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Growth to Value: Option Exercise and the Cross Section of Equity Returns

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Abstract

We put forward a general equilibrium model to study the link between the cross section of expected returns and book-to-market characteristics. We model two primitive assets: value assets, and growth assets that are options on assets in place. The cost of option exercise, which is endogenously determined in equilibrium, is highly procyclical and acts as a hedge against risks in assets in place. Consequently, growth options are less risky than value assets, and the model features a value premium. Our model incorporates long-run risks in aggregate consumption (as in Bansal and Yaron (2004)) and replicates the empirical failure of the conditional CAPM prediction. We calibrate the model and show that it is able to quantitatively account for the observed pattern in mean returns on book-to-market sorted portfolios, the magnitude of the CAPM-alphas, and other silent features of the cross-sectional data.

Introduction

Historically, stocks with a high book-to-market ratio, on average, earn higher returns than those with a low ratio of book-to-market equity. The difference in mean returns on value and growth stocks, the so-called value premium, is known to pose a serious challenge to the standard asset-pricing models such as the Capital Asset Pricing Model (CAPM). This paper offers a rational explanation of why growth stocks, which effectively are options on assets in place, are less risky and, consequently, carry a low premium relative to value stocks. We put forward a general equilibrium model that accounts for the dispersion in average returns on value and growth stocks as well as the failure of the CAPM predictions in the data.

We model growth assets as options on assets in place. Exercising an option requires one unit of installed capital and results in the creation of a new asset in place. Thus, growth options are long positions in assets in place and short positions in installed capital. If the cost of option exercise is sensitive to macroeconomic risks, it will act as a hedge against risks in assets in place, making growth options less risky than value assets. We show that in equilibrium, the marginal cost of installed capital is indeed highly procyclical if the supply of physical capital is scarce relative to the existing options and aggregate risk is mean reverting. The intuition of our result can be explained as follows. If the economy is currently above its long-run trend, it will likely slow down in the future due to mean reversion, which makes delaying option exercise less attractive. Owners of growth assets who are trying to expedite option exercise in good times will drive up the price of installed capital. By the same logic, the cost of option exercise is lower when macroeconomic conditions are unfavorable. The procyclical dynamics of the equilibrium price of installed capital partially offsets cyclical fluctuations in assets in place, which makes growth assets less vulnerable to aggregate risks and, consequently, results in the value premium.

We embed the above mechanism in a long-run risks economy (Bansal and Yaron (2004)) and provide an endogenous link between dividend exposure to persistent risks in consumption at the aggregate level and differential exposure to long-run risks across book-to-market sorted portfolios. We show that our model, calibrated to match time-series properties of aggregate consumption and the stock market, is able to capture key properties of the cross section of asset dividends and prices. Quantitatively, the difference in returns on value and growth firms inside the models averages about 4.3%, which is roughly 80% of the historical value premium. We further show that in the model, value assets that have high exposure to long-run consumption risks will always have high α's in the conditional CAPM regressions. In simulations, the model-implied average conditional α of high book-to-market stocks is almost 3%, whereas that of growth portfolio is about -1.7%, which is

consistent with the pattern of the CAPM mispricing in the data.[1]

The economic mechanism of our model is supported by several empirical observations. First, as shown in Hansen, Heaton, and Li (2006), and Bansal, Dittmar, and Kiku (2009) among others, value stocks are highly sensitive to low-frequency fluctuations in aggregate consumption. We show that value premium arises in equilibrium if assets in place are highly exposed to long-run consumption risks. Second, our model rationalizes the importance of long-run consumption risks in explaining cross-sectional differences in the observed risk premia, highlighted recently in Bansal, Dittmar, and Lundblad (2005), Hansen, Heaton, and Li (2006), Kiku (2006), Malloy, Moskowitz, and Vissing-Jorgensen (2006), Bansal, Dittmar, and Kiku (2009), and Bansal, Kiku, and Yaron (2007).

Our paper contributes to the literature that studies the relation between expected returns and firms' investment decisions, in particular, Berk, Green, and Naik (1999), Gomes, Kogan, and Zhang (2003), Carlson, Fisher, and Giammarino (2004), Kyle (2004), Cooper (2006), Gârleanu, Panageas, and Yu (2009), and Novy-Marx (2008). However, our paper differs from the existing research along several dimensions.

First, real option-based models typically imply that growth options are riskier than assets in place either because the price of the strike asset is assumed to be constant in partial equilibrium settings, or because unexercised options expire immediately, or both. [2] Given that growth options are long positions in assets in place and short positions in a risk-free asset, they are riskier than value assets in the above models. In contrast, in our model, growth options are long lived and compete for installed capital, the scarce resource needed for option exercise. Consequently, they are less risky because the price of the strike asset is endogenously procyclical. This implication is consistent with recent evidence in Kogan and Papanikolaou (2010) who show that, in the data, option-intensive firms yield lower returns on average than firms with high stock of physical capital.

Second, in order to account for the value premium, real option based models typically hypothesize that high book-to-market firms, rather than growth firms are option intensive. In our model, value firms are assets in place intensive, and growth firms have higher loadings on options. Empirical evidence on cash-flow duration and growth intensity of book-to-market sorted portfolios suggests that value firms appear to have fewer growth opportunities relative to growth (or low book-to-market) firms. Dechow, Sloan, and Soliman (2004) and Da (2006) show that growth stocks tend to pay off far in the future whereas value stocks are characterized by short duration of

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[1]The failure of the standard market and consumption betas has been illustrated in Mankiw and Shapiro (1986), Fama and French (1992), Lewellen and Nagel (2006), Petkova and Zhang (2005). The conditional versions of the CAPM, their testable implications and pertinent econometric issues are studied in Jagannathan and Wang (1996), Ferson and Harvey (1991, 1999), Lettau and Ludvigson (2001), Santos and Veronesi (2006), and Brandt and Chapman (2008), among others.

[2]In Berk, Green, and Naik (1999), growth options could be riskier or less risky than assets in place depending on parameter values.

their assets. Analysts' forecasts of long-term growth are also systematically lower for value firms than they are for growth firms. In addition, compared to growth firms, high book-to-market firms in the data tend to have a lower ratio of capital expenditure to sales, which is typically viewed as a proxy for growth options available to firms (Da, Guo, and Jagannathan (2009)).

Finally, many existing models are built on only one source of risk.[3] Therefore, although they are able to generate a high expected return for value firms, in all these models, the conditional CAPM still holds. In contrast, our model is able to account for the failure of the conditional CAPM in the data. We allow for two sources of risks: long-run and short-run fluctuations in aggregate consumption, and consider a representative agent with the recursive preferences as in Kreps and Porteus (1978), Epstein and Zin (1989), and Weil (1989). Since the two consumption risks in the recursive utility framework carry different risk compensations, a one-factor model such as the CAPM will fail to account for the equilibrium asset prices.

Our paper is also related to a broader literature on firms' decisions and their asset pricing implications. McDonald and Siegel (1986) study optimal timing decisions of investment under uncertainty. Abel and Eberly (1994) provide a unified framework of investment under frictions. Abel, Dixit, Eberly, and Pindyck (1996) study the link between the option pricing approach and q-theory approach to optimal investment. Zhang (2005) and Gala (2005) are recent applications of investment theory to the study of the cross section of equity returns. Gourio (2006) focuses on implications of operating leverage for the value premium.

On a technical level, we obtain closed form solutions of a general equilibrium model with aggregate uncertainty and a nontrivial cross section of assets. Similar techniques of solving for the cross-sectional distribution of firms have been used by Miao (2005) in a partial equilibrium model, and Luttmer (2007) in a general equilibrium economy without aggregate uncertainty. Optimal stopping problems in economies with recursive preferences and long-run risks are studied by Bhamra, Kuehn, and Strebulaev (2007) and Chen (2008).

The paper is organized as follows. In section I, we set up the model and define the appropriate notion of equilibrium. Section II presents the solution of the model and characterizes the cross-sectional distribution of assets. Section III describes the condition for the value premium to exist in equilibrium and provides analysis of the failure of the conditional CAPM. We calibrate the model and discuss its quantitative implications in Section IV. Finally, Section V concludes. All technical details and proofs are provided in the Appendix.

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[3]With the exception of Berk, Green, and Naik (1999) and Kyle (2004) who allow for two sources of risks.

I Set-up of the Model

I.A Preferences

Consider an economy with a continuum of households who have identical intertemporal preferences described by the Kreps and Porteus (1978) utility with a constant relative risk aversion parameter γ and a constant intertemporal elasticity of substitution (IES) ψ. Time is continuous and infinite. We follow Duffie and Epstein (1992a, 1992b) and represent the preferences as stochastic differential utility.[4] The functional form of the stochastic differential utility can be found in Appendix I. Consistent with the long-run risks literature (Bansal and Yaron (2004)), we assume preference for early resolution of uncertainty, and an IES higher than 1, that is $\gamma > \psi > 1$.

I.B Endowments

The economy has two types of endowments: endowment of investment options and endowment of installed capital. Investment options arrive exogenously at the rate m_t per unit of time. The growth rate of m_t is given by

$$\frac{dm_t}{m_t} = \theta_t dt + \sigma_C\left(\theta_t\right) dB_t. \tag{1}$$

In the above equation, $\{B_t\}_{t\geq 0}$ is a one-dimensional standard Brownian motion, and $\{\theta_t\}_{t\geq 0}$ is a two-state Markov process with state space $\Theta = \{\theta_H, \theta_L\}$, where $\theta_H > \theta_L$. We further assume $\sigma_C\left(\theta_H\right) \geq \sigma_C\left(\theta_L\right)$, which guarantees that the equilibrium consumption growth volatility is counter-cyclical. The transition probability of θ_t over an infinitesimal time interval Δ is given by

$$\left[\begin{array}{cc} e^{-\lambda_H \Delta} & 1 - e^{-\lambda_H \Delta} \\ 1 - e^{-\lambda_L \Delta} & e^{-\lambda_L \Delta} \end{array}\right].$$

The endowment of installed capital arrives exogenously at rate δm_t per unit of time, where $\delta < 1$. That is, the arrival rate of installed capital and that of investment options are proportional to each other, and the supply of investment options exceeds that of the installed capital at all times. Here we do not explicitly model the technology of producing new investment options and new installed capital and simply assume that they have the same cyclical properties. Investment options are the result of technology innovations, and installed capital is produced from physical investment. Technology innovations are typically believed to be the engine of economic growth.[5] From this point of view, the arrival of investment options must be pro-cyclical. Although creation

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[4]The discrete time version of recursive preferences is discussed in Kreps and Porteus (1978), Epstein and Zin (1989), and Weil (1989).

[5]Aghion and Howitt (1998) provide an excellent summary of this literature.

of installed capital is hard to measure without explicitly modeling the technology that transforms investment goods into installed capital, the procyclicality of physical investment in the data suggests that installed capital is also procyclical.

Investment options are storable. An investment option by itself does not produce any consumption good, but starts producing consumption goods once implemented. Implementing an investment option requires one unit of installed capital. We call an implemented option — an investment option matched with one unit of installed capital — an asset in place. Investment options capture the essence of growth assets. They do not carry any installed capital and are long-duration assets in the sense that they do not generate any cash flow immediately but are expected to generate a cash-flow stream in the future. For investment option i, we use the notation t^i to denote the calendar time at which it acquires a unit of installed capital.

Investment options differ by their quality. The initial quality of all investment options is assumed to be a constant X_0 upon birth. After birth, an investment option dies at Poisson rate $\kappa > 0$. Conditional on survival, the quality of the investment option, denoted X_t^i, evolves according to the following stochastic differential equation:

$$dX_t^i = X_t^i \left[\mu_O dt + \sigma_O dB_t^i \right], \quad \forall t \leq t^i \tag{2}$$

until the option is implemented and becomes an asset in place, or is hit by a Poisson death shock. $\{B_t^i\}_{t \geq 0}$ is a standard Brownian motion and is assumed to be independent across options. μ_O and σ_O are constants. An investment option replicates itself at the rate $\frac{dm_t}{m_t}$.[6] Therefore, accounting for the Poisson death shock, the expected number of offsprings produced by an investment option from time t to $t + s$ (including the original option itself), denoted $G(t, t + s)$, is given by

$$G(t, t + s) = \exp \left\{ \int_t^{t+s} \left[\theta_u - \frac{1}{2} \sigma_C^2 (\theta_u) \right] du + \int_t^{t+s} \sigma_C (\theta_u) dB_u - \kappa s \right\}. \tag{3}$$

Here we assume that the rate at which an investment option replicates itself is proportional to the rate of arrival of new investment options and installed capital. This will ensure that the equilibrium distribution of investment options is stationary after an appropriate normalization.

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[6]If $\frac{dm_t}{m_t} < 0$, it can be interpreted as a death shock or depreciation.

I. C Technology

Installed capital is not storable unless it is matched with an option.[7] Each investment option can be matched with at most one unit of installed capital, after which it becomes an asset in place. Assets in place are value assets – they carry one unit of installed capital and produce consumption goods, which are paid right away as dividend to shareholders. Hence, cash flows generated by assets in place have a shorter duration relative to investment options.

The initial level of the cash flow generated by asset in place i, denoted D_t^i, depends on the quality of the investment option it implements upon option exercise:

$$D_{t^i}^i = X_{t^i}.$$

The law of motion of D_t^i is given by:

$$dD_t^i = D_t^i \left[\left(\mu_A dt + \sigma_A dB_t^i \right) + \frac{dm_t}{m_t} \right], \qquad \forall \, t \geq t^i. \tag{4}$$

The dividend growth rate is exposed to two sources of shocks: the idiosyncratic shock dB_t^i and the aggregate shock $\frac{dm_t}{m_t}$. Similar to options, assets in place also die at Poisson rate κ.

Let $\mathcal{I}_t$ denote the set of assets in place that are actively operating at time t, the aggregate consumption of the economy at time t is given by:

$$C_t = \int_{i \in \mathcal{I}_t} D_t^i di. \tag{5}$$

A profit maximization firm that owns an investment option can choose to purchase a unit of installed capital and turn the investment option into an asset in place at any time. The decision to match installed capital with an investment option is irreversible. Once matched, installed capital cannot be combined productively with a different investment option later on. The price of investment options, installed capital and assets in place will be determined by equilibrium conditions. We now turn to the definition of equilibrium.

I. D Definition of Equilibrium

Let $\{\pi_t\}_{t \geq 0}$ denote the state price density of the economy. Let $V_A(D, \theta)$ be the value function of assets in place, and $V_O(X, \theta)$ denote the value of investment options. Given $\{\pi_t\}_{t \geq 0}$, the value of

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[7]We assume that the depreciation rate of installed capital is 100%. This guarantees that all installed capital is immediately consumed to build assets in place. This assumption is made for simplicity – all equilibrium calculations will go through as long as the depreciation rate is high enough.

an asset in place is determined by:

$$V_A \left(D_t^i, \theta_t \right) = E_t \left[\int_t^\infty \frac{\pi_s}{\pi_t} e^{-\kappa(s-t)} D_s^i ds \right], \ t \geq t^i. \tag{6}$$

In the class of equilibria we construct, $V_A (D, \theta)$ is linear in D, i.e., $V_A (D, \theta) = a (\theta) D$. The functional form of $a (\theta)$ is given in Appendix II.

The optimal stopping problem of investment option i at calendar time $t \leq t^i$ can be written as

$$V_O(X_t^i, \theta_t) \equiv \max_\tau E_t \left[\frac{\pi_\tau}{\pi_t} G (t, \tau) \left\{ V_A \left(X_\tau^i, \theta_t \right) - q (\theta_\tau) \right\} \right], \tag{7}$$

where $q (\theta)$ is the equilibrium price of installed capital, and the optimization is taken over all stopping times τ (adapted to an appropriately defined filtration). In the class of equilibria we consider, the optimal decision rule for an investment option can be characterized by an optimal exit threshold X^*. It is optimal to exercise the option if $X_t^i \geq X^*$. Since an investment option will be immediately exercised once its quality reaches the threshold level, X^* is also the absorbing barrier of the cross-sectional distribution of the quality of unexercised investment options. Figure 1 depicts the dynamics of a cohort of newly arrived investment options. At time t, measure m_t of investment options with initial quality X_0 arrives. Some of them will hit the threshold level X^* in the future and become assets in place; the others will die prematurely.

The equilibrium price of installed capital is determined by the market clearing condition, which requires that during any time interval, the total measure of investment options that exercise their growth options must equal the total supply of installed capital. We focus on equilibria in which the measure of unexercised investment options grows at the same rate as the exogenous arrival rate of new options. In particular, we conjecture that there exists a density function $\Phi (\cdot)$, such that $\Phi (X) m_t$ is the density of unexercised options with quality X for all $X > 0$. We use the notation $m_{EXIT} [\Phi, X]$ to denote the absorbing rate of density Φ at the absorbing barrier X. The precise definition of the equilibrium is stated below.

Definition 1 *Competitive Equilibrium with Balanced Growth*

A competitive equilibrium with balanced growth is a collection of equilibrium prices and quantities that satisfies the following conditions:

a) Shareholder value maximization for investment options: The option exercise threshold X^* solves the optimal stopping problem (7).

b) Market clearing for installed capital: The rate of option exercise equals the rate of the arrival

of installed capital. That is,

$$\forall t, \ m_t \times m_{EXIT} \left[\Phi, X^* \right] = \delta m_t. \tag{8}$$

c) Market clearing for consumption goods: aggregate consumption is the sum of all divided paid by assets in place as in equation (5).

d) Consistency of macro- and micro-variables: The normalized density Φ is consistent with the law of motion of the quality of individual investment options in equation (2).

Condition d) requires that the variables describing the macroeconomic quantities be consistent with the individual behavior of investment options. Technically, this implies that Φ has to satisfy a version of the Komogorov forward equation. Details of the technical conditions are discussed in Appendix III.

II Characterization of Equilibrium

In this section, we construct a competitive equilibrium with balanced growth defined above. We first conjecture that the equilibrium growth rate of aggregate consumption is exactly $\frac{dm_t}{m_t}$. Given this conjecture, we solve the optimal stopping problem of investment options and derive the (normalized) stationary distribution of investment options and assets in place. We determine the equilibrium price of installed capital, $q(\theta)$, using the market clearing condition in equation (8). Finally, we verify our conjecture and prove that total dividend payment of the economy sums up to aggregate consumption.

II. A The Optimal Stopping Problem

We conjecture that along the balanced growth path, aggregate consumption grows at the same rate as m_t.

Conjecture 1 *Aggregate consumption, C_t, evolves as*

$$\frac{dC_t}{C_t} = \theta_t dt + \sigma_C \left(\theta_t \right) dB_t. \tag{9}$$

Given the law of motion of aggregate consumption, the state price density of the economy can be determined as in the Lucas(1978)–Breeden(1979) framework. The functional form of $\{\pi_t\}_{t \geq 0}$ can be found in Appendix I. We make the following assumptions on the parameters of the model

to guarantee that the values of investment options and assets in place are finite:

$$\beta - \left(1 - \frac{1}{\psi}\right)\theta + \frac{1}{2}\gamma\left(1 - \frac{1}{\psi}\right)\sigma_C^2(\theta) + \kappa - \mu > 0, \quad for \quad \theta = \theta_H, \ \theta_L, \ and \ \mu = \mu_O, \mu_A \quad (10)$$

Under the above assumption, the optimal stopping problem in equation (7) has a well-defined solution for any $q(\theta)$. In general, given any functional form of the price of installed capital, $q(\theta)$, the solution to the optimal stopping problem of investment options can be summarized by a pair of thresholds, $X^*(\theta_H)$, $X^*(\theta_L)$, such that it is optimal to exercise an investment option in state θ if X_t reaches $X^*(\theta)$ from below, for $\theta = \theta_H, \theta_L$. The solution to the optimal stopping problem is summarized in the following proposition.

Proposition 1 *Optimal Stopping for Investment Options*

Given the equilibrium price of installed capital, $q(\theta)$, the value function of investment options is

$$V_O(X, \theta_H) = K_1 e_1 X^{\zeta_1} + K_2 e_2 X^{\zeta_2}, \quad V_O(X, \theta_L) = K_1 X^{\zeta_1} + K_2 X^{\zeta_2}, \quad (11)$$

where the parameters $1 < \zeta_1 < \zeta_2$, and $e_1 > 0$, $e_2 < 0$ are given in Appendix II.

The option exercise thresholds, $X^*(\theta_H)$ and $X^*(\theta_L)$, along with the two constants K_1, K_2, are jointly determined by the following value matching and smooth pasting conditions:

$$\begin{bmatrix} V_O(X^*(\theta_H), \theta_H) \\ V_O(X^*(\theta_L), \theta_L) \end{bmatrix} = \begin{bmatrix} V_A(X^*(\theta_H), \theta_H) - q(\theta_H) \\ V_A(X^*(\theta_L), \theta_L) - q(\theta_L) \end{bmatrix} \quad (12)$$

$$\begin{bmatrix} \frac{\partial V_O}{\partial X}(X^*(\theta_H), \theta_H) \\ \frac{\partial V_O}{\partial X}(X^*(\theta_L), \theta_L) \end{bmatrix} = \begin{bmatrix} \frac{\partial V_A}{\partial X}(X^*(\theta_H), \theta_H) \\ \frac{\partial V_A}{\partial X}(X^*(\theta_L), \theta_L) \end{bmatrix} \quad (13)$$

Proof: See Appendix II ∎

Equilibrium requires that $X^*(\theta_H) = X^*(\theta_L) = X^*$, which will determine $q(\theta)$.

II.B Distribution of Investment Options and Assets in Place

In this section, for a given absorbing barrier X^*, we derive the stationary distribution Φ such that $\Phi(\cdot) m_t$ is the density of the cross-sectional distribution of investment options. This will allow us to impose the market clearing condition and solve for the unique X^* that equates supply and demand of installed capital. The equilibrium prices and quantities will be completely characterized once

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the market clearing X^* is determined. Finally, we close the model by verifying Conjecture 1.

In the equilibrium we construct, the measure of unexercised investment options in the economy grows at the same rate as the arrival of new options. At any time t, the total measure of investment options that enter the economy is m_t. The total measure of options being exercised equals the arrival rate of installed capital, δm_t, and the total measure of options that die of the Poisson shock is $(1 - \delta) m_t$. The cross-sectional distribution of the quality of investment options is time-invariant after being normalized by m_t.

We show in Appendix III that both the invariant distribution Φ and the exit rate of investment options at a given absorbing barrier X, $m_{EXIT}(\Phi, X)$ can be solved in closed form. This allows us to solve equation (8) for the market clearing X^*, which is summarized by the following proposition.

Proposition 2 *Market Clearing Option Exercise Threshold*

The unique option exercise threshold X^* determined by the market clearing condition (8) is given by

$$X^* = \delta^{\frac{1}{\eta_2}} X_0, \tag{14}$$

where $\eta_2 < 0$ is given in equation (54) in Appendix III.

Proof: See Appendix III ∎

The equilibrium prices and quantities can be obtained through the following procedure. Given the supply of investment options and installed capital, the market clearing condition determines a unique level of option exercise threshold, X^*. Given X^*, we use Proposition 1 to solve for the equilibrium price $q(\theta)$ and the value function of investment options. Note that once we impose $X^*(\theta_H) = X^*(\theta_L) = X^*$, the two value matching conditions and the smooth pasting conditions in equations (12) and (13) can be used to jointly determine $q(\theta_H)$ and $q(\theta_L)$ and the constants K_1 and K_2. Finally, given X^*, one can verify that aggregate consumption grows at the same rate as m_t. The details of the construction of the balanced growth path are discussed in Appendix III.

III Asset-Pricing Implications

This section studies the asset-pricing implications of the model. We derive a general formula for the equilibrium risk premium and establish conditions under which options are less risky than assets in place. In addition, we show that value assets are not only characterized by high risk premia but also high α's in the conditional CAPM regressions.

III. A   Risk Premium

Denote the return of asset i over the time interval $[t, t + \Delta]$ by $R^i_{t,t+\Delta}$, then

$$R^i_{t,t+\Delta} = \frac{V_O \left(X^i_{t+\Delta}, \theta_{t+\Delta} \right) G \left(t, t + \Delta \right)}{V_O \left(X^i_t, \theta_t \right)},$$

if asset i is an investment option, and

$$R^i_{t,t+\Delta} = \frac{V_A \left(D^i_{t+\Delta}, \theta_{t+\Delta} \right) + D^i_t \Delta}{V_A \left(D^i_t, \theta_t \right)},$$

if asset i is an asset in place. We use $RP(i, t)$ to denote the risk premium of asset i at time t:

$$RP(i, t) = \lim_{\Delta \to 0} \frac{1}{\Delta} E_t \left[R^i_{t,t+\Delta} - r(\theta_t) \Delta \right],$$

where $r(\theta_t)$ is the instantaneous risk-free rate of the economy.

To simplify notation, we denote the price of asset i at time t as $V(i, t, \theta_t)$, with the understanding that $V(i, t, \theta_t) = V_O \left(X^i_t, \theta_t \right)$, if asset i is an investment option, and $V(i, t, \theta_t) = V_A \left(D^i_t, \theta_t \right)$, if asset i is an asset in place. The risk premium of a generic asset i is given by the following proposition.

Proposition 3 *Risk Premium*

The risk premium of asset i is

$$
\begin{aligned}
RP(i, t) &= \gamma \sigma^2_C (\theta_L) + \lambda_L (1 - \hat{\omega}) \left[\frac{V(i, t, \theta_H)}{V(i, t, \theta_L)} - 1 \right], \quad if \;\; \theta_t = \theta_L & (15) \\
RP(i, t) &= \gamma \sigma^2_C (\theta_H) + \lambda_H (1 - \hat{\omega}^{-1}) \left[\frac{V(i, t, \theta_L)}{V(i, t, \theta_H)} - 1 \right], \quad if \;\; \theta_t = \theta_H, & (16)
\end{aligned}
$$

where $\hat{\omega} < 1$ is defined in Appendix I.

Proof: See Appendix IV ∎

The risk premium of an asset has two components: compensation for short-run risks and compensation for long-run risks. The first term in (15) and (16) can be written as

$$\gamma \sigma^2_C (\theta_t) = \gamma \lim_{\Delta \to 0} Cov_t \left(R^i_{t,t+\Delta}, \; \frac{C_{t+\Delta} - C_t}{C_t} \right),$$

which is just the compensation for the covariation of the asset return with contemporaneous innovations in consumption growth, that is, compensation for the asset's exposure to short-run

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consumption risk. We will call this term the short-run risk premium.

The second component of the risk premium is the compensation for the covariation with innovations in the expected consumption growth, θ_t, i.e., compensation for long-run risks. In fact, the second component of the risk premium is proportional to

$$\lim_{\Delta \to 0} Cov_t \left(R^i_{t,t+\Delta}, \ \frac{\theta_{t+\Delta} - \theta_t}{\theta_t} \right).$$

Since $\lambda_L (1 - \hat{\omega}) > 0$ and $\lambda_H \left(1 - \hat{\omega}^{-1}\right) < 0$, long-run risks carry a positive risk premium.[8] We will label this component of the risk premium the long-run risk premium.

An asset's exposure to long-run risks will depend on its characteristics, which are summarized by $V(i, t, \theta_t)$. It follows from Proposition 3 that the long-run risk premium of asset i is higher than that of asset j in both states of the world if

$$\frac{V(i, t, \theta_H)}{V(i, t, \theta_L)} > \frac{V(j, t, \theta_H)}{V(j, t, \theta_L)}. \tag{17}$$

Intuitively, the higher is the size of the jump in asset value in case of a regime switch in the expected consumption growth, the higher is the exposure to long-run risks and, consequently, the higher is the compensation for risks associated with changes in θ_t.

III. B   Value Premium

The following proposition provides a sufficient condition under which investment options are less risky than assets in place.

Proposition 4 *Value Premium*

There exists $\bar{\mu}$, such that if $\mu_A > \bar{\mu}$, then all assets in place have higher risk premia than all growth options, that is

$$\frac{V_A(D, \theta_H)}{V_A(D, \theta_H)} > \frac{V_O(X, \theta_H)}{V_O(X, \theta_L)}, \quad for \ \ all \ \ D \ and \ X. \tag{18}$$

$\bar{\mu}$ is a function of the parameters of the model given in Appendix IV.

Proof: See Appendix IV ∎

Proposition 4 implies that value assets are riskier than growth options as long as assets in place have sufficiently high exposure to long-run risks. The exact technical condition under recursive

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[8]$\lambda_L (1 - \hat{\omega})$ and $\lambda_H \left(\hat{\omega}^{-1} - 1\right)$ can be interpreted as the risk-neutral probabilities of a regime switch in θ.

preference is quite involved and is detailed in Appendix IV. Here, we illustrate the basic intuition of the above proposition under the assumption of risk-neutrality of the representative agent and zero replication rate of investment options. We provide a heuristic agrement that if $\frac{V_A(X^*,\theta_H)}{V_A(X^*,\theta_H)} > 0$ then condition (18) holds at the optimal option exercise threshold X^*.

Note that value assets are riskier than at-the-money options if the price of installed capital is more procyclical than assets in place. That is, condition

$$\frac{V_A\left(X^*,\theta_H\right)}{V_A\left(X^*,\theta_H\right)} \geq \frac{V_O\left(X^*,\theta_H\right)}{V_O\left(X^*,\theta_L\right)}. \tag{19}$$

is equivalent to:

$$\frac{q\left(\theta_H\right)}{q\left(\theta_L\right)} \geq \frac{V_A\left(X^*,\theta_H\right)}{V_A\left(X^*,\theta_H\right)}. \tag{20}$$

The equivalence follows directly from the value matching condition at the option exercise threshold X^*: $V_O\left(X^*,\theta\right) = V_A\left(X^*,\theta\right) - q\left(\theta\right)$, for $\theta = \theta_H$, θ_L.

We first show that if the probability of regime switch is zero, then equation (20) holds with equality. Consequently, installed capital, assets in place and options all have the same exposure to long-run risks.

At the threshold level X^*, an option owner must be indifferent between exercising it right away and waiting for an infinitesimal amount of time, Δ before making the option exercise decision. Suppose the interest rate between time t and $t + \Delta$ is r. Further, suppose that $\theta_t = \theta_H$. Then the payoff of exercising the option immediately is equal to $V_A\left(X^*,\theta_H\right) - q\left(\theta_H\right)$. At time $t + \Delta$, with probability $e^{-\lambda_H\Delta}$, $\theta_{t+\Delta} = \theta_H$ and the payoff of option exercise is given by $V_A\left(X^+,\theta_H\right) - q\left(\theta_H\right)$, where $X^+ = E\left[\max\left(X_{t+\Delta}, X^*\right)\right]$. With probability $1 - e^{-\lambda_H\Delta}$, $\theta_{t+\Delta} = \theta_L$ and the payoff of option exercise is $V_A\left(X^*,\theta_L\right) - q\left(\theta_L\right)$. Therefore, the indifference condition of an option with quality X^* is described by:

$$V_A\left(X^*,\theta_H\right) - q\left(\theta_H\right) = e^{-r\Delta}\left\{e^{-\lambda_H\Delta}\left[V_A\left(X^+,\theta_H\right) - q\left(\theta_H\right)\right] + \left(1 - e^{-\lambda_H\Delta}\right)\left[V_A\left(X^+,\theta_L\right) - q\left(\theta_L\right)\right]\right\}. \tag{21}$$

Similarly, if $\theta_t = \theta_L$, the indifference condition is:

$$V_A\left(X^*,\theta_L\right) - q\left(\theta_L\right) = e^{-r\Delta}\left\{e^{-\lambda_L\Delta}\left[V_A\left(X^+,\theta_L\right) - q\left(\theta_L\right)\right] + \left(1 - e^{-\lambda_L\Delta}\right)\left[V_A\left(X^+,\theta_H\right) - q\left(\theta_H\right)\right]\right\}. \tag{22}$$

If $\lambda_H = \lambda_L = 0$, then equations (21) and (22) reduce to:

$$V_A\left(X^*,\theta_H\right) - q\left(\theta_H\right) = e^{-r\Delta}\left[V_A\left(X^+,\theta_H\right) - q\left(\theta_H\right)\right], \tag{23}$$

and

$$V_A\left(X^*, \theta_L\right) - q\left(\theta_L\right) = e^{-r\Delta}\left[V_A\left(X^+, \theta_L\right) - q\left(\theta_L\right)\right]. \tag{24}$$

It follows that, in this case,

$$\frac{q\left(\theta_H\right)}{q\left(\theta_L\right)} = \frac{V_A\left(X^*, \theta_H\right)}{V_A\left(X^*, \theta_H\right)}.$$

Consider now the case of $\lambda_H, \lambda_L > 0$. Note that the left-hand side of equation (21) is the benefit of exercising the option at t when the economy in the good state, while the right-hand side is the payoff associated with waiting until $t + \Delta$. Note also that $[V_A\left(X^+, \theta_H\right) - q\left(\theta_H\right)] > [V_A\left(X^+, \theta_L\right) - q\left(\theta_L\right)]$ since the option payoff is higher in the good state. The possibility of a regime shift in θ lowers the benefit of waiting by $\left(1 - e^{-\lambda_H\Delta}\right)\left\{[V_A\left(X^+, \theta_H\right) - q\left(\theta_H\right)] - [V_A\left(X^+, \theta_L\right) - q\left(\theta_L\right)]\right\}$ — the option becomes less valuable in the case of a regime switch to θ_L. If $q\left(\theta_H\right)$ stays the same as that in equation (23), options with quality X^* will strictly prefer immediate exercise to waiting. Thus, $q\left(\theta_H\right)$ has to increase to deter entrance. By the same logic, as follows from equations (22) and (24), the possibility of a regime switch increases the benefit of waiting in the bad state. To induce entrance and restore the equilibrium, $q\left(\theta_L\right)$ must be lower than that in equation (24). In summary, the possibility of a regime switch in θ creates an incentive to expedite option exercise in the good state (θ_H), and encourages waiting in the bad state (θ_L). The equilibrium market clearing condition requires the price of installed capital vary more with the aggregate state than the value of assets in place.

The fact that options and assets in place have different exposure to long-run risks depends crucially on the assumption that θ_t is a mean-reverting process. As discussed above, mean reversion creates incentives for option owners to expedite exercise whenever θ_t is above its mean, because waiting in the good state is associated with a potential loss in the value of the option as θ_t is likely to revert to its average. Similarly, mean reversion in aggregate risk encourages waiting whenever θ_t is below its mean. In contrast, the price of installed capital and options have the same exposure to short-run risks since the latter are assumed to be i.i.d.

The above mechanism is illustrated in Figure 2.[9] We use dashed lines to represent value functions of assets in place in the high state (thick line) and in the low state (thin line). Solid lines are value functions of investment options. At the option exercise threshold X^*, the distance between the two thick lines and the distance between the two thin lines is the price of installed capital in states θ_H and θ_L, respectively. Note that the percentage change in the price of installed capital as the economy shifts from one state to another is higher than that of assets in place. That is, the price of installed capital is more procyclical than the value of assets in place. Risks in installed capital partially offset risks in assets in place, making growth options less risky relative to

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[9]Figures 2 and 3 are computed under the parameter values calibrated in Section IV.

value assets. Indeed, as the figure shows, the value of assets in place changes by more than that of investment options upon a shift in θ. Figure 3 plots the model-implied risk premia of investment options and assets in place against the quality of investment options and the dividend level of assets in place. The dashed lines are the risk premia of assets in place, and the solid lines represent the risk premia of investment options. The thick lines correspond to the risk premia in the high state, and the two thin lines are the risk premia when $\theta_t = \theta_L$. As the figure shows, uniformly, assets in place (or value assets) are riskier than investment options (or growth assets) and have to provide investors with higher risk premia.

Since our model is a general equilibrium, aggregate dividends equal to aggregate consumption and, consequently, have the same exposure to long-run risks. If we depart from general equilibrium, the model will always feature a value premium as long as dividends are sufficiently exposed to long-run risks in consumption.[10] This requirement is consistent with empirical evidence in the long-run risks literature. In the data, value firms' cash flows respond more strongly to permanent or near-permanent growth shocks than aggregate consumption (see, for example, Bansal, Dittmar, and Lundblad (2005), and Hansen, Heaton, and Li (2006)). At the cross-sectional level, recent research shows that value assets are more exposed to long-run consumption risks than growth assets (Bansal, Dittmar, and Lundblad (2005), Hansen, Heaton, and Li (2006), Kiku (2006), Malloy, Moskowitz, and Vissing-Jorgensen (2006), Bansal, Kiku, and Yaron (2007), Bansal, Dittmar, and Kiku (2009)). Our results thus provide an endogenous link between the above empirical observations: high exposure of cash flows of assets in place to persistent consumption risks translates into relatively low risk exposure of growth options through the equilibrium adjustment of the price of installed capital.

The discussed mechanism is absent in the real option-based models of Berk, Green, and Naik (1999) and Gomes, Kogan, and Zhang (2003) as they assume that options depreciate fully if not exercised immediately. Consequently, there are no unexercised options in their equilibria. In Carlson, Fisher, and Giammarino (2004) and Gârleanu, Panageas, and Yu (2009), the supply of the resources needed for option exercise is assumed to be infinitely elastic. Thus, the presence of unexercised options in these models has no pricing implication for the cost of option exercise, and the latter is always risk-free. In contrast, in our model, investment options are always in excess supply in equilibrium and have to compete for installed capital, a scarce resource needed for option exercise. As shown above, this competition and mean-reversion in aggregate risk will result in the equilibrium value premium.

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[10]In fact, assuming IES equals 1, the model will generate a value premium if dividends have higher exposure to long-run risks than consumption.

III. C Failure of Conditional CAPM

Suppose an econometrician observes return series (generated under the null of our model) and aims to test the pricing restrictions implied by the standard CAPM. In order to draw inferences about the conditional CAPM, consider the following regression:

$$R^i_{t,t+\Delta} - r(\theta_t)\Delta = \alpha^i_{t,\Delta} + \left(R^W_{t,t+\Delta} - r_{t,t+\Delta}\right)\beta^i_{t,\Delta} + \varepsilon_{t,t+\Delta}, \tag{25}$$

where $R^i_{t,t+\Delta}$ is the rate of return of asset i over $[t, t+\Delta]$, $R^W_{t,t+\Delta}$ is the return of aggregate wealth, and $r(\theta_t)\Delta$ is the risk-free rate over the same time interval.

If the CAPM were true, all the α's in specification (25) would be zero. This could happen only if the aggregate wealth portfolio were perfectly correlated with the true stochastic discount factor. In our model, innovations in the stochastic discount factor are driven by two sources: long-run and short-run risks in consumption. Importantly, the two risks carry very different risk premia. A one-factor model, such as the CAPM, in general, will not be able to convey all the necessary pricing information to account for the cross-sectional differences in expected returns. To formalize this claim, consider theoretical values of α and β in the CAPM specification:

$$\alpha^i_{t,\Delta} = \left(E_t\left[R^i_{t,t+\Delta}\right] - r(\theta_t)\Delta\right) - \beta^i_{t,\Delta}\left(E_t\left[R^W_{t,t+\Delta}\right] - r(\theta_t)\Delta\right), \tag{26}$$

and

$$\beta^i_{t,\Delta} = \frac{Cov_t\left(R^i_{t,t+\Delta}, R^W_{t,t+\Delta}\right)}{Var_t\left(R^W_{t,t+\Delta}\right)}. \tag{27}$$

We consider a continuous time limit of $\alpha^i_{t,\Delta}$ and $\beta^i_{t,\Delta}$, defined as

$$\alpha^i_t = \lim_{\Delta \to 0}\frac{1}{\Delta}\alpha^i_{t,\Delta}, \quad \beta^i_t = \lim_{\Delta \to 0}\beta^i_{t,\Delta}. \tag{28}$$

Using equations (26) and (27), and Proposition 3, we obtain the following proposition that characterizes the CAPM α's in our model economy.[11]

Proposition 5 *Failure of the CAPM*

Assets with high exposure to long-run risks obtain high α's in the conditional CAPM.

Proof: See Appendix IV ■

Given that value assets have larger exposure to long-run risks than growth options, they will

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[11]Population values of the CAPM $\alpha's$ are presented in Appendix IV.

obtain high α's in the conditional CAPM regressions. Growth firms, on the other hand, will feature low alphas. We will assess the magnitude of alphas in both conditional and unconditional CAPM's in our calibration exercise below.

IV   Quantitative Evaluation of the Model

To evaluate the ability of the model to account for the observed value premium and other features of book-to-market sorted portfolios, we run the following simulation exercise. We choose the model parameters to match key properties of aggregate consumption, the stock market index, and the risk-free rate. Once the calibrated model ensures reasonable dynamics of aggregate quantities, we examine its performance in the cross section. Our calibration is guided solely by time-series dynamics of the observed aggregate data and does not exploit any cross-sectional information.

We simulate the model on the monthly frequency but target the dynamics of annual data. We focus on annual moments in order to avoid any seasonal and measurement biases in the data. More specifically, we simulate monthly series over 80 years, aggregate the simulated variables to the annual frequency, and report various moments of the resulting annual data. To remove the effect of initial conditions, we effectively simulate 160 years of data and discard the first half of the sample. We find that increasing the size of the initial simulated sample does not alter the results. We repeat simulations 100 times and report the medians of various statistics of interest across simulations.

IV. A   Data Sources

Our targeted data in calibration consist of real per capita consumption of non-durables and services, the stock market index of the NYSE, AMEX, and NASDAQ traded firms, and the three-month Treasury bill. Consumption data are taken from the NIPA tables published by the Bureau of Economic Analysis. The stock market and risk-free rate data come from the Center for Research in Securities Prices (CRSP). Cross-sectional data that the model is confronted with at the evaluation stage comprise three book-to-market sorted portfolios: "Growth" and "Value" portfolios that consist of firms in the lowest and highest 30th percentile respectively, and firms in the middle portfolio that we label "Neutral". The breakpoints are determined by the 30th and 70th percentiles of the book-to-market sort of the NYSE-listed stocks. Our portfolio construction follows the standard procedure of Fama and French (1992) using the data from the Compustat and CRSP databases. For each book-to-market portfolio as well as the aggregate stock market index, we construct value-weighted returns and per-share dividend series as in Campbell and Shiller (1988) and Bansal, Dittmar, and Lundblad (2005). Asset data are converted to real using the personal

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consumption deflator. All data are annual and span the period from 1930 to 2007.

IV. B Parameter Configuration and Targeted Moments

Our preference and time-series configuration is presented in Table I. We choose preference parameters following the long-run risks literature. Similar to Bansal-Yaron (2004), we set β at 0.007, use a risk-aversion parameter of 10, and set the elasticity of intertemporal substitution at 1.5. This choice of preferences, together with technology parameters to be discussed below, allows the model to match the dynamics and the level of the risk-free rate, as well as the magnitude of the risk premium in the economy.

To match the observed consumption and dividend dynamics, we relax the general equilibrium restriction that dividends sum up to aggregate consumption and calibrate them separately. We continue to assume that aggregate consumption obeys the law of motion described in equation (9), but use a more flexible parameterization of the dividend dynamics:

$$\frac{dD_t^i}{D_t^i} = \mu_A\left(\theta_t\right) dt + \sigma_A\left(\theta_t\right) dB_t^i + \sigma_C\left(\theta_t\right) dB_t.^{12}$$

Consumption growth parameters are chosen to match time-series dynamics of observed consumption data. Table II compares the moments of the simulated growth rates with the corresponding sample statistics. Point estimates along with the Newey and West (1987) standard errors are presented in the left column; model-implied statistics are reported on the right. Our calibration is able to easily match the mean, volatility, and first two autocorrelations of consumption growth. For example, the model-implied first-order autocorrelation of consumption growth is 0.42, which agrees well with the observed persistence of 0.44. In addition to the four moments reported in the table, our calibration is designed to capture evidence on the NBER-dated business cycle fluctuations. In particular, we allow for a longer duration of expansions of approximately 5.5 years relative to recessions, which on average last for about 2.5 years. Expansions in the model are defined by a high mean and a low volatility of consumption growth. Similarly, recessions are associated with both low growth and high uncertainty about future consumption. Thus, our calibration also accounts for a negative covariation between growth and uncertainty observed in the data.[13] Finally, our calibration matches the low-frequency dynamics of the volatility of consumption growth. We estimate consumption volatility by a moving average of absolute residuals from an AR(1) fitted

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[12]The model solution in this case follows closely that of the general quilibrium set-up and is available upon request.

[13]In the data, consumption uncertainty is persistently high when the economy is in contraction. For instance, the average absolute AR(1)-residual of consumption growth is about 0.02 during the NBER-defined recessions and only 0.01 when the economy expands. We account for this empirical evidence by allowing for high consumption volatility in low states and low volatility when consumption growth is expected to rise.

to consumption growth rates. We take a three-year moving average to separate the long-run component from transitory movements in economic uncertainty. The extracted low-frequency volatility component in the data is quite persistent, with the first-order autocorrelation of 0.68. Our calibration leads to a similar persistence of about 0.67.

Option quality and cash-flow parameters are chosen to match key moments of dividend growth rates of the aggregate stock market portfolio. In addition to the time-series parameters specified in Table I, we set X_0 at 1 (a pure normalization that has no qualitative or quantitative effect), and use $\delta = 0.75$, which implies that 75% of new investment options eventually obtain capital and become assets in place and the rest die prematurely. Finally, we set the annualized values of μ_O and σ_O to 0.05 and 0.25 respectively, and assume that options and assets in place depreciate at an annual rate of 8%, i.e., $\kappa = 0.08$. We find the model implications to be generally robust to the choice of parameters that govern the entry and the evolution of unexercised investment options.

Overall, our calibration is consistent with the dynamics of the observed aggregate dividend.[14] As shown in Table III, we are able to match the mean growth rate of aggregate dividends, their volatility, as well as their correlation with consumption growth. The only dimension that the model has some difficulty addressing is the persistence of dividend growth. In the data, the first-order autocorrelation of dividend growth rates is 0.22, whereas the corresponding statistic in the model is about 0.57. Although undesirable, this implication in our view is not critical. The issue of a relatively high serial correlation of dividend growth rates induced by the channel of long-run risks can easily be resolved by introducing another common (orthogonal to consumption risks) component into firms' cash flows as in Bansal and Yaron (2004). We do not entertain such an extension as it will have no effect on prices.

The bottom panel of Table III illustrates that the model can successfully account for the historically high equity premium, as highlighted in Bansal-Yaron (2004). The model-implied average excess return of the market portfolio is about 6.81%. As in the data, the correlation between equity returns and consumption growth is quite low. The model also correctly predicts higher volatility of asset prices relative to dividends – the standard deviation of the market return is about 20% in the data as well as in the model. As mentioned above, the model generates plausible dynamics of the risk-free rate with a mean of 1.5% and a volatility of about 1.1%.

Note that, in the model, asset prices are strongly procyclical: the correlation between the log of the market-to-book ratio and consumption growth is about 38%. This implication confirms well with the cyclical properties of the observed series – in the data, this correlation is approximately 30%. In addition, the model-implied market-to-book ratio is highly persistent, with the first-order

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[14]As in the data, the market portfolio in simulations is a value-weighted combination of individual firms. We will discuss the construction of the cross section in the next section.

autocorrelation of about 0.88, which provides a good match for the sample persistence of 0.84.

IV. C   Cross Section of Dividends and Returns

Relying on the assumed time-series dynamics and model solutions, we simulate a pool of investment options and assets in place, which we use as building blocks for creating firms. We view a firm as a collection of investment options and assets in place that we randomly sample from the simulated pool. Guided by the cross-sectional dispersion of sales and market capitalization in the data, we assume that the initial distribution of investment options and assets in place across firms is Pareto. After stapling investment options and assets in place, we track each firm over time, replacing extinct units with brand-new investment options.[15] Our cross section consists of 2,000 firms. We sort the simulated sample of firms into three book-to-market portfolios following the same sorting procedure as in the data.

Table IV illustrates the dynamics of the per-share dividend growth rates of the book-to-market sorted portfolios. Note that in the data, the value portfolio is characterized by high unconditional growth, whereas firms in growth portfolio, on average, exhibit low per-share growth. Our model captures this feature of the data. In particular, as the book-to-market ratio increases, the unconditional growth of the per-share dividends raises from 0.45% to about 4.70% in the data, and from -0.38% to about 3.52% in the model. The model-implied volatilities of growth rates are similar to their data counterparts, except for the highest book-to-market portfolio. This, however, speaks in favor of rather than against the model, as the high volatility of observed growth rates in the value portfolio is driven by few virtually zero dividend observations in the beginning of the sample, when the data quality is somewhat suspect. Similar to the stock market index, the model-implied correlations between portfolios' growth rates and aggregate consumption are somewhat high relative to the sample estimates. Nevertheless, the two are not far apart from a statistical point of view. Overall, our calibration seems to capture the key moments of the cross-section of dividend growth rates quite well, although the model parameters are not chosen to target any moments of the cross-sectional data.

From the perspective of our model, a more interesting dynamic characteristic of cash flows is their low-frequency (rather then contemporaneous) covariation with consumption. Empirically, the exposure of dividend growth rates to long-run consumption risks is increasing from low to high book-to-market portfolio as documented in Bansal, Dittmar, and Lundblad (2005), Bansal, Dittmar, and

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[15]Note that the number of firms in our simulation is fixed, but the composition of investment options and assets in place inside each firm changes over time. We have also entertained a more complex way of creating a cross section, with the number of firms increasing over time as in the data. This alternative simulation requires additional assumptions about firms' entrance but does not particularly affect the dynamics of the resulting portfolios.

Kiku (2009), and Hansen, Heaton, and Li (2006). Our model features a similar positive relationship between long-run cash-flow betas and book-to-market characteristics. Given that in simulations, the long-run risk variable is conveniently available, we measure long-run exposures inside the model by simply regressing monthly dividend growth rates onto the expected growth component, θ_t. We find a monotonically increasing pattern in the model-implied long-run betas, starting from 5.3 for the growth portfolio and reaching 8.1 for the value portfolio.

The average returns of the simulated portfolios along with their empirical counterparts are presented in Table V. Consistent with the data, the model generates a sizable value premium: the mean return of high book-to-market firms is much higher than the average compensation for holding growth firms. Quantitatively, the model-implied value premium is about 4.3% per annum. For comparison, the difference in mean returns on high and low book-to-market portfolios in the data amounts to about 5.5%. Thus, the model is able to account for a sizable portion of the cross-sectional variation in mean returns on book-to-market sorted portfolios.

Note that, as in Bansal and Yaron (2004), we allow for time variation in the conditional volatility of consumption growth and, consequently, variation in the risk premium. The model-implied spread in expected returns on value and growth firms similarly exhibits countercyclical dynamics. In particular, inside the model, the correlation between the conditional value premium and the volatility of consumption growth is 0.25, whereas its correlation with expected growth in consumption amounts to -0.43. To compute these numbers, we construct the conditional value premium by regressing the spread in realized annual returns of value and growth portfolios on their lagged price-dividend ratios. The expected growth in consumption is measured by fitting an AR(1) process to annual consumption growth, and its volatility is constructed by taking a three-year moving average of absolute residuals from the above regression. The countercyclical dispersion in expected returns of value and growth firms implied by the model is consistent with empirical evidence in Kiku (2006), and Chen, Petkova, and Zhang (2008).

IV.D CAPM Implications

In the model, as in the data, the standard CAPM fails. We illustrate the magnitude of the model deviations from the unconditional and conditional CAPM predictions in Tables VI and VII.

Table VI reports the unconditional-CAPM alphas and the corresponding t-statistics for each of the book-to-market portfolios. The unconditional CAPM inside the model is strongly rejected: all three alphas are economically and statistically significant. Similar to the data, the CAPM tends to underprice growth stocks (by about 1.7% per annum) and overprice value stocks (by about 3.6%). Overall, both the pattern and the magnitude of simulated alphas are consistent with the CAPM

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mispricing in the data. In addition, the model-implied CAPM betas display almost no, if not negative, relationship with average returns. For example, the low-premium growth portfolio has a beta of about 1.04, whereas the high-premium value portfolio has a beta of 0.92.

The conditional CAPM also falls short in explaining the cross-sectional dispersion in risk premia inside the model. To evaluate the performance of the conditional market betas, we run three-year rolling window regressions of monthly excess returns of book-to-market portfolios on monthly excess returns of the aggregate stock market. Average alphas and their robust t-statistics are presented in Table VII. On average, the model-implied conditional alphas are monotonically increasing, from about -1.7% for growth firms to almost 3% for the value portfolio, replicating an empirically strong positive relationship between alphas and book-to-market characteristics. Quantitatively, average conditional alphas in simulations conform to the failure of the conditional CAPM in the actual data. To understand the failure of the standard CAPM, recall that in our model the market return does not proxy for the true stochastic discount factor and, consequently, is not able to correctly price any other assets as highlighted in Proposition 5.

IV.E Additional Cross-Sectional Characteristics

Panel B of Table VIII summarizes the transitional dynamics of the simulated cross section. For comparison and completeness, transition frequencies across book-to-market portfolios in the data are reported in Panel A. We define transition probabilities by a fraction of firms that migrate from one bin in the current year to another bin next year, when portfolios are rebalanced. Frequencies reported in the table are time-series averages of transition probabilities in the simulated and observed sample.[16] It is apparent that, in the model, firms are likely to stay in their current bin. For example, about 85% of firms in the growth portfolio are still classified as growth the following year, and nearly 80% of value firms remain in the top book-to-market bin next year. The corresponding numbers in the data are 78% and 82% for growth and value portfolios, respectively. Similar to the data, the model-implied transitions to nearby portfolios are more frequent than migration to more distant portfolios. Although this simulated cross section is somewhat sluggish relative to the data (except for the value portfolio), overall the model implies very reasonable transitions across portfolios.

Table IX compares average market shares of book-to-market portfolios between the model and the data. Market shares represent a fraction of the market value of a given portfolio in the total market capitalization. As the table shows, the model-implied distribution of market values across

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[16]Note that in the data, unlike the model, a fraction of firms exits the market every period. We ignore firms that disappear in the observed sample, and rescale all the transition probabilities so that they sum to one. This has only a minor effect on the sample statistics but facilitates the comparison between the model and the data.

portfolios matches well the observed pattern in market shares. In the model, the growth portfolio contributes about 60% to the total market portfolio; in the data, its share fluctuates around 54%. The mean share of the value portfolio is 10% in the model and 12% in the data. Consistent with the data, the model-implied shares are quite persistent: the cross-sectional average of the first-order autocorrelations in market shares is about 0.85.

To summarize, we show that our theoretical model calibrated to match the observed time-series dynamics of aggregate consumption and the stock market is able to generate a cross section of firms that is consistent with key properties of the observed book-to-market portfolios and is able to simultaneously reconcile the value premium and empirical failure of the standard CAPM.

V   Conclusion

We present a general equilibrium model of investment options that rationalizes the value premium. Growth opportunities in our model are options on assets in place (i.e., value assets). If physical capital required for option exercise is in scarce supply and aggregate risk is mean reverting, growth options (under some general conditions) are always less risky than value assets and yield lower expected returns.

Growth options are modelled as long positions in assets in place and short positions in installed capital. Heterogeneity in risk exposure of growth and value assets, therefore, depends on risk properties of the price of installed capital. In good times, option exercise is especially valuable as it results in creation of a highly productive value asset. Consequently, more options compete for scarce capital and drive its price up. Similarly, production assets are less profitable in bad times and option owners prefer delaying exercise until the economy recovers. To encourage investment and clear the market, the cost of option exercise in recessions has to decrease. Thus, in equilibrium, the price of installed capital is highly procyclical and acts as a hedge against risks in assets in place, making growth options less risky.

To match key properties of aggregate cash flows and returns, we consider an economy with recursive preferences and long-run risks. We calibrate the model using time series data on the US consumption and stock market index and show that it can accounts for a significant portion of the cross-sectional dispersion in mean returns on value and growth firms. In addition, the model is able to replicate the failure of the conditional and unconditional CAPM regressions, as well as other silent features of book-to-market sorted portfolios.

Appendix

Appendix I: State Price Density

Following Duffie and Epstein (1992a, 1992b), we represent preferences as stochastic differential utility. We use $\{U_t\}_{t \geq 0}$ to denote the utility process of the representative agent. Given the consumption process $\{C_s : s \geq 0\}$, for every $t \geq 0$, the date-t utility of the agent, denoted U_t, is defined recursively by[17]

$$U_t = E_t \left[\int_t^\infty F(C_s, U_s) \, ds \right]. \tag{29}$$

In the above equation, $F(C, U)$ is the aggregator of the recursive preferences, given by[18]

$$F(C, U) = \frac{\beta}{1 - 1/\psi} \frac{C^{1 - 1/\psi} - ((1 - \gamma) U)^{\frac{1 - 1/\psi}{1 - \gamma}}}{((1 - \gamma) U)^{\frac{1 - 1/\psi}{1 - \gamma} - 1}}. \tag{30}$$

It is convenient to represent the Markov chain $\{\theta_t\}_{t \geq 0}$ as stochastic integrals with respect to Poisson processes. In particular, let $\{N_{H,t}\}_{t \geq 0}$ be a Poisson process with intensity λ_H, and $\{N_{L,t}\}_{t \geq 0}$ be a Poisson process with intensity λ_L. Let $I_{\{x\}}$ be the indicator function, that is,

$$I_{\{x\}}(y) = \left\{ \begin{array}{ll} 1 & if \ y = x \\ 0 & if \ y \neq x \end{array} \right. .$$

$\{\theta_t\}_{t \geq 0}$ can be represented as

$$d\theta_t = (\theta_H - \theta_L) \times \eta \left(\theta_t^- \right)^T dN_t, \tag{31}$$

where $\eta(\theta)$ and N_t are vector notations:

$$\eta(\theta) = \left[-I_{\{\theta_H\}}(\theta), \ I_{\{\theta_L\}}(\theta) \right]^T, \tag{32}$$

$$N_t = [N_{Ht}, \ N_{Lt}]^T. \tag{33}$$

Here we adopt the convention that $\{\theta_t\}$ is right-continuous with left limit, and use the notation

$$\theta_t^- = \lim_{s \to t, s < t} \theta_s.$$

We conjecture that the equilibrium consumption of the representative agent satisfies equation (9). To

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[17]Representation of stochastic differential utility in the infinite horizon case is discussed in Duffie and Epstein (1992b). The existence and uniqueness of stochastic differential utility of the Kreps and Porteus (1978) type are discussed in Duffie and Lions (1992) and Schroder and Skiadas (1999).

[18]In general, recursive preferences are characterized by a pair of aggregators $(F, \mathcal{A})$. Duffie and Epstein (1992b) show that one can always normalize $\mathcal{A} = 0$. The aggregator F used here is the normalized aggregator.

guarantee that the life-time utility of the representative agent is finite, we assume

$$\beta + \frac{1}{2}\gamma\left(1 - \frac{1}{\psi}\right)\sigma_C^2\left(\theta\right) - \left(1 - \frac{1}{\psi}\right)\theta > 0, \quad for \ \theta = \theta_H, \theta_L.$$

To solve for the pricing kernel of the economy, we first need to derive the equilibrium utility process of the representative agent. This is given by the following lemma.

Lemma 1 *The utility function of the representative agent is given by*

$$U_t = \frac{1}{1-\gamma}H\left(\theta_t\right)C_t^{1-\gamma}. \tag{34}$$

The function $H\left(\theta\right)$ is defined by

$$H\left(\theta_H\right) = \left\{\frac{1}{\beta}\left[\beta + \frac{1}{2}\gamma\left(1 - \frac{1}{\psi}\right)\sigma_C^2\left(\theta_H\right) - \left(1 - \frac{1}{\psi}\right)\theta_H - \frac{1 - 1/\psi}{1 - \gamma}\lambda_H\left(\omega^{-1} - 1\right)\right]\right\}^{-\frac{1-\gamma}{1-1/\psi}} \tag{35}$$

$$H\left(\theta_L\right) = \left\{\frac{1}{\beta}\left[\beta + \frac{1}{2}\gamma\left(1 - \frac{1}{\psi}\right)\sigma_C^2\left(\theta_L\right) - \left(1 - \frac{1}{\psi}\right)\theta_L - \frac{1 - 1/\psi}{1 - \gamma}\lambda_L\left(\omega - 1\right)\right]\right\}^{-\frac{1-\gamma}{1-1/\psi}}, \tag{36}$$

where $\omega < 1$ is the unique solution on $\left(0, \infty\right)$ to the following equation:

$$\omega^{-\frac{1-1/\psi}{1-\gamma}} = \frac{\beta + \frac{1}{2}\gamma\left(1 - \frac{1}{\psi}\right)\sigma_C^2\left(\theta_H\right) - \left(1 - \frac{1}{\psi}\right)\theta_H - \frac{1-1/\psi}{1-\gamma}\lambda_H\left(\omega^{-1} - 1\right)}{\beta + \frac{1}{2}\gamma\left(1 - \frac{1}{\psi}\right)\sigma_C^2\left(\theta_L\right) - \left(1 - \frac{1}{\psi}\right)\theta_L - \frac{1-1/\psi}{1-\gamma}\lambda_L\left(\omega - 1\right)}. \tag{37}$$

In addition, ω satisfies

$$\omega = \frac{H\left(\theta_H\right)}{H\left(\theta_L\right)}. \tag{38}$$

Proof: We first show that equation (37) has a unique solution on $\left(\omega^*, 1\right)$, where

$$\omega^* = 1 - \min\left\{1, \frac{\beta + \frac{1}{2}\gamma\left(1 - \frac{1}{\psi}\right)\sigma_C^2\left(\theta_L\right) - \left(1 - \frac{1}{\psi}\right)\theta_L}{\frac{1-1/\psi}{\gamma-1}\lambda_L}\right\}.$$

Denote

$$LHS\left(\omega\right) = \omega^{-\frac{1-1/\psi}{1-\gamma}}$$

and

$$RHS\left(\omega\right) = \frac{\beta + \frac{1}{2}\gamma\left(1 - \frac{1}{\psi}\right)\sigma_C^2\left(\theta_H\right) - \left(1 - \frac{1}{\psi}\right)\theta_H - \frac{1-1/\psi}{1-\gamma}\lambda_H\left(\omega^{-1} - 1\right)}{\beta + \frac{1}{2}\gamma\left(1 - \frac{1}{\psi}\right)\sigma_C^2\left(\theta_L\right) - \left(1 - \frac{1}{\psi}\right)\theta_L - \frac{1-1/\psi}{1-\gamma}\lambda_L\left(\omega - 1\right)}.$$

Note that $LHS\left(\omega\right)$ is strictly increasing on $\left(\omega^*, 1\right)$, and $LHS\left(0\right) = 0$ and $LHS\left(1\right) = 1$. Also, $RHS\left(\omega\right)$ is strictly decreasing on $\left(\omega^*, 1\right)$. $RHS\left(\omega\right) \to +\infty$ as $\omega \to 0$, and $RHS\left(1\right) < 1$. This establishes the existence and uniqueness of the solution to (37) on $\left(\omega^*, 1\right)$. Using equations (34)–(37), one can show that U_t in equation (34) satisfies the defining properties of the stochastic differential utility in the infinite horizon case, as given in Appendix C in Duffie and Epstein (1992b). ■

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Using the results in Duffie and Epstein (1992b), the state price density of this economy is given by

$$\frac{d\pi_t}{\pi_t} = \frac{dF_C(C_t, V_t)}{F_C(C_t, V_t)} + F_V(C_t, V_t) \, dt.$$

Applying the generalized Ito's formula (Øksendal and Sulem (2004)), we can derive the expression of the pricing kernel and the risk-free interest rate of the economy. This is summarized in the following lemma.

Lemma 2 *The state price density of the economy, denoted by* $\{\pi_t\}_{t \geq 0}$, *is a Levy process of the form*

$$d\pi_t = \pi_t \left[-\bar{r}(\theta_t) \, dt - \gamma \sigma_C dB_t - \eta_\pi \left(\theta_t^-\right)^T dN_t \right]. \tag{39}$$

The risk-free interest rate, $r(\theta)$ *is given by:*

$$
\begin{aligned}
r(\theta) &= \beta + \frac{1}{\psi}\theta - \frac{1}{2}\gamma\left(1 + \frac{1}{\psi}\right)\sigma_C^2(\theta) + \lambda_H I_{\{\theta_H\}}(\theta)\left[1 - \hat{\omega}^{-1} + \frac{\frac{1}{\psi} - \gamma}{1 - \gamma}\left(\omega^{-1} - 1\right)\right] \\
&\quad + \lambda_L I_{\{\theta_L\}}(\theta)\left[1 - \hat{\omega} + \frac{\frac{1}{\psi} - \gamma}{1 - \gamma}(\omega - 1)\right]
\end{aligned} \tag{40}
$$

The notations in the above equations are defined as follows:

$$\eta_\pi(\theta) = \left[\left(1 - \hat{\omega}^{-1}\right) I_{\{\theta_H\}}(\theta), \left(1 - \hat{\omega}\right) I_{\{\theta_L\}}(\theta)\right], \tag{41}$$

$$\hat{\omega} = \omega^{\frac{1/\psi - \gamma}{1 - \gamma}}, \tag{42}$$

$$
\begin{aligned}
\bar{r}(\theta_H) &= \beta + \frac{1}{\psi}\theta_H - \frac{1}{2}\gamma\left(1 + \frac{1}{\psi}\right)\sigma_C^2(\theta_H) + \frac{\frac{1}{\psi} - \gamma}{1 - \gamma}\lambda_H\left(\omega^{-1} - 1\right), \\
\bar{r}(\theta_L) &= \beta + \frac{1}{\psi}\theta_L - \frac{1}{2}\gamma\left(1 + \frac{1}{\psi}\right)\sigma_C^2(\theta_L) + \frac{\frac{1}{\psi} - \gamma}{1 - \gamma}\lambda_L(\omega - 1).
\end{aligned}
$$

Appendix II: Valuation of Options and Assets in Place

We first solve for the value function of assets in place. By equation (6), using generalized Ito's formula (Øksendal and Sulem (2004)), the value function $V_A(D, \theta)$ has to satisfy:

$$Z - [\kappa + r(\theta_H)] V_A(D, \theta_H) + [\mu_A + \theta_H - \gamma\sigma_C^2(\theta_H)] DV'(D, \theta_H) \tag{43}$$

$$+ \frac{1}{2}\left[\sigma_A^2 + \sigma_C^2(\theta_H)\right] V''(D, \theta_H) + \lambda_H\left[\hat{\omega}^{-1}V(D, \theta_L) - V(D, \theta_H)\right] = 0$$

$$Z - [\kappa + r(\theta_L)] V_A(D, \theta_L) + [\mu_A + \theta_L - \gamma \sigma_C^2(\theta_L)] DV'(D, \theta_L)$$
$$+ \frac{1}{2} [\sigma_A^2 + \sigma_C^2(\theta_L)] V''(D, \theta_L) + \lambda_L [\hat{\omega} V(D, \theta_H) - V(D, \theta_L)] = 0$$

Using linearity, we have

$$a(\theta_H) = \frac{\lambda_H \hat{\omega}^{-1} + \beta_L + \kappa - \mu_A}{(\beta_H + \kappa - \mu_A)(\beta_L + \kappa - \mu_A) - \lambda_H \lambda_L}, \quad a(\theta_L) = \frac{\lambda_L \hat{\omega} + \beta_H + \kappa - \mu_A}{(\beta_H + \kappa - \mu_A)(\beta_L + \kappa - \mu_A) - \lambda_H \lambda_L}, \quad (44)$$

where we denote

$$\beta_H = \beta - \left(1 - \frac{1}{\psi}\right)\theta_H + \frac{1}{2}\gamma\left(1 - \frac{1}{\psi}\right)\sigma_C^2(\theta_H) + \frac{1/\psi - \gamma}{1 - \gamma}\lambda_H(\omega^{-1} - 1) + \lambda_H,$$
$$\beta_L = \beta - \left(1 - \frac{1}{\psi}\right)\theta_L + \frac{1}{2}\gamma\left(1 - \frac{1}{\psi}\right)\sigma_C^2(\theta_L) + \frac{1/\psi - \gamma}{1 - \gamma}\lambda_L(\omega - 1) + \lambda_L.$$

Note that in the special case $\kappa = \mu_A = 0$, $a(\theta)$ is the wealth-to-consumption ratio. Let $a_W(\theta_t)$ denote the wealth-to-consumption ratio, we have:

$$\frac{a_W(\theta_H)}{a_W(\theta_L)} = \frac{\lambda_H \hat{\omega}^{-1} + \beta_L}{\lambda_L \hat{\omega} + \beta_H} = \omega^{\frac{1 - 1/\psi}{1 - \gamma}}, \quad (45)$$

where the first equality follows direction from (44) and the second follows from the definition of ω in (37).

We next consider the optimal stopping problem of investment options in equation (7), subject to the law of motion of the quality of options, (2), that of the replication rate of options, (3) and that of the pricing kernel, (39), respectively.

Lemma 3 *The value function of investment options is of the following form in equation(11), where $e_1 > 0$, $e_2 < 0$ are the two solutions to the following quadratic equation:*

$$\lambda_L \hat{\omega} e^2 + (\beta_H - \beta_L) e - \lambda_H \hat{\omega}^{-1} = 0, \quad (46)$$

and ξ_1, $\xi_2 > 0$ are the unique positive solutions to the quadratic equation:

$$\frac{1}{2}\sigma_O^2 \zeta_i^2 + \left[\mu_O - \frac{1}{2}\sigma_O^2\right]\zeta_i - (\kappa + \beta_L) + \lambda_L \hat{\omega} e_i = 0,$$

for $i = 1, 2$, respectively.

Proof: Using generalized Ito's formula (Øksendal and Sulem (2004)), the value function of the optimization problem in (7) must satisfy the following coupled ordinary differential equations (ODE):

$$-V_O(X, \theta_H)[\beta_H + \kappa - \lambda_H] + \frac{\partial V_O}{\partial X}(X, \theta_H) X\mu_O + \frac{1}{2}\frac{\partial^2 V_O}{\partial^2 X}(X, \theta_H) X^2 \sigma_O^2 \quad (47)$$
$$+ \lambda_H [\hat{\omega}^{-1} V_O(X, \theta_L) - V_O(X, \theta_H)] = 0$$

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and

$$-V_O\left(X,\theta_L\right)\left[\beta_L+\kappa-\lambda_L\right]+\frac{\partial V_O}{\partial X}\left(X,\theta_L\right)X\mu_O+\frac{1}{2}\frac{\partial^2 V_O}{\partial^2 X}\left(X,\theta_L\right)X^2\sigma_O^2 \qquad (48)$$

$$+\lambda_L\left[\hat{\omega}V_O\left(X,\theta_H\right)-V_O\left(X,\theta_L\right)\right]=0.$$

We seek a solution of the following form:

$$V_O\left(X,\theta_H\right)=eX^{\varsigma};\ \ V_O\left(X,\theta_L\right)=X^{\varsigma}.$$

It follows that equations (47) and (48) are written as:

$$-\left[\beta_H+\kappa-\lambda_H\right]eX^{\varsigma}+\mu_O e\zeta X^{\varsigma}+\frac{1}{2}\sigma_O^2 e\zeta\left(\zeta-1\right)X^{\varsigma}+\lambda_H\left[\hat{\omega}^{-1}X^{\varsigma}-eX^{\varsigma}\right]\ \ =\ \ 0$$

$$-\left[\beta_L+\kappa-\lambda_L\right]X^{\varsigma}+\mu_O\zeta X^{\varsigma}+\frac{1}{2}\sigma_O^2\zeta\left(\zeta-1\right)X^{\varsigma}+\lambda_L\left[\hat{\omega}eX^{\varsigma}-X^{\varsigma}\right]\ \ =\ \ 0.$$

Dividing both sides by X^{ς} and rearranging, we obtain

$$\frac{1}{2}\sigma_O^2\zeta^2+\left[\mu_O-\frac{1}{2}\sigma_O^2\right]\zeta-\left(\kappa+\beta_H\right)+\lambda_H\hat{\omega}^{-1}e^{-1}\ \ =\ \ 0 \qquad (49)$$

$$\frac{1}{2}\sigma_O^2\zeta^2+\left[\mu_O-\frac{1}{2}\sigma_O^2\right]\zeta-\left(\kappa+\beta_L\right)+\lambda_L\hat{\omega}e\ \ =\ \ 0. \qquad (50)$$

There e and ζ can be obtained by the following procedure. Let e^* be a solution to the quadratic equation:

$$\lambda_H\hat{\omega}^{-1}e^{-1}-\beta_H=\lambda_L\hat{\omega}e-\beta_L, \qquad (51)$$

then ξ can be obtained by solving equation (50) with $e=e^*$.

We denote the two solutions to (51) as:

$$e_1\ \ =\ \ \frac{1}{2\lambda_L\hat{\omega}}\left\{\left(\beta_L-\beta_H\right)+\sqrt{\left(\beta_L-\beta_H\right)^2+4\lambda_H\lambda_L}\right\}>0,$$

$$e_2\ \ =\ \ \frac{1}{2\lambda_L\hat{\omega}}\left\{\left(\beta_L-\beta_H\right)-\sqrt{\left(\beta_L-\beta_H\right)^2+4\lambda_H\lambda_L}\right\}<0.$$

For each of the two solutions, there are two solutions for ζ to the quadratic equation (50), one of which is negative. Note that the boundary condition

$$\lim_{X\to0}V_O\left(X,\theta\right)\geq0,\ \ \theta=\theta_H,\theta_L$$

rules out the possibility of negative ζ. Let ζ_1 be the positive solution to (50) with $e=e_1$, and ζ_2 be the positive solution that corresponds to $e=e_2$. One can verify $\zeta_2>\zeta_1>1$. This completes the proof. ∎

Appendix III: The Cross-Sectional Distribution of Investment Options and Assets in Place

We first set up some notations. Consider the following quadratic equation:

$$\kappa + \left(\mu_O - \frac{1}{2}\sigma_O^2\right)\eta - \frac{1}{2}\sigma_O^2\eta^2 = 0. \tag{52}$$

Denote the two roots of (52) as

$$\eta_1 = \left(\frac{\mu_O}{\sigma_O^2} - \frac{1}{2}\right) + \sqrt{\left(\frac{\mu_O}{\sigma_O^2} - \frac{1}{2}\right)^2 + \frac{2\kappa}{\sigma_O^2}} > 0 \tag{53}$$

$$\eta_2 = \left(\frac{\mu_O}{\sigma_O^2} - \frac{1}{2}\right) - \sqrt{\left(\frac{\mu_O}{\sigma_O^2} - \frac{1}{2}\right)^2 + \frac{2\kappa}{\sigma_O^2}} < 0. \tag{54}$$

We will also frequently refer to the following two equations. The first is the forward equation for a family of density functions indexed by the time variable, l:

$$\nu_l(l, y) = -\kappa\nu(l, y) - \mu\nu_y(l, y) + \frac{1}{2}\sigma^2\nu_{yy}(l, y). \tag{55}$$

The second is an equation that define the operator $T\nu$:

$$[T\nu](y) = \int_0^\infty \nu(s, y)\, ds. \tag{56}$$

Under appropriate conditions, which we discuss below, $T\nu$ is the stationary distribution associated with the family of transition densities, $\{\nu(l, y)\}_{l>0}$.

It is convenient to consider the distribution of log quality. We use lower case to denote logs:

$$x = \ln X, \quad x^* = \ln X^*, \quad x_0 = \ln X_0. \tag{57}$$

Consider an investment option with quality $x_s = x$ at time s. For $l > 0$, let $\nu_O(l, \cdot | x)\frac{m_{s+l}}{m_s}$ be the density of the log quality of the investment option at time $s + l$, that is, for any interval (y_1, y_2), $\int_{y_1}^{y_2} \nu_O(l, y | x)\, dy \times \frac{m_{s+l}}{m_s}$ is the probability of the event $x_{s+l} \in (y_1, y_2)$. We first show that $\forall l > 0$, $\forall y \neq x$, $\nu_O(l, y | x)$ has to satisfy the forward equation (55) with $\mu = \mu_O - \frac{1}{2}\sigma_O^2$, $\sigma = \sigma_O$. In addition, $\nu_O(l, y | x)$ also satisfy the following boundary conditions:

$$\forall l > 0, \quad \nu_O(l, x^* | x) = 0; \quad \lim_{y \to -\infty} \nu_O(l, y | x) = 0, \tag{58}$$

and

$$\forall y \neq x, \quad \lim_{l \to 0} \nu_O(l, y | x) = 0. \tag{59}$$

The above claim is formalized in the following lemma.

Lemma 4 $\nu_O\left(l, y \mid x\right)$ *satisfies conditions (55), (58), and (59).*

Proof: By Ito's formula, the law of motion of log quality, x_{s+l} is given by:

$$dx_{s+l} = \left(\mu_O - \frac{1}{2}\sigma_O^2\right) dl + \sigma_O dB_{s+l}^i. \tag{60}$$

We use the following Markov Chain approximation (Reference) of the Brownian motion in (60). For an infinitesimal h, let

$$\Delta = \frac{h^2}{\mu_O h + \sigma_O^2}.$$

The transition probability of the Markov Chain is:

$$\Pr\left(y, y+h\right) = \frac{\mu_O h + \frac{1}{2}\sigma_O^2}{\mu_O h + \sigma_O^2}; \quad \Pr\left(y, y-h\right) = \frac{\frac{1}{2}\sigma_O^2}{\mu_O h + \sigma_O^2}.$$

Suppose $\forall y$, the total measure of options at location y at time $s + l$ is $\nu_O\left(l, y\right) \frac{m_{s+l}}{m_s}$, then at time $s + l + \Delta$, the total measure at location y comes from two sources: the mass at location $y - h$ at time $s + l$, and the mass at location $y + h$ at time $s + l$. The total measure of options at location $y - h$ at time $s + l$ is $\nu_O\left(l, y-h\right) \frac{m_{s+l}}{m_s}$. These options are expected to replicate themselves at rate $G\left(s + l, s + l + \Delta\right)$ and will visit location y with probability $\Pr\left(y - h, x\right)$; therefore the total measure of options that come from this source is $\nu_O\left(l, y-h\right) \frac{m_{s+l}}{m_s} G\left(s + l, s + l + \Delta\right) \Pr\left(y - h, y\right)$. We can do the same calculation for the mass at location $y + h$ at time $s + l$ and get:

$$\begin{aligned}
\nu_O\left(l + \Delta, y\right) \frac{m_{s+l+\Delta}}{m_s} &= \nu_O\left(l, y-h\right) \frac{m_{s+l}}{m_s} G\left(s + l, s + l + \Delta\right) \Pr\left(y - h, y\right) \\
&\quad + \nu_O\left(l, y+h\right) \frac{m_{s+l}}{m_s} G\left(s + l, s + l + \Delta\right) \Pr\left(y + h, y\right).
\end{aligned} \tag{61}$$

Note

$$m_{s+l} G\left(s + l, s + l + \Delta\right) = m_{s+l+\Delta} e^{-\kappa \Delta}.$$

Equation (61) can be written as:

$$\nu_O\left(l + \Delta, y\right) = e^{-\kappa \Delta} \left[\nu_O\left(l, y-h\right) \Pr\left(y - h, y\right) + \nu_O\left(l, y+h\right) \Pr\left(y + h, y\right)\right]. \tag{62}$$

Subtracting $\nu_O\left(l, y \mid x\right)$, dividing by Δ, and taking the limit as $\Delta \to 0$ on both sides of equation (62), we obtain equation (55). $\blacksquare$

The solution to the partial differential equation (55) with the boundary conditions, (58) and (59), is

given by:

$$\nu\left(l,y\right) = e^{-\kappa a} n\left(t\sigma^2,\ y-x-\mu a\right)$$
$$-e^{-\frac{2\mu(x-\bar{x})}{\sigma^2}} n\left(t\sigma^2,\ y+x-\mu a-2\bar{x}\right), \tag{63}$$

where

$$n\left(t,y\right) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{y^2}{2t}},$$

and $\mu = \mu_O - \frac{1}{2}\sigma_O^2$, $\sigma = \sigma_O$, $\bar{x} = \ln X^*$. This solution can be found in Luttmer (2007).

Suppose the density of the log quality of options at time 0 is $\phi\left(x\right) m_0$, then density of log quality at time t is given by:

$$\int_{-\infty}^{+\infty} \nu\left(t,y|\,x\right) \frac{m_t}{m_0} \times \phi\left(x\right) m_0 dx + \int \nu\left(t-u,y|\,x_0\right) \frac{m_t}{m_u} \times m_u du$$
$$= m_t \left[\int_{-\infty}^{+\infty} \nu\left(t,y|\,x\right) \phi\left(x\right) dx + \int \nu\left(t-u,y|\,x_0\right) du\right],$$

where the first term is the density of options existing at time 0, and the second term is the integral of the density of all options arrive during the time interval $(0,t)$. We define a stationary density $\phi_O\left(y|\,x_0\right) = T\nu_O\left(y|\,x_0\right)$, where the operator T is defined in (56). Then along the balance growth path, the density of the log quality of options is given by $\phi_O\left(y|\,x_0\right) m_t$. This claim is formalized by the following lemma.

Lemma 5 *Suppose* $\nu\left(l,y|\,x\right)$ *satisfies the boundary conditions (58) and (59). Suppose also* $\kappa > 0$, *then the integral in (56) exists, and* $T\nu\left(y|\,x\right)$ *is given by:*

$$T\nu\left(y|\,x\right) = \begin{cases} \frac{1}{\sqrt{\mu^2+2\kappa\sigma^2}} e^{-\eta_2(x-\bar{x})} \left[e^{\eta_2(y-\bar{x})} - e^{\eta_1(y-\bar{x})}\right] & if \quad y \geq x \\ \frac{1}{\sqrt{\mu^2+2\kappa\sigma^2}} \left[e^{-\eta_1(x-\bar{x})} - e^{\eta_2(x-\bar{x})}\right] e^{\eta_1(y-\bar{x})} & if \quad y < x \end{cases} . \tag{64}$$

Also, $T\nu\left(y|\,x\right)$ *satisfies:*

$$T\nu\left(y|\,x\right) = \int_{-\infty}^{+\infty} \nu\left(t,y|\,x'\right) T\nu\left(x'|\,x\right) dx + \int \nu\left(t-u,y|\,x\right) du, \tag{65}$$

where in equations (64) and (65), $x = x_0$, $\bar{x} = x^*$, $\mu = \mu_O - \frac{1}{2}\sigma_O^2$, $\sigma = \sigma_O$, η_1 *and* η_2 *are defined in (53) and (54).*

Proof of Proposition 2

Let $\phi_O\left(y|\,x_0\right) = T\nu_O\left(y|\,x_0\right)$ by the stationary density of the log quality of options, and let Φ be the corresponding stationary density of the quality of options. The absorbing rate of Φ at the absorbing barrier X^* can be calculated as

$$m_{EXIT}\left[\Phi, X^*\right] = \frac{1}{2}\sigma_O^2 \left|\phi'\left(x^*\right)\right| = \left(\frac{X_0}{X^*}\right)^{-\eta_2}.$$

We can then solve equation (8) and show that market clearing X^* is given by equation (14).

Proof of Conjecture 1

Consider an asset in place with log dividend level $\ln D_s^i = x$ at time s. let $\nu_A\left(l, y\,|\, x\right)$ be the density of $\ln\left(D_{s+l}^i \frac{m_s}{m_{s+l}}\right)$. Using the same argument as in the proof of lemma 4, we can show that $\nu_A\left(l, y\,|\, x\right)$ is given by (63) with $\mu = \mu_O - \frac{1}{2}\sigma_O^2$, $\sigma = \sigma_O$, and $\bar{x} = \infty$. Define $\phi_A\left(y\right) = T\nu_A\left(y\,|\, x^*\right)$, and

$$D_0 = \delta \times m_0 \times \int_{-\infty}^{\infty} e^y \phi_A\left(y\right) dy. \tag{66}$$

Suppose at time 0, the economy is endowed with measure D_0 of an initial generation of assets in place with dividend level 1, and the law of motion of the dividend paid by the assets in place is the same as that in (4). Then at time t, the total dividend paid by the initial generation of assets in place is

$$
\begin{aligned}
& D_0 \times \int_{-\infty}^{+\infty} e^y \frac{m_t}{m_0} \nu\left(t, y\,|\, 0\right) dy \\
=\ & D_0 \times \int_{-\infty}^{+\infty} e^{y-x} \frac{m_t}{m_0} \nu\left(t, y\,|\, x\right) dy \\
=\ & \delta \times m_0 \times \int_{-\infty}^{\infty} e^x \hat{\phi}\left(x\right) dx \times \int_{-\infty}^{+\infty} e^{y-x} \frac{m_t}{m_0} \nu\left(t, y\,|\, x\right) dy \\
=\ & \delta \times m_t \times \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^y \nu\left(t, y\,|\, x\right) \hat{\phi}\left(x\right) dy dx,
\end{aligned}
\tag{67}
$$

where the first equality uses a change of variable:

$$\forall x, \quad \int_{-\infty}^{+\infty} e^y \nu\left(t, y\,|\, 0\right) dy = \int_{-\infty}^{+\infty} e^{y-x} \nu\left(t, y\,|\, x\right) dy,$$

and the rest of the argument follows Fubini's theorem, provided that the integrals are finite.

Note for any $u \in (0, t)$, assets in place enter the economy with initial log dividend level x^* at rate δm_u. Therefore, at time t, the total dividend produced by generation u assets in place is:

$$\delta m_u \times \int_{-\infty}^{\infty} e^y \frac{m_t}{m_u} \nu\left(t - u, y\,|\, x^*\right) dy = \delta m_t \times \int_{-\infty}^{\infty} e^y \nu\left(t - u, y\,|\, x^*\right) dy.$$

The total consumption goods produced by assets in place arrived during the interval $(0, t)$ at time t is therefore:

$$\delta m_t \times \int_0^t \int_{-\infty}^{\infty} e^y \nu\left(t - u, y\,|\, x^*\right) dy du. \tag{68}$$

Using equations (67) and (68), the total amount of consumption goods produced at time t is therefore

$$\delta \times m_t \times \left\{ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^y \nu\left(t, y|\, x\right) \hat{\phi}\left(x\right) dy dx + \int_0^t \int_{-\infty}^{\infty} e^y \nu\left(t - u, y|\, x^*\right) dy du \right\}$$

$$= \delta \times m_t \times \left\{ \int_{-\infty}^{\infty} e^y \left[\int_{-\infty}^{\infty} \nu\left(t, y|\, x\right) \hat{\phi}\left(x\right) dx + \int_0^t \nu\left(t - u, y|\, x^*\right) du \right] dy \right\}$$

$$= \delta \times m_t \times \left\{ \int_{-\infty}^{\infty} e^y \hat{\phi}\left(y\right) dy \right\}$$

$$= D_0 \times \frac{m_t}{m_0},$$

where the second to last equality follows from property (65) applied to $T\nu_A\left(y|\, x\right)$. This proves the conjecture. Note the above analysis relies on the fact that the integral in equation (66) is finite. The following lemma provides conditions under which this is true.

Lemma 6 *Suppose $\mu_A - \kappa < 0$, then the integral in (66) is finite. Furthermore, if we assume that $\exists x_{MAX} < \infty$, such that $\forall s > 0$, for all assets in place of generation s, the following condition holds:*

$$\forall t > 0, \quad \ln\left(D_{s+t}^i \frac{m_s}{m_{s+t}} \right) \leq x_{MAX}$$

then the integral in (66) is always finite.[19]

Appendix IV: Asset Pricing Implications

Proof of Proposition 4

Here we show that if the parameters of the model satisfy the following condition:

$$\frac{\lambda_H \hat{\omega}^{-1} + \beta_L + \kappa - \mu_A}{\lambda_L \hat{\omega} + \beta_H + \kappa - \mu_A} > \frac{1}{2\lambda_L \hat{\omega}} \left\{ (\beta_L - \beta_H) + \sqrt{(\beta_L - \beta_H)^2 + 4\lambda_H \lambda_L} \right\}, \tag{69}$$

then the conclusion of the proposition holds.

Proof: Note that by the definition of $a\left(\theta\right)$ and e_1, condition (69) is equivalent to: $\frac{a(\theta_H)}{a(\theta_L)} > e_1$. We first prove that the above condition implies $K_2 < 0$, where K_2 is the contant in the value function in (11). Given the functional form of $V_O\left(X, \theta\right)$ in equation (11) K_2 is determined by the two smooth pasting conditions in (13). We have:

$$K_2 = \frac{1}{\zeta_2} \left(X^*\right)^{1-\zeta_2} \frac{e_1 a\left(\theta_L\right) - a\left(\theta_H\right)}{e_1 - e_2}.$$

By lemma 3, $e_1 - e_2 > 0$, therefore, condition (69) implies that $K_2 < 0$.

---

[19] As shown in the lemma, if the condition $\mu_A - \kappa < 0$ is not satisfied, we can impose an upper bound on the detrended dividend process so that the integral (66) is always finite. This modification will affect the valuation of assets in this economy. However, as long as we choose x_{MAX} to be large enough, the asset pricing implications of the model will go through without change. Extension of the model to this case is available from authors upon request.

Now consider an option with quality $0 < X \le X^*$, we have:

$$\frac{V_O\left(X,\theta_H\right)}{V_O\left(X,\theta_L\right)} = \frac{K_1 e_1 X^{\zeta_1} + K_2 e_2 X^{\zeta_2}}{K_1 X^{\zeta_1} + K_2 X^{\zeta_2}} < \frac{K_1 e_1 X^{\zeta_1}}{K_1 X^{\zeta_1}} = e_1 < \frac{a\left(\theta_H\right)}{a\left(\theta_L\right)},$$

where the first inequality is due to $e_2 < 0$, and the second inequality follows from condition (69). This completes the proof.

Finally, note that under the assumptions of the consumption growth and preference parameters, the left-hand side of (69) is increasing in μ_A. We can therefore choose $\bar{\mu}$ to be the greatest lower bound of μ_A such that condition (69) holds. ∎

Proof of Proposition 5

Proof: First note that by equation (26) and (27), the theoretical values of α_t^i is given by the following equations:

$$\alpha_t^i = \left(\gamma - \chi\left(\theta_H\right)\right)\sigma_C^2\left(\theta_H\right) + \lambda_H\left[\left(1 - \hat{\omega}^{-1}\right) - \chi\left(\theta_H\right)\left(\frac{a_W\left(\theta_L\right)}{a_W\left(\theta_H\right)} - 1\right)\right]\left[\frac{V\left(i,t,\theta_L\right)}{V\left(i,t,\theta_H\right)} - 1\right] \quad if \ \theta_t = \theta_H \ , \quad (70)$$

$$\alpha_t^i = \left(\gamma - \chi\left(\theta_L\right)\right)\sigma_C^2\left(\theta_L\right) + \lambda_L\left[\left(1 - \hat{\omega}\right) - \chi\left(\theta_L\right)\left(\frac{a_W\left(\theta_H\right)}{a_W\left(\theta_L\right)} - 1\right)\right]\left[\frac{V\left(i,,t\theta_H\right)}{V\left(i,t,\theta_L\right)} - 1\right] \quad if \ \theta_t = \theta_L, \quad (71)$$

where

$$\chi\left(\theta_t\right) = \lim_{\Delta \to 0}\frac{E_t\left[R_{t,t+\Delta}^{ref}\right] - r_{t,t+\Delta}}{Var_t\left(R_{t,t+\Delta}^{ref}\right)}. \tag{72}$$

To prove that assets with higher exposure to long-run obtain higher $\alpha's$, we need to show

$$\left(\hat{\omega}^{-1} - 1\right) > \chi\left(\theta_H\right)\left(1 - \frac{a_W\left(\theta_L\right)}{a_W\left(\theta_H\right)}\right), \quad and \quad \left(1 - \hat{\omega}\right) > \chi\left(\theta_L\right)\left(\frac{a_W\left(\theta_H\right)}{a_W\left(\theta_L\right)} - 1\right). \tag{73}$$

We only prove the first inequality in (73) here, the second inequality can be established by the same argument. Take the continuous time limit of (72) and use (45), we have:

$$\chi\left(\theta_H\right) = \frac{\gamma\sigma_C^2\left(\theta_H\right) + \lambda_H\left(\hat{\omega}^{-1} - 1\right)\left(1 - \omega^{-\frac{1-1/\psi}{1-\gamma}}\right)}{\sigma_C^2\left(\theta_H\right) + \lambda_H\left(\omega^{-\frac{1-1/\psi}{1-\gamma}} - 1\right)^2}.$$

Therefore the right hand side of the first inequality in (73) can be written as:

$$\chi\left(\theta_H\right)\left(1 - \frac{a_W\left(\theta_L\right)}{a_W\left(\theta_H\right)}\right) = \frac{\gamma\sigma_C^2\left(\theta_H\right)\left(1 - \omega^{-\frac{1-1/\psi}{1-\gamma}}\right) + \lambda_H\left(\hat{\omega}^{-1} - 1\right)\left(1 - \omega^{-\frac{1-1/\psi}{1-\gamma}}\right)^2}{\sigma_C^2\left(\theta_H\right) + \lambda_H\left(1 - \omega^{-\frac{1-1/\psi}{1-\gamma}}\right)^2}.$$

34

The above implies

$$\min\left\{\gamma\left(1-\omega^{-\frac{1-1/\psi}{1-\gamma}}\right),\hat{\omega}^{-1}-1\right\} \leq \chi\left(\theta_H\right)\left(1-\frac{a_W\left(\theta_L\right)}{a_W\left(\theta_H\right)}\right) \leq \max\left\{\gamma\left(1-\omega^{-\frac{1-1/\psi}{1-\gamma}}\right),\hat{\omega}^{-1}-1\right\}$$

To establish the first inequality in (73), it is enough to show

$$\left(\hat{\omega}^{-1}-1\right) > \gamma\left(1-\omega^{-\frac{1-1/\psi}{1-\gamma}}\right). \tag{74}$$

By equation (42), (74) is equivalent to:

$$\omega^{\frac{\gamma-1/\psi}{1-\gamma}}-1-\gamma\left(1-\omega^{-\frac{1-1/\psi}{1-\gamma}}\right) > 0. \tag{75}$$

Note (75) is always true because $\omega < 1$ (The function $\omega^{\frac{\gamma-1/\psi}{1-\gamma}}-1-\gamma\left(1-\omega^{-\frac{1-1/\psi}{1-\gamma}}\right)$ is decreasing on $(0,1)$ and reaches 0 at 1 under the parameter restriction $\gamma > \psi > 1$). This completes the proof. ∎

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Table I

Table I

Model Parameters

Preferences		Consumption		Cash Flows	
β	0.007	θ_H	0.03	$\mu_A(\theta_H)$	0.145
γ	10	θ_L	-0.01	$\mu_A(\theta_L)$	-0.100
ψ	1.5	λ_H	0.18	$\sigma_A(\theta_H)$	0.25
		λ_L	0.40	$\sigma_A(\theta_L)$	0.35
		$\sigma_C(\theta_H)$	0.006		
		$\sigma_C(\theta_L)$	0.010		

Table I presents the chosen configuration of preferences and time-series parameters of the model. All the parameters of consumption and cash-flow growth dynamics are expressed in annual terms.

Table II

Consumption Growth Dynamics

Statistic	— **Data** —		— **Model** —
$E[\Delta c]$	1.94	(0.32)	1.72
$\sigma(\Delta c)$	2.14	(0.52)	2.58
$AC(1)$	0.44	(0.11)	0.42
$AC(2)$	0.16	(0.22)	0.13

Table II presents moments of annual consumption growth in the data and in the model. $E[\Delta c]$ denotes the unconditional mean of consumption growth, $\sigma(\Delta c)$ is the standard deviation, and $AC(1)$ and $AC(2)$ are the first- and second-order autocorrelations of growth rates, respectively. Means and volatilities are expressed in percentage terms. Consumption data, taken from the BEA, are real, annual, per capita series of non-durable expenditure and services from 1930 to 2007. Standard errors of the sample statistics, reported in parentheses, are calculated using the Newey and West (1987) variance-covariance estimator with four lags. The model-implied moments represent the medians of the corresponding statistics across 100 simulations. The model is simulated on a monthly frequency, each simulation is 80-year long.

Table III

Dynamics of Aggregate Market Portfolio

Statistic	— **Data** —		— **Model** —
Dividend Growth			
$E[\Delta d]$	0.81	(1.48)	0.78
$\sigma(\Delta d)$	11.99	(2.48)	11.08
$AC(1)$	0.22	(0.14)	0.57
$Corr(\Delta d, \Delta c)$	0.66	(0.23)	0.68
Return			
$E[R]$	8.46	(1.88)	8.31
$\sigma(R)$	19.52	(1.98)	21.80
$AC(1)$	-0.03	(0.11)	-0.03
$Corr(R, \Delta c)$	0.07	(0.15)	0.07

Table III presents data- and model-based moments of annual dividend growth rates and returns of the aggregate stock market portfolio. $E[\cdot]$ and $\sigma(\cdot)$ denote the unconditional mean and standard deviation, respectively. $AC(1)$ is the first-order autocorrelation coefficient, and $Corr(\cdot, \Delta c)$ denotes the unconditional correlation between the corresponding variable and consumption growth. Means and volatilities are expressed in percentage terms. The aggregate stock market index is the value-weighted portfolio of NYSE, AMEX, and NASDAQ traded firms recorded by CRSP. All data are annual, expressed in real terms, and cover the period from 1930 to 2007. Standard errors of the data statistics, reported in parentheses, are calculated using the Newey and West (1987) variance-covariance estimator with four lags. The model-implied moments represent the medians of the corresponding statistics across 100 simulations. The model is simulated on a monthly frequency, each simulation is 80-year long. The market portfolio from inside the model is constructed from a simulated pool of 2,000 firms.

Table IV

Table IV

Dividend Growth Rate Dynamics of Book-to-Market Portfolios

Portfolio	— **Data** —		— **Model** —
	$E[\Delta d]$		
Growth	0.45	(1.40)	-0.38
Neutral	1.98	(1.63)	1.39
Value	4.70	(2.30)	3.52
	$\sigma(\Delta d)$		
Growth	13.67	(2.10)	14.58
Neutral	14.58	(3.85)	10.91
Value	21.24	(4.59)	10.15
	$Corr(\Delta d, \Delta c)$		
Growth	0.40	(0.16)	0.55
Neutral	0.66	(0.23)	0.73
Value	0.55	(0.20)	0.68

Table IV presents sample and model-implied means ($E[\Delta d]$) and standard deviations ($\sigma(\Delta d)$) of the annual per-share dividend growth rates for book-to-market sorted portfolios, as well as their correlations with aggregate consumption growth ($Corr(\Delta d, \Delta c)$). Means and volatilities are expressed in percentage terms. "Growth" and "Value" portfolios represent firms in the lowest and highest 30th percentile of the book-to-market sort, respectively. "Neutral" portfolio consists of firms in the middle of the distribution. All data are annual, expressed in real terms, and cover the period from 1930 to 2007. Standard errors of the data statistics, reported in parentheses, are calculated using the Newey and West (1987) variance-covariance estimator with 4 lags. The model-implied moments represent the medians of the corresponding statistics across 100 simulations. The model is simulated on a monthly frequency, each simulation is 80-year long, the size of the simulated cross section is 2,000 firms.

Table V

Average Returns of Book-to-Market Portfolios

Portfolio	— **Data** —		— **Model** —
Growth	7.75	(1.94)	6.99
Neutral	9.47	(1.99)	9.16
Value	13.21	(2.21)	11.24

Table V presents average annual returns on book-to-market sorted portfolios in the data and in the model. "Growth" and "Value" portfolios represent firms in the lowest and highest 30th percentile of the book-to-market sort, respectively. "Neutral" portfolio consists of firms in the middle of the distribution. All data are annual, expressed in real percentage terms, and cover the period from 1930 to 2007. Standard errors of the data statistics, reported in parentheses, are calculated using the Newey and West (1987) variance-covariance estimator with four lags. The model-implied moments represent the medians of the corresponding statistics across 100 simulations. The model is simulated on a monthly frequency, each simulation is 80-year long, the size of the simulated cross section is 2,000 firms.

Table VI

Unconditional-CAPM Alphas of Book-to-Market Portfolios

Portfolio	— **Data** —		— **Model** —	
	α	t-stat	α	t-stat
Growth	-0.34	-0.64	-1.73	-2.72
Neutral	0.74	0.77	1.83	2.21
Value	2.82	2.09	3.59	4.47

Table VI illustrates the performance of the unconditional CAPM in the data and in the model. The entries represent intercepts and the corresponding t-statistics from an OLS regression of annual excess returns of book-to-market portfolios on the excess return of the aggregate stock market. "Growth" and "Value" portfolios represent firms in the lowest and highest 30th percentile of the book-to-market sort, respectively. "Neutral" portfolio consists of firms in the middle of the distribution. The observed aggregate stock market index is the value-weighted portfolio of NYSE/AMEX/NASDAQ traded firms. The risk-free rate in the data is measured using the three-month Treasury bill. The data set covers the period from 1930 to 2007. The model-implied moments represent the medians of the corresponding statistics across 100 simulations. The model is simulated on a monthly frequency, each simulation is 80-year long, the size of the simulated cross section is 2,000 firms.

Table VII

Average Conditional-CAPM Alphas of Book-to-Market Portfolios

Portfolio	— **Data** —		— **Model** —	
	α	t-stat	α	t-stat
Growth	-0.73	-2.07	-1.74	-2.94
Neutral	1.21	2.89	1.37	2.06
Value	3.36	3.78	2.95	4.17

Table VII illustrates the performance of the conditional CAPM in the data and in the model. The entries are average (annualized) alphas and their t-statistics from the 36-month rolling regressions of excess returns of book-to-market portfolios on the excess return of the aggregate stock market. "Growth" and "Value" portfolios represent firms in the lowest and highest 30th percentile of the book-to-market sort, respectively. "Neutral" portfolio consists of firms in the middle of the distribution. The observed aggregate stock market index is the value-weighted portfolio of NYSE/AMEX/NASDAQ traded firms. The risk-free rate in the data is measured using the three-month Treasury bill. The data set covers the period from 1930 to 2007. The model-implied moments represent the medians of the corresponding statistics across 100 simulations. The model is simulated on a monthly frequency, each simulation is 80-year long, the size of the simulated cross section is 2,000 firms.

Table VIII

Transition Probabilities across Book-to-Market Portfolios

Panel A: Data

		— To —		
		Growth	Neutral	Value
From	Growth	0.78	0.20	0.02
	Neutral	0.15	0.65	0.20
	Value	0.01	0.17	0.82

Panel B: Model

		— To —		
		Growth	Neutral	Value
From	Growth	0.85	0.08	0.07
	Neutral	0.08	0.82	0.10
	Value	0.04	0.16	0.80

Table VIII reports average frequencies of transition across book-to-market portfolios in the data (Panel A) and in the model (Panel B). Transition probabilities are defined as a fraction of firms that migrate from one bin in year t to another bin in year $t+1$. "Growth" and "Value" portfolios represent firms in the lowest and highest 30th percentile of the book-to-market sort, respectively. "Neutral" portfolio consists of firms in the middle of the distribution. The observed data cover the period from 1930 to 2007. The model-implied moments represent the medians of the corresponding statistics across 100 simulations. The model is simulated on a monthly frequency, each simulation is 80-year long, the size of the simulated cross section is 2,000 firms.

Table IX

**Market Shares of Book-to-Market Portfolios**

Portfolio	— **Data** —	— **Model** —
Growth	0.54	0.60
Neutral	0.34	0.30
Value	0.12	0.10

Table IX reports average market shares of book-to-market portfolios in the simulated and observed data. "Growth" and "Value" portfolios represent firms in the lowest and highest 30th percentile of the book-to-market sort, respectively. "Neutral" portfolio consists of firms in the middle of the distribution. The observed data cover the period from 1930 to 2007. The model-implied moments represent the medians of the corresponding statistics across 100 simulations. The model is simulated on a monthly frequency, each simulation is 80-year long, the size of the simulated cross section is 2,000 firms.
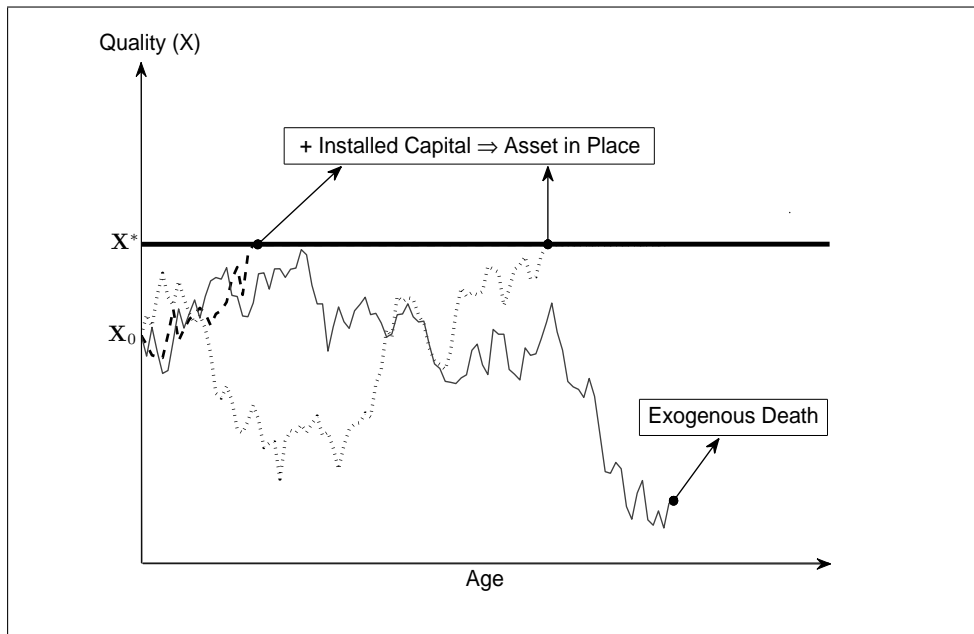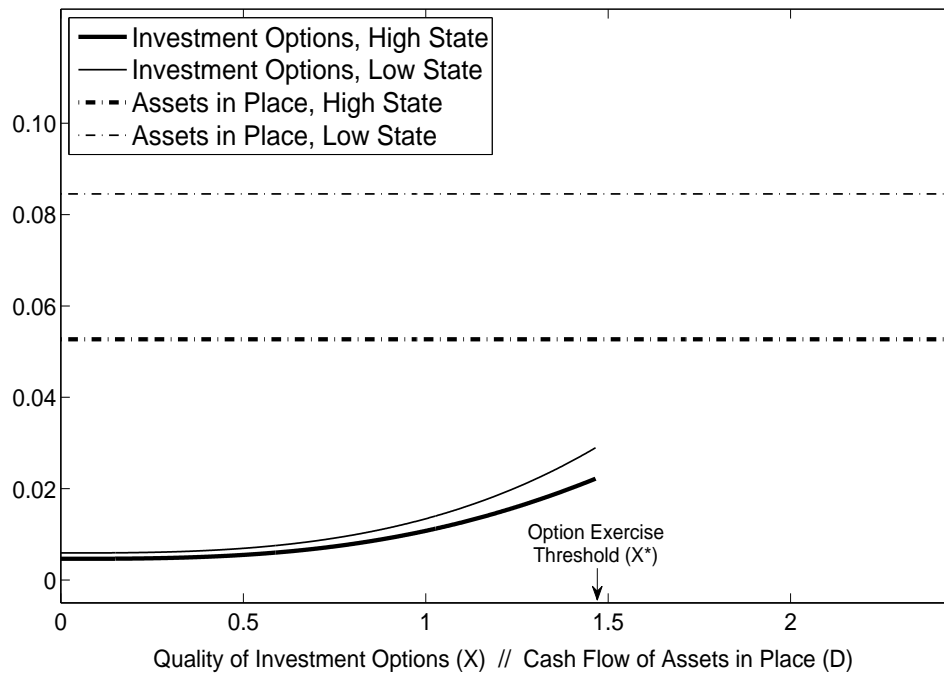
Figure 1. Dynamics of Investment Options

This figure illustrates the dynamics of a cohort of investment options. X_0 represents the initial quality of a newly born option. X^* is the equilibrium option exercise threshold. Options whose quality reaches X^* obtain a unit of installed capital and become assets in place (dashed and dotted lines), others exit the economy once they are hit by the exogenous death shock (solid line).
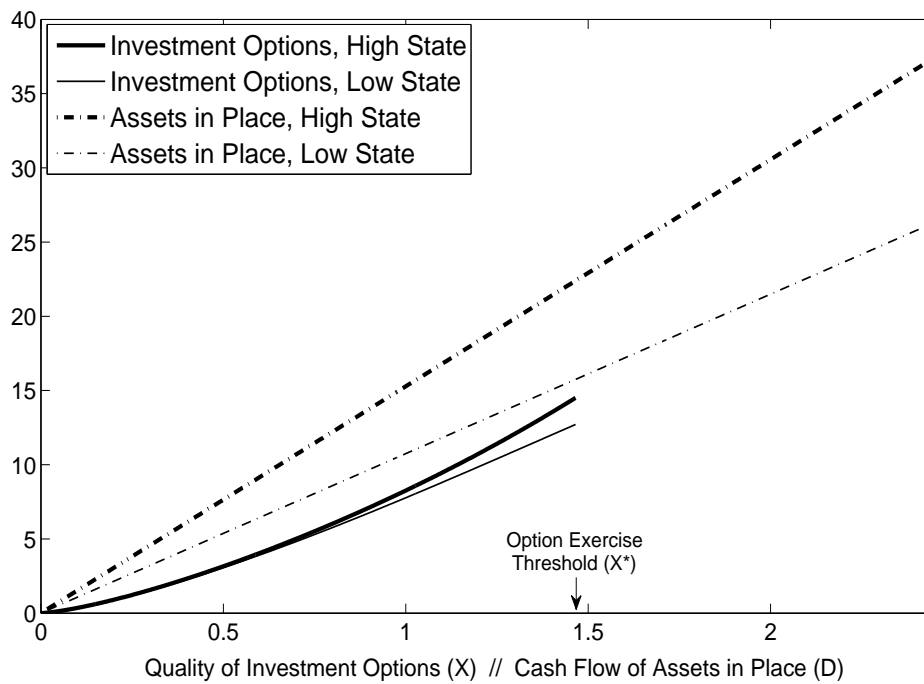
Figure 2. Value Functions of Investment Options and Assets in Place

This figure plots the value functions of investment options and assets in place for the two states, High (θ_H) and Low (θ_L). At the option exercise threshold, X^*, the distances between the values of assets in place (dashed lines) and those of options (solid lines) are the equilibrium prices of installed capital.

Figure 3. Equity Risk Premia of Investment Options and Assets in Place

This figure plots the risk premia of investment options against their quality (X) and the risk premia of assets in place as a function of their dividend (D) for the two states, High (θ_H) and Low (θ_L). Risk premia are expressed in annual percentage terms.

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