Pricing long-lived securities in dynamic endowment economies*

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Abstract

We solve for asset prices in a general affine representative-agent economy with isoelastic recursive utility and rare events. Our novel solution method is exact in two special cases: no preference for early resolution of uncertainty and elasticity of intertemporal substitution equal to one. Our results clarify model properties governed by the elasticity of intertemporal substitution, by risk aversion, and by the preference for early resolution of uncertainty. Our results also highlight that a covariance-based factor structure arises as a very special case, rather than as a general property, of equilibrium models.

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1 Introduction

The framework of representative-agent asset pricing, in which complete markets allows for the diversification of idiosyncratic risks, has for many years delivered benchmark models of the cross-section and time-series of stock prices and returns.[1] These models are at the same time simple and rich in the types of economic intuition they capture. In this paper, we focus on two nested sub-classes of the dynamic representative-agent framework with the goal of clarifying important implications for risk premia and asset prices.

In the first part of the paper, we extend classic cross-sectional results of Merton (1973) and Breeden (1979) to a dynamic setting with recursive utility (Epstein and Zin, 1989) and rare events. This section assumes only isoelastic recursive utility and a Markov structure. Widening the class of models beyond the traditional diffusion framework of Merton (1973) has dramatic implications for the cross-section. The intertemporal capital asset pricing model (ICAPM) of Merton is a standard justifications for the near-universal use of covariance-based factor models in finance. However, the ICAPM relies on conditional log-normality. Without the assumption of conditional log-normality, a factor structure may not hold.

In this section, we also show that the dynamics of the wealth-consumption ratio are principally governed by the elasticity of intertemporal substitution (EIS) and the risk premia in the cross-section relative to a consumption-based model are governed by the preference for early resolution of uncertainty. However, relative to a wealth-based model, risk premia in the cross-section are governed by risk aversion. When risk aversion is equal to one, a rare-event wealth CAPM holds regardless of the EIS. In contrast, a rare-event consumption CAPM holds if there is no preference for early resolution of uncertainty, regardless of risk aversion. Some of these comparative statics have appeared elsewhere in the literature, but the advantage of our frameowrk is that we can show them in a more general setting.

In the second part of the paper, we derive approximate analytical solutions for the

[1] Textbook treatments include Campbell, Lo, and MacKinlay (1997) and Cochrane (2001).

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pricing of long-lived assets. Our solution method takes as its starting point the widely-used method of Campbell and Shiller (1988) which involves a first-order approximation of the price-dividend ratio by a log-linear function. Previous studies use this method to compute the wealth-consumption ratio (which is necessary for computing other asset prices under recursive utility), and then to compute prices on other assets. While we use the method to compute the wealth-consumption ratio, we then, given the approximation, compute prices on other assets exactly. As a consequence, our method, unlike others, is exact both when the elasticity of intertemporal substitution is equal to one, and when utility is time-additive. The reason is that the approximation for the wealth-consumption ratio is exact when the EIS is equal to one, and a log-linear wealth-consumption ratio is not necessary for closed-form solutions for asset prices under time-additive utility.

In an example, we extend the model of Wachter (2013) to a case of non-unitary EIS. We show that our technique is notably closer to the solution when the exact problem is solved numerically, than the standard approximation. Moreover, because our solution is closer to the true solution in a formal sense, it delivers insight into the economics of the problem.

The remainder of the paper proceeds as follows. Section 2 describes our general set-up and derives results for the cross-section. Section 3 describes the affine set-up with analytical solutions. Section 4 quantitatively evaluates the solution method under an example economy.

2 General Model

2.1 Assumptions

Let B_{ct} be a unidimensional Brownian motion and B_{Xt} an n-dimensional Brownian motion, such that B_{ct} and B_{Xt} are independent. For $j = 1, \ldots, m$, let N_{jt} be independent Poisson processes. Consider functions $\mu_c : \mathbf{R}^n \to \mathbf{R}$, $\sigma_c : \mathbf{R}^n \to \mathbf{R}$, $\mu_X : \mathbf{R}^n \to \mathbf{R}^n$, $\sigma_X : \mathbf{R}^n \to \mathbf{R}^n$, and $\lambda : \mathbf{R}^n \rightarrow \mathbf{R}^m$. Assume the endowment follows the process

$$\frac{dC_t}{C_{t^-}} = \mu_c(X_{t^-})\,dt + \sigma_c(X_{t^-})dB_{ct} + \sum_{j=1}^{m}(e^{Z_{cjt}} - 1)dN_{jt}, \tag{1}$$

where X_t is a vector of state variables following the process

$$dX_t = \mu_X(X_{t^-})\,dt + \sigma_X(X_{t^-})dB_{Xt} + \sum_{j=1}^{m} Z_{Xjt}dN_{jt}, \tag{2}$$

and where, for all $j = 1, \ldots, m$ and t, Z_{cjt} are scalar random variables and Z_{Xjt} an $n \times 1$ vector of random variables. We assume the joint distribution of Z_{cjt} and Z_{Xjt} is time-invariant, and thus suppress the t subscript when not essential for clarity. The intensity for Poisson process N_{jt} is time-varying and given by $\lambda_j(x)$, the j-th element of $\lambda(x)$. We adopt the convention that B_{Xt}, and therefore X_t, are column vectors and that σ_c is a row vector.

Consider an asset that pays cash flows determined by the outcome of a dividend process

$$\frac{dD_t}{D_{t^-}} = \mu_d(X_t) + \sigma_d(X_t)dB_{ct} + \sum_{j=1}^{m}\left(e^{Z_{djt}} - 1\right)dN_{jt}. \tag{3}$$

There may be many such assets, but because we will assume complete markets, it will suffice to consider each asset in isolation, and therefore it is not necessary to add a subscript to D_t. Similarly to the rare-event outcomes for consumption and for X_t, Z_{djt} is a random variable with time-invariant distribution for all j.[2] In what follows, let $B_t = [B_{ct}, B_{Xt}^\top]^\top$. For a generic function $h(C_t, D_t, X_t)$, define

$$\mathcal{J}_j(h(C_t, D_t, X_t)) = h(C_t, D_t, X_t) - h(C_{t^-}, D_{t^-}, X_{t^-}) \qquad \text{if } dN_{jt} = 1$$

[2]Dividends are assumed to be perfectly correlated with consumption during normal times. Normal-times dividend risk that is uncorrelated with consumption and the state variables will have no impact on risk premia or on asset prices themselves. Our analysis can easily be extended to allow for dividend risk that is correlated with the state variables.

and we will use

$$\bar{\mathcal{J}}(h(C_t, D_t, X_t)) = \left[E_{\nu_1}\left[\mathcal{J}_1(h(C_t, D_t, X_t))\right], \cdots, E_{\nu_m}\left[\mathcal{J}_m(h(C_t, D_t, X_t))\right]\right]^\top$$

where E_{ν_j} denotes expectations taken with respect to the joint distribution of Z_{cj}, Z_{dj}, Z_{Xj}, and conditional on information just prior to t.

Assume a representative agent with recursively-defined utility

$$V_t = E_t\left[\int_t^\infty f(C_s, V_s)ds\right], \tag{4}$$

with

$$f(C,V) = \frac{\beta}{1-\frac{1}{\psi}}\left((1-\gamma)V\right)\left(\left(C\left((1-\gamma)V\right)^{-\frac{1}{1-\gamma}}\right)^{1-\frac{1}{\psi}} - 1\right). \tag{5}$$

where ψ is the elasticity of intertemporal substitution and γ is risk aversion (Duffie and Epstein, 1992b). When $\gamma = 1/\psi$, the recursion in (5) is linear, and (4) reduces to the time-additive case.

We are interested in two limiting cases of (5). When $\psi = 1$, (5) reduces to

$$f(C,V) = \beta\left((1-\gamma)V\right)\left(\log C - \frac{1}{1-\gamma}\log\left((1-\gamma)V\right)\right). \tag{6}$$

(Duffie and Epstein, 1992a). When $\gamma = 1$, (5) reduces to

$$f(C,V) = \frac{\beta}{1-\frac{1}{\psi}}\left(e^{\left(1-\frac{1}{\psi}\right)(\log C - V)} - 1\right). \tag{7}$$

When $\gamma = \psi = 1$, V_t equivalent to time-additive log utility. In what follows, let $\theta = (1-\gamma)/(1-\frac{1}{\psi})$. Then $\theta = 1$ corresponds to time-additive utility, and $\frac{1}{\theta} = 0$ will, in a formal sense, correspond to $\psi = 1$.

2.2 General characterization of the solution

We first characterize the value function, the wealth-consumption ratio, and the state-price density in terms of the state variables. Here and in the remainder of the paper, we follow the convention that partial derivatives with respect to a vector are row vectors; for example, $\partial I/\partial X = [\partial I/\partial X_1, \cdots, \partial I/\partial X_n]$. Proofs not given in the main text are contained in Appendix A.

Proposition 1 (**Value function**). *Suppose the representative agent's preference is defined by (4)–(6), where the consumption growth process follows (1) and the state variable process follows (2). In equilibrium,* $J(C_t, X_t) = V_t$, *where*

$$J(C_t, X_t) = \frac{C_t^{1-\gamma} I(X_t)^{1-\gamma}}{1-\gamma}, \tag{8}$$

for $\gamma \neq 1$ *and*

$$J(C_t, X_t) = \log C_t + \log I(X_t)$$

for $\gamma = 1$, *where* $I(\cdot)$ *satisfies the partial differential equation*

$$\begin{aligned}
&\frac{\beta}{1-\frac{1}{\psi}}\left[I(x)^{\frac{1}{\psi}-1} - 1\right] + \mu_c(x) + \frac{1}{2}\mathrm{tr}\left[\left(\frac{1}{I}\frac{\partial^2 I}{\partial x^2} - \frac{\gamma}{I^2}\left(\frac{\partial I}{\partial x}\right)^\top \left(\frac{\partial I}{\partial x}\right)\right)\sigma(x)\sigma(x)^\top\right] \\
&+ \frac{1}{I}\frac{\partial I}{\partial x}\mu_X(x) - \frac{1}{2}\gamma\sigma_c(x)^2 + \frac{1}{1-\gamma}\sum_{j=1}^{m}\lambda_j(x) E_{\nu_j}\left[e^{(1-\gamma)Z_{cj}}\left(\frac{I(x+Z_{Xj})}{I(x)}\right)^{1-\gamma} - 1\right] = 0.
\end{aligned} \tag{9}$$

for $\psi \neq 1$ *and*

$$\begin{aligned}
&-\beta \log I + \mu_c(x) + \frac{1}{2}\mathrm{tr}\left[\left(\frac{1}{I}\frac{\partial^2 I}{\partial x^2} - \frac{\gamma}{I^2}\left(\frac{\partial I}{\partial x}\right)^\top \left(\frac{\partial I}{\partial x}\right)\right)\sigma(x)\sigma(x)^\top\right] \\
&+ \frac{1}{I}\frac{\partial I}{\partial x}\mu_X(x) - \frac{1}{2}\gamma\sigma_c(x)^2 + \frac{1}{1-\gamma}\sum_{j=1}^{m}\lambda_j(x) E_{\nu_j}\left[e^{(1-\gamma)Z_{cj}}\left(\frac{I(x+Z_{Xj})}{I(x)}\right)^{1-\gamma} - 1\right] = 0.
\end{aligned} \tag{10}$$

for $\psi = 1$.

Equation 10 is a special case of (9) as can be seen by taking the limit as $\psi \to 1$.

Given the value function (8), we can now express the wealth-consumption ratio and the state price density in terms of $I(X_t)$. The consumption-wealth ratio will play an important role in our solution method for the affine case.

Corollary 2 (**Wealth-consumption ratio**). *Let W_t denote the wealth of the representative agent at time t. Then the wealth-to-consumption ratio $G^c(X_t) \equiv W_t/C_t$ is a function of X_t and is given by*

$$G^c(X_t) = \begin{cases} \beta^{-1} I(X_t)^{1-\frac{1}{\psi}} & \psi \neq 1 \\ \beta^{-1} & \psi = 1. \end{cases} \tag{11}$$

Proof of Corollary 2 Conjecture that the equilibrium wealth-consumption ratio is a function of X_t, namely $G^c(X_t) \equiv W_t/C_t$. Optimality requires that the derivative of f with respect to C_t equals the derivative of J with respect to W_t (Duffie, 1996, Chapter 9). By the chain rule, $\partial J/\partial W = (\partial J/\partial C) G^c(X_t)^{-1}$, so that

$$\frac{\partial f}{\partial C} = \frac{\partial J}{\partial C} \frac{1}{G^c(X_t)}. \tag{12}$$

Furthermore, $V_t = J(C_t, X_t)$. Taking the derivative of (5) with respect to C, and substituting (8) in for V implies

$$\frac{\partial f}{\partial C} = \beta C^{-\gamma} I(X_t)^{\frac{1}{\psi}-\gamma}. \tag{13}$$

Combining (13) with (12), and applying (8) to calculate the right hand side of (12), verifies the conjecture and implies (11). $\square$

Corollary 3 (**State-price density**). *The state-price density is given by:*

$$\pi_t = \begin{cases} \exp\left\{-\beta \int_0^t \left((1-\theta) I(X_s)^{\frac{1}{\psi}-1} + \theta\right) ds\right\} \beta C_t^{-\gamma} I(X_t)^{\frac{1}{\psi}-\gamma} & \psi \neq 1 \quad (14) \\ \exp\left\{-\beta \int_0^t \left((1-\gamma)\log I(X_s) + 1\right) ds\right\} \beta C_t^{-\gamma} I(X_t)^{1-\gamma} & \psi = 1 \quad (15) \end{cases}$$

Proof of Corollary 3 Duffie and Skiadas (1994) characterize the state-price density as

$$\pi_t = \exp\left\{\int_0^t \frac{\partial f}{\partial V}(C_s, V_s)ds\right\} \left.\frac{\partial f}{\partial C}\right|_{C_t, V_t}$$

The results follow from substituting $V_t = J(C_t, X_t)$ using (8) into (5) and (6) and taking partial derivatives. □

2.3 Risk premia

From Corollary 3, we can write the SDE for π_t in terms of the underlying shocks:

$$\frac{d\pi_t}{\pi_{t^-}} = \mu_{\pi t}\, dt + \sigma_{\pi t} dB_t + \sum_{j=1}^{m} \frac{\mathcal{J}_j(\pi_t)}{\pi_{t^-}} dN_{jt}, \tag{16}$$

where expressions for μ_{π_t}, $\sigma_{\pi t}$ and $\mathcal{J}_j(\pi_t)/\pi_{t^-}$ follow from Ito's Lemma.[3] Let S_t be an asset paying D_t, such that $S_t = S(D_t, X_t)$. Ito's Lemma implies an SDE for S_t:

$$\frac{dS_t}{S_{t^-}} = \mu_{St}\, dt + \sigma_{St} dB_t + \sum_{j=1}^{m} \frac{\mathcal{J}_j(S_t)}{S_{t^-}} dN_{jt}, \tag{17}$$

for a scalar process μ_{St} and (row) vector process σ_{St}.

Given the state-price density, risk premia and prices follow from no-arbitrage pricing. This is equivalent to solving for an equilibrium with a representative agent, provided the state prices are as in Corollary 3. For convenience, let $\lambda_{jt} = \lambda_j(X_t)$, and $\lambda_t = \lambda(X_t)$.

Lemma 4 (**No arbitrage**). *Assume there is no arbitrage with state prices given by (16). Suppose an asset has price process given by (17). Then*

$$\mu_{\pi t} + \mu_{St} + \frac{D_t}{S_t} + \sigma_{\pi t}\sigma_{St}^\top + \lambda_t^\top \frac{\bar{\mathcal{J}}(\pi_t S_t)}{\pi_{t^-} S_{t^-}} = 0. \tag{18}$$

[3] The state-price density π_t is not a function of C_t and X_t, and thus $\mathcal{J}_j(\pi_t)$ is not strictly speaking defined. To be precise, define $\hat{\pi}_t = C_t^{-\gamma} I(X_t)^{\frac{1}{\psi}-\gamma}$, and replace $\mathcal{J}_j(\pi_t)$ in (16) with $\mathcal{J}_j(\hat{\pi}_t)\frac{\pi_t}{\hat{\pi}_t}$.

With the no arbitrage condition given in Lemma 4, we can calculate the risk premium of the asset S_t. Note that the expected return on this asset is given by:

$$r_t^S = \mu_{St} + \frac{D_t}{S_t} + \lambda_t^\top \frac{\bar{\mathcal{J}}(S_t)}{S_{t^-}}. \tag{19}$$

We now state a result that holds under the general form for the state-price density (16) and for an asset price (17).

Theorem 5 (Risk premia). *Let r_t denote the continuously compounded risk-free rate. The continuous-time limit of the risk premium for the asset with price process (17) is*

$$r_t^S - r_t = -\sigma_{\pi t}\sigma_{St}^\top - \sum_{j=1}^m \lambda_{jt} E_{\nu_j}\left[\frac{\mathcal{J}_j(\pi_t)}{\pi_{t^-}}\frac{\mathcal{J}_j(S_t)}{S_{t^-}}\right]. \tag{20}$$

When there are no Poisson shocks, (20) is a standard pricing result that inspires tests of factor models of the cross-section. Elements of $-\sigma_{\pi t}$ are often called risk prices, while elements of σ_{St} are referred to as risk quantities.[4] As will be shown below, the elements of $\sigma_{\pi t}$ are determined based on consumption growth, the state-variables and the primitive parameters of the utility function. The elements of $\sigma_{\pi t}$ can then be uncovered, up to scaling factors, through OLS regression of stock returns on consumption growth and on the state variables. A large empirical literature in asset pricing tests (20), under various specifications for the underlying processes, without the Poisson terms, sometimes with the model restrictions discussed below and sometimes without. Theorem 5 suggests that such tests are mis-specified. If risk is not purely Brownian, risk premia need not be linear functions of normal-times covariances. In a recent paper (Tsai and Wachter, 2016), we calibrate a model of rare events to show that this result can account for the value premium. Note also that the convenient separation between prices and quantities of risk, which holds for diffusion processes, fails in the case Poisson risk.[5]

[4]Below, we will argue that it may make sense to refer to $(\sigma_{\pi t})_k$ rather than $-(\sigma_{\pi t})_k$ as the risk price for for Brownian k depending on whether or not an increase in B_{kt} improves utility.

[5]The above discussion associates risk prices with Brownian shocks. Alternatively, one may associate risk

We now specialize Theorem 5 to the model in Section 2.1. For the remaining results in this section, we assume the endowment process given in (1) and (2) with utility given by (4–7).[6]

Corollary 6 (Rare-event Consumption CAPM). *Risk premia are given by (20) with*

$$\sigma_{\pi t} = \left[-\gamma \sigma_{ct}, \left(\frac{1}{\psi} - \gamma \right) \frac{1}{I(X_t)} \frac{\partial I}{\partial X} \sigma_{Xt} \right] \tag{21}$$

and

$$\frac{\mathcal{J}_j(\pi_t)}{\pi_{t^-}} = \left(\frac{I(X_{t^-} + Z_{Xj})}{I(X_{t^-})} \right)^{\frac{1}{\psi} - \gamma} e^{-\gamma Z_{cj}} - 1.$$

Consider the case of no preference for early resolution of uncertainty $\frac{1}{\psi} = \gamma$ and no rare events $\lambda_j = 0$. The model reduces to that of Breeden (1979). If we allow for rare events but continue to assume no preference for early resolution of uncertainty, the model reduces to a "rare-event" consumption CAPM, in which what matters is not only covariance with consumption, but covariance with the marginal utility which itself is driven exclusively by consumption. Because these shocks need not be small, however, they will not be captured by normal-times covariances.

We now specify an intuitive definition of whether a shock increases or decreases investment opportunities, based on whether it makes the agent beter off. In what follows, we refer to a shock to the kth component of the vector Brownian motion B_{Xt} as Brownian shock k.

Definition 1. *Brownian shock k constitutes an improvement of the investment opportunity set if it leads to an increase in the value function. It constitutes a deterioration of the investment opportunity set if it leads to a decrease in the value function. That is,* $(dB_{Xt})_k >$

prices with state variables and with consumption. Going between the two is straightforward. To find the prices of risk associated with the state variables and consumption, project σ_S and σ_π on the $(n+1)\times(n+1)$ matrix $\sigma = [\sigma_c e_1, \sigma_X^\top]^\top$, where e_1 is the $(n+1)\times 1$ vector with 1 as the first element and zero elsewhere. Thus the $(n+1)\times 1$ vector of risk prices is $\sigma_\pi \sigma^\top (\sigma\sigma^\top)^{-1}$ and the $1 \times (n+1)$ vector of risk quantities is $\sigma_S \sigma^\top (\sigma\sigma^\top)^{-1}$. While we continue to use risk prices associated with shocks for convenience, are results hold for this alternative definition.

[6] We assume that $I(X_t) > 0$ for all realizations of X_t. This is a natural assumption given the form of the value function in (8), and it is true in the affine jump-diffusion case explored in the following section.

0 *is an improvement if and only if* $\left(\frac{\partial J}{\partial X}\sigma_X\right)_k > 0$, *where* $\left(\frac{\partial J}{\partial X}\sigma_X\right)_k$ *is the* k*th component of the row vector* $\frac{\partial J}{\partial X}\sigma_X$.

Definition 1 allows for a shock to affect multiple state variables through the matrix of loadings σ_X. In the special case where state variables are uniquely identified with a Brownian motion, then σ_X is diagonal, and a shock to a variable is an improvement if and only if it increases the value function.

Definition 2. *If* $(dB_t)_k > 0$ *constitutes an improvement of the investment opportunity set, then* $-(\sigma_{\pi t})_k$ *is the price of risk for Brownian shock* k*. If* $(dB_t)_k > 0$ *constitues a deterioration, then,* $(\sigma_{\pi t})_k$ *is the price of risk.*

While the definition doesn't formally cover the case of shocks to B_{ct}, clearly positive Brownian shocks to consumption increase utility, and hence have a positive price of risk.

Given these definitions, there is a general result linking the utility function parameters to the risk prices.

Corollary 7. *Brownian shock* j *has a positive price of risk in the consumption-based model if and only if* $\gamma > 1/\psi$.

Proof. The result follows from the prices of risk (21), and from the fact that, given (8), the sign of $(\frac{\partial J}{\partial X}\sigma_X)_k$ is equal to the sign of $\frac{1}{I_t}(\frac{\partial I}{\partial X}\sigma_X)_k$. □

Namely, if the agent has a preference for an early resolution of uncertainty, then shocks to the distribution of consumption are associated with a positive risk premium. If there is no preference for the timing of the resolution of uncertainty then these shocks do not have a risk premium.

The above theorems show additional terms relative to the consumption CAPM. These additional terms depend on the preference for the early resolution of uncertainty. However, many studies use asset returns rather than consumption in cross-sectional regressions because asset returns are less noisy. Indeed, the original ICAPM of Merton (1973) is written purely in terms of asset returns. Campbell (1993) derives an ICAPM in a discrete-time homoskedastic setting with recursive utility. An ICAPM also holds in our general setting. The sign of prices of risk on the state variables no longer depends on a preference for an early resolution of uncertainty, but rather on whether risk aversion exceeds one.

Using (8) and (11) we rewrite the value function as a function of wealth and of X_t:

$$J(C_t, X_t) = J\left(W_t \frac{1}{G^c(X_t)}, X_t\right) = \beta^{1-\gamma} \frac{W_t^{1-\gamma}}{1-\gamma} I(X_t)^{\frac{1}{\psi}(1-\gamma)}. \tag{22}$$

Likewise, from (14) it follows that

$$\begin{aligned} \pi_t &= \exp\left\{-\beta \int_0^t \left((1-\theta) I(X_s)^{\frac{1}{\psi}-1} + \theta\right) ds\right\} \beta^{1-\gamma} W_t^{-\gamma} G^c(X_t)^\gamma I(X_t)^{\frac{1}{\psi}-\gamma} \\ &= \exp\left\{-\beta \int_0^t \left((1-\theta) I(X_s)^{\frac{1}{\psi}-1} + \theta\right) ds\right\} \beta^{1-\gamma} W_t^{-\gamma} I(X_t)^{\frac{1}{\psi}(1-\gamma)} \end{aligned} \tag{23}$$

Moreover, from $W_t = C_t G^c(X_t)$, it follows that wealth evolves according to

$$\frac{dW_t}{W_{t^-}} = \mu_{wt}\, dt + \sigma_{wt} dB_{wt} + \sum_{j=1}^m (e^{Z_{wjt}} - 1) dN_{jt},$$

where

$$\sigma_{wt} dB_{wt} = \sigma_{ct} dB_{ct} + \left(1 - \frac{1}{\psi}\right) \frac{1}{I(X_t)} \frac{\partial I}{\partial X} \sigma_{Xt} dB_{Xt}$$

and

$$Z_{wjt} = Z_{cjt} + \left(1 - \frac{1}{\psi}\right) \log(\mathcal{J}(I(X_t)) + 1).$$

While Z_{wjt} need not have a time-invariant distribution in general, it will have a time-invariant distribution in the affine model of the next section. A rare-event ICAPM then follows from Theorem 5:

Corollary 8 (Rare-event ICAPM). *Define $B_t = [B_{wt}, B_{Xt}^\top]^\top$ but assume the conditions of Corollary 6 hold otherwise. Then risk premia are given as in (20), with*

$$\sigma_{\pi t} = \left[-\gamma \sigma_{wt}, \frac{1}{\psi}(1-\gamma) \frac{1}{I(X_t)} \frac{\partial I}{\partial X} \sigma_{Xt}\right] \tag{24}$$

and

$$\frac{\mathcal{J}_j(\pi_t)}{\pi_{t^-}} = \left(\frac{I(X_{t^-} + Z_{Xj})}{I(X_{t^-})}\right)^{\frac{1}{\psi}(1-\gamma)} e^{-\gamma Z_{wjt}} - 1. \tag{25}$$

Consider first the case with no Poisson shocks. Then Corollary 8 is precisely the ICAPM of Merton (1973), derived under the more general condition of recursive utility. If we allow for Poisson shocks, the ICAPM no longer holds, but note that the preference for early resolution of uncertainty plays no special role. Finally, when $\gamma = 1$, a rare-event CAPM holds. That is, risk premia depend only on the covariance with wealth during normal times and during rare events. This holds regardless of the value of the EIS.

We derive a wealth-based analogue of Corollary 7:

Corollary 9. *Brownian shock j has a positive price of risk in the wealth-based model if and only if risk aversion is greater than 1.*

Why the difference between the comparative statics in Corollary 7 and Corollary 9? It is because wealth, or, depending on one's point of view, consumption, already contains an endogenous response of the agent to changes in investment opportunities. This change is mediated through the elasticity of intertemporal substitution. The following corollary follows directly from (11) and from (8).

Corollary 10. *When $\psi > 1$, a shock representing an improvement in investment opportunities decreases the wealth-consumption ratio and a shock representing a deterioration increases the ratio. The opposite holds when $\psi < 1$*

3 Affine Model

Further assume the drift and volatility in the C_t and X_t processes, as well as the jump probability, are affine functions of the state variables X_t. That is, for a column vector x of length n, define

$$\mu_c(x) = k_0 + k_1 x \tag{26a}$$

$$\sigma_c^2(x) = u_0 + u_1 x \tag{26b}$$

$$\mu_X(x) = K_0 + K_1 x \tag{26c}$$

$$(\sigma_X(x)\sigma_X(x)^\top)_{ij} = (U_0)_{ij} + (U_1)_{ij} x \tag{26d}$$

$$\lambda(x) = l_0 + l_1 x, \tag{26e}$$

where k_0 and u_0 are scalars, k_1 and u_1 are $1 \times n$, l_0 is $m \times 1$, l_1 is $m \times n$, K_0 is $n \times 1$, K_1 and U_0 are $n \times n$ matrices, and U_1 can be thought of as an $n \times n \times n$ matrix in a sense that will be made more precise below.

Finally, l_0 is a column vector of length m. This is similar to the affine structure defined by Duffie, Pan, and Singleton (2000), except in that case it is a specification of the endowment process rather than the discount rate. This structure can accomodate time-varying rare disasters as in Wachter (2013), as well as time-variation in the mean and standard deviation of consumption growth, as in Bansal and Yaron (2004). The model can accomodate rare events that affect the moments of the consumption growth process (Benzoni, Collin-Dufresne, and Goldstein, 2011; Drechsler and Yaron, 2011; Tsai and Wachter, 2016), and self-exciting jumps (Nowotny, 2011). The model can also accomodate a stationary dividend-consumption ratio, while still allowing dividends to temporary respond more to disasters (Longstaff and Piazzesi, 2004).

Assumption (26) describe conditions on conditional means, variances, and covariances for the processes C_t and X_t. Specifically, consider (26d). From (2), it follows that $\sigma_X(x)\sigma_X(x)^\top$ is the normal-times conditional variance of X_t. That is, it is the instantaneous variance over an interval without rare events. Given two linear combinations of X_t, $a^\top X_t$ and $b^\top X_t$, for column vectors a, b, the normal-times conditional covariance of $a^\top X_t$ with $b^\top X_t$ is $a^\top \sigma_X(x)\sigma_X(x)^\top b$. Assumption (26d) implies that this conditional covariance is linear in x. To see this, note that from (26d) it follows that

$$a^\top \sigma(x)\sigma(x)^\top b = \sum_{i,j} a_i(\sigma(x)\sigma(x)^\top)_{i.j} b_j \tag{27}$$

$$= \sum_{i,j} a_i (U_0)_{ij} b_j + \sum_{i,j} a_i \left((U_1)_{ij} x\right) b_j. \tag{28}$$

For a fixed $i, j = 1, \ldots, N$, $(U_1)_{ij}$ is a row vector. Let $(U_1)_{ij,k}$ be the kth element of this row vector so that

$$a_i \left((U_1)_{ij} x\right) b_j = \sum_k a_i (U_1)_{ij,k} x_k b_j = \sum_k a_i (U_1)_{ij,k} b_j x_k$$

From (28), it then follows that

$$a^\top \sigma(x)\sigma(x)^\top b = a^\top U_0 b + a^\top U_1 b x,$$

where $a^\top U_1 b$ is formally defined to be the column vector with kth element $a_i (U_1)_{ij,k} b_j$.

In what follows, define a $m \times 1$ vector $Z_c = [Z_{c1}, \cdots, Z_{cm}]^\top$ and an $m \times n$ matrix $Z_X = [Z_{X1}, \cdots Z_{Xm}]^\top$. Given a vector x, we use e^x to denote the exponential of each element in x. To evaluate expressions at $\gamma = 1$, apply $\lim_{\gamma \to 1} \frac{1}{1-\gamma} e^{(1-\gamma)y} - 1 = y$.

3.1 Value function

We first characterize the value function.

Theorem 11. *The value function takes the form (8), with*

$$I(x) \simeq \exp\{a + b^\top x\}, \tag{29}$$

where a is a scalar and b is $n \times 1$. When $\psi \neq 1$,

$$a = \frac{1}{i_1}\left(\frac{1}{1-1/\psi}(i_1 \log\beta + i_0 - \beta) + k_0 - \frac{1}{2}\gamma u_0 + b^\top K_0 + \frac{1}{2}(1-\gamma)\, b^\top U_0 b \right.$$
$$\left. + \frac{1}{1-\gamma}\left(E_\nu\left[e^{(1-\gamma)(Z_c + Z_X b)} - 1\right]\right)^\top l_0\right), \quad (30)$$

and

$$\frac{1}{2}(1-\gamma)\, b^\top U_1 b - i_1 b^\top + b^\top K_1 + k_1 - \frac{1}{2}\gamma u_1 + \frac{1}{1-\gamma}\left(E_\nu\left[e^{(1-\gamma)(Z_c+Z_X b)} - 1\right]\right)^\top l_1 = 0, \quad (31)$$

with $i_1 = e^{E\left[\log\left(\beta I(X_t)^{\frac{1}{\psi}-1}\right)\right]}$ and $i_0 = i_1(1 - \log i_1)$. For $\psi = 1$, (29) is exact, and (30) and (31) hold with $\lim_{\psi \to 1} i_1 = \beta$ and $\lim_{\psi\to 1} \frac{1}{1-\frac{1}{\psi}}(i_1 \log\beta + i_0 - \beta) = 0$.

It follows immediately from (31) that the vector b is nonzero if and only if at least one state variable affects the consumption distribution directly, either through the mean (k_1), the variance u_1, or the jump probability l_1. Given one such state variable, however, other state variables could enter into the value function even if they don't affect consumption directly; they just need to affect the distribution of the state variable that does affect consumption.[7]

3.2 State price density

We next characterize the state-price density and the riskfree rate.

Theorem 12. *The state-price density is given by*

$$\frac{d\pi_t}{\pi_t} = \mu_{\pi t} dt + \sigma_{\pi t} dB_t + \sum_{j=1}^{m}(e^{Z_{\pi j}} - 1) dN_{jt}, \quad (32)$$

[7]For example, in Seo and Wachter (2016), the state variable ξ_t does not affect consumption directly, but it affects the time-varying mean of the disaster probability λ_t.

where

$$\sigma_{\pi t} \simeq \left[-\gamma \sigma_c(X_t), \; \left(\frac{1}{\psi} - \gamma \right) b^\top \sigma_X(X_t) \right]^\top \tag{33}$$

and

$$Z_{\pi j} \simeq -\gamma Z_{cj} + \left(\frac{1}{\psi} - \gamma \right) b^\top Z_{Xj}. \tag{34}$$

where b is given in Theorem 11. Furthermore,

$$\mu_{\pi t} = -r_t - \sum_{j=1}^{m} \lambda_j(X_t) E_\nu[e^{Z_{\pi j}} - 1]$$

where r_t, the riskfree rate, is given by

$$r_t \simeq \beta + \frac{1}{\psi}(k_0 + k_1 X_t) - \frac{1}{2}\gamma \left(1 + \frac{1}{\psi} \right)(u_0 + u_1 X_t) - \frac{1}{2}\left(\gamma - \frac{1}{\psi} \right)\left(1 - \frac{1}{\psi} \right)\left(b^\top U_0 b + \left(b^\top U_1 b \right) X_t \right)$$
$$+ \left(E_\nu \left[\left(1 - \frac{1}{\theta} \right) \left(e^{(1-\gamma)(Z_c + Z_X b)} - 1 \right) - \left(e^{-\gamma Z_c + \left(\frac{1}{\psi} - \gamma \right) Z_X b} - 1 \right) \right] \right)^\top (l_0 + l_1 X_t). \tag{35}$$

The approximations are exact in the case of $\psi = 1$ and $\gamma = \frac{1}{\psi}$.

This theorem shows that derivatives with respect to the value function in Corollaries 6 and 8 can be replaced with the simpler constant vector b. Moreover, the rare events' impact on marginal utility π_t can be replaced by the simpler expression (34).

3.3 Equity prices in the affine model

Given (3) and equilibrium, equity prices are functions of X_t and D_t. We further specify

$$\mu_d(x) = k_0^d + k_1^d x, \tag{36}$$

$$\sigma_c(x)\sigma_d(x) = u_0^{cd} + u_1^{cd} x, \tag{37}$$

where k_0^d and u_0^{cd} are scalars, and k_1^d and u_1^{cd} are $1 \times n$. We simplify the problem by considering equity strips, namely equity that pays a dividend at a single point in time (Lettau and Wachter, 2007). From this point in the argument forward, no further approximations are necessary.

Theorem 13. *Let $H(D,x,\tau)$ denote the price of an asset that pays dividend D, τ years in the future. Then*

$$H(D,x,\tau) \simeq D \exp\left\{a_\phi(\tau) + b_\phi(\tau)^\top x\right\}, \tag{38}$$

where functions $a_\phi(\tau) : [0,\infty) \to \mathbf{R}$ and $b_\phi(\tau) : [0,\infty) \to \mathbf{R}^n$ solve

$$\begin{aligned}\frac{\partial a_\phi(\tau)}{\partial \tau} &= k_0^d - \frac{1}{\psi}k_0 - \beta + \frac{1}{2}\gamma\left(1+\frac{1}{\psi}\right)u_0 - \gamma u_0^{cd} + b_\phi(\tau)^\top K_0 \\ &\quad + \frac{1}{2}\left(1-\frac{1}{\psi}\right)\left(\gamma - \frac{1}{\psi}\right)b^\top U_0 b + \frac{1}{2}b_\phi(\tau)^\top U_0 b_\phi(\tau) + \left(\frac{1}{\psi}-\gamma\right)b_\phi(\tau)^\top U_0 b \\ &\quad + \left(E_\nu\left[\left(\frac{1}{\theta}-1\right)\left(e^{(1-\gamma)(Z_c+Z_X b)}-1\right) + \left(e^{-\gamma Z_c + Z_d + Z_X\left(b_\phi(\tau)+(1/\psi-\gamma)b\right)}-1\right)\right]\right)^\top l_0,\end{aligned} \tag{39}$$

and

$$\begin{aligned}\left(\frac{\partial b_\phi(\tau)}{\partial \tau}\right)^\top &= k_1^d - \frac{1}{\psi}k_1 + \frac{1}{2}\gamma\left(1+\frac{1}{\psi}\right)u_1 - \gamma u_1^{cd} + b_\phi(\tau)^\top K_1 \\ &\quad + \frac{1}{2}\left(1-\frac{1}{\psi}\right)\left(\gamma - \frac{1}{\psi}\right)b^\top U_1 b + \frac{1}{2}b_\phi(\tau)^\top U_1 b_\phi(\tau) + \left(\frac{1}{\psi}-\gamma\right)b_\phi(\tau)^\top U_1 b \\ &\quad + \left(E_\nu\left[\left(\frac{1}{\theta}-1\right)\left(e^{(1-\gamma)(Z_c+Z_X b)}-1\right) + \left(e^{-\gamma Z_c + Z_d + Z_X\left(b_\phi(\tau)+(1/\psi-\gamma)b\right)}-1\right)\right]\right)^\top l_1.\end{aligned} \tag{40}$$

The approximations are exact if utility is time-additive or if $\psi = 1$.

Given the above result, the price of the claim to all future dividends follows immediately.

Corollary 14. *Let $S(D_t, X_t)$ denote the time t price of an asset that pays the stream of dividends given by (3), then*

$$S(D_t, X_t) = \int_0^\infty H(D_t, X_t, \tau)d\tau = D_t G(X_t), \tag{41}$$

where H is the function defined in (38) and $G(X_t)$ is the price-dividend ratio which can be *expressed as:*

$$G(X_t) \simeq \int_0^\infty e^{a(\tau)+b(\tau)^\top X_t} d\tau. \tag{42}$$

The approximation is exact if utility is time-additive or if $\psi = 1$.

While we have written Theorems 12 and 13, as well as Corollary 14 in terms of approximations, both results are exact given the (approximate) value function in Theorem 11. Furthermore, these results are all exact in the case of $\psi = 1$ and time-additive utility.[8]

Corollary 14 applies to the asset that pays aggregate consumption as dividend, that is, $\mu_d = \mu_c$, $\sigma_d = \sigma_c$ and $Z_{dj} = Z_{cj}$ for all $j = 1, \cdots, m$. It follows that this theorem provides an alternative way to solve for the consumption-wealth ratio $G^c(X_t)$. Indeed, when we set $\psi = 1$ in the equations in this theorem, we find $a_\phi(\tau) = -\beta\tau$ and $b_\phi(\tau) = 0$, verifying that $G^c(X_t) = \beta^{-1}$. However, when $\psi \neq 1$, the wealth consumption ratio calculated using (42), is not the same as (11). Which one is a better approximation? Because (42) does not require an additional approximation, it is probably no worse than the approximation in (11). Moreover, the wealth-consumption ratio is exact in the case of Corollary 14 under time-additive utility, which is a reason to think it might be better. While precise statements are not available, in our experience, approximating the level of the wealth-consumption ratio is especially tricky. Corollary 14 does not obtain information on the level from (11), but rather only uses the slopes b, and the equilibrium conditions to determine the level. In practice, this appears to be more robust.

We now turn to risk premia. Despite the potential complexity of this model, risk premia for equity strips always take a strikingly simple form. First note that the expected return on zero-coupon equity is

$$r_t^{H,\tau} = \mu_{Ht} + \lambda_t^\top \frac{\bar{\mathcal{J}}(H_t)}{H_{t^-}}.$$

Corollary 15. *Consider the claim to the dividend τ years in the future. The risk premium*

[8] Note, however, that Theorem 11 is approximate in the case of time-additive utility. The distinction is that it is not necessary to obtain the value function to price securities when utility is time-additive. The value function does have an exact solution in this case, but we do not give it here.

on this claim equals

$$r_t^{H,\tau} - r_t = \gamma\sigma_{ct}\sigma_{dt} - \left(\frac{1}{\psi} - \gamma\right) b^\top \sigma_{Xt}\sigma_{Xt}^\top b_\phi(\tau) \\ - \sum_j \lambda_{jt} E_{\nu_j}\left[\left(e^{-\gamma Z_{cj} + \left(\frac{1}{\psi}-\gamma\right) b^\top Z_{\pi j}} - 1\right)\left(e^{Z_{dj} + b_\phi(\tau)^\top Z_{Xj}} - 1\right)\right] \quad (43)$$

Proof The result follows from the general expression for risk premia in Theorem 5, and the expression for the state-price density (Theorem 12) and the price of the asset (Theorem 13) in the affine case.

Risk premia for complex long-lived assets are simply a weighted sum of the risk premia on zero-coupon equity.

4 Example

We apply the findings in the previous section to generalize the Wachter (2013) disaster risk model to the case of non-unitary ψ. This model is a special case of that in Section 3 with $n = m = 1$, $X_t = \lambda_t$, the disaster probability. Because the Poisson shocks are disasters, $Z_{ct} < 0$. $\mu_X(x) = \kappa_\lambda(\bar{\lambda} - x)$, while $\sigma_X(x) = \sigma_\lambda\sqrt{x}$. The functions $\mu_c(x)$, $\sigma_c(x)$, $\mu_d(x)$ and $\sigma_d(x)$ are constants, $Z_{Xt} = 0$, and $Z_{dt} = \phi Z_{ct}$. Equations 30 and 31 have closed-form solutions with

$$a = \frac{1}{i_1}\left(\left(1 - \frac{1}{\psi}\right)^{-1}(i_1 \log\beta + i_0 - \beta) + \mu - \frac{1}{2}\gamma\sigma^2 + b\kappa\bar{\lambda}\right), \quad (44)$$

$$b = \frac{(\kappa + i_1) - \sqrt{(\kappa + i_1)^2 - 2\sigma_\lambda^2 E_\nu\left[e^{(1-\gamma)Z_c} - 1\right]}}{(1-\gamma)\sigma_\lambda^2}. \quad (45)$$

When $\psi = 1$, these equations reduce to those in Wachter (2013), as described in Theorem 11.[9]

[9] The notation is slightly different from that of Wachter (2013). In order to accomodate the $\gamma = 1$ case, a and b in this paper are equal to a and b in the previous paper divided by $1 - \gamma$.

It follows from (45) that $b < 0$ regardless of the preference parameters. Therefore, an increase in the probability of a rare disaster always decreases the investor's utility. Applying the definitions from Section 2.3, the price of risk for λ_t (relative to the CCAPM) is equal to $(\sigma_{\pi t})_2 = \left(\frac{1}{\psi} - \gamma\right) b\sigma_\lambda\sqrt{\lambda_t}$, and thus is positive if and only if $\gamma > \frac{1}{\psi}$.

The wealth-consumption ratio, by Corollary 2, can be approximated by

$$G^c(\lambda_t) \simeq \beta^{-1} \exp\left\{\left(1 - \frac{1}{\psi}\right)(a + b\lambda_t)\right\}$$

which is decreasing in λ_t if and only if $\psi > 1$. It follows from Theorem 5 that the premium for bearing λ_t-risk is positive if $\psi > 1$ and $\gamma > 1/\psi$ or if $\psi < 1$ and $\gamma < 1/\psi$. In the former case, the wealth-consumption ratio decreases in λ_t, and the agent prefers an early resolution of uncertainty (so the price of λ_t-risk is positive). In the latter case, the wealth-consumption ratio increases in λ_t and the agent prefers a late resolution of uncertainty (so the price of λ_t-risk is negative).

To evaluate the numerical properties of the solution, we choose a calibration fit to the first two moments of equity and Treasury bill returns.[10] The appealing aspect of this rare disaster model is its ability to explain high equity return volatility with a low volatility of consumption growth, without counterfactually generating predictive relations between consumption growth and stock prices. Raising the elasticity of intertemporal substitution above 1 helps the model explain stock market volatility relative to the model of Wachter (2013).

To gauge the accuracy of the solution, we first compare the wealth-consumption ratio under our log-linearization method, to one calculated using Chebyshev inequalities. Panel A of Figure 1 shows the solution as a function of the disaster probability, under two different assumptions for the disaster size. Panel A shows that the exact and approximate wealth-consumption ratios are nearly indistinguishable, indicating that our approximation is highly

[10]See Tsai and Wachter (2015): discount rate $\beta = 0.01$, risk aversion $\gamma = 3$, normal-times consumption growth $\mu = 0.0195$, consumption growth volatility $\sigma = 0.0125$, dividend growth $\mu_D = 0.04$, leverage $\phi = 3$, and mean-reversion $\kappa_\lambda = 0.12$, volatility $\sigma_\lambda = 0.081$ average disaster probability $\bar{\lambda} = 0.0286$.

accurate, even when rare disasters are large. In Panel B we examine the price-dividend ratio for the asset with leverage $\phi = 3$ (the wealth-consumption ratio is, by definition, the price-dividend ratio with $\phi = 1$). Here, we compare three solution methods: (1) An exact numerical calculation, (2) our method, which we call "integration," and which is exact given the approximation for the value function, and (3) log-linearizing the price-dividend ratio as described in Appendix B and which is frequently done in the literature. While both approximations seem adequate, approximation (3) is noticeably coarser than (2), at all levels of the disaster probability.

Figure 2 shows the price-dividend ratio (Panel A), and the component of the risk premium that compensates for the risk of variation in the disaster probability (Panel B). We show this for unitary EIS and for our benchmark calibration of EIS equal to 2. The reason to focus on a specific component of the risk premium is that, in the special case described in this section, it is this component alone that differs between the log-linearization and the integration approach.[11] This component is economically sizable, and represents about half of the total risk premium.

Specifically, corollary 15 shows that, in the case of equity strips, risk premia have three terms: a consumption CAPM term from the correlation between consumption and dividends in normal times, a term that arises from the correlation between the price of the asset and the state variables in normal times, and finally a term that arises directly from rare events. The same is true for the market as a whole, or any asset with a price $S(D_t, \lambda_t)$. For the special case of the model in this section, the second term, which is the one we are interested in, equals $\left(\frac{1}{\psi} - \gamma\right) b\sigma_\lambda(\frac{1}{S_t}\frac{\partial S}{\partial \lambda})\lambda_t$, which follows from Theorem 5, where we apply the expression for the state-price density in Theorem 12.

Log-linearizing the price-dividend ratio, as described in Appendix B, implies that this term is linear λ_t. This log-linearization is misleading however. When we calculate an exact expression for this term by creating a weighted average of the terms in Corollary 15 (we

[11] The reason is that the disaster only affects consumption and the state variables, not the probability of a rare disaster.

refer to this as "integration" in the figure):

$$\frac{1}{S_t}\frac{\partial S}{\partial \lambda} = \int_0^\tau \frac{e^{a_\phi(\tau)+b_\phi(\tau)\lambda_t}}{\int_0^\tau e^{a_\phi(u)+b_\phi(u)\lambda_t}\,du} b'_\phi(\tau)\,d\tau$$

we see that it is strikingly non-linear, and is in fact concave in the disaster probability. The concavity is a reflection of an important economic effect that the log-linearization approach leaves out. When the disaster probability increases, claims to dividends in the long-term fall in price by more than claims to dividends in the short-term because of duration. However, these long-term claims have greater risk premia, again, because of duration. As the disaster probability increases, risk premia on all claims increase, but claims on the long-term assets increase by more. At the same time, these assets have a lower weight in the overall market. For this reason, the log-linearization approach over-predicts the risk premium, and under-predicts the size of the price-dividend ratio itself, as shown in Panel A.

5 Conclusion

In this paper, we have extended classic results on the cross-section to the setting of rare events. When there are no rare events, and utility is time-additive, our results reduce to the consumption CAPM of Breeden (1979). When there are no rare events and risk aversion is equal to one, our results reduce to the wealth CAPM of Sharpe (1964). In the rare-event versions of these models, risk premia are not necessarily determined by covariances with consumption in the first case, nor in the second case are risk premia necesssarily determined by covariances with wealth. Moving beyond these knife-edge cases, the sign of risk premia relative to the consumption CAPM is determined by the agent's preference for early resolution of uncertainty, while the sign of risk premia relative to the wealth CAPM is determined by whether risk aversion is below or above one. While versions of these models without rare events lead to the usual factor structure, when rare events can occur, there is again no reason to assume the general factor structure holds. This is perhaps surprising given that that the factor structure has dominated empirical asset pricing for many years.

In the second part of the paper we specialized to an affine structure and solve explicitly for the prices of long-lived assets. These assets are integrals of prices of equity strips: claims to dividends at specific points in time. Our solution relies on an approximation for the wealth-consumption ratio. It is fully exact in two special cases: EIS of one and time-additive utility. In all other cases, asset prices are exact given the approximate solution of the wealth-consumption ratio. Despite the richness of the problem, our formulas for prices and risk premia are quite simple. Besides being highly accurate, our approach preserves the important intuition that long-lived assets are sums (or integrals) of individual claims, a fact that should not be forgotten in the focus on the overall market portfolio.

Appendix

A Proof of Theorems

Proof of Proposition 1 For convenience, let $J_t = J(C_t, X_t)$. The Hamilton-Jacobi-Bellman (HJB) equation is given by:

$$\mathcal{D}J_t + f(C_t, J_t) = 0. \tag{A.1}$$

Substituting (8) into (5)–(6) yields:

$$f(C_t, J_t) = \begin{cases} J_t \beta\theta \left[I(X_t)^{\frac{1}{\psi}-1} - 1 \right] & \psi \neq 1, \quad \text{(A.2)} \\ -J_t \beta(1-\gamma) \log I(X_t) & \psi = 1. \quad \text{(A.3)} \end{cases}$$

By Ito's Lemma:

$$\frac{\mathcal{D}J}{J} = \frac{1}{J}\Bigg(\frac{\partial J}{\partial C} C\mu_c(x) + \frac{\partial J}{\partial X}\mu_X(x) + \frac{1}{2}\frac{\partial^2 J}{\partial C^2} C^2 \sigma_c^2(x) + \frac{1}{2}\mathrm{tr}\left[\frac{\partial^2 J}{\partial X^2}\sigma(x)\sigma(x)^\top \right] + \sum_{j=1}^m \lambda_j E_{\nu_j}\left[J\left(c\, e^{Z_{cj}}, x + Z_{Xj}\right) - J(c,x) \right] \Bigg), \tag{A.4}$$

where $\frac{\partial J}{\partial X}$ and $\frac{\partial^2 J}{\partial X^2}$ are the gradient and Hessian matrix of J. Equation (8) implies:

$$\frac{1}{J}\frac{\partial J}{\partial C} = \frac{1-\gamma}{C}, \quad \frac{1}{J}\frac{\partial^2 J}{\partial C^2} = \frac{-\gamma(1-\gamma)}{C^2}, \quad \frac{1}{J}\frac{\partial J}{\partial X} = \frac{1-\gamma}{I}\frac{\partial I}{\partial X}, \tag{A.5}$$

$$\frac{1}{J}\frac{\partial^2 J}{\partial X^2} = (1-\gamma)\left(\frac{1}{I}\left(\frac{\partial^2 I}{\partial X^2} \right)^2 - \frac{\gamma}{I^2}\left(\frac{\partial I}{\partial X} \right)^\top \left(\frac{\partial I}{\partial X} \right) \right), \tag{A.6}$$

and

$$\frac{J\left(c\, e^{Z_{cj}}, x + Z_{Xj}\right)}{J} = e^{(1-\gamma)Z_{cj}} \left(\frac{I(x + Z_{Xj})}{I(x)} \right)^{1-\gamma}. \tag{A.7}$$

Substituting (A.5–A.7) into (A.4) yields:

$$
\begin{aligned}
\frac{\mathcal{D}J}{J} = (1-\gamma)\,\mu_c(x) + \frac{1-\gamma}{I}\frac{\partial I}{\partial X}\mu_X(x) - \frac{1}{2}\gamma\,(1-\gamma)\,\sigma_c(x)^2 \\
+ \frac{1-\gamma}{2}\mathrm{tr}\left[\left(\frac{1}{I}\left(\frac{\partial^2 I}{\partial X^2}\right)^2 - \frac{\gamma}{I^2}\left(\frac{\partial I}{\partial X}\right)^\top\left(\frac{\partial I}{\partial X}\right)\right)\sigma(x)\sigma(x)^\top\right] \\
+ \sum_{j=1}^{m}\lambda_j E_{\nu_j}\left[e^{(1-\gamma)Z_{cj}}\left(\frac{I(x+Z_{Xj})}{I(x)}\right)^{1-\gamma} - 1\right].
\end{aligned}
\quad \text{(A.8)}
$$

Finally, substituting (A.2) and (A.8) into (A.1) yields (9) and verifies the form (8) for $\psi \neq 1$. Analogously, substituting (A.3) and (A.8) into (A.1) yields (10), and verifies (8) for $\psi = 1$. □

Proof of Lemma 4 Let S_t be the price of the asset that pays a continuous dividend stream D_t . Then by no arbitrage,

$$S_t = E_t\left[\int_t^\infty \frac{\pi_s}{\pi_t}D_s ds\right]. \quad \text{(A.9)}$$

Multiplying each side of (A.9) by π_t implies

$$\pi_t S_t = E_t\left[\int_t^\infty \pi_u D_u du\right]. \quad \text{(A.10)}$$

The same equation must hold at any time $s > t$:

$$\pi_s S_s = E_t\left[\int_s^\infty \pi_u D_u du\right]. \quad \text{(A.11)}$$

Combining (A.10) and (A.11) implies

$$\pi_t S_t = E_t\left[\pi_s S_s + \int_t^s \pi_u D_u du\right]. \quad \text{(A.12)}$$

Adding $\int_0^t \pi_u D_u du$ to both sides of (A.12) implies

$$\pi_t S_t + \int_0^t \pi_u D_u du = E_t \left[\pi_s S_s + \int_0^s \pi_u D_u du \right]. \tag{A.13}$$

Therefore, $\pi_t S_t + \int_0^t \pi_u D_u du$ is a martingale. By Ito's Lemma:

$$\begin{aligned}\pi_t S_t + \int_0^t \pi_u D_u du = \pi_0 S_0 + \int_0^t \pi_s S_s \left(\mu_{\pi,u} + \mu_{S,u} + \frac{D_u}{S_u} + \sigma_{\pi,u} \sigma_{S,u}^\top \right) du \\ + \int_0^t \pi_u S_u (\sigma_{S,u} + \sigma_{\pi,u}) dB_u + \sum_j \sum_{0<u_{ij}\le t} \left(\pi_{u_{ij}} S_{u_{ij}} - \pi_{u_{ij}^-} S_{u_{ij}^-} \right),\end{aligned} \tag{A.14}$$

where $u_{ij} = \inf\{u : N_{ju} = i\}$ (namely, the time that the ith type j jump occurs). Adding and subtracting the jump compensation term from (A.14) yields:

$$\begin{aligned}\pi_t S_t + \int_0^t \pi_u D_u du = \pi_0 S_0 + \int_0^t \pi_u S_u \left(\mu_{\pi,u} + \mu_{S,u} + \frac{D_u}{S_u} + \sigma_{\pi,u} \sigma_{S,u}^\top + \lambda_u^\top \frac{\bar{\mathcal{J}}(\pi_u S_u)}{\pi_u S_u} \right) du \\ + \int_0^t \pi_u S_u (\sigma_{S,u} + \sigma_{\pi,u}) dB_u \\ + \sum_j \sum_{0<u_{ij}\le t} \left(\pi_{u_{ij}} S_{u_{ij}} - \pi_{u_{ij}^-} S_{u_{ij}^-} \right) - \int_0^t \lambda_u^\top \bar{\mathcal{J}}(\pi_u S_u)\, du.\end{aligned} \tag{A.15}$$

The second and third terms on the right-hand side of (A.15) have zero expectation. Therefore the first term in (A.15) must also have zero expectation, and it follows that the integrand of this term must equal zero. □

Proof of Theorem 5 Equation 19 implies

$$\mu_{St} + \frac{D_t}{S_t} = r_t^S - \lambda_t^\top \frac{\bar{\mathcal{J}}(S_t)}{S_{t^-}}. \tag{A.16}$$

While, in equilibrium, the drift in the state-price density and the riskfree rate are linked through

$$\mu_{\pi t} = -r_t - \lambda_t^\top \frac{\bar{\mathcal{J}}(\pi_t)}{\pi_{t^-}}. \tag{A.17}$$

Finally, note that

$$E_{\nu_j}\left[\frac{\mathcal{J}_j(\pi_t)}{\pi_t}\frac{\mathcal{J}_j(S_t)}{S_t}\right] = E_{\nu_j}\left[\frac{\mathcal{J}_j(\pi_t S_t)}{\pi_{t^-}S_{t^-}} - \frac{\mathcal{J}_j(\pi_t)}{\pi_{t^-}} - \frac{\mathcal{J}_j(S_t)}{S_{t^-}}\right], \tag{A.18}$$

Substituting (A.16–A.18) into (18) and rearranging gives (20). □

Proof of Theorem 11 We follow Chacko and Viceira (2005) and conjecture that $I(x)$ is (approximately) exponential affine. Then

$$\frac{1}{I}\frac{\partial I}{\partial x} \simeq [b_1, \cdots, b_n] = b^\top, \tag{A.19}$$

$$\frac{1}{I}\frac{\partial^2 I}{\partial x^2} \simeq \begin{bmatrix} b_1^2 & \cdots & b_1 b_n \\ \vdots & \cdots & \vdots \\ b_n b_2 & \cdots & b_n^2 \end{bmatrix} = b\,b^\top. \tag{A.20}$$

For $\psi \neq 1$, substitute (26), (A.19), and (A.20) into (9) of Proposition 1 to find:

$$\beta I(x)^{\frac{1}{\psi}-1} = \beta - \left(1-\frac{1}{\psi}\right)(k_0+k_1x) + \frac{1}{2}\gamma\left(1-\frac{1}{\psi}\right)(u_0+u_1x) - \left(1-\frac{1}{\psi}\right)\left(b^\top K_0 + b^\top K_1 x\right)$$
$$-\frac{1}{2}\left(1-\frac{1}{\psi}\right)(1-\gamma)\left(b^\top U_0 b + \left(b^\top U_1 b\right)x\right) - \frac{1}{\theta}\left(E_\nu\left[e^{(1-\gamma)(Z_c+Z_X)b} - 1\right]\right)^\top (l_0 + l_1 x). \tag{A.21}$$

Note that

$$\beta I(X_t)^{\frac{1}{\psi}-1} \simeq i_0 - i_1 \log\left(\beta^{-1} I(X_t)^{1-\frac{1}{\psi}}\right), \tag{A.22}$$

where $i_1 = e^{E\left[\log\left(\beta I(X_t)^{1/\psi - 1}\right)\right]}$, and $i_0 = i_1(1-\log i_1)$. Substituting (A.22) into (A.21) and matching coefficients yields (31) and (30), verifying the conjecture.

For $\psi = 1$ we follow a similar derivation and note that $\log I(x) = a + b^\top x$. The HJB

(10) can be rewritten as:

$$\beta\left(a+b^\top x\right)=(k_0+k_1x)-\frac{1}{2}\gamma(u_0+u_1x)+\left(b^\top K_0+b^\top K_1x\right)$$
$$+\frac{1}{2}(1-\gamma)\left(b^\top U_0b+\left(b^\top U_1b\right)x\right)+\frac{1}{1-\gamma}\left(E_\nu\left[e^{(1-\gamma)(Z_c+Z_Xb)}-1\right]\right)^\top(l_0+l_1x). \quad \text{(A.23)}$$

We then match coefficients as above. To show that the limits work out as stated, see the Lemma below. □

Lemma A.1. *Let* $y=(k_0,k_1,u_0,u_1,K_0,K_1,U_0,U_1,l_0,l_1,\nu,\gamma)$*. Let* $I(X,\psi;y)=\exp\left\{a(\psi)+b(\psi)^\top X\right\}$ *denote the value function as a function of* ψ *with* $\psi\neq 1$*. Suppose* $I(X,\psi;y)$ *is well-defined at* (y,ψ) *for* $\psi\in(1-\epsilon,1+\epsilon)\setminus\{1\}$ *with solutions* $b(\psi)$ *and* $a(\psi)$ *given by (31) and (30). Let* $\tilde{I}(X;y)$ *denote the value function with* $\psi=1$*,* $\tilde{I}(X;y)$ *is well defined at* y*, with solutions* $\tilde{b}$ *and* $\tilde{a}$ *as described in Theorem 11. Furthermore, assume* $\lim_{\psi\to 1}\frac{\partial I(X,\psi;y)}{\partial\psi}<\infty$ *exists. Then,* $\lim_{\psi\to1}a(\psi)=\tilde{a}$ *and* $\lim_{\psi\to1}b(\psi)=\tilde{b}$*.*

Proof of Lemma A.1 Note that

$$i_1=\exp\left(E\left[\log\left(\beta I(X_t,\psi;y)^{\frac{1}{\psi}-1}\right)\right]\right)=\beta\exp\left(\left(\frac{1}{\psi}-1\right)E[\log I(X_t,\psi;y)]\right).$$

Since $\lim_{\psi\to1}1/\psi-1=0$, the above expression converges to β. Next, we look at the limit of $(1-1/\psi)^{-1}(i_1\log\beta+i_0-\beta)$. For convenience, we denote $I(X,\psi;y)$ by I_y

$$\frac{1}{1-1/\psi}(i_1\log\beta+i_0-\beta)=\beta\left(E[\log I_y]e^{\left(\frac{1}{\psi}-1\right)E[\log I_y]}+\frac{1}{1-1/\psi}\left(e^{\left(\frac{1}{\psi}-1\right)E[\log I_y]}-1\right)\right)$$

When $\psi\to1$, $1/\psi-1\to0$ and the first term in the bracket converges to $E[\log I_y]$. Apply l'Hopital's rule to the second term:

$$\lim_{\psi\to1}\frac{1}{1-1/\psi}\left(e^{\left(\frac{1}{\psi}-1\right)E[\log I_y]}-1\right)=\lim_{\psi\to1}-\frac{\exp(E[\log I_y])^{\frac{1}{\psi}-1}-1}{\frac{1}{\psi}-1}$$
$$=-E[\log I_y].$$

Therefore, as $\psi \to 1$, $\theta(i_1 \log\beta + i_0 - \beta) = 0$, that is, $\lim_{\psi\to 1} a(\psi) = \tilde{a}$. $\square$

Proof of Theorem 12 Equation 32 follows from Ito's Lemma applied to Corollary 3. Equations 33 and 34 follow from Corollary 6, substituting for $I(x)$ from Theorem 11.

We directly calculate $\mu_{\pi t}$ and then back out the riskfree rate from the no-arbitrage condition (A.17). First consider $\psi \neq 1$. We apply Ito's lemma to (14) to find

$$\mu_{\pi t} = -\beta\left((1-\theta)I(X_t)^{\frac{1}{\psi}-1} + \theta\right) - \gamma\mu_c(X_t) + \frac{1}{2}\gamma(\gamma+1)\sigma_c^2(X_t) \\ + \left(\frac{1}{\psi} - \gamma\right)\frac{1}{I}\frac{\partial I}{\partial X}\mu_X(X_t) + \frac{1}{2}\left(\frac{1}{\psi}-\gamma\right)^2 \operatorname{tr}\left(\frac{1}{I}\frac{\partial^2 I}{\partial X^2}\sigma_X(X_t)\sigma_X(X_t)^\top\right).$$

Substituting in for $I(X_t)$ and its derivatives using (A.20–A.21), together with (26), we find

$$\mu_{\pi t} \simeq -\beta - \frac{1}{\psi}(k_0 + k_1 x) + \frac{1}{2}\gamma\left(1+\frac{1}{\psi}\right)(u_0 + u_1 x) + \frac{1}{2}\left(1-\frac{1}{\psi}\right)\left(\gamma - \frac{1}{\psi}\right)\left(b^\top U_0 b + \left(b^\top U_1 b\right)x\right) \\ - \left(1 - \frac{1}{\theta}\right)\left(E_\nu\left[e^{(1-\gamma)(Z_c + Z_X b)} - 1\right]\right)^\top (l_0 + l_1 x). \quad \text{(A.24)}$$

For $\psi = 1$, apply the same argument using (15) to find:

$$\begin{aligned} \mu_{\pi t} &= -\beta\left((1-\gamma)\log I(X_t) + 1\right) - \gamma\mu_c(X_t) + (1-\gamma)\frac{1}{I}\frac{\partial I}{\partial X}\mu_X(X_t) + \frac{1}{2}\gamma(\gamma+1)\sigma_c^2(X_t) \\ &\qquad + \frac{1}{2}(1-\gamma)^2 \operatorname{tr}\left(\frac{1}{I}\frac{\partial^2 I}{\partial X^2}\sigma_X(X_t)\sigma_X(X_t)^\top\right) \\ &= -\beta - (k_0 + k_1 x) + \gamma(u_0 + u_1 x) - \left(E_\nu\left[e^{(1-\gamma)(Z_c+Z_X b)} - 1\right]\right)^\top (l_0 + l_1 x). \qquad \text{(A.25)} \end{aligned}$$

The risk-free rate then follows from the no-arbitrage condition (A.17). The exact result for time-additive utility follows from the fact that (14) reduces to

$$\pi_t = e^{-\int_0^t \beta\, ds}\beta C_t^{-\gamma}$$

when $\theta = 0$. This is the standard form of the state-price density under time-additive utility and constant relative risk aversion.

$\square$

We prove a no-arbitrage theorem for zero-coupon assets, analogous to the result for long-lived assets (Lemma 4).

Lemma A.2. *Let $H(D_t, X_t, T-t)$ denote the time-t price of a single future dividend payment at time $T > t$. For fixed T, define $H_t = H(D_t, X_t, T-t)$. Define $\mu_{H,t}$ and $\sigma_{H,t}$ such that*

$$\frac{dH_t}{H_{t^-}} = \mu_{H,t}dt + \sigma_{H,t}dB_t + \sum_{j=1}^{m} \frac{\mathcal{J}(H_t)}{H_t} dN_{jt}. \tag{A.26}$$

Then no-arbitrage implies that

$$\mu_{\pi,t} + \mu_{Ht} + \sigma_{\pi,t}\sigma_{Ht}^{\top} + \sum_j \lambda_{jt} E_{\nu_j}\left[\frac{\mathcal{J}_j(\pi_t H_t)}{\pi_t H_t}\right] = 0. \tag{A.27}$$

Proof No-arbitrage implies that $H(D, x, 0) = D$ and that

$$\pi_t H(D_t, X_t, T-t) = E_t\left[\pi_s H(D_s, X_s, T-s)\right]$$

for $s > t$. Ito's Lemma applied to $\pi_t H_t$ implies

$$\pi_t H_t = \pi_0 H_0 + \int_0^t \pi_s H_s \left(\mu_{H,s} + \mu_{\pi,s} + \sigma_{\pi,s}\sigma_{H,s}^{\top}\right) + \int_0^t \pi_s H_s(\sigma_{H,s} + \sigma_{\pi,s})dB_s$$
$$+ \sum_j \sum_{0 < s_{ij} \leq t} \left(\pi_{s_{ij}} H_{s_{ij}} - \pi_{s_{ij}^-} H_{s_{ij}^-}\right), \tag{A.28}$$

where $s_{ij} = \inf\{s : N_{js} = i\}$ (namely, the time that the ith type j jump occurs). Adding and subtracting the jump compensation term from (A.28) yields:

$$\pi_t H_t = \pi_0 H_0 + \int_0^t \pi_s H_s \left(\mu_{H,s} + \mu_{\pi,s} + \sigma_{\pi,s}\sigma_{H,s}^\top + \lambda_s^\top \frac{\bar{\mathcal{J}}(\pi_s H_s)}{\pi_s H_s} \right) ds$$
$$+ \int_0^t \pi_s H_s (\sigma_{H,s} + \sigma_{\pi,s}) dB_s$$
$$+ \sum_j \sum_{0<s_{ij}\leq t} \left(\pi_{s_{ij}} H_{s_{ij}} - \pi_{s_{ij}^-} H_{s_{ij}^-} \right) - \int_0^t \lambda_s^\top \bar{\mathcal{J}}(\pi_s H_s)\, ds. \quad \text{(A.29)}$$

The second and third terms on the right-hand side of (A.29) have zero expectation. Therefore the first term in (A.29) must also have zero expectation, and it follows that the integrand of this term must equal zero. □

Proof of Theorem 13 Conjecture (38). As in the proof of Lemma A.2, fix T and define $H_t = H(D_t, X_t, T-t)$, which follows (A.26). Let $\tau = T - t$. It follows from Ito's Lemma that

$$\sigma_{Ht}(\tau) \simeq \left[\sigma_d,\ b_\phi(\tau)^\top \sigma_X(x)\right]^\top, \quad \text{(A.30)}$$

and

$$\mu_{Ht}(\tau) \simeq (k_0^d + k_1^d x) + b_\phi(\tau)^\top (K_0 + K_1 x) - \left(a'_\phi(\tau) + b'_\phi(\tau)^\top x\right)$$
$$+ \frac{1}{2}\left(b_\phi(\tau)^\top U_0 b_\phi(\tau) + \left(b_\phi(\tau)^\top U_1 b_\phi(\tau)\right) x \right), \quad \text{(A.31)}$$

where $b'_\phi(\tau) = [b'_{\phi 1}(\tau),\ \cdots,\ b'_{\phi n}(\tau)]^\top$ denotes the vector of derivatives with respect to τ. Also, by Ito's Lemma,

$$\frac{\mathcal{J}_j(H_t)}{H_t} \simeq e^{Z_{Hj}} - 1,$$

with

$$Z_{Hj} = Z_{dj} + b_\phi(\tau)^\top Z_{Xj}. \quad \text{(A.32)}$$

Substituting (33–35) and (A.30–A.32) into the no-arbitrage condition (A.27) implies:

$$
\begin{aligned}
0 = &-\beta - \frac{1}{\psi}(k_0 + k_1 x) + \frac{1}{2}\gamma\left(1 + \frac{1}{\psi}\right)(u_0 + u_1 x) + (k_0^d + k_1^d x) + b_\phi(\tau)^\top (K_0 + K_1 x) \\
&- \left(\frac{\partial a_\phi(\tau)}{\partial \tau} + \left(\frac{\partial b_\phi(\tau)}{\partial \tau}\right)^\top x\right) + \frac{1}{2}\left(1 - \frac{1}{\psi}\right)\left(\gamma - \frac{1}{\psi}\right)\left(b^\top U_0 b + \left(b^\top U_1 b\right) x\right) - \gamma(u_0^{cd} + u_1^{cd} x) \\
&+ \frac{1}{2}\left(b_\phi(\tau)^\top U_0 b_\phi(\tau) + \left(b_\phi(\tau)^\top U_1 b_\phi(\tau)\right) x\right) + \left(\frac{1}{\psi} - \gamma\right)\left(b_\phi(\tau)^\top U_0 b + \left(b_\phi(\tau)^\top U_1 b\right) x\right) \\
&+ \left(\left(\frac{1}{\theta} - 1\right) E_\nu\left[e^{(1-\gamma)(Z_c + Z_X b)} - 1\right] + E_\nu\left[e^{-\gamma Z_c + Z_d + Z_X\left(b_\phi(\tau) + (1/\psi - \gamma) b\right)} - 1\right]\right)^\top (l_0 + l_1 x).
\end{aligned}
$$

Matching the constant terms implies (39) and matching the terms multiplying x implies (40), satisfying the conjecture.

For $\psi = 1$ and $\psi = 1/\gamma$, (33–35) hold with equality. The conjecture that (38) holds with equality is therefore satisfied. $\square$

B Approximating the price-dividend ratio by a log-linear function

An alternative approximation approach involves log-linearizing the price-dividend ratio. Let $G(X_t) = S_t/D_t$ and conjecture

$$
\frac{S_t}{D_t} = G(X_t) \simeq e^{\hat{a}_\phi + \hat{b}_\phi^\top X_t},
$$

where $\hat{a}_\phi$ is a scalar $\hat{b}_\phi = \left[\hat{b}_{\phi 1}, \cdots, \hat{b}_{\phi n}\right]$ is a column vector. Ito's Lemma then implies that

$$
\sigma_{St} = \left[\sigma_d,\ \hat{b}_\phi\, \sigma_X(x)\right]^\top, \tag{B.1}
$$

and

$$
\mu_{St} = (k_0^d + k_1^d x) + \hat{b}_\phi^\top (K_0 + K_1 x) + \frac{1}{2}\left(\left(\hat{b}_\phi^\top U_0 \hat{b}_\phi\right) + \left(\hat{b}_\phi^\top U_1 \hat{b}_\phi\right) x\right), \tag{B.2}
$$

If $\psi \neq 1$,

$$\frac{\bar{\mathcal{J}}\left(\pi_t S_t\right)}{\pi_t S_t} = e^{-\gamma Z_c + Z_d + Z_X\left(\left(\frac{1}{\psi}-\gamma\right)b + \hat{b}_\phi\right)}. \tag{B.3}$$

Substituting (A.24), (33) along with (B.1), (B.2) and (B.3) into the no-arbitrage condition (18) implies:

$$\begin{aligned}
0 = & -\beta - \frac{1}{\psi}(k_0 + k_1 x) + \frac{1}{2}\gamma\left(1 + \frac{1}{\psi}\right)(u_0 + u_1 x) + (k_0^d + k_1^d x) + \hat{b}_\phi^\top (K_0 + K_1 x) \\
& + \frac{1}{2}\left(1 - \frac{1}{\psi}\right)\left(\gamma - \frac{1}{\psi}\right)\left(b^\top U_0 b + \left(b^\top U_1 b\right)x\right) + \frac{1}{2}\left(\hat{b}_\phi^\top U_0 \hat{b}_\phi + \left(\hat{b}_\phi^\top U_1 \hat{b}_\phi^\top\right)x\right) + \frac{1}{G_t} \\
& - \gamma(u_0^{cd} + u_1^{cd} x) + \left(\frac{1}{\psi} - \gamma\right)\left(\hat{b}_\phi^\top U_0 b + \left(\hat{b}_\phi^\top U_1 b\right)x\right) \\
& + E_\nu\left[\left(\frac{1}{\theta} - 1\right)\left(e^{(1-\gamma)(Z_c + Z_X b)} - 1\right) + \left(e^{-\gamma Z_c + Z_d + Z_X\left(\hat{b}_\phi + \left(\frac{1}{\psi}-\gamma\right)b\right)} - 1\right)\right]^\top (l_0 + l_1 x).
\end{aligned}$$

And we log-linearize

$$\frac{1}{G_t} = g_0 - g_1 \log(G(x_t)), \tag{B.4}$$

where $g_1 = e^{E_\nu[-\log G]}$ and $g_0 = g_1(1 - \log g_1)$. We can now match coefficient and find that $\hat{b}_\phi$ solves

$$\begin{aligned}
0 = & -g_1 \hat{b}_\phi^\top - \frac{1}{\psi}k_1 + \frac{1}{2}\gamma\left(1 + \frac{1}{\psi}\right)u_1 + k_1^d + \hat{b}_\phi^\top K_1 - \gamma u_1^{cd} + \frac{1}{2}\left(1 - \frac{1}{\psi}\right)\left(\gamma - \frac{1}{\psi}\right)b^\top U_1 b + \left(\frac{1}{\psi} - \gamma\right)\hat{b}_\phi^\top U_1 b \\
& + \frac{1}{2}\hat{b}_\phi^\top U_1 \hat{b}_\phi^\top + \left(E_\nu\left[\left(\frac{1}{\theta} - 1\right)\left(e^{(1-\gamma)(Z_c + Z_X b)} - 1\right) + e^{-\gamma Z_c + Z_d + Z_X\left(\hat{b}_\phi + \left(\frac{1}{\psi}-\gamma\right)b\right)} - 1\right)\right]^\top l_1,
\end{aligned}$$

and $\hat{a}_\phi$ is given by

$$\begin{aligned}
\hat{a}_\phi = & \frac{1}{g_1}\Bigg(g_0 - \beta - \frac{1}{\psi}k_0 + \frac{1}{2}\gamma\left(1 + \frac{1}{\psi}\right)u_0 + k_0^d + \hat{b}_\phi^\top K_0 - \gamma u_0^{cd} + \frac{1}{2}\left(1 - \frac{1}{\psi}\right)\left(\gamma - \frac{1}{\psi}\right)b^\top U_0 b + \frac{1}{2}\hat{b}_\phi^\top U_0 \hat{b}_\phi \\
& + \left(\frac{1}{\psi} - \gamma\right)\hat{b}_\phi^\top U_0 b + E_\nu\left[\left(\frac{1}{\theta} - 1\right)\left(e^{(1-\gamma)(Z_c + Z_X b)} - 1\right) + \left(e^{-\gamma Z_c + Z_d + Z_X\left(\hat{b}_\phi + \left(\frac{1}{\psi}-\gamma\right)b\right)} - 1\right)\right]^\top l_0\Bigg).
\end{aligned}$$

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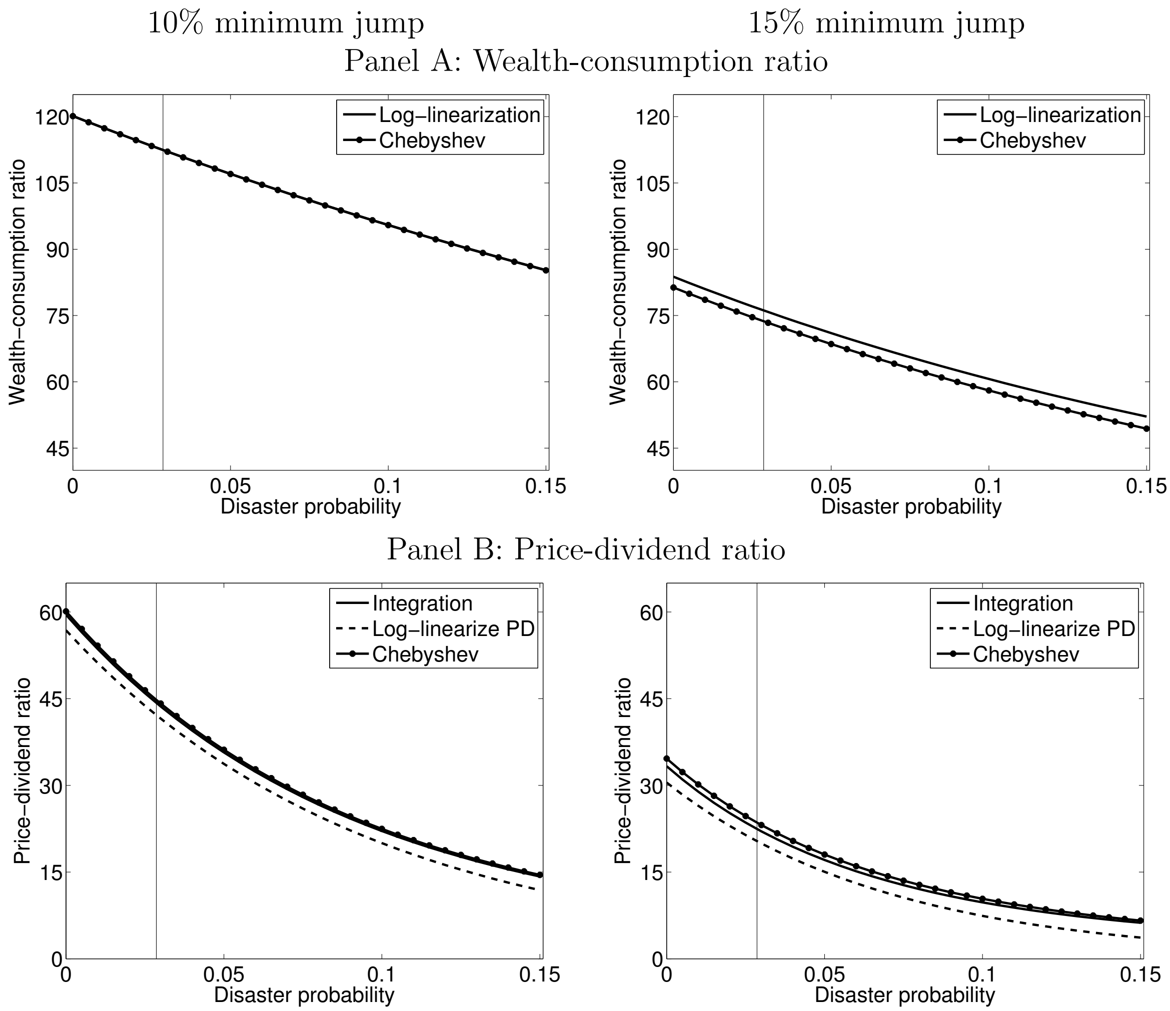
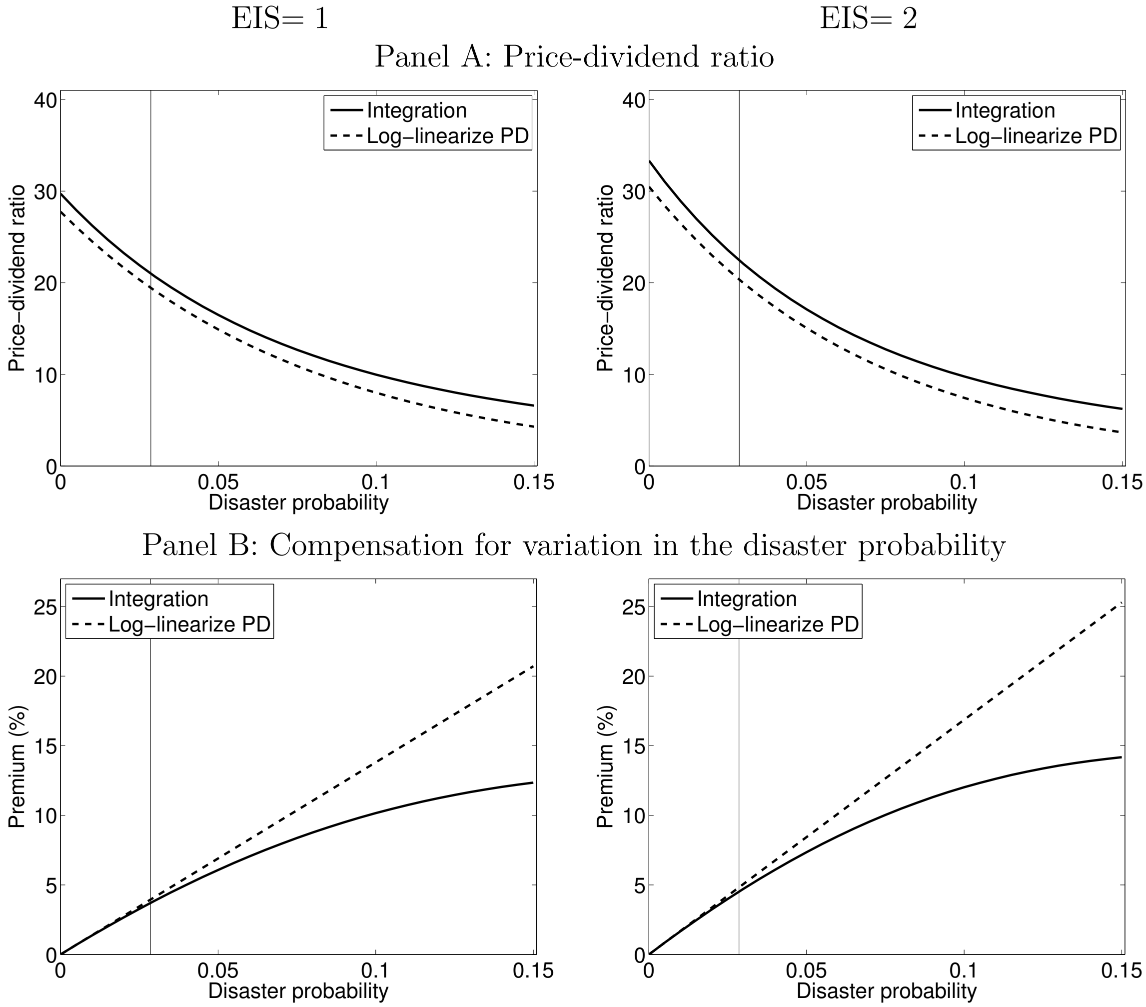
Figure 1: Log-linear Approximation vs Exact Solutions



This figure plots the wealth-consumption ratio (Panel A) and the price-dividend ratio (Panel B) under the time-varying disaster risk model. We calculate the value function by log-linearization or by Chebyshev polynomials. Given the solutions using the log-linearization method, the price-dividend ratio is calculated using either the integration or log-linearization method. We use the Barro and Ursúa (2008) data to for the distribution for Z_c and evaluate the model at different cutoffs. The price-dividend and wealth-consumption ratios are expressed in annual terms.

Figure 2: Integration vs Log-linearization Method



This figure plots the price-dividend ratio (Panel A) and a component of the equity premium (Panel B) under the time-varying disaster risk model. The price dividend ratio is calculated either by integrating over future dividend claims (Integration), or by log-linearization (Log-linearize PD). We focus on the component of the equity premium that compensates for the risk of time-varying λ_t. The price-dividend ratio is in annual terms. The λ-premium is in annual percentage terms. We use the Barro and Ursúa (2008) data to for the distribution for Z_c with 15% cutoffs.