l(u_l)\right) \tag{71}$$

so that local deviations are unprofitable if

$$\mu_h\pi f_l\left(u_l\right)\left[v_h-u_l\right]-\mu_h\pi f_l(u_l)\left(\bar{v}-u_l\right)\frac{v_h-c_l}{c_h-c_l}\leqslant 0.$$

Since F_l is continuous in our constructed equilibrium, we may simplify this condition using straightforward algebra as

$$u_l\left(v_h-c_h\right)\leqslant\bar{v}\left(v_h-c_l\right)-v_h\left(c_h-c_l\right).$$

Consequently, we see that it suffices to check that this deviation is unprofitable at $\max\text{Supp}(F_l)$. Using $u_l=\bar{v}-\left(1-\pi\right)\left(\bar{v}-c_h\right)$, simple algebraic manipulations show that this local deviation is unprofitable as long as

$$\frac{\bar{v}-v_l}{\left(1-\phi_l\right)\left(\bar{v}-c_h\right)}\leqslant 1-\pi \tag{72}$$

which is guaranteed by Lemma 29 since $\phi_l\leqslant\phi_2$.

In order to rule out global deviations, we show that for any value of $u_h'\in\text{Supp}\left(F_l\right)$, the profit function in increasing in u_l' for all $u_l'\leqslant u_h'$. Thus, profits are maximized at the pooling menu $u_l'=u_h'$ so that there are no profitable deviations.

Profits associated with any global deviation (u_l', u_h') with $u_l' \leqslant u_h'$ and $u_h' \in \text{Supp}(F_l)$ are given by

$$\mu_l \left(1-\pi+\pi F_l(u_l')\right)\left(\nu_l - u_l'\right) + \mu_h \left(1-\pi+\pi F_l\left(u_h'\right)\right)\Pi_h\left(u_l', u_h'\right).$$

Differentiating, we obtain

$$\begin{aligned}
&\mu_l \pi f_l\left(u_l'\right)\left(\nu_l - u_l'\right) - \mu_l\left(1-\pi+\pi F_l(u_l')\right) + \mu_h\left(1-\pi+\pi F_l\left(u_h'\right)\right)\frac{\nu_h - c_h}{c_h - c_l} \geqslant \\
&\mu_l \pi f_l\left(u_l'\right)\left(\nu_l - u_l'\right) - \mu_l\left(1-\pi+\pi F_l(u_l')\right) + \mu_h\left(1-\pi+\pi F_l\left(u_l'\right)\right)\frac{\nu_h - c_h}{c_h - c_l} = \\
&\mu_l \pi f_l\left(u_l'\right)\left(\nu_l - u_l'\right) - \mu_l \phi_l\left(1-\pi+\pi F_l(u_l')\right) \qquad (73)
\end{aligned}$$

where the inequality follows from the fact that $u_l' \leqslant u_h'$ so that $F_l(u_h') \geqslant F_l(u_l')$. Using (71) to substitute for $\pi f_l(u_l')$, we can write the last line of (73) as

$$\mu_l(1-\pi+\pi F_l(u_l'))\left[1+\frac{\nu_l-\bar{\nu}}{\bar{\nu}-u_l'} - \phi_l\right].$$

Since $u_l' \leqslant u_h' \leqslant \max \text{Supp}\,(F_l)$, the expression in the bracket takes its minimum value at $u_l' = \max \text{Supp}\,(F_l)$ so that

$$1+\frac{\nu_l-\bar{\nu}}{\bar{\nu}-u_l'} - \phi_l \geqslant 1+\frac{\nu_l-\bar{\nu}}{(1-\pi)(\bar{\nu}-c_h)} - \phi_l \geqslant 0$$

where the second inequality follows from (72). This implies that the expression in (73) is positive so that profits are globally maximized at $u_l' = u_h'$ for all $u_h' \in \text{Supp}(F_l)$. ■

Lemma 32. *Suppose $\phi_2 < \phi_l < \phi_1$. There exists a mixed equilibrium.*

Proof. Recall that the threshold $\hat{u}_l$ is such that the constructed equilibrium features pooling contracts for $u_l \in [\min \text{Supp}(F_l), \hat{u}_l]$ and separating menus for $u_l \in (\hat{u}_l, \max \text{Supp}(F_l))$. First, we claim that when $\phi_2 < \phi_l < \phi_1$, then $\hat{u}_l$ is interior in the sense that $c_h < \hat{u}_l < \bar{u}(\hat{u}_l)$. Second, we prove that no local or global deviations are profitable.

To see that $\hat{u}_l$ is interior, conjecture that $\hat{u}_l > c_h$ (we will verify it later), in which case $\hat{u}_l$ must satisfy[48]

$$\bar{\nu} - \left\{\nu_l + (\hat{u}_l - \nu_l)\left[(1-\pi)\frac{\bar{\nu}-c_h}{\bar{\nu}-\hat{u}_l}\right]^{\frac{1}{\phi_l}}\right\} - (1-\pi)(\bar{\nu}-c_h) = 0. \qquad (74)$$

Let $H(\hat{u}_l)$ denote the left-hand side of (74). We will prove that when $\phi_2 < \phi_l < \phi_1$, there are two solutions to $H(\hat{u}_l) = 0$ with $\hat{u}_l > c_h$.

First, observe that one solution to $H(\hat{u}_l) = 0$ is given by

$$\hat{u}_l = \bar{u} = \bar{\nu} - (1-\pi)(\bar{\nu}-c_h).$$

This solution coincides with the conjecture that all menus are pooling and therefore $\bar{u}(\hat{u}_l) = \hat{u}_l$.

We argue that there exists another solution $\hat{u}_l \in (c_h, \bar{u})$. We show this by proving that $H(\cdot)$ is convex, $H(c_h) > 0$ and $H'(\bar{u}) > 0$ so that an additional solution in the interval $(c_h, \bar{u})$ must exist.

Note that

$$H'(u) = -\left[(1-\pi)\frac{\bar{\nu}-c_h}{\bar{\nu}-u}\right]^{\frac{1}{\phi_l}} - (u-\nu_l)\frac{1}{\phi_l}\left[(1-\pi)\frac{\bar{\nu}-c_h}{\bar{\nu}-u}\right]^{\frac{1}{\phi_l}-1}(1-\pi)(\bar{\nu}-c_h)(\bar{\nu}-u)^{-2}.$$

[48] Recall that equilibrium profits satisfy $\bar{\Pi} = (1-\pi)(\bar{\nu}-c_h)$ when the worst menu offered in equilibrium is the pooling, monopsony menu.

By differentiating $H'(\cdot)$ and applying extensive algebraic manipulations (available upon request), one can show that $H''(\cdot) \geqslant 0$. Recall that $\bar{u}$ is defined so that $H(\bar{u}) = 0$ and

$$H'(\bar{u}) = -1 - \frac{1}{\phi_l}\frac{\bar{u} - v_l}{\bar{v} - \bar{u}} = H'(\bar{u}) = \frac{1-\phi_l}{\phi_l}\left[1 - \frac{\bar{v} - v_l}{(1-\pi)(1-\phi_l)(\bar{v} - c_h)}\right]$$

where the second equality is obtained by substituting for $\bar{u}$ and re-arranging terms. When $\phi_l > \phi_2$, the term in brackets is negative, by Lemma 29, so that $H'(\bar{u}) > 0$. Finally, one can show that $H(c_h)$ satisfies

$$H(c_h) \quad = \quad \frac{1}{(1-\pi)^{\frac{1}{\phi_l}} - (1-\pi)}\left[v_l + \frac{\pi(\bar{v} - v_l)}{(1-\pi)^{\frac{1}{\phi_l}} - (1-\pi)} - c_h\right].$$

From Lemma 29, since $\phi_l < \phi_1 < 0$, the term in brackets is strictly positive and since the leading fraction is also positive we must have $H(c_h) > 0$.

Hence, when $\phi_2 < \phi_l < \phi_1 < 0$, there must exist a solution to $H(\hat{u}_l) = 0$ with $\hat{u}_l \in (c_h, \bar{u})$. When $\hat{u}_l < \bar{u}$, one can show that $F_l(\hat{u}_l) < 1$ when F_l is determined by (22) on the interval $[c_h, \hat{u}_l]$ which confirms the conjecture that $\hat{u}_l$ is the interior of the support of F_l.

We now show that buyers cannot improve their profits by deviating from the constructed mixed allocation. As in Lemma 30 with only separation, the distribution F_l for $u_l \in [\hat{u}_l, \max \operatorname{Supp}(F_l)]$ is chosen to ensure local deviations are not profitable. It remains to show, then, that local deviations are not profitable in the pooling region and that no global deviations are profitable. As in Lemma 31 with only pooling menus, it suffices to ensure that at the upper bound of the pooling region, $\hat{u}_l$, no local deviations are profitable, or

$$\hat{u}_l(v_h - c_h) \leqslant \bar{v}(v_h - c_l) - v_h(c_h - c_l). \tag{75}$$

To prove that (75) holds, first note that since $\phi_2 < \phi_l < \phi_1$, we have $c_h < \hat{u}_l < \bar{u}(\hat{u}_l)$. We now prove that (75) is satisfied at $\hat{u}_l$. Algebra, available upon request shows that (75) may be written as

$$\hat{u}_l \leqslant \frac{-\phi_l}{1-\phi_l}\bar{v} + \frac{1}{1-\phi_l}v_l.$$

Since $H(\hat{u}_l) = 0$, if $H\left(\frac{-\phi_l}{1-\phi_l}\bar{v} + \frac{1}{1-\phi_l}v_l\right) \leqslant 0$ then since $H(\cdot)$ is convex, (75) must be satisfied.

Using the form of $H(\cdot)$ implied by the left-hand side of (74), one can show that

$$H\left(\frac{-\phi_l}{1-\phi_l}\bar{v} + \frac{1}{1-\phi_l}v_l\right) = (\bar{v} - v_l)\left[\frac{\bar{v} - v_l - (1-\pi)(\bar{v} - c_h)}{\bar{v} - v_l} + \phi_l\frac{(1-\phi_l)^{\frac{1}{\phi_l}-1}(1-\pi)^{\frac{1}{\phi_l}}(\bar{v} - c_h)^{\frac{1}{\phi_l}}}{(\bar{v} - v_l)^{\frac{1}{\phi_l}}}\right]. \tag{76}$$

We now show that the term in brackets on the right side of (76) is negative. To simplify notation, define $\xi = (1-\pi)(\bar{v} - c_h)/(\bar{v} - v_l)$ so that the term in brackets can be written compactly as

$$1 - \xi + \phi_l(1-\phi_l)^{\frac{1}{\phi_l}-1}\xi^{\frac{1}{\phi_l}}.$$

Let $G(\xi) = 1 - \xi + \phi_l(1-\phi_l)^{\frac{1}{\phi_l}-1}\xi^{\frac{1}{\phi_l}}$ and observe that for $\xi \leqslant 1/(1-\phi_l)$, we have

$$G'(\xi) = -1 + [(1-\phi_l)\xi]^{\frac{1}{\phi_l}-1} \geqslant 0$$

so that for low values of ξ, $G(\xi)$ is an increasing function.

Since $\phi_l > \phi_2$, (63) implies that $\xi < 1/(1-\phi_l)$. Moreover, since $G(1/(1-\phi_l)) = 0$, it must be that $G(\xi) \leqslant G(1/(1-\phi_l)) \leqslant 0$ which ensures the term in brackets in (76) is indeed negative as desired.

To rule out global deviations, one can use the arguments provided in the proofs of Lemmas 30 and 31 in each region of the Supp (F_l). Since the arguments are exact replicas of the arguments above, we omit them here. ■

A.6 Proof of Theorem 13

We begin with a lemma which ensures that when $\phi_l \neq 0$, the marginal distribution F_l is continuous—that is, F_l has no mass points. We then prove uniqueness of the equilibrium first for $\phi_l > 0$ and then for $\phi_l < 0$. (In Appendix A.9, we demonstrate uniqueness for $\phi_l = 0$.)

Lemma 33. *If $\phi_l \neq 0$, then F_l is continuous.*

Proof. Recall from Lemma 28 that if F_l has a mass point then it occurs at $\hat{u}_l = v_l$. As well, from Lemma 26, there must exist a positive measure set $S = \{\hat{u}_l, u_h)\}$ such that each equilibrium menu $(\hat{u}_l, u_h)$ has $\Pi_l = 0$. Let $\underline{u}_h$ denote the lowest value of u_h for which $(\hat{u}_l, u_h)$ belongs to the closure of the set S and let $\bar{u}_h$ denote the highest such value. Without loss of generality, we may assume that $(\hat{u}_l, \bar{u}_h)$ and $(\hat{u}_l, \underline{u}_h)$ belong to S and thus deliver the same profits to a buyer as the equilibrium level of profits.

Consider then the value of two different deviations: $(\hat{u}_l - \varepsilon, \underline{u}_h)$ and $(\hat{u}_l + \varepsilon, \bar{u}_h)$, for a small value of $\varepsilon > 0$, both of which must be feasible. The profits from these deviations are given by

$$\begin{aligned}
\Pi(\hat{u}_l - \varepsilon, \underline{u}_h) &= \mu_h(1 - \pi + \pi F_h(\underline{u}_h))\Pi_h(\hat{u}_l - \varepsilon, \underline{u}_h) + \mu_l(1 - \pi + \pi F_l(\hat{u}_l - \varepsilon))\varepsilon \\
\Pi(\hat{u}_l + \varepsilon, \bar{u}_h) &= \mu_h(1 - \pi + \pi F_h(\bar{u}_h))\Pi_h(\hat{u}_l + \varepsilon, \bar{u}_h) - \mu_l(1 - \pi + \pi F_l(\hat{u}_l + \varepsilon))\varepsilon
\end{aligned}$$

These equalities are valid because F_h does not have a mass point and F_l does not have a mass point for $u_l > v_l$ or $u_l < v_l$. Since F_l is then left or right differentiable at $\hat{u}_l$, we have that

$$\begin{aligned}
\frac{d}{d\varepsilon}\Pi(\hat{u}_l - \varepsilon, \underline{u}_h)\Big|_{\varepsilon=0} &= -\mu_h(1 - \pi + \pi F_h(\underline{u}_h))\frac{v_h - c_h}{c_h - c_l} + \mu_l\left(1 - \pi + \pi F_l^-(\hat{u}_l)\right) \\
\frac{d}{d\varepsilon}\Pi(\hat{u}_l + \varepsilon, \bar{u}_h)\Big|_{\varepsilon=0} &= \mu_h(1 - \pi + \pi F_h(\bar{u}_h))\frac{v_h - c_h}{c_h - c_l} - \mu_l\left(1 - \pi + \pi F_l^+(\hat{u}_l)\right)
\end{aligned}$$

The optimality of menus in S implies that the above expressions must both be non-positive. Since the equilibrium distributions are well-behaved above and below v_l, the equilibrium necessarily exhibits the strict rank-preserving property by Theorem 6 and therefore, $F_l^-(\hat{u}_l) = F_h(\underline{u}_h)$ and $F_l^+(\hat{u}_l) = F_h(\bar{u}_h)$. As a result, the above inequalities imply that

$$\begin{aligned}
-\mu_h\frac{v_h - c_h}{c_h - c_l} + \mu_l &\leqslant 0 \\
\mu_h\frac{v_h - c_h}{c_h - c_l} - \mu_l &\leqslant 0
\end{aligned}$$

When $\phi_l \neq 0$, one of the above is violated. Hence, a profitable deviation exists yielding the necessary contradiction. ■

Case 1: $\phi_l > 0$. As we have shown, any separating equilibrium is uniquely determined. Thus, in order to show the uniqueness of the equilibrium in this case, it remains to show that any equilibrium is separating. To see this, suppose to the contrary that $u_l = u_h$ for some menu offered in equilibrium. Now, consider the following alternative menu $(u_l - \epsilon, u_h)$ for a small and positive value of ϵ. This menu is feasible and the change in the profits for a small value of ϵ is given by

$$\mu_l(1 - \pi + \pi F_l(u_l))\epsilon - \mu_h(1 - \pi + \pi F_h(u_h)\frac{v_h - c_h}{c_h - c_l}\epsilon - \mu_l\pi f_l^-(u_l)(v_l - u_l)\epsilon$$

where f_l^- is the left-derivative of F_l at u_l—recall from Appendix A.3 that F_l must be differentiable.

Using the definition of ϕ_l and strict rank preserving property, we can write the above as

$$\mu_l \phi_l (1 - \pi + \pi F_l(u_l))\epsilon - \mu_l \pi f_l^-(u_l)(v_l - u_l)\epsilon.$$

The above expression must be positive: $\phi_l > 0$, F_l and $f_l^-(u_l)$ are weakly positive, and $u_l > v_l$ since $u_l = u_h \geqslant c_h > v_l$ where $c_h > v_l$ by the lemons assumption. Therefore, this alternative menu is a profitable deviation which yields the necessary contradiction.

Case 2: $\phi_l < 0$. To prove the equilibrium characterized in Proposition 12 is unique, we first prove that in any equilibrium with $\phi_l < 0$, if $\bar{u} = \max \text{Supp}(F_l)$, then $U_h(\bar{u}) = \bar{u}$ so that the best menu in equilibrium is a pooling menu. Next, we prove that if the equilibrium has a pooling region, the region begins at the lower bound of the support of F_l or ends at the upper bound of F_l. Additionally, if the equilibrium features a separating region, this region must end at the upper bound of the support of F_l. These results imply that any equilibrium must take one of the three forms described in Proposition 12: only separating, only pooling, or mixed. Finally, we show that the necessary conditions for each type of equilibrium to exist are mutually exclusive so that at most one type of equilibrium exists for each region of the parameter space, ensuring our equilibrium is unique for all $\phi_l < 0$. We prove these results in the sequence of following lemmas.

Lemma 34. *If $\phi_l < 0$, then the best equilibrium menu is a pooling menu.*

Proof. Let $\bar{u} = \max \text{Supp}(F_l)$ and suppose for contradiction that $U_h(\bar{u}) > \bar{u}$. Consider a deviation menu with $(u_l', u_h') = (\bar{u} + \varepsilon, U_h(\bar{u}))$. Since $U_h(\bar{u}) > \bar{u}$, this menu is incentive compatible and has $F_l(u_l') = F_l(u_h') = 1$. This menu increases the buyer's profits relative to the menu $(\bar{u}, U_h(\bar{u}))$ by the amount

$$-\mu_l \varepsilon + \mu_h \frac{v_h - c_h}{c_h - c_l} = -\mu_l \phi_l \varepsilon > 0$$

where the inequality follows from $\phi_l < 0$. This profitable deviation yields the necessary contradiction so that we must have $U_h(\bar{u}) = \bar{u}$. ■

Lemma 35. *If $\phi_l < 0$ and an equilibrium features $[u_1, u_2] \subseteq \text{Supp}(F_l)$ such that $U_h(u_l) = u_l$ for $u_l \in [u_1, u_2]$, then either $u_1 = \min \text{Supp}(F_l)$ or $u_2 = \max \text{Supp}(F_l)$.*

Proof. Suppose for contradiction that a pooling interval with $u_1 > \min \text{Supp}(F_l)$ and $u_2 < \max \text{Supp}(F_l)$ exists. Then there must exist intervals sufficiently close to and below u_1 and above u_2 respectively in which the equilibrium menus feature separation. Since in these intervals, $U_h(u_l) > u_l$ but $U_h(u_1) = u_1$ and $U_h(u_2) = u_2$, we must have $\lim_{u_l \nearrow u_1} U_h'(u_l) \leqslant 1$ and $\lim_{u_l \searrow u_2} U_h'(u_l) \geqslant 1$. In any region with $U_h(u_l) > u_l$, the distribution F_l must also satisfy

$$\frac{\pi f_l(u_l)}{1 - \pi + \pi F_l(u_l)} = \frac{-\phi_l}{u_l - v_l}$$

since local deviations must be unprofitable. Moreover, in any such region, by the equal profit condition, U_h must satisfy

$$\bar{v} - \mu_l \phi_l u_l - \mu_h \frac{v_h - c_l}{c_h - c_l} U_h(u_l) = \bar{\Pi} (1 - \pi + \pi F_l(u_l))^{-1}$$

where $\bar{\Pi}$ denotes the level of equilibrium profits.

Using these features of the conjectured equilibrium, in the separating regions, $U_h'(u_l)$ satisfies

$$-\mu_l \phi_l - (1 - \mu_l \phi_l) U_h'(u_l) = \frac{\bar{\Pi}}{1 - \pi + \pi F_l(u_l)} \frac{\phi_l}{u_l - v_l}$$

and so U_h'' satisfies

$$-(1-\mu_l\phi_l)\,U_h''(u_l) = \frac{\bar{\Pi}\pi f_l(u_l)}{[1-\pi+\pi F_l(u_l)]^2}\frac{\phi_l}{u_l-v_l} + \frac{\bar{\Pi}}{1-\pi+\pi F_l(u_l)}\frac{-\phi_l}{[u_l-v_l]^2}$$

which implies that U_h is concave when $\phi_l < 0$. However, the existence of the pooling region implies that $U_h'^{+}(u_2) \geqslant 1 \geqslant U_h'^{-}(u_1)$ which contradicts the concavity of U_h given that $u_1 < u_2$. Hence, either $u_1 = \min \mathrm{Supp}(F_l)$ or $u_2 = \max \mathrm{Supp}(F_l)$. ■

Lemma 36. *If $\phi_l < 0$ and an equilibrium features $[u_1, u_2] \subseteq \mathrm{Supp}(F_l)$ such that $U_h(u_l) > u_l$ for $u_l \in (u_1, u_2)$, then $u_2 = \max \mathrm{Supp}(F_l)$.*

Proof. Suppose by way of contradiction that an equilibrium features separation ($U_h(u_l) > u_l$) on an interval $[u_1, u_2] \subseteq \mathrm{Supp}(F_l)$ with $u_2 < \max \mathrm{Supp}(F_l)$. Then there must exist a pooling interval $[u_2, \bar{u}]$ for some $\bar{u}$. Since $u_2 > \min \mathrm{Supp}(F_l)$, Lemma 35 implies that $\bar{u} = \max \mathrm{Supp}\,(F_l)$. Since the conjectured equilibrium features separation in $[u_1, u_2]$ with $U_h(u_l) \to u_l$ as $u_l \to u_2$, we must have $U_h'^{-}(u_2) \leqslant 1$. As the conjectured equilibrium satisfies

$$\frac{\pi f_l(u_l)}{1-\pi+\pi F_l(u_l)} = \frac{-\phi_l}{u_l - v_l}$$

on the interval $[u_1, u_2]$, $U_h'(u_2) \leqslant 1$ implies

$$\frac{1}{1-\mu_l\phi_l}\left[-\mu_l\phi_l + \frac{\bar{\Pi}}{1-\pi+\pi F_l(u_2)}\frac{-\phi_l}{u_2-v_l}\right] \leqslant 1$$

or

$$-\phi_l\bar{\Pi} \leqslant [1-\pi+\pi F_l(u_2)]\,(u_2 - v_l)\,.$$

Since $u_2 < \bar{u}$, $F(u_2) < 1$ so that

$$-\phi_l\bar{\Pi} < u_2 - v_l. \tag{77}$$

Moreover, Lemma 34 ensures that the best equilibrium menu is pooling with utility $\bar{u}$ and therefore equilibrium profits satisfy $\bar{\Pi} = \bar{v} - \bar{u}$. Using the fact that $u_2 < \bar{u}$, substituting for $\bar{\Pi}$ in (77), and rearranging terms, we obtain

$$0 < \phi_l - \frac{v_l - \bar{u}}{\bar{v} - \bar{u}}. \tag{78}$$

We will show that (78) implies a cream skimming deviation must be a profitable deviation from the best (pooling) menu, yielding the necessary contradiction. Since the conjectured equilibrium features pooling in the interval $[u_2, \bar{u}]$, for u_l in this interval, the equilibrium satisfies

$$(1-\pi+\pi F_l(u_l))(\bar{v} - u_l) = (1-\pi)(\bar{v} - \bar{u})$$

so that

$$f_l(u_l) = \frac{1-\pi+\pi F_l(u_l)}{\pi(\bar{v} - u_l)}. \tag{79}$$

Consider then a cream-skimming deviation of the form $(u_l', u_h') = (\bar{u} - \varepsilon, \bar{u})$ which yields profits equal to

$$(1-\pi+\pi F_l(\bar{u}-\varepsilon))\mu_l(v_l - \bar{u} + \varepsilon) + (1-\pi+\pi F_h(\bar{u}))\mu_h\Pi_h(\bar{u}-\varepsilon, \bar{u}). \tag{80}$$

Differentiating (80) with respect to ε, and evaluating it at $\varepsilon = 0$, we obtain

$$(1-\pi+\pi F_l(\bar{u}))\mu_l - \pi f_l(\bar{u})\mu_l(v_l - \bar{u}) - (1-\pi+\pi F_h(\bar{u}))\mu_h\frac{v_h - c_h}{c_h - c_l}$$

which, given that $F_l(\bar{u}) = 1$ and $f_l(\bar{u}) = 1/[\pi(\bar{v} - \bar{u})]$, can be written as

$$\mu_l \left[\phi_l - \frac{v_l - \bar{u}}{\bar{v} - \bar{u}} \right] > 0 \tag{81}$$

where the inequality follows from (78). Hence, this cream-skimming deviation strictly increases the buyers' profits relative to the conjectured equilibrium level, a contradiction. ■

Since the only possible equilibria when $\phi_l < 0$, then, are fully separating (except at the upper bound of the support of F_l), fully pooling, or mixed, we need only prove that only one of these equilibria may exist for any value of ϕ_l. We have already shown in the proof of Proposition 12 that $\phi_2 < \phi_1 < 0$. Recall that a necessary condition for a fully pooling equilibrium is that $\phi_l \leqslant \phi_2$. Hence, there is no fully pooling equilibrium when $\phi_l > \phi_2$. Similarly, a necessary condition for a fully separating equilibrium is that $\phi_l \geqslant \phi_1$ so that when $\phi_l < \phi_1$, no fully separating equilibrium exists. This means that in the interval $\phi_2 < \phi_l < \phi_1$, the only possible equilibrium is a mixed equilibrium. Moreover, the threshold in the mixed equilibrium is interior to the support of F_l only if ϕ_l lies between ϕ_2 and ϕ_1. Hence, at most one of these types of equilibria may exist for any value of $\phi_l < 0$, proving that the equilibrium described in Proposition 13 is unique. ■

A.7 Proof of Proposition 14 and Lemma 15

Proof of Proposition 14. When $\phi_l < 0$, it is immediate that welfare is (weakly) maximized when $\pi = 0$. To prove that welfare is maximized for $\pi \in (0,1)$ when $\phi_l > 0$ we prove that our measure of welfare is strictly increasing in π at $\pi = 0$ and strictly decreasing in π at $\pi = 1$.

Given the function forms for $x_h(u_l), \bar{u}_l$ and $F_l(u_l)$, we can compute the definite integrals in (23) exactly. Extensive algebraic calculations available upon request reveal that welfare is equal to

$$\begin{aligned} W(\pi, \mu_h) = {} & (1-\mu_h)v_l + \mu_h c_h \\ & + 2\frac{(v_l - c_l)(v_h - c_h)}{\pi(v_h - c_l)} \left[(1-\mu_h)(1-\pi)\pi + \frac{(1-\pi)^{1/\phi_l(\mu_h)} - (1-\pi)^2}{\left(\frac{1}{\phi_l(\mu_h)} - 2\right)} + \mu_h \frac{1-(1-\pi)^2}{2} \right] \\ & - \frac{(v_l - c_l)(v_h - c_h)}{\pi(v_h - c_l)} \left[-(1-\mu_h)(1-\pi)^2 \log(1-\pi) + \frac{(1-\pi)^{\frac{1+\phi_l(\mu_h)}{\phi_l(\mu_h)}} - (1-\pi)^2}{\left(\frac{1}{\phi_l(\mu_h)} - 1\right)} + \mu_h \pi(1-\pi) \right]. \end{aligned} \tag{82}$$

where we have written ϕ_l explicitly as a function of μ_h as we will explore properties with respect to μ_h below. Differentiating (82) and rearranging terms, one can show that $W_\pi(\pi, \mu_h)$ satisfies

$$\begin{aligned} \frac{1}{\xi} W_\pi(\pi, \mu_h) = {} & -2(1-\mu_h) - \frac{2}{\pi^2} \frac{\phi_l}{1-2\phi_l} \left[(1-\pi)^{\frac{1}{\phi_l}} - 1 + \pi^2 + \frac{1}{\phi_l} \pi (1-\pi)^{\frac{1-\phi_l}{\phi_l}} \right] \\ & - \frac{(1-\mu_h)}{\pi^2} \left[\pi(1-\pi) + (1-\pi^2)\log(1-\pi) \right] \\ & + \frac{1}{\pi^2} \frac{\phi_l}{1-\phi_l} \left[(1-\pi)^{\frac{1+\phi_l}{\phi_l}} - (1-\pi^2) + \frac{1+\phi_l}{\phi_l} \pi (1-\pi)^{\frac{1}{\phi_l}} \right] \end{aligned} \tag{83}$$

where

$$\xi = \frac{(v_l - c_l)(v_h - c_h)}{v_h - c_l} > 0.$$

As $\pi \to 1$, using (83), it is straightforward to show that

$$\frac{1}{\xi} \lim_{\pi \to 1} W_\pi(\pi, \mu_h) = -2(1-\mu_h) < 0$$

so that $W_\pi(\pi, \mu_h) < 0$ for π sufficiently close to 1. Instead, as $\pi \to 0$, (83) implies that

$$\frac{1}{\xi} \lim_{\pi \to 0} W_\pi(\pi, \mu_h) = \frac{1}{2}\left[\frac{1}{\phi_l} - (1-\mu_h)\right] > 0$$

where the inequality follows since $\phi_l \in (0,1]$. We have thus shown that welfare is increasing in π for π near 0 and decreasing in π for π near 1 as desired. ■

Proof of Lemma 15. Since $\phi_l > 0$, one can show that the introduction of the price floor only changes the characterization of equilibrium in Proposition 10 by changing the boundary condition and the profits from offering the worst equilibrium menu. Formally, Proposition 10 continues to hold, with adjusted boundary condition $F_l(p) = 0$ and adjusted equal profit condition

$$(1-\pi+\pi F_l(u_l))\left[\mu_h \Pi_h(u_l, U_h(u_l)) + \mu_l(v_l - u_l)\right] = (1-\pi)\mu_l\left[v_l - c_l - \phi_l(p - c_l)\right]. \tag{84}$$

Using the adjusted boundary and equal profit conditions, we can determine $F_l(u_l; p)$ and $x_h(u_l; p)$, which allows us to define welfare as a function of π, μ_h, and p according to

$$W(\pi, \mu_h, p) = (1-\mu_h)v_l + \mu_h c_h + \mu_h(v_h - c_h)\int x_h(u_l; p) f_l(u_l; p)[1-\pi+2\pi F_l(u_l; p)]du_l. \tag{85}$$

Straightforward, but tedious calculus (available on request) then proves that W is increasing in p for π near 0, and decreasing in p near $\pi = 1$. ■

A.8 Proof of Proposition 16

Proof. We start with the form of $W(\pi, \mu_h)$ given by (82). Tedious, but straightforward, manipulations can then be used to derive

$$W_{\mu_h}(\pi, \mu_h) = c_h - v_l + \frac{(v_l - c_l)(v_h - c_h)}{(v_h - c_l)}\left[2\pi - 1 + \frac{-(1-\pi)^2 \log(1-\pi) + \phi_l'(\mu_h)\hat{H}(\pi, \mu_h)}{\pi}\right].$$

where $\hat{H}(\pi, \mu_h)$ is a continuous function given on $\phi_l(\mu_h) \in (0,1)$ given by

$$\begin{aligned}
\hat{H}(\pi, \mu_h) = {} & \log(1-\pi)\frac{(1-\pi)^{\frac{1}{\phi_l(\mu_h)}}}{\phi_l(\mu_h)}\left[\frac{1-\pi}{(1-\phi_l(\mu_h))^2} - \frac{2}{(1-2\phi_l(\mu_h))^2}\right] \\
& + \log(1-\pi)(1-\pi)^{\frac{1}{\phi_l(\mu_h)}}\left[\frac{4}{(1-2\phi_l(\mu_h))^2} - \frac{(1-\pi)}{(1-\phi_l(\mu_h))^2}\right] \\
& + \frac{2\left[(1-\pi)^{1/\phi_l(\mu_h)} - (1-\pi)^2\right]}{(1-2\phi_l(\mu_h))^2} - \frac{\left[(1-\pi)^{\frac{1}{\phi_l(\mu_h)}}(1-\pi) - (1-\pi)^2\right]}{(1-\phi_l(\mu_h))^2}
\end{aligned}$$

We will argue that when π is sufficiently small, then

$$\lim_{\mu_h \to 0} W_{\mu_h}(\pi, \mu_h) < \lim_{\mu_h \to \mu_0} W_{\mu_h}(\pi, \mu_h) \tag{86}$$

so that the W_{μ_h} must be increasing on an interval of μ_h – that is, W must be convex on an interval of μ_h. In contrast, the above inequality is reversed when π is sufficiently close to 1.

It is straightforward to show that

$$\begin{aligned}&\lim_{\mu\to\mu_0} W_\mu(\pi,\mu_h) - \lim_{\mu\to 0} W_\mu(\pi,\mu_h)\\ =\ &\left(\frac{\nu_h-c_h}{c_h-c_l}\right)\frac{(\nu_l-c_l)(\nu_h-c_h)}{(\nu_h-c_l)\pi}\left[2(1-\pi)\log(1-\pi)+2(1-\pi)\pi+\frac{1}{2}(1-\pi)^2[\log(1-\pi)]^2+\frac{(1-\pi)^2}{(1-\bar{\mu})^2}\right].\end{aligned}$$

Now, define $M(\pi)$ as

$$M(\pi)=2\log(1-\pi)+2\pi+\frac{1}{2}(1-\pi)[\log(1-\pi)]^2+\frac{(1-\pi)}{(1-\bar{\mu})^2}.$$

Then, inequality (86) is satisfied if and only if $M(\pi)>0$. Note that as $\pi\to 0$, $M(\pi)\to 1/(1-\mu_0)^2>0$ so that for π sufficiently close to 0, inequality (86) is satisfied which implies that $W(\pi,\mu_h)$ is convex in μ_h in some interval of $\mu_h\in(0,\mu_0)$. However, as $\pi\to 1, M(\pi)\to-\infty$ so that inequality (86) is violated, implying that $W(\pi,\mu_h)$ is concave in μ_h in some interval of $\mu_h\in(0,\mu_0)$. ■

A.9 Masspoint Equilibria

Proposition 37. *Suppose $\phi_l=0$. The unique equilibrium of the game is described by the pair of distribution functions, with $F_l(u_l)$ degenerate at ν_l and $F_h(u_h)$ satisfying*

$$(1-\pi+\pi F_h(u_h))\,\mu_h\Pi_h(\nu_l,u_h)=(1-\pi)\,\mu_l(\nu_l-c_l) \tag{87}$$

with $\text{Supp}(F_h)=[c_h,c_h+\pi(\nu_l-c_l)(\nu_h-c_h)/(\nu_h-c_l)]$.

Proof of Proposition 37. To show that the constructed distributions constitute an equilibrium, we show that there are no profitable deviations. In other words,

$$\forall(u_h',u_l'):\mu_h(1-\pi+\pi F_l(u_l'))\,\Pi_h(u_l',u_h')+\mu_l(1-\pi+\pi F_l(u_l'))(\nu_l-u_l')\leqslant(1-\pi)\,\mu_l(\nu_l-c_l)$$

We consider two cases:

1. $u_h'>\max\text{Supp}(F_h)=\bar{u}_h$: In this case, when $u_l'>\nu_l$, the profit function is given by

   $$\mu_h\Pi_h(u_l',u_h')+\mu_l(\nu_l-u_l')$$

   Since $\phi_l=0$, the above function is invariant to changes in u_h' and is strictly decreasing in u_l'. Therefore, its value must be less than its value evaluated at $(\bar{u}_h,\nu_l)$ which gives the equilibrium profits. When, $u_l'\leqslant\nu_l$, the profits are given by $\mu_h\Pi_h(u_l',u_h')$ which is decreasing in u_h' and therefore

   $$\mu_h\Pi_h(u_l',u_h')+\mu_l(1-\pi)(\nu_l-u_l')<\mu_h\Pi_h(u_l',\bar{u}_h)+\mu_l(1-\pi)(\nu_l-u_l')$$

   Note that the right hand side of the above inequality is a linear function of u_l' whose derivative is given by

   $$\begin{aligned}\mu_h\frac{\nu_h-c_h}{\nu_l-c_l}-\mu_l(1-\pi) &= \mu_h\frac{\nu_h-c_h}{c_h-c_l}-\mu_l+\mu_l\pi\\ &= -\mu_l\phi_l+\mu_l\pi=\mu_l\pi>0.\end{aligned}$$

Therefore, we must have that

$$\mu_h \Pi_h\left(u_l', \bar{u}_h\right) + \mu_l\left(1-\pi\right)\left(v_l - u_l'\right) \leqslant \mu_h \Pi_h\left(v_l, \bar{u}_h\right) = \left(1-\pi\right)\mu_l\left(v_l - c_l\right)$$

where the last equality follows from (87).

2. $u_h' \in [c_h, \bar{u}_h]$. In this case, when $u_l' > v_l$, profits are given by

$$\begin{aligned}\mu_h\left(1-\pi+\pi F_l\left(u_h'\right)\right)\Pi_h\left(u_l', u_h'\right) + \mu_l\left(v_l - u_l'\right) &\leqslant \mu_h\left(1-\pi+\pi F_l\left(u_h'\right)\right)\Pi_h\left(v_l, u_h'\right)\\ &= \left(1-\pi\right)\mu_l\left(v_l - c_l\right)\end{aligned}$$

where the inequality is satisfied since $u_l' > v_l$ and the last equality follows from (87).

When $u_l' \leqslant v_l$, profits are given by

$$\mu_h\left(1-\pi+\pi F_l\left(u_h'\right)\right)\Pi_h\left(u_l', u_h'\right) + \mu_l\left(1-\pi\right)\left(v_l - u_l'\right)$$

The above function is linear in u_l' and its derivative is given by

$$\begin{aligned}\mu_h\left(1-\pi+\pi F_l\left(u_h'\right)\right)\frac{v_h - c_h}{c_h - c_l} - \mu_l\left(1-\pi\right) &= \left(1-\pi\right)\left(\mu_h\frac{v_h - c_h}{c_h - c_l} - \mu_l\right) + \pi F_l\left(u_h'\right)\frac{v_h - c_h}{c_h - c_l}\\ &= \pi F_l\left(u_h'\right)\frac{v_h - c_h}{c_h - c_l} \geqslant 0\end{aligned}$$

Therefore it is maximized at $u_l' = v_l$. This establishes that there are no profitable deviations.

To conclude the proof,we show that the equilibrium constructed is the unique equilibrium when $\phi_l = 0$.

In order to show uniqueness of equilibrium, it is sufficient to show that the in any equilibrium, F_l, must be degenerate at v_l. When F_l is degenerate at v_l, from Lemmas 24 and 27, we know that F_h must be continuous and strictly increasing and therefore it must satisfy (87).

Suppose that $u_l \neq v_l$ exists that belongs to the support of F_l. Then the proof of Lemma 28 can be used to show that for values of $u_l \neq v_l$, F_l must have no flat and mass points and consequently equilibrium must exhibit Strict Rank Preserving. Now consider any menu for which $u_l < v_l$ and a deviation that increases the value of u_l by a small amount. In this case, F_l is differentiable and we can write the change in profits from such a deviation as

$$\begin{aligned}&\mu_l \pi f_l^+(u_l)(v_l - u_l) - \mu_l(1-\pi+\pi F_l(u_l)) + \mu_h\frac{v_h - c_h}{c_h - c_l}(1-\pi+\pi F_h(u_h)) =\\ &\mu_l \pi f_l^+(u_l)(v_l - u_l) - \mu_l\phi_l(1-\pi+\pi F_l(u_l)) > 0\end{aligned}$$

where in the above f_l^+ is the right derivative of F_l and we have used SRP. The above implies that increasing u_l must be a profitable deviation which proves the contradiction. The case with $u_l > v_l$ is ruled out in a similar fashion. This concludes the proof. ■

A.10 Model with Many Types

A.10.1 Proof of Lemma 17

This proof is a direct extension of the proof of Lemma 1, and hence is omitted for brevity.

A.10.2 Proof of Proposition 19

To show the strict rank preserving property, we first show that F_j's are continuous and strictly increasing. The argument for this claim is inductive.

Step 1: F_N is strictly increasing and continuous.

F_N *is strictly increasing.* Suppose, towards a contradiction, that there is an interval $\left[u'_N, u''_N\right]$ where F_N is constant and takes a value between 0 and 1. Without loss of generality, we can assume that u''_N belongs to some contract that is offered in equilibrium. Let one such menu be given by $\mathbf{u}'' = \left(u''_1, \cdots, u''_N\right)$. Given our assumption that the equilibrium is separating, this menu must maximize $\sum_{i=1}^{N} \mu_i \left(1 - \pi + \pi F_i\left(u_i\right)\right) \Pi_i\left(u_{i-1}, u_i\right)$ over the set of menus that subject to the participation constraints. Now consider a menu given by $\left(u''_1, \cdots, u''_{N-1}, u''_N - \varepsilon\right)$ for a small ε. Since $u''_N > u'_N \geqslant c_N$, this menu satisfies the participation constraint. Moreover, this menu keeps the fraction of non-captive N types constant while increasing profits per N-th type, thus yielding higher profits, a contradiction.

F_N *is continuous.* Suppose, towards a contradiction, that F_N has a mass point at $\hat{u}_N$. Let $\mathbf{u} = (u_1, \cdots, u_{N-1}, \hat{u}_N)$ be an arbitrary equilibrium menu with its N-th element given by $\hat{u}_N$. Note that we must have $\Pi_N\left(u_{N-1}, \hat{u}_N\right) \leqslant 0$ and $\hat{u}_N = c_N$. The fact that $\Pi_N\left(u_{N-1}, \hat{u}_N\right) \leqslant 0$ is immediate, since otherwise a small increase in $\hat{u}_N$ would attain a higher level of profits. Additionally, if $\hat{u}_N > c_N$, then a small decrease in $\hat{u}_N$ would attain higher profits. Such a change increases profits because either $\Pi_N < 0$—in which case this change decreases the probability that an N type accepts the offer discretely—or $\Pi_N = 0$—in which case this change makes profits per N type strictly positive.

Non-positivity of profits together with $\hat{u}_N = c_N$, implies that

$$v_N - \frac{v_N - c_{N-1}}{c_N - c_{N-1}} c_N + \frac{v_N - c_N}{c_N - c_{N-1}} u_{N-1} \leqslant 0 \Rightarrow \frac{v_N - c_N}{c_N - c_{N-1}} u_{N-1} \leqslant \frac{v_N - c_N}{c_N - c_{N-1}} c_{N-1} \Rightarrow u_{N-1} \leqslant c_{N-1}.$$

This inequality, together with the participation constraint, $c_{N-1} \leqslant u_{N-1}$, implies that u_{N-1} must equal c_{N-1} and $\Pi_N = 0$. That is, any menu $\mathbf{u}$ with $\hat{u}_N$ as its N-th element must also satisfy $u_{N-1} = c_{N-1}$, so that F_{N-1} must also have a mass point at c_{N-1}. Repetition of this argument implies that any menu containing a mass point at $\hat{u}_N$ must also satisfy $u_j = c_j$, and thus F_j must have a mass point at c_j. However, then a small increase in u_1 from $u_1 = c_1$ must increase profits, as F_1 puts a mass at c_1 and profits from type 1 sellers are positive. This yields the necessary contradiction.

Step 2: If $\{F_k\}_{k=j+1}^{N}$ are strictly increasing and continuous, then F_j must have the same properties.

To prove this claim, we first prove the following lemma:

Lemma 38. *Suppose that, for some $j \leqslant N-1$, the distributions $\{F_k\}_{k=j}^{N}$ are continuous and strictly increasing. Then there exists a sequence of strictly increasing and continuous functions $\left\{U_{k,j}\left(u_j\right)\right\}_{k=j+1}^{N}$ such that for any menu $\hat{\mathbf{u}}$ offered in equilibrium with its j-th element given by $\hat{u}_j$, $\left(\hat{u}_{j+1}, \cdots, \hat{u}_N\right) = \left(U_{j+1,j}\left(\hat{u}_j\right), \cdots, U_{N,j}\left(\hat{u}_j\right)\right)$.*

Proof. We prove this claim by induction. For any value of u_{N-1}, let $U_N^+\left(u_{N-1}\right)$ be the set of values of u_N such that equilibrium menus exists with $(N-1)$-th and N-th elements given by (u_{N-1}, u_N).

We first show that $U_N^+\left(u_{N-1}\right)$ is a strictly increasing function. Using exactly the same arguments as in the two-type case, it is straightforward to show that: (*i*) $U_N^+\left(u_{N-1}\right)$ must be a strictly increasing correspondence; and (*ii*) if $u, u' \in U_N^+\left(u_{N-1}\right)$, then $[u, u'] \subseteq U_N^+\left(u_{N-1}\right)$. These results are direct implications of strict supermodularity of the function $\mu_N\left(1 - \pi + \pi F_N\left(u_N\right)\right) \Pi_N\left(u_{N-1}, u_N\right)$ and the strict monotonicity of F_N.

Now suppose that for some $\hat{u}_{N-1}$, $U_N^+\left(\hat{u}_{N-1}\right)$ is a correspondence and so contains an interval given by $[u', u'']$. Then

$$\Pr\left(u_{N-1} = \hat{u}_{N-1}\right) = \int_{\left\{\left(u_1, \cdots, u_{N-2}, \hat{u}_{N-1}, u_N\right) \in \mathrm{Supp}(\Phi)\right\}} d\Phi \geqslant F_N\left(u''\right) - F_N\left(u'\right) > 0$$

where the last inequality follows from the fact that F_N is strictly increasing. This inequality implies that F_{N-1} has a mass point at $\hat{u}_{N-1}$, in contradiction with the assumption that F_{N-1} is continuous. Hence, U_N^+ must be a single-valued function.

One can also adapt our arguments from the two-type case to show that $U_N^+(u_{N-1})$ is strictly increasing. If it were constant on an interval, then F_N must have a mass point, contradicting the continuity of F_N. Thus $U_N^+(u_{N-1})$ is a strictly increasing function and we may write profits from the N-th type as function of u_{N-1} only. Let this function be given by $\Pi_N^+(u_{N-1})$.

Next, let $U_{N-1}^+(u_{N-2})$ be defined in a similar fashion as above. Since the profit function,

$$\mu_{N-1}\left(1-\pi+\pi F_{N-1}(u_{N-1})\right)\Pi_{N-1}(u_{N-2},u_{N-1})+\Pi_N^+(u_{N-1})$$

is strictly supermodular and F_{N-1} and F_{N-2} are strictly increasing and continuous, U_{N-1}^+ must be a strictly increasing, single-valued function. Exact repetition of this argument implies that for all $k \in \{j,\ldots,N-1\}$, U_j^+ is a strictly increasing function. Therefore, we must have that

$$U_{k,j}(\hat{u}_j) = U_k^+\left(U_{k-1}^+\left(\cdots\left(U_{j+1}^+(\hat{u}_j)\right)\right)\right)$$

for all $k \in \{j+1,\ldots,N\}$ and this concludes the proof. ■

We now return to proving step 2 of the induction argument.

F_j *is strictly increasing.* Suppose, by way of contradiction, that F_j has a flat over an interval $[u_j', u_j'']$. Much as in Lemma 28, we prove that if F_j is flat on the interval $[u_j', u_j'']$, then the marginal benefit of delivering one additional unit of surplus to type $j+1$ (incorporating the impact on all types $i > j+1$) changes with $u_j \in [u_j', u_j'']$. This fact allows us to show alternative menus with higher levels of profits than the conjectured equilibrium level must exist.

To see this, first let $U_{j+1}^+(u_j)$ be the correspondence defined in the proof of Lemma 38. By our induction assumption and Lemma 38, profits from types $\{j+1,\ldots,N\}$ can be written as

$$\mu_{j+1}\left(1-\pi+\pi F_{j+1}(u_{j+1})\right)\Pi_{j+1}(u_j,u_{j+1})+\Pi_{j+2}^+(u_{j+1})$$

where $\Pi_{j+2}^+(u_{j+1})$ are equilibrium profits constructed by applying $U_{k,j+1}$ as defined in Lemma 38. Note that these profits are strictly supermodular in (u_j,u_{j+1}), and, as a result, $U_{j+1}^+(u_j)$ is a strictly increasing correspondence. Additionally, since F_j is flat over the interval $[u_j', u_j'']$, we must have that $U_{j+1}^+(u_j')$ and $U_{j+1}^+(u_j'')$ must have a common element (as in the proof of Lemma 28). Let $\overline{u}_{j+1}$ be this common element.

Let $\mathbf{u}'$ be an equilibrium menu with j-th element given by u_j' and $(j+1)$-th element given by $\overline{u}_{j+1}$ and $\mathbf{u}''$ be an equilibrium menu with j-th element given by u_j'' and $j+1$-th element given by $\overline{u}_{j+1}$. Note that a perturbation of $\mathbf{u}'$ which increases u_j' by a small amount must not increase profits. Similarly, a perturbation of $\mathbf{u}''$ which decreases $u_j{}''$ by a small amount must not increase profits. Since F_j is flat on $[u_j', u_j'']$, non-positivity of these two perturbations imply

$$-\mu_j F_j(u_j')\frac{\nu_j-c_{j-1}}{c_j-c_{j-1}}+\mu_{j+1}F_{j+1}(\overline{u}_{j+1})\frac{\nu_{j+1}-c_{j+1}}{c_{j+1}-c_j}=0. \tag{88}$$

As a consequence, profits obtained from any menu $\hat{\mathbf{u}}$, which is the same as $\mathbf{u}'$ except at its j-th element and has j-th element equal to $u_j \in [u_j', u_j'']$, must yield the same profits as $\mathbf{u}'$.

We now show that a perturbation from some such $\hat{\mathbf{u}}$ must strictly increase profits. In particular, consider a perturbation from $\hat{\mathbf{u}}$ which increases $u_{j+1} = \bar{u}_{j+1}$ by a small amount, ε. Since F_{j+1} is strictly increasing and continuous, the change in profits from this perturbation is given by

$$\mu_{j+1}f_{j+1}(\overline{u}_{j+1})\Pi_{j+1}(u_j,\overline{u}_{j+1})+\mu_{j+1}\left(1-\pi+\pi F_{j+1}(\overline{u}_{j+1})\right)\frac{\nu_{j+1}-c_j}{c_{j+1}-c_j}+\frac{d}{du_{j+1}}\Pi_{j+2}^+(\overline{u}_{j+1}). \tag{89}$$

Since $f_{j+1}\left(\overline{u}_{j+1}\right) > 0$ and Π_{j+1} is linear in u_j the expression in (89) must be non-zero for some $u_j \in (u'_j, u''_j)$. This implies some menu can strictly raise profits above the conjectured equilibrium level and is a contradiction. Thus, F_j cannot have a flat.

F_j *is continuous.* Now suppose that F_j has a discontinuity at $\hat{u}_j$. As in step 1, it must be that $\Pi_j\left(\hat{u}_{j-1}, \hat{u}_j\right) \leqslant 0$. There are two possibilities: $\hat{u}_j = c_j$ or $\hat{u}_j > c_j$. If $\hat{u}_j = c_j$, then a straightforward adaptation of the argument in step 1—where we proved F_N is continuous—can be applied to yield a contradiction. Hence, consider the second case with $\hat{u}_j > c_j$. Notice immediately that $\Pi_j\left(\hat{u}_{j-1}, \hat{u}_j\right)$ must equal zero, since otherwise a small decrease in $\hat{u}_j$ would strictly increase profits. Since there is a unique value $\hat{u}_{j-1}$ such that $\Pi_j\left(\hat{u}_{j-1}, \hat{u}_j\right) = 0$, if F_j has a mass point at $\hat{u}_j$, F_{j-1} must also have a mass point at some $\hat{u}_{j-1}$. Repeating this argument implies that F_1 must have a mass point, and this mass point must be at v_1 since $u_1 = v_1$ is the unique value such that $\Pi_1(u_1) = 0$.

Let $\mathbf{u} = \left(v_1, \ldots \hat{u}_{j-1}, \hat{u}_j, u_{j+1}, U_{j+2,j+1}\left(u_{j+1}\right), \ldots, U_{N,j+1}\left(u_{j+1}\right)\right)$. Since the distribution functions $F_{j+1}, \ldots, F_N$ have no mass points, $U^+_{j+1}(\hat{u}_j) = [u'_{j+1}, u''_{j+1}]$ for some values u'_{j+1} and u''_{j+1}.

Let $1 \leqslant k \leqslant j$ be the highest index for which $\phi_k \neq 0$—recall, by assumption $\phi_1 \neq 0$ so that $k \geqslant 1$. Now consider two different perturbations from $\mathbf{u}$ where we perturb elements k through j according to

$$\begin{aligned}
\mathbf{u}^- &= \left(v_1, \ldots, \hat{u}_{k-1}, \hat{u}_k - \epsilon, \ldots, u'_j - \epsilon, u'_{j+1}, U_{j+2,j+1}\left(u'_{j+1}\right), \ldots, U_{N,j+1}\left(u'_{j+1}\right)\right), \\
\mathbf{u}^+ &= \left(v_1, \ldots, \hat{u}_{k-1}, \hat{u}_k + \epsilon, \ldots, u'_j + \epsilon, u''_{j+1}, U_{j+2,j+1}\left(u''_{j+1}\right), \ldots, U_{N,j+1}\left(u''_{j+1}\right)\right).
\end{aligned}$$

For small ϵ, the change in the profits from the above perturbations are, respectively, given by

$$\begin{aligned}
&\mu_k(1-\pi+\pi F^-_k(\hat{u}_k))\frac{v_k - c_{k-1}}{c_k - c_{k-1}} + \mu_{k+1}(1-\pi+\pi F^-_{k+1}(\hat{u}_{k+1})) + \cdots + \mu_j(1-\pi+\pi F^-_j(\hat{u}_j)) \\
&\qquad -\mu_{j+1}(1-\pi+\pi F^-_{j+1}(u'_{j+1}))\frac{v_{j+1} - c_{j+1}}{c_{j+1} - c_j}, \\
&-\mu_k(1-\pi+\pi F^+_k(\hat{u}_k))\frac{v_k - c_{k-1}}{c_k - c_{k-1}} - \mu_{k+1}(1-\pi+\pi F^+_{k+1}(\hat{u}_{k+1})) - \cdots - \mu_j(1-\pi+\pi F^+_j(\hat{u}_j)) \\
&\qquad +\mu_{j+1}(1-\pi+\pi F^-_{j+1}(u''_{j+1}))\frac{v_{j+1} - c_{j+1}}{c_{j+1} - c_j}.
\end{aligned}$$

Since the distributions F_i are well behaved above and below each $\hat{u}_i$, the strict rank preserving property implies $F^-_i(\hat{u}_i) = F_{j+1}(u'_{j+1})$, and $F^+_i(\hat{u}_i) = F_{j+1}(u''_{j+1})$ for all values of $i \leqslant j$. We may then write the the change in profits from the above perturbations respectively as

$$\begin{aligned}
&(1-\pi+\pi F^-_k(\hat{u}_k))\sum_{i=k}^{j}\mu_i\phi_i, \\
&-(1-\pi+\pi F^+_k(\hat{u}_k))\sum_{i=k}^{j}\mu_i\phi_i.
\end{aligned}$$

Since k is the highest index below j for which $\phi_k \neq 0$, one of the above expressions must be positive. Therefore, one of the constructed menus increases profits, yielding a contradiction. The claim that equilibrium is strictly rank-preserving then follows immediately from Lemma 38. ■

A.10.3 Proof of Lemma 20

The monopsonist maximizes

$$\mu_1\left(v_1 - u_1\right) + \sum_{i=2}^{N}\mu_i\left[v_i - \frac{v_i - c_{i-1}}{c_i - c_{i-1}}u_i + \frac{v_i - c_i}{c_i - c_{i-1}}u_{i-1}\right] = \sum_{i=1}^{N}\mu_i\left(v_i - \phi_i u_i\right)$$

subject to the monotonicity constraint

$$1 \geqslant \frac{u_n - u_{n-1}}{c_n - c_{n-1}} \geqslant \cdots \geqslant \frac{u_{i+1} - u_i}{c_{i+1} - c_i} \geqslant \frac{u_i - u_{i-1}}{c_i - c_{i-1}} \ldots > 0. \tag{90}$$

Given the linearity in payoffs and constraints, the solution to this problem is a single price offer, i.e. $u_i = c_J$, $i \leqslant J$ and $u_i = c_i$ for $i > J$ for some $J \in \{1, 2...N\}$; see arguments in Myerson (1985) and Samuelson (1984). To see why J must be the *largest* integer such that $\sum_{i=1}^{J-1} \mu_i \phi_i < 0$, suppose otherwise, i.e. $\exists\ k < J$ such that $\sum_{i=1}^{k-1} \mu_i \phi_i < 0$ and the monopsonist sets $u_i = c_k$ for $i \leqslant k$ and $u_i = c_i$ for $i > k$. Then, a deviation which increases all u_i for $i < J$ by ε changes profits by $-\varepsilon \sum_{i=1}^{J-1} \mu_i \phi_i > 0$. ■

A.10.4 Proof of Lemma 21

To show that the best equilibrium menu satisfies $u_i = u_J$ for $i < J$, suppose by way of contradiction that for some $i < J$, $u_i < u_J$. The monotonicity constraint then implies that $u_J > u_{J-1}$; if $u_J = u_{J-1}$, then we must have $u_i = u_{i-1}$ for all $i < J$. Now, consider an alternative menu that increases all the utilities of types below J by ε. The probability of trade with any type does not change (since this is already the best menu), the change in profits is given by $-\varepsilon \sum_{i=1}^{J-1} \mu_i \phi_i$, which is strictly positive by the definition of J in (37).

To show that the worst equilibrium menu satisfies $u_i = c_i$ for $i \geqslant J$, suppose by way of contradiction that $u_{J+k} > c_{J+k}$ for some $k \geqslant 0$. This inequality, together with repeated application of the monotonicity constraint, implies that $u_i > c_i$ for all $i \leqslant J + k$. Now consider an alternative menu that lowers the utility of all types below and including $J + k$ by ε. This does not change the probability of trade as the original menu is the worst menu. However, the change in profits from captive types is $\varepsilon \sum_{i=1}^{J+k} \mu_i \phi_i$, which is positive by the definition of J in (37). ■

A.10.5 The Solution to the System of ODEs in (36)

The general solution to this system of equations depends on the sign of the profits from the lowest types, $v_1 - u_1$. From (35), when $\phi_1 > 0$, this profit is positive while when $\phi_1 < 0$, this profit is negative. In what follows, we assume that the sequence $\gamma_i = \frac{v_i - c_{i-1}}{c_i - c_{i-1}} \frac{\phi_1}{\phi_j}$ takes on different values for all $i \geqslant 2$, i.e., $\gamma_i \neq \gamma_j$.[49] We thus have the following general solution:

$$U_i = \sum_{k=0}^{i} a_{k,i} \left(|v_1 - u_1|\right)^{\gamma_k}$$

with

$$\gamma_0 = 0, \gamma_1 = 1$$

where

$$\begin{aligned} a_{0,i} &= \frac{v_i (c_i - c_{i-1})}{v_i - c_{i-1}} + \frac{v_i - c_i}{v_i - c_{i-1}} a_{0,i-1} \\ a_{k,i} &= \frac{v_i - c_i}{v_i - c_{i-1}} \frac{\gamma_i}{\gamma_i - \gamma_k} a_{k,i-1} \end{aligned}$$

with

$$\begin{aligned} a_{0,1} &= v_1 \\ a_{1,1} &= \mathbf{sgn}(v_1 - u_1) \end{aligned}$$

[49] While it is possible to provide the general solution of the ODEs, this assumption greatly simplifies the formulation.

where **sgn** is 1 if its argument positive while its -1 when its argument is negative.

In the above formulation, the variables $\{a_{i,i}\}_{i=2}^{N}$ are unknown and have to be determined by the boundary conditions in Lemma 21. To do this, for any value of $\underline{u}_1 = \min \text{Supp}(F_1)$, we can use equation (35) to solve for F_1, with the boundary condition that $\underline{u}_1$. We can then find the value of $\overline{u}_1$, i.e., the upper bound of the support of F_1, using $F_1(\overline{u}_1) = 1$. We refer to this value as $\tilde{u}_1(\underline{u}_1)$ as a function of $\underline{u}_1$. The boundary conditions then are given by:

$$\begin{aligned} &U_J(\underline{u}_1) = c_J, \cdots, U_N(\underline{u}_1) = c_N \\ &U_2(\tilde{u}_1(\underline{u}_1)) = \tilde{u}_1(\underline{u}_1), \cdots, U_J(\tilde{u}_1(\underline{u}_1)) = \tilde{u}_1(\underline{u}_1) \end{aligned}$$

The above is a system of $N - J + 1 + J - 1 = N$ equations with N unknowns given by $a_{i,i}{}_{i=2}^{N}$ and $\underline{u}_1$. Solving this system of equations determines the equilibrium.

A.11 Construction of equilibrium for the insurance model

The construction of equilibrium follows the logic of Section 4. For brevity, we focus on the region of the parameter space where all equilibrium menus are separating and involve no cross-subsidization. This obtains when the fraction of type-b agents, μ_b, is sufficiently large. The optimality conditions with respect to u_b and u_g in this case are

$$\frac{\pi f_b(u_b)}{1-\pi+\pi F_b(u_b)}\Pi_b(u_b) = C'(u_b) - \frac{\mu_g}{\mu_b}\left[\frac{\theta_g(1-\theta_g)}{\theta_b-\theta_g}C'\left(u_g^n\right) - \frac{\theta_g(1-\theta_g)}{\theta_b-\theta_g}C'\left(u_g^a\right)\right] \tag{91}$$

$$\frac{\pi f_g(u_g)}{1-\pi+\pi F_g(u_g)}\Pi_g(u_b,u_g) = \frac{(1-\theta_g)\theta_b}{\theta_b-\theta_g}C'\left(u_g^n\right) - \frac{\theta_g(1-\theta_b)}{\theta_b-\theta_g}C'\left(u_g^a\right). \tag{92}$$

These two differential equations, along with the boundary conditions $F_j(\underline{u}_j) = 0$ with $\underline{u}_j \equiv \theta_j w(y-d) + (1-\theta_j)w(y)$, characterize the equilibrium. Note that these are similar in structure to (20), except that the marginal cost of delivering utility varies with the level of utility (this was constant in the linear model). To solve this system, we make use of the SRP relationship, $F_b(u_b) = F_g(U_g(u_b))$, which implies $f_b(u_b) = f_g(U_g(u_b))U_g'(u_b)$. Dividing the first differential equation by the second and using the SRP identities, we obtain

$$\frac{\Pi_b(u_b)U_g'(u_b)}{\Pi_g(u_b,U_g(u_b))} = \frac{C'(u_b) - \frac{\mu_g}{\mu_b}\left[\frac{\theta_g(1-\theta_g)}{\theta_b-\theta_g}C'\left(u_g^n\right) - \frac{\theta_g(1-\theta_g)}{\theta_b-\theta_g}C'\left(u_g^a\right)\right]}{\frac{(1-\theta_g)\theta_b}{\theta_b-\theta_g}C'\left(u_g^n\right) - \frac{\theta_g(1-\theta_b)}{\theta_b-\theta_g}C'\left(u_g^a\right)}, \tag{93}$$

where u_g^n and u_g^a are related to u_b and U_g through (38). Equation (93) is thus an ordinary differential equation in U_g, along with the boundary condition $U_g(\underline{u}_b) = \underline{u}_g$. Note that this does not depend on π. Given U_g, equations (91) – (92) can be solved for the distribution functions.

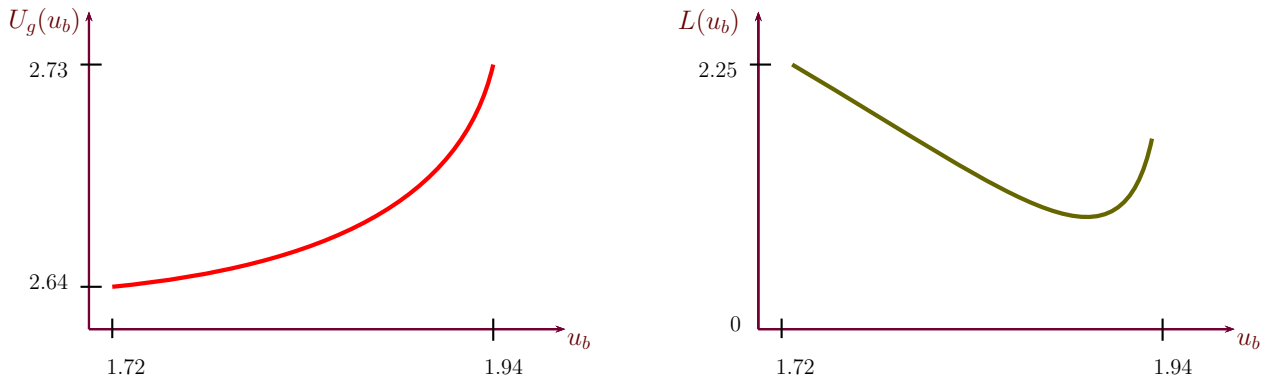
Given a functional form for the utility function, w, this system can be solved numerically. Figure 10 depicts the solution for the following parameterization: $w(c) = \sqrt{2c}$, $y = 10$, $d = 9$, $\theta_b = 0.9$, $\theta_g = 0.6$, $\mu_g = 0.3$. The left panel plots the equilibrium U_g, while the right panel shows the resource losses associated with imperfect insurance—specifically, the function $L(u_b)$ from (39).

A.12 Type-specific π

Since our proofs that F_h and F_l have no flat regions and F_h has no mass points immediately extend to the case when $\pi_l \neq \pi_h$, we omit them in the interest of brevity. Hence, we begin by analyzing the potential for mass point equilibria—that is, for $F_l(\cdot)$ to feature a mass point—to emerge when $\pi_l \neq \pi_h$.

Proposition 39. *Suppose $\pi_l < \pi_h$. Then $F_l(\cdot)$ does not have a mass point.*

Figure 10: Effect of varying competition



Proof. We prove a profitable deviation exists much as in the case when $\pi_l = \pi_h$. In particular, in any such equilibrium with a mass point, $\Pi_l = 0$ and the following inequalities must hold

$$\begin{aligned} -\mu_h \left(1 - \pi_h + \pi_h F_l^- (\hat{u}_l)\right) \frac{v_h - c_h}{c_h - c_l} + \mu_l \left(1 - \pi_l + \pi_l F_l^- (\hat{u}_l)\right) &\leqslant 0 \\ \mu_h \left(1 - \pi_h + \pi_h F_l^+ (\hat{u}_l)\right) \frac{v_h - c_h}{c_h - c_l} - \mu_l \left(1 - \pi_l + \pi_l F_l^+ (\hat{u}_l)\right) &\leqslant 0 \end{aligned}$$

Rearranging the above, we must have

$$\frac{1 - \pi_l + \pi_l F_l^- (\hat{u}_l)}{1 - \pi_h + \pi_h F_l^- (\hat{u}_l)} \leqslant \frac{\mu_h}{\mu_l} \frac{v_h - c_h}{c_h - c_l} \leqslant \frac{1 - \pi_l + \pi_l F_l^+ (\hat{u}_l)}{1 - \pi_h + \pi_h F_l^+ (\hat{u}_l)} \tag{94}$$

Since $F_l^+ (\hat{u}_l) > F_l^- (\hat{u}_l)$ and $\pi_l < \pi_h$, then we must have that

$$\frac{1 - \pi_l + \pi_l F_l^- (\hat{u}_l)}{1 - \pi_h + \pi_h F_l^- (\hat{u}_l)} > \frac{1 - \pi_l + \pi_l F_l^+ (\hat{u}_l)}{1 - \pi_h + \pi_h F_l^+ (\hat{u}_l)}$$

which is a contradiction. ■

Proposition 40. *Suppose $\pi_l > \pi_h$. If a mass points exists, then $F_l(v_l) = 1$.*

Proof. First, it is immediate that a mass point cannot exist for any $u_l \neq v_l$. Hence, suppose by way of contradiction that there is a mass on v_l that is not full. Then either $F_l^- (v_l) > 0$ or $F_l^+ (v_l) < 1$. Since above and below v_l, the equilibrium features no mass points, the equilibrium must also satisfy the strict rank-preserving property. Let $S = \{(v_l, u_h)\}$ and note that S must have positive measure. Furthermore, the set S must be of the form $\{(v_l, u_h) : u_h \in [\underline{u}_h, \bar{u}_h]\}$. Note that we have, $\bar{u}_h > \underline{u}_h \geqslant c_h > v_l$.

Therefore, in a neighborhood around S, all equilibrium menus should be separating. As a result, they must satisfy the optimality condition with respect to u_l—for values of $u_l \in [v_l - \varepsilon, v_l + \varepsilon] \setminus \{v_l\}$ for small but positive ε (depending on whether mass is above or below v_l):

$$-\mu_l \left(1 - \pi_l + \pi_l F_l (u_l)\right) + \mu_l \pi_l f_l (u_l) (v_l - u_l) + \mu_h \left(1 - \pi_h + \pi_h F_h (u_h)\right) \frac{v_h - c_h}{c_h - c_l} = 0$$

Using SRP,

$$-\mu_l \left(1 - \pi_l + \pi_l F_l (u_l)\right) + \mu_l \pi_l f_l (u_l) (v_l - u_l) + \mu_h \left(1 - \pi_h + \pi_h F_l (u_l)\right) \frac{v_h - c_h}{c_h - c_l} = 0$$

Therefore, if positive mass is above v_l, we must have that

$$\mu_h \left(1-\pi_h+\pi_h F_l\left(u_l\right)\right)\frac{v_h-c_h}{c_h-c_l}-\mu_l\left(1-\pi_l+\pi_l F_l\left(u_l\right)\right)>0$$

and if it is below,

$$\mu_h \left(1-\pi_h+\pi_h F_l\left(u_l\right)\right)\frac{v_h-c_h}{c_h-c_l}-\mu_l\left(1-\pi_l+\pi_l F_l\left(u_l\right)\right)<0$$

From above, if mass point is to be an equilibrium property, the inequality (94) must hold:

$$\frac{1-\pi_l+\pi_l F_l^-\left(v_l\right)}{1-\pi_h+\pi_h F_l^-\left(v_l\right)}\leqslant\frac{\mu_h}{\mu_l}\frac{v_h-c_h}{c_h-c_l}\leqslant\frac{1-\pi_l+\pi_l F_l^+\left(v_l\right)}{1-\pi_h+\pi_h F_l^+\left(v_l\right)}<\frac{\pi_l}{\pi_h}. \tag{95}$$

Now suppose that $F_l^+\left(v_l\right)<1$, then from the differential equation above,

$$F_l\left(u_l\right)\left[\mu_h\pi_h\frac{v_h-c_h}{c_h-c_l}-\pi_l\mu_l\right]-\mu_l\pi_l f_l\left(u_l\right)\left(u_l-v_l\right)+\mu_h\left(1-\pi_h\right)\frac{v_h-c_h}{c_h-c_l}-\mu_l\left(1-\pi_l\right)=0$$

The general solution to the above differential equation is given by

$$F_l\left(u_l\right)=A_1\left(u_l-v_l\right)^{\frac{\mu_h\pi_h\frac{v_h-c_h}{c_h-c_l}-\pi_l\mu_l}{\mu_l\pi_l}}+A_2$$

Since $\frac{\mu_h\pi_h\frac{v_h-c_h}{c_h-c_l}-\pi_l\mu_l}{\mu_l\pi_l}<0$ from (95), the above expression approaches either $\pm\infty$ as u_l approaches v_l from above. Hence, $F_l^+\left(v_l\right)<1$ cannot hold.

Now suppose that $F_l^-\left(v_l\right)>0$. Then similar to above, we must have that

$$F_l\left(u_l\right)=A_1\left(v_l-u_l\right)^{\frac{\mu_h\pi_h\frac{v_h-c_h}{c_h-c_l}-\mu_l\pi_l}{\mu_l\pi_l}}+A_2$$

As u_l converges to v_l, the above expression converges to ∞ which is in contradiction with $F_l^-\left(v_l\right)<1$. This proves the claim.

■

A.12.1 Proof of Proposition 22

We have already shown a masspoint equilibrium, if it exists, must full mass at v_l. Now, the worst menu in a masspoint equilibrium (i.e. the one with the lowest u_h) must set $u_h=c_h$ (otherwise, lowering u_h strictly raises profits). By construction, a function F_h that satisfies (41) ensures equal profits at all points in the support. To rule out other deviations, consider the payoff from offering $u_l'=v_l-\varepsilon,\quad u_h'\in\left[\underline{u}_h,\bar{u}_h\right]$. The change in profits (per ε) satisfy

$$\mu_l\left(1-\pi_l\right)-\left(1-\pi_h+\pi_h F_h\right)\mu_h\frac{v_h-c_h}{c_h-c_l}=\left[1-\frac{\left(1-\pi_h+\pi_h F_h\right)}{\left(1-\pi_l\right)}\frac{\mu_h}{\mu_l}\frac{v_h-c_h}{c_h-c_l}\right]\mu_l\left(1-\pi_l\right).$$

It is sufficient to show that this is negative at the bottom, i.e. when $F_h=0$, which leads to

$$1-\frac{\left(1-\pi_h\right)}{\left(1-\pi_l\right)}\frac{\mu_h}{\mu_l}\frac{v_h-c_h}{c_h-c_l}<0\qquad\Rightarrow\qquad\frac{1-\pi_l}{1-\pi_h}<1-\phi.$$

To rule out equilibria without masspoints, note that, in such an environment, the equilibrium is strictly rank-preserving, so there must be a worst menu, i.e. one with $F_l=F_h=0$. Suppose it is a pooling menu - then, it must offer $u_h=u_l=c_h$. In other words, $\Pi_l=v_l-c_h<0$. On the other hand, if it is a

separating one, it must satisfy the FOC for u_l :

$$\frac{\pi_l f_l}{1-\pi_l+\pi_l F_l}\Pi_l = 1-\left(\frac{1-\pi_h}{1-\pi_l}\right)(1-\phi_l) \quad <0 \qquad \Rightarrow \qquad \Pi_l<0$$

i.e. the worst menu in a non-masspoint equilibrium must necessarily lose money on the low type. But then, the best menu must also lose money, because

$$\Pi_l(\bar{u}_l) = \nu_l - \bar{u}_l < \nu_l - \underline{u}_l < 0$$

Now, consider a deviation of the form $(\bar{u}_l-\varepsilon, \bar{u}_h)$ changes profits, relative to $(\bar{u}_l, \bar{u}_h)$, by

$$\mu_l - \mu_h \frac{\nu_h - c_h}{c_h - c_l} - \mu_l f_l \Pi_l(\bar{u}_l) = \mu_l \phi - f_l \Pi_l(\bar{u}_l) \quad > \quad 0$$

yielding the desired contradiction. Thus, in a masspoint equilibrium, the distribution of u_l is degenerate at ν_l, i.e. buyers make zero profits from type-l sellers. A buyer can deviate and offer a lower u_l, but that brings higher profits only from the *captive* l—types at the expense of lower profits from both captive and non-captive h—types. When the condition in part (1) of the proposition is satisfied, π_l is sufficiently high or equivalently, the fraction of captive l—types is too low to make such a deviation attractive.

A.13 Equilibrium with vertical differentiation

Here, we conjecture and characterize an equilibrium with vertical differentiation. We restrict attention to the region of the parameter space where both buyers offer separating contracts without cross-subsidization. First, note that the upper and lower bounds of the distributions of both buyers must coincide, i.e. the distributions of offers by both buyers have the same support. This then implies that F_l^2 has mass of α at its lowest point c_l. To see this, consider the equal profit condition for each buyer (recall that all ties are resolved in favor of buyer 1):

$$\begin{aligned}(1-\pi)(\nu_l - c_l) &= \Pi(\bar{u}_l, \bar{u}_h)\\ (1-\pi+\pi\alpha)(\nu_l - c_l + B) &= \Pi(\bar{u}_l, \bar{u}_h) + B.\end{aligned}$$

Solving, we obtain $\alpha = \frac{B}{B+\nu_l-c_l}$. Next, we posit that (i) $U_h^1(u_l)$ is strictly increasing everywhere in the support (ii) $U_h^2(u_l) = c_h$ for $u_l \in [c_l, c_l+s]$, $s \geqslant 0$. In the interval $(c_l+s, \overline{u}_l]$, $U_h^2(u_l)$ is strictly increasing. Formally, the distributions F_j^k satisfy the strict rank-preserving conditions

$$F_l^1(u_l) = F_h^1(U_h^1(u_l)) \qquad u_l \in [\underline{u}_l, \overline{u}_l] \tag{96}$$

$$F_l^2(u_l) = F_h^2(U_h^2(u_l)) \qquad u_l \in (c_{l+s}, \overline{u}_l]. \tag{97}$$

The optimality conditions for u_l and u_h for the two buyers yield:

$$\frac{\pi f_l^2(u_l)}{1-\pi+\pi F_l^2(u_l)}\Pi_l^1(u_l) = 1-\frac{\mu_h}{\mu_l}\left(\frac{1-\pi+\pi F_h^2\left(U_h^1(u_l)\right)}{1-\pi+\pi F_l^2(u_l)}\right)\frac{\nu_h-c_h}{c_h-c_l} \tag{98}$$

$$\frac{\pi f_h^2(u_h)}{1-\pi+\pi F_h^2\left(U_h^1(u_l)\right)}\Pi_h^1\left(u_l, U_h^2(u_l)\right) = \frac{\nu_h-c_l}{c_h-c_l} \tag{99}$$

$$\frac{\pi f_l^1(u_l)}{1-\pi+\pi F_l^1(u_l)}(\nu_l-u_l) = 1-\frac{\mu_h}{\mu_l}\left(\frac{1-\pi+\pi F_h^1\left(U_h^2(u_l)\right)}{1-\pi+\pi F_l^1(u_l)}\right)\frac{\nu_h-c_h}{c_h-c_l} \tag{100}$$

$$\frac{\pi f_h^1(u_h)}{1-\pi+\pi F_h^1\left(U_h^2(u_l)\right)}\Pi_h^2\left(u_l, U_h^2(u_l)\right) = \frac{\nu_h-c_l}{c_h-c_l}. \tag{101}$$

This system of equations (96) – (101), along with the boundary conditions

$$\begin{aligned}
F_l^1(c_l) &= F_h^1(c_h) = 0 \\
F_l^2(c_l) &= \alpha \\
F_l^1(\overline{u}_l) &= F_l^2(\overline{u}_l) = 1 \\
F_h^1(\overline{u}_h) &= F_h^2(\overline{u}_h) = 1 \\
(1-\pi)(v_l - c_l) &= \left(1 - \pi + \pi F_l^1(c_l + s)\right)(v_l - c_l - s) + (1-\pi)\Pi_h(c_l + s, c_h) \ .
\end{aligned}$$

characterize the 6 unknown functions $F_l^1, F_l^2, F_h^1, F_h^2, U_h^1$, and U_h^2.