The Equity Premium Implied by Production

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Abstract

This paper studies the determinants of the equity premium as implied by producers' first-order conditions. A simple closed form expression is presented for the Sharpe ratio as a function of investment volatility and technology parameters. Calibrated to the U.S. postwar economy, the model can match the historical first and second moments of the market return and the risk free interest rate. The market's Sharpe ratio and the market price of risk are very volatile.

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In the twenty years since Mehra and Prescott's paper on the equity premium puzzle many studies have proposed and evaluated utility functions for their ability to explain the most salient aggregate asset pricing facts. Several specifications have demonstrated their ability to improve considerably over a basic time-separable constant relative risk aversion setup. Despite the progress, however, it seems that we have not yet reached the state where there would be a widely accepted replacement for the standard time-separable utility specification.

Contrary to the consumption side, the production side of asset pricing has received considerably less attention. Focusing on the production side shifts the burden towards representing production technologies and interpreting production data. While a number of asset pricing studies have considered nontrivial production sectors, these have generally been studied jointly with some specific preference specification. Thus, the analysis could not escape the constraints imposed by the preference side. A *pure* production asset pricing literature has emerged from the Q-theory of

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investment. However, typically, these studies consider a limited set of implications for the links between investment and stock returns, but not the equity premium.[1]

The more limited attention given to production-based versus consumption-based models can seem surprising in light of some views widely held by economists. For instance, a reasonably strong case can be made for firms behaving rationally. Friedman (1953) and others have pointed out that competition among firms creates a strong driving force for profit maximization lest they go out of business. Contrary to that, the popularity of behavioral finance and behavioral economics suggests a more pessimistic and complex view about consumer rationality.

In this paper I am interested in studying the macroeconomic determinants of asset prices given by a multi-input aggregate production technology. The focus is exclusively on the producers' first-order conditions that link production variables and state prices, with investment in different capital goods playing the key role. Two sets of questions are considered. First, what properties of investment and production technologies are important for the first and second moments of risk free rates and aggregate equity returns? Second, does a model plausibly calibrated to the U.S. economy have the ability to replicate first and second moments of risk free rates and aggregate equity returns?

The work most closely related to mine are Cochrane's contributions on production-based asset pricing (1988, 1991). Some of the features that differentiate my work are that I focus explicitly on the equity premium, use more general functional forms for adjustment cost, and base the empirical evaluation on the two main types of U.S. fixed capital investment, namely equipment & software as well as structures. Cochrane (1993) derives a set of asset pricing implications of a production function where the productivity level can be selected in a state-contingent way.

My model pictures the problem of a representative producer that selects multiple fixed input factors facing a full set of state prices. In order to be able to recover the state prices from the production decisions, the production possibilities need to be rich enough to allow substitution between all states of nature. A necessary condition for this in my model is that there have to be as many predetermined factors of production as there are states of nature. This assumption of "complete technologies" is not satisfied in most asset pricing studies with nontrivial production sectors. Of course, in a general equilibrium environment it doesn't usually play such an important

[1] An incomplete list of contributions include: for successful utility functions, Abel (1990), Campbell and Cochrane (1999), Constantinides (1990); for models with nontrivial production sectors Jermann (1998) and Rouwenhorst (1995); for production asset pricing studies, Cochrane (1988, 1991), Li, Vassalou and Xing (2003), Gomes, Yaron and Zhang (2002), and Belo (2007). Other examples of related asset pricing studies with rich production structures are Berk, Green and Naik (1999), Carlson, Fisher and Giammarino (2003), Hugonnier, Morellec and Sundaresan (2005), Novy-Marx (2005), and Tuzel (2007).

role.

The model is calibrated to a two sector representation. I use U.S. data on investment for equipment & software, as well as for structures. These two types of investments sum to the total of U.S. fixed nonresidential investment. With this representation the two sectors also have some natural asymmetries. As becomes clear below, asymmetries across sectors play an important role in the analysis.

One of the paper's main contributions is to characterize the determinants of the equity premium. Specifically, a closed form expression is presented for the Sharpe ratio at steady-state as a function of investment volatility and adjustment cost curvature. The key quantitative findings are the following. For unconditional moments, the model can match the historical first and second moments of the market return and the risk free interest rate. For conditional moments, the expected excess equity return, the market's Sharpe ratio and the market price of risk are very volatile.

The paper is organized as follows. Section 1 presents the model and section 2 some general asset pricing implications. Section 3 introduces functional forms. Section 4 characterizes theoretical links between asset prices and investment. Section 5 contains the calibration and section 6 the quantitative analysis.

1 Model

The model represents the producer's choice of capital inputs for a given state price process. Key ingredients are capital adjustment cost and stochastic productivity.

Assume an environment where uncertainty is modelled as the realization of s, one out of a finite set $S = (\mathfrak{s}_1, \mathfrak{s}_2, ...\mathfrak{s}_N)$, with s_t the current period realization and $s^t \equiv (s_0, s_1, ...s_t)$ the history up to and including t. Assume an aggregate production function

$$Y\left(s^t\right) = F\left(\left\{K_j\left(s^{t-1}\right)\right\}_{j\in J}, s^t, N\left(s^t\right)\right),$$

where the presence of s^t allows for a technology shock, K_j is the j-th capital stock, and N labor. Note that in the analysis of the model, labor will not play an active role. Capital accumulation for capital good of type j is represented by

$$K_j\left(s^t\right) = K_j\left(s^{t-1}\right)(1-\delta_j) + Z_j\left(s^t\right) I_j\left(s^t\right), \tag{1}$$

where δ_j is the depreciation rate and $Z_j\left(s^t\right)$ the technology for producing capital goods. Assume $Z_j\left(s^t\right) = Z_j\left(s^{t-1}\right)\cdot\lambda^{Z_j}\left(s_t\right)$, with $\lambda^{Z_j}\left(s_t\right)$ following a N-state Markov process. The total cost of investing in capital good of type j is given by

$$H_j\left(K_j\left(s^{t-1}\right), I_j\left(s^t\right), Z_j\left(s^t\right)\right).$$

This specification will be further specialized below.

Taking as given state prices $P\left(s^t\right)$, the representative firm solves the following problem

$$\max_{\{I,K',N\}} \sum_{t=0}^{\infty} \sum_{s^t} P\left(s^t\right) \left[\begin{array}{c} F\left(\left\{K_j\left(s^{t-1}\right)\right\}_{j\in J}, s^t, N\left(s^t\right)\right) - w\left(s^t\right) N\left(s^t\right) \\ -\sum_j H_j\left(K_j\left(s^{t-1}\right), I_j\left(s^t\right), Z_j\left(s^t\right)\right) \end{array} \right]$$

$$\text{s.t. } K_j\left(s^{t-1}\right)\left(1-\delta_j\right) + Z_j\left(s^t\right) I_j\left(s^t\right) - K_j\left(s^t\right) = 0, \ \forall s^t, j$$

with s_0 and $K_j\left(s_{-1}\right)$ given, and $P\left(s_0\right) = 1$ without loss of generality.

Labeling the multiplier on the capital accumulation equations by $P\left(s^t\right) q_j\left(s^t\right)$, q represents the marginal value of one unit of installed capital in terms of the numeraire of the same period. In equilibrium, it is also the cost of installing one unit of capital including adjustment cost. Given the homogeneity assumptions made below qZ is the ratio of the market value over the book value of capital, that is, Tobin's Q. Indeed, $1/Z$ is equal to the price of a unit of capital in terms of the final good. The book value (or replacement cost) of the capital stock is then K/Z. The introduction of the investment specific technology Z allows the model to capture the historical downward trend observed in U.S. equipment prices.

First-order conditions are summarized by

$$0 = -H_{j,2}\left(K_j\left(s^{t-1}\right), I_j\left(s^t\right), Z_j\left(s^t\right)\right) + Z_j\left(s^t\right) q_j\left(s^t\right),$$

$$q_j\left(s^t\right) = \sum_{s_{t+1}} \frac{P\left(s^t, s_{t+1}\right)}{P\left(s^t\right)} \left[\begin{array}{c} F_{K_j}\left(\left\{K_i\left(s^t\right)\right\}_{i\in J}, s^t, s_{t+1}, N\left(s^t, s_{t+1}\right)\right) \\ -H_{j,1}\left(K_j\left(s^t\right), I_j\left(s^t, s_{t+1}\right), Z_j\left(s^t, s_{t+1}\right)\right) + \left(1-\delta_j\right) q_j\left(s^t, s_{t+1}\right) \end{array} \right],$$

and

$$F_N\left(\left\{K_j\left(s^{t-1}\right)\right\}, s^t, N\left(s^t\right)\right) - w\left(s^t\right) = 0.$$

Substituting out shadow prices, we have

$$\sum_{s_{t+1}} P\left(s_{t+1}|s^t\right) \left[\begin{array}{c} F_{K_j}\left(\left\{K_j\left(s^t\right)\right\}, s^t, s_{t+1}, N\left(s^t, s_{t+1}\right)\right) \\ -H_{j,1}\left(K_j\left(s^t\right), I_j\left(s^t, s_{t+1}\right), Z_j\left(s^t, s_{t+1}\right)\right) \\ +\left(1-\delta_j\right) \frac{H_{j,2}\left(K_j\left(s^t\right), I_j\left(s^t, s_{t+1}\right), Z_j\left(s^t, s_{t+1}\right)\right)}{Z_j\left(s^t, s_{t+1}\right)} \end{array} \right] \left(\frac{Z_j\left(s^t\right)}{H_{j,2}\left(K_j\left(s^{t-1}\right), I_j\left(s^t\right), Z_j\left(s^t\right)\right)} \right) = 1$$

for each j, where the notation $P\left(s_{t+1}|s^t\right)$ shows the price of the numeraire in s_{t+1} conditional on s^t and in units of the numeraire at s^t. From this condition, define the investment return $R_j^I\left(s^t, s_{t+1}\right)$ implicitly through $\sum_{s_{t+1}} P\left(s_{t+1}|s^t\right) R_j^I\left(s^t, s_{t+1}\right) = 1$. $R_j^I\left(s^t, s_{t+1}\right)$ is the rate of return realized in s_{t+1} from adding a marginal amount of capital of type j in state s^t. The first-order condition shows that at the maximum investing one unit in a given type of capital produces a change in the profit plan that is worth one unit.[2]

2 From investment returns to state prices and asset returns

In order to recover state prices uniquely from the producers first-order conditions it is necessary to have as many types of capital inputs as there are states of nature. This "complete technologies" requirement represents the producers' ability to move resources across all states of nature. Representing the first-order conditions in matrix form yields for the case with two states of nature and two capital inputs

$$\begin{bmatrix} R_1^I\left(s^t, \mathfrak{s}_1\right) & R_1^I\left(s^t, \mathfrak{s}_2\right) \\ R_2^I\left(s^t, \mathfrak{s}_1\right) & R_2^I\left(s^t, \mathfrak{s}_2\right) \end{bmatrix} \begin{bmatrix} P\left(\mathfrak{s}_1 | s^t\right) \\ P\left(\mathfrak{s}_2 | s^t\right) \end{bmatrix} = \mathbf{1}, \tag{2}$$

or more compactly $R^I\left(s^t\right) \cdot p\left(s^t\right) = \mathbf{1}$. The state price vector is obtained by the matrix inversion

$$p\left(s^t\right) = \left(R^I\left(s^t\right)\right)^{-1} \mathbf{1}.$$

Clearly, it isn't necessarily the case that this matrix inversion is feasible nor that state prices are necessarily positive for any chosen set of returns. As further discussed below, the requirement for positive state prices will constrain my empirical implementation.

In this environment, the risk free return is given by

$$1/R^f\left(s^t\right) = \mathbf{1} p\left(s^t\right) = P\left(\mathfrak{s}_1 | s^t\right) + P\left(\mathfrak{s}_2 | s^t\right).$$

Starting with equation 2, it is easy to check that if one of the investment returns is not state-contingent, that is $R_j^I\left(s^t, s_{t+1}\right) = R_j^I\left(s^t\right)$, then, as is implied by no-arbitrage, it equals the risk free rate, $R_j^I\left(s^t\right) = R^f\left(s^t\right)$.

Consider aggregate capital returns

$$R\left(s^t, s_{t+1}\right) \equiv \frac{D\left(s^t, s_{t+1}\right) + V\left(s^t, s_{t+1}\right)}{V\left(s^t\right)},$$

where $D\left(s^t, s_{t+1}\right) = F\left(\left\{K_j\left(s^{t-1}\right)\right\}, s^t, N\left(s^t\right)\right) - w\left(s^t\right) N\left(s^t\right) - \sum_j H_j\left(K_j\left(s^{t-1}\right), I_j\left(s^t\right), Z_j\left(s^t\right)\right)$ represents the dividends paid by the firm and $V\left(s^t, s_{t+1}\right)$ the ex-dividend value of the firm. Assuming constant returns to scale in $F(.)$ and $H_j(.)$, Hayashi's (1982) result applies, and this return will be equal to a weighted average of the investment returns:

$$R\left(s^t, s_{t+1}\right) = \sum_j \frac{q_j\left(s^t\right) K_j\left(s^t\right)}{\sum_i q_i\left(s^t\right) K_i\left(s^t\right)} \cdot R_j^I\left(s^t, s_{t+1}\right). \tag{3}$$

[2] Strict concavity will be assumed below, so that first-order and transversality conditions (given below) are sufficient for a maximum.

The market price of risk, aka the highest Sharpe ratio, also has a simple expression. Let us, introduce the stochastic discount factor $m\left(s_{t+1}|s^t\right)$ by dividing and multiplying through by the probabilities $\pi\left(s_{t+1}|s^t\right)$, so that

$$P\left(s_{t+1}|s^t\right)=\left(\frac{P\left(s_{t+1}|s^t\right)}{\pi\left(s_{t+1}|s^t\right)}\right)\pi\left(s_{t+1}|s^t\right)=m\left(s_{t+1}|s^t\right)\pi\left(s_{t+1}|s^t\right).$$

Ruling out arbitrage implies $E_t\left(m\left(s_{t+1}|s^t\right)R^e\left(s^t,s_{t+1}\right)\right)=0$, for $\forall R^e\left(s^t,s_{t+1}\right)$ defined as an excess return. It is then easy to see that

$$\max\frac{E\left[R^e\left(s^t,s_{t+1}\right)|s^t\right]}{Std\left[R^e\left(s^t,s_{t+1}\right)|s^t\right]}=\frac{Std\left[m\left(s^t,s_{t+1}\right)|s^t\right]}{E\left[m\left(s^t,s_{t+1}\right)|s^t\right]}=\sqrt{\frac{\sum_{s_{t+1}}P\left(s_{t+1}|s^t\right)^2/\pi\left(s_{t+1}|s^t\right)}{\left[\sum_{s_{t+1}}P\left(s_{t+1}|s^t\right)\right]^2}-1}.$$

3 Functional Forms

This section presents the functional forms and the simulation strategies.

3.1 Investment cost function

The investment cost function plays a crucial role in the analysis. Its form is chosen to satisfy two criteria. First, I require investment returns to be stationary. This is achieved through a particular type of homogeneity. Second, I want the curvature of the cost function to be slightly more general than the standard quadratic specification.

A simple functional form that satisfies these criteria is

$$H\left(K,I,Z\right)=\left\{\frac{b}{\nu}\left(ZI/K\right)^{\nu}+c\right\}\left(K/Z\right),$$

with $b,c>0$, $\nu>1$. For each capital stock, different parameter values will be allowed. For compactness, the notation doesn't express that. As can easily be seen, this function is convex in I for $\nu>1$. Adjustment cost and the direct cost for additional capital goods are separable, trivially so because $H\left(K,I,Z\right)=\left[H\left(1,ZI/K\right)-ZI/K+ZI/K\right]\cdot\left(K/Z\right)=\left[H\left(1,ZI/K\right)-ZI/K\right]\cdot\left(K/Z\right)+I\equiv C\left(1,ZI/K\right)\cdot\left(K/Z\right)+I$. I impose restrictions on the parameters of $H\left(.\right)$ so that $C\left(1,ZI/K\right)\geq 0$, that is, the pure adjustment cost is nonnegative.

The cost function is homogenous of degree 1 in I and K/Z. This is required for balanced growth. Indeed, given the capital accumulation equation, IZ and K are cointegrated, and so are I and K/Z. With this homogeneity assumption, the investment cost $H(.)$ will share the same trend as I and K/Z. As further discussed below, additional balanced growth requirements will contribute to making investment returns stationary.

For a given investment process, the curvature parameter ν determines the volatility of the market price of capital. This parameter will be a crucial contributor to return volatility and risk premiums. From the first-order conditions the following relationship between the investment rate and the marginal cost of capital is obtained

$$qZ = b\,(IZ/K)^{\nu-1}\,.$$

The elasticity of qZ with respect to IZ/K is

$$\frac{\partial qZ}{\partial\,(IZ/K)}\frac{IZ/K}{qZ} = \nu - 1.$$

In addition to having the degree of freedom to chose this curvature parameter, the ability to have different curvatures across sectors will be important.

The parameters b and c are less important for asset pricing implications. They provide the flexibility to center the adjustment cost function and to minimize the amount of resources lost due to adjustment cost. It is easy to see that by setting $\nu = b = 1$, and $c = 0$, the case without adjustment cost is obtained

$$H\,(K, I, Z) = I.$$

3.2 Production function

I chose a production function that is consistent with stationary investment returns and that is easily tractable. Specifically, output, after payments to labor, is a linearly separable function of the capital stocks

$$F\left(\left\{K_j\left(s^t\right)\right\}_{j\in J}, s^t, s_{t+1}, N\left(s^{t+1}\right)\right) - w\left(s^{t+1}\right)N\left(s^{t+1}\right) = \sum_j \frac{A_j\left(s^{t+1}\right)}{Z_j\left(s^{t+1}\right)}K_j\left(s^t\right).$$

Marginal products of capital are then

$$F_{K_j}\left(\left\{K_j\left(s^t\right)\right\}_{j\in J}, s^t, s_{t+1}, N\left(s^t, s_{t+1}\right)\right) = \frac{A_j\left(s^{t+1}\right)}{Z_j\left(s^{t+1}\right)}.$$

The term Z_j is introduced to guarantee stationary returns. It implies, for instance, that as a given type of capital gets cheaper to produce, that is as Z increases, it also becomes less productive. This is related to one of the properties implied by Greenwood, Hercowitz and Krusell's (1997) balanced growth path. $A_j\left(s^{t+1}\right)$ can be thought of as a productivity shock.

3.3 Simulation strategy and stationarity of returns

For the quantitative analysis, the optimal investment process is taken as given. The implied investment returns and state prices can then easily be derived. As mentioned above, I require returns to be stationary. This imposes additional restrictions on the investment process. These issues are discussed here in detail.

I will assume a stochastic process for investment growth rates $\lambda^{I_j}\left(s^{t+1}\right)$, implicitly defined by $I_j\left(s^t, s_{t+1}\right) = I_j\left(s^t\right)\lambda^{I_j}\left(s^{t+1}\right)$. Under the assumed functional forms, investment returns can then be written as

$$
\begin{aligned}
R_j^I\left(s^t, s_{t+1}\right) \quad = \quad & \left(1/\lambda_{t+1}^{Z_j}\right) \cdot \frac{A_{j,t+1}}{b\left(Z_{jt} I_{j,t}/K_{j,t}\right)^{\nu-1}} \\
& + \left(1/\lambda_{t+1}^{Z_j}\right) \cdot \frac{b\left(1-\frac{1}{\nu}\right)\left(Z_{j+1t} I_{jt+1}/K_{jt+1}\right)^{\nu} - c}{b\left(Z_{jt} I_{j,t}/K_{j,t}\right)^{\nu-1}} \\
& + \left(1/\lambda_{t+1}^{Z_j}\right) \cdot \left(1-\delta_j\right) \cdot \frac{b\left(Z_{jt+1} I_{j,t+1}/K_{j,t+1}\right)^{\nu-1}}{b\left(Z_{jt} I_{j,t}/K_{j,t}\right)^{\nu-1}},
\end{aligned} \tag{4}
$$

where for compactness the state-dependence is not explicit.

The dynamic structure of the variables of interest can be summarized in the following expressions. Realized investment returns displayed in equation 4 can be written as a function of four elements:

$$
R_j^I\left(s^t, s_{t+1}\right) = R_j^I\left(\frac{Z_j\left(s^t\right) I_j\left(s^t\right)}{K_j\left(s^{t-1}\right)}; \lambda^{I_j}\left(s^{t+1}\right), \lambda^{Z_j}\left(s^{t+1}\right), A_j\left(s^{t+1}\right)\right) \text{ for } j = 1, 2. \tag{5}
$$

For the simulations, I can generate realizations of all the quantities of interest based on a probability matrix describing the law of motion for the exogenous state s_{t+1}. In particular, combining the capital accumulation equations, (1) , with the specifications for $I_j\left(s^t, s_{t+1}\right)$ and $Z_j\left(s^t, s_{t+1}\right)$, the investment-capital ratios evolve as

$$
\frac{Z_j\left(s^{t+1}\right) I_j\left(s^{t+1}\right)}{K_j\left(s^t\right)} = \left(\frac{\frac{Z_j\left(s^t\right) I_j\left(s^t\right)}{K_j\left(s^{t-1}\right)}}{\left(1-\delta_j\right) + \frac{Z_j\left(s^t\right) I_j\left(s^t\right)}{K_j\left(s^{t-1}\right)}}\right) \lambda^{I_j}\left(s^{t+1}\right) \lambda^{Z_j}\left(s^{t+1}\right) \text{ for } j = 1, 2. \tag{6}
$$

In order to compute the aggregate return defined in equation 3, it is also necessary to keep track of the ratio of the book values of the two types of capital. It is easy to show that this ratio evolves as

$$
\frac{K_1\left(s^t\right)}{Z_1\left(s^t\right)} / \frac{K_2\left(s^t\right)}{Z_2\left(s^t\right)} = \left(\frac{K_1\left(s^{t-1}\right)}{Z_1\left(s^{t-1}\right)} / \frac{K_2\left(s^{t-1}\right)}{Z_2\left(s^{t-1}\right)}\right) \frac{\left(1-\delta_1 + \frac{Z_1\left(s^t\right) I_1\left(s^t\right)}{K_1\left(s^{t-1}\right)}\right)}{\left(1-\delta_2 + \frac{Z_2\left(s^t\right) I_2\left(s^t\right)}{K_2\left(s^{t-1}\right)}\right)} \frac{\lambda^{Z_{j2}}\left(s^t\right)}{\lambda^{Z_1}\left(s^t\right)}.
$$

Inspection of equation (4) reveals that given the various assumptions made on the exogenous processes and functional forms, and assuming stationary shocks $\lambda^{I_j}\left(s^{t+1}\right), \lambda^{Z_j}\left(s^{t+1}\right)$ and $A_j\left(s^{t+1}\right)$, investment returns are stationary. However, stationarity of the investment returns is not sufficient for the stationarity of the aggregate asset return. Indeed, as shown in equation (3), the aggregate return equals a weighted average of the investment returns. For stationarity, the weights need to be stationary too. Aggregate returns are given by

$$
R\left(s^t, s_{t+1}\right) = \sum_j \frac{\frac{b\left(Z_{jt} I_{j,t}/K_{j,t}\right)^{\nu-1}}{Z_{j,t}} K_{j,t+1}}{\sum_i \frac{b\left(Z_{it} I_{i,t}/K_{i,t}\right)^{\nu-1}}{Z_{i,t}} K_{i,t+1}} R_j^I\left(s^t, s_{t+1}\right).
$$

A sufficient (and necessary) condition for stationarity, given the previous assumptions, is that $K_{1,t+1}/Z_{1,t}$ and $K_{2,t+1}/Z_{2,t}$ are cointegrated. Given that the investment capital ratios $Z_{jt}I_{j,t}/K_{j,t}$ are stationary, this is equivalent to $I_{1,t}$ and $I_{2,t}$ being cointegrated. Setting investment expenditure growth rates equal across sectors, that is $\lambda^{I1}(s_{t+1}) = \lambda^{I2}(s_{t+1})$, guarantees that $I_{1,t}$ and $I_{2,t}$ are cointegrated. While investment expenditure growth realizations are assumed to be equal across the two types of capital, I remain free to choose the realizations for λ_t^{Z1} and λ_t^{Z2} independently. This is less restrictive than it might appear. As seen above, what matters for the investment returns is the behavior of the product $\lambda_t^{I1}\lambda_t^{Z1}$, and not λ_t^{I1} individually. That is, in general, it would be more important to fit the process of real investment growth rather than the growth in investment expenditure. Moreover, for the considered empirical counterparts, as shown below, the historical volatilities of $\lambda^{I1}$and λ^{I2} are nearly identical, and realizations of the two growth rates are strongly positively correlated. Alternatively, one could introduce additional components for each process that have purely transitory effects and would thus not need to be restricted to ensure balanced growth. However, given the requirement to keep the number of states small, the additional flexibility introduced in this way would be rather limited.

4 Analytical results

This section contains a series of analytical results that explain key model mechanisms. First, the determinants of the equity premium are considered. I present simple closed form expressions for the Sharpe ratio and the risk free rate depending on the technology parameters and investment volatility. Second, I describe the measures taken to insure that the simulations are consistent with nonnegative state prices and finite firm values. An additional result is included. It is shown that in a model without technology shocks interest rates cannot be constant if state prices are to be recovered from the producers first-order conditions.

4.1 What determines the equity premium?

I consider here the relationship between investment and asset prices, and in particular the equity premium. For this analysis, a continuous-time representation will be more transparent than the discrete-time model used sofar. The analysis proceeds in two steps. First, I show that in order to have a positive equity premium, the investment return that is expected to be higher needs to be the more volatile. Second, I show that under some conditions, asymmetries in the investment cost curvature ν can generate this property, and present a simple expression for the Sharpe ratio.

As a counterpart to the two-state representation in discrete time, consider a one-dimensional

Brownian motion. Investment returns for the two types of capital are given by

$$\frac{dR_j}{R_j} = \mu_j\left(.\right) dt + \sigma_j\left(.\right) dz, \text{ for } j = 1, 2, \tag{7}$$

and the state-price process also has this form

$$\frac{d\Lambda}{\Lambda} = -r^f\left(.\right) dt + \sigma\left(.\right) dz. \tag{8}$$

Assume that the two returns are positively (perfectly) correlated so that $sign\left(\sigma_1\right) = sign\left(\sigma_2\right)$. The drift and diffusion coefficients are allowed to change with the state of the economy. For compactness, from now on, the notation will not explicitly acknowledge this.

The objective is to derive the drift and diffusion terms of the state-price process, $-r^f$ and σ, from the given return processes, that is from the four values μ_j and σ_j for $j = 1, 2$. In this environment, the absence of arbitrage implies that

$$0 = E_t\left(\frac{d\Lambda_t}{\Lambda_t}\right) + E_t\left(\frac{dR_{jt}}{R_{jt}}\right) + E_t\left(\frac{d\Lambda_t}{\Lambda_t}\frac{dR_{jt}}{R_{jt}}\right), \tag{9}$$

so that

$$0 = -r^f dt + \mu_i dt + \sigma_i \sigma dt,$$

and thus there are 2 equations and 2 unknowns. The solution of this system is

$$\begin{aligned} r^f &= \frac{\sigma_2\mu_1 - \sigma_1\mu_2}{\sigma_2 - \sigma_1} \\ -\sigma &= \frac{\mu_2 - \mu_1}{\sigma_2 - \sigma_1}. \end{aligned}$$

Clearly, in order to be able to recover the state price process from the two returns, the two volatility terms have to be different, that is $\sigma_2 - \sigma_1 \neq 0$. This is an invertibility requirement similar to the one for the discrete time case. However, there is no issue here about possibly negative state prices. Indeed, a process such as (8) cannot become negative if it is initially positive.

From the pricing equation (9), the volatility term equals the Sharpe ratios

$$-\sigma = \frac{\mu_1 - r^f}{\sigma_1} = \frac{\mu_2 - r^f}{\sigma_2},$$

and using the solutions derived above

$$\mu_j - r^f = -\sigma\sigma_j = \sigma_j\left[\frac{\mu_2 - \mu_1}{\sigma_2 - \sigma_1}\right]. \tag{10}$$

Clearly, with positively correlated returns, that is $sign\left(\sigma_1\right) = sign\left(\sigma_2\right)$, the signs of both risk premiums are identical, and thus the sign of the aggregate equity premium, a weighted average of the two premiums, will be the same as for the two premiums. From equation (10) it is easy to see that there is a positive equity premium in the aggregate if, and only if, the return with the higher risk premium is more volatile.[3]

Let us now make the link to the production side of the model. I consider a model without technology shocks, where the only source of uncertainty are the state prices. Technology shocks could be added for this analysis, but given their relatively minor quantitative impact, as shown later in the paper, keeping the expressions simple seems preferable. As shown in the appendix, the realized return to a given capital stock equals

$$\left\{\frac{A-c}{b\left(\frac{I_t}{K_t}\right)^{\nu-1}}+\left(1-\frac{1}{\nu}\right)I_t/K_t-\delta+(\nu-1)\left[\left(\lambda^I-1\right)-(I_t/K_t-\delta)+\frac{1}{2}(\nu-2)\,\sigma_I^2\right]\right\}dt+(\nu-1)\,\sigma_I dz, \tag{11}$$

where $\left(\lambda^I-1\right)$ and σ_I are drift and diffusion terms of investment. Given the previous analysis, if we consider an investment policy for which $\sigma_{I_1}=\sigma_{I_2}$, then in order to be able to recover the state price process $\nu_1\neq\nu_2$ is required.

In order to obtain more transparent expressions, consider this return when $I_t/K_t=\left(\lambda^I-1\right)+\delta$. This holds at the deterministic steady state for a given $\left(\lambda^I-1\right)$, assuming $\left(\lambda^I-1\right)+\delta>0$.[4] The return then simplifies to

$$\left\{\left(\bar{R}-1\right)+\frac{1}{2}(\nu-1)(\nu-2)\,\sigma_I^2\right\}dt+(\nu-1)\,\sigma_I dz, \tag{12}$$

Where $\bar{R}$ is the return in a deterministic model at the steady state with the same technology parameters and with investment growth equal to λ^I.[5] Focusing on the return at this steady state point, should be informative about average model behavior. An example at the end of the quantitative analysis illustrates this.

Consider how ν and δ contribute to the sign and magnitude of the equity premium, given that these are asymmetries between equipment and structures that are considered in the quantitative analysis. As is clear from equation (12), for a given $\bar{R}$, there is no separate role for depreciation rates δ at steady state. For differences in the curvature parameters we have the following result:

Proposition 1 *Assume $\sigma_{I_j}=\sigma_I$, $\bar{R}_j=\bar{R}$ and $\nu_1\neq\nu_2$, then the equity premium at the steady state is positive if $\nu_j>1.5$ for $\forall j$.*

[3] Indeed, if $\sigma_1,\sigma_2>0$, this implies that if $\mu_2-\mu_1>0$, one needs $\sigma_2-\sigma_1>0$, and it can be seen that $\mu_j-r^f>0$. Alternatively, if $\sigma_1,\sigma_2<0$, this condition implies that if $\mu_2-\mu_1>0$ one needs $\sigma_2-\sigma_1<0$ (sector 2 is more volatile), and then again $\mu_j-r^f>0$.

[4] In particular, consider a path where $dz=0$ for all future values. Then, under the assumptions made here, for a given constant λ^I, I_t/K_t will converge to $\lambda^I-1+\delta$.

[5] $\bar{R}=\frac{A-c}{b\left(\lambda^I-(1-\delta)\right)^{v-1}}+\left(1-\frac{1}{v}\right)\lambda^I+\frac{1}{v}(1-\delta)$.

The proof builds on the results derived earlier in this section. Because $(\nu - 1)$ multiplies $\sigma_I dz$ in the return equations, the capital with the higher v will have the more volatile return. As shown above, this asymmetry can generate a positive equity premium if ν has a positive effect on the drift of the return 12. It is then easy to see that differentiating the drift term (for a fixed $\bar{R}$) yields

$$\frac{\partial (\nu - 1)(\nu - 2)}{\partial \nu} = 2(\nu - 1.5) \rightarrow \text{ if } \nu > 1.5 \text{ then} \frac{\partial (\nu - 1)(\nu - 2)}{\partial \nu} > 0.$$

To summarize, starting from a common curvature parameter $\nu > 1.5$ and increasing the curvature for one type of capital, the capital with the higher curvature will have a higher drift, everything else equal.[6]

Proposition 2 *Assume $\sigma_{I_j} = \sigma_I$ and $\nu_1 \neq \nu_2$, then steady state values for Sharpe ratios and the risk free rate are given by*

$$\frac{\mu_j - r^f}{\sigma_j}|_{ss} = \frac{\bar{R}_2 - \bar{R}_1}{(\nu_2 - \nu_1)\sigma_I} + \frac{\nu_1 + \nu_2 - 3}{2}\sigma_I, \tag{13}$$

and

$$r^f|_{ss} = \frac{\left(\bar{R}_1 - 1\right)(\nu_2 - 1) - \left(\bar{R}_2 - 1\right)(\nu_1 - 1)}{\nu_2 - \nu_1} - (\nu_1 - 1)(\nu_1 - 2)\frac{\sigma_I^2}{2}. \tag{14}$$

The first term in equation 13 shows how a difference in the deterministic returns $\bar{R}_j$ contributes to an increase in the Sharpe ratio if the higher deterministic return corresponds to the more volatile return. Clearly, in a deterministic model, $\bar{R}_j = \bar{R}$ would be required to rule out arbitrage; however, in a model with uncertainty, there is no such requirement.

In case $\bar{R}_j = R$, because σ_j and σ_I have the same sign (given $\nu_j > 1$), a necessary and sufficient condition for a positive equity premium is that $\nu_1 + \nu_2 > 3$. Clearly, the equity premium is then increasing in the sum of the curvature parameters. The equation suggests that the curvature parameters ν have a similarly important role as the risk aversion coefficient in the basic consumption-based model. However, the equation for the Sharpe ratio, together with the return equations (11) and (12), highlight a fundamental trade-off in the model's ability to explain asset returns. Increasing the curvature parameters ν increases the equity premium, but this also makes returns more volatile. Therefore, asset prices alone will impose a clear limit on how much curvature can be used to generate large risk premiums. In standard consumption-based asset pricing models this trade-off is much less present. In fact, as is well known, in a basic constant relative risk aversion environment, for the benchmark case with IID consumption growth, increasing risk aversion increases the equity premium without affecting return volatility.

[6] The case of identical investment volatility across types of capital considered here should have some independent appeal. Moreover, for the types of capital considered below, historical investment growth volatilities are roughly the same.

Equation 14 for the risk free rate shows how investment uncertainty contributes to a lower steady state interest rate by an extent that is affected by the amount of the adjustment cost curvature. This parallels the precautionary saving effect on interest rates in standard consumption-based models. The equation for the risk free rate further simplifies if it is assumed that $\bar{R}_j = R$:

$$r^f|_{ss} = \left(\bar{R} - 1\right) - (v_1 - 1)(v_2 - 1)\frac{\sigma_I^2}{2}. \tag{15}$$

4.2 What is an admissible investment process?

In this section I consider the requirements for an investment process to be admissible, in the sense that it has to represent a solution to the firm's problem for the implied state price process, and that this price process is itself well behaved. The two key requirements are that the derived state prices have to be positive and that the implied firm value has to be finite. While a large set of investment processes are admissible, these requirements nevertheless impose constraints on the investment process and on the specification of model. For this reason, this section also provides the motivation for some of the choices made in the empirical analysis.

4.2.1 Positive state prices

Solving equation (2) gives the state prices in the two-state case as

$$P\left(\mathfrak{s}_1|s^t\right) = \frac{R_2^I\left(s^t, \mathfrak{s}_2\right) - R_1^I\left(s^t, \mathfrak{s}_2\right)}{|R|}, \text{ and } P\left(\mathfrak{s}_2|s^t\right) = \frac{R_1^I\left(s^t, \mathfrak{s}_1\right) - R_2^I\left(s^t, \mathfrak{s}_1\right)}{|R|}, \tag{16}$$

with

$$|R| = R_1^I\left(s^t, \mathfrak{s}_1\right) R_2^I\left(s^t, \mathfrak{s}_2\right) - R_2^I\left(s^t, \mathfrak{s}_1\right) R_1^I\left(s^t, \mathfrak{s}_2\right).$$

As equation 16 makes clear, state prices in this model are state-non-separable. That is, the price for goods delivered in a given state depends on the investment returns of the other state, in addition to return of the same state. This is unlike CRRA-implied state prices that depend solely on consumption growth of the same state. Considering the ratio of the state prices offers some intuitive insights about what is required for positive state prices

$$\frac{P\left(s^t, \mathfrak{s}_1\right)}{P\left(s^t, \mathfrak{s}_2\right)} = \frac{R_2^I\left(s^t, \mathfrak{s}_2\right) - R_1^I\left(s^t, \mathfrak{s}_2\right)}{R_1^I\left(s^t, \mathfrak{s}_1\right) - R_2^I\left(s^t, \mathfrak{s}_1\right)}. \tag{17}$$

A necessary condition for positive state prices is that the terms in the numerator and in the denominator of the right hand side of (17) have the same sign. Each of these two terms represents the spread between the two investment returns in a given state. As is clear from (17), the two terms can only have the same sign if each type of investment dominates the other in one of the two states. Indeed, optimal choice with positive prices would imply that if one type of investment were to generate a higher return in both states, then resources would be reallocated into this type of capital from the other.

To see some of the properties needed to satisfy this positivity requirement, consider a second-order Taylor-series approximation of the investment return around the deterministic steady state. To focus on the quantitatively important channels, I again consider a model without technology shocks where the only source of uncertainty are the state prices. A second-order Taylor approximation is obtained by assuming that the investment-capital ratio is at its steady state $I_t\left(s^t\right)/K_t\left(s^{t-1}\right)=\bar{\lambda}-1+\delta$, for a given steady state growth rate $\bar{\lambda}$, so that

$$R^I_{t,t+1}=\bar{R}+(\nu-1)\,\Delta\lambda'+\frac{B}{2}\left(\Delta\lambda'\right)^2+o\left(\left(\Delta\lambda'\right)^2\right) \tag{18}$$

where $\Delta\lambda'=\lambda'-\bar{\lambda}$ and

$$B=\frac{\nu-1}{\lambda}\left\{\nu-1-\frac{1-\delta}{\lambda}\right\}.$$[7]

Assume equally sized up and down movements in a two-state setting so that

$$\Delta\lambda_j\left(\mathfrak{s}_2\right)=-\Delta\lambda_j\left(\mathfrak{s}_1\right)\equiv\overline{\Delta\lambda_j},\text{ for each } j\in(1,2)\,.$$

Assume also, like in subsection 4.1, that the investment growth volatilities are equal in the two sectors and positively correlated, so that

$$\overline{\Delta\lambda_1}=\overline{\Delta\lambda_2}=\overline{\Delta\lambda}.$$

With this approximation, the ratio determining relative state prices is given as

$$\frac{P\left(.,\mathfrak{s}_1\right)}{P\left(.,\mathfrak{s}_2\right)}=\frac{\left[\nu_2-\nu_1\right]\overline{\Delta\lambda}+\left[\left(\bar{R}_2-\bar{R}_1\right)+\frac{1}{2}\left(B_2-B_1\right)\left(\overline{\Delta\lambda}\right)^2\right]+o\left(\left(\overline{\Delta\lambda}\right)^2\right)}{\left[\nu_2-\nu_1\right]\overline{\Delta\lambda}-\left[\left(\bar{R}_2-\bar{R}_1\right)+\frac{1}{2}\left(B_2-B_1\right)\left(\overline{\Delta\lambda}\right)^2\right]+o\left(\left(\overline{\Delta\lambda}\right)^2\right)} \tag{19}$$

As shown by equation (19), in order to have positive prices at steady state, the first term in the fraction, $\left[\nu_2-\nu_1\right]\overline{\Delta\lambda}$, needs to dominate the second. In general, this will require a minimum amount of asymmetry in the curvature parameters ν_j across types of capital.

Away from steady state, in particular when investment-capital ratios reach lower levels, some state prices in my quantitative setup have a tendency to eventually turn negative. That this might happen is suggested by equation 4. As the current investment-capital ratio gets close to 0, returns can get arbitrarily large, and the spread between two returns for a given state can switch sign. In order to deal with this in the simulations, the marginal product term, A, is allowed to be state-contingent with the objective to rule out negative state prices. I describe the exact approach

[7] The only difference compared to the continuous-time equation derived above is the second-order term. With $(1-\delta)=\lambda=1$, we would have $B=(v-1)(v-2)$, which is the term in the continuous time counterpart.

in the calibration section below. Below it is also shown that shocks to A have only second-order effects on the considered asset price implications. This is because the level of A is small relative to the other terms in the return equation (4).

4.2.2 Finite value and transversality condition

In my model it is feasible to generate a sequence of investment returns and state prices without fully specifying the process for investment and possible technology shocks. Indeed, as shown in equation 5, returns are fully determined by the current period investment-capital ratios and next periods investment growth and technology shock realizations. However, because I fully specify investment growth and technology shocks processes, it needs be made sure that these processes imply a finite firm value and satisfy the transversality condition.

The transversality condition that guarantees optimality of the path satisfying the first-order condition is

$$\lim_{t\to\infty}\sum_{s^t}\frac{P\left(s^t\right)}{P\left(s_0\right)}\left\{A\left(s^t\right)+H_I\left(s^t\right)(1-\delta)-H_K\left(s^t\right)\right\}K_t\left(s^{t-1}\right)=0.$$

In the simulations, I check numerically that firm values are finite. Given the setup used, it can be shown that if firm values are finite, the transversality condition is also satisfied. Typically, the finiteness requirement is satisfied by bounding the investment-capital ratios. Specifically, consider a two-state process for the growth rates in investment and Z, where the realizations are fixed functions of the two realizations of s. Then, the extreme paths of repeating forever either the higher or the lower of the two growth rates, $\lambda^{I_j}(s)$ and $\lambda^{Z_j}(s)$, will generate natural upper and lower bounds for the investment-capital ratios, as is clear from 6. For the parameterizations considered, such two-state processes do not satisfy finiteness. However, finiteness can be achieved with tighter bounds. I implement this by making the investment growth rates λ^{I_j} a function of not only the current realization of s, but also of the current investment-capital ratios, as described in more detail in the calibration section. Intuitively, to have a finite firm value, I need to rule out paths for which the growth rates of the capital stocks are very high.

4.2.3 Models with no technology shocks: Interest rates cannot be constant

Let's consider again the benchmark environment where the only source of uncertainty that firms face are stochastic state prices. That is to say, there are no shocks to the production technology. An important result can be shown: even without technology shocks, investment returns can be optimally state-contingent as long as interest rates are not constant. In an environment where interest rates are constant forever, investment returns are constant too. Thus, with constant interest rates it is not possible to recover the state price process from producers' first-order conditions.

The basic economic idea in this section is that if a firm is subject to convex capital adjustment costs, it will not find it optimal to choose a volatile investment plan unless forced by changing prospects in future valuations.

Consider the discrete-time model with no technology shocks. Assuming a general two-state environment where state prices do not necessarily have a Markov representation. The firm's problem is given by

$$\max_{\{I(s^t),K(s^{t+1})\}} \sum_{t=0}^{T-1} \sum_{s^t} \left[AK_t(s_{t-1}) - \left\{ \frac{b}{\nu} \left(I(s^t)/K_t(s^{t-1}) \right)^{\nu} + c \right\} K_t(s^{t-1}) \right] \times \left(\prod_{j=0}^{t-1} P(s_{j+1}|s^j) \right)$$
$$+ \sum_{s^T} \Psi K_T(s^{T-1}) \left(\prod_{j=1}^{T-1} P(s_{j+1}|s^j) \right),$$

subject to

$$0 = (1-\delta) K_t(s^{t-1}) + I(s^t) - K_{t+1}(s^t),$$

with $K(s_0)$ given, $P(s_0|s^0) = 1$, and $\Psi > 0$ a parameter; assuming that $s_t \in (\mathfrak{s}_1, \mathfrak{s}_2)$.

The solution to this problem for $T \to \infty$ is equivalent to the solution of the general version of the problem with enough regularity so that the firm value is finite. However, it is easy to see in this model why interest rate volatility is needed. Indeed, from $T-1$ to T, without technology shocks, the return to capital equals the risk free rate. For the second-to-last return-period, that is, from $T-2$ to $T-1$, it can be checked that the return is given by

$$R_{T-2,T-1}(s^{T-2}, \mathfrak{s}_j) = \frac{\alpha\left(\frac{\Psi}{R^f_{T-1,T}(s^{T-2},\mathfrak{s}_j)}\right)}{\alpha\left(\frac{\Psi}{R^f_{T-1,T}(s^{T-2},\mathfrak{s}_1)}\right) P(\mathfrak{s}_1|s^{T-2}) + \alpha\left(\frac{\Psi}{R^f_{T-1,T}(s^{T-2},\mathfrak{s}_2)}\right) P(\mathfrak{s}_2|s^{T-2})}, \text{ for } j = 1, 2.$$

with $\alpha(x) = (A-c) + (1-\delta)x + \left(1 - \frac{1}{\nu}\right) \left(\frac{1}{b}\right)^{\frac{1}{\nu-1}} x^{\frac{\nu}{\nu-1}}$. Clearly, if the interest rate $R^f_{T-1,T}(s^{T-2}, \mathfrak{s}_j)$ is constant, that is if it does not depend on $\mathfrak{s}_j$, then, $R_{T-2,T-1}(s^{T-2}, \mathfrak{s}_j) = R_{T-2,T-1}(s^{T-2}) = R^f_{T-2,T-1}(s^{T-2})$. However, to the extent that one-period interest rates are state-contingent at $T-1$, the return to the firm from $T-2$ to $T-1$ will be state-contingent, and it will depend on the technology of the firm, in particular, the parameters of the adjustment cost function. Going backwards in time, this same argument can be made if all future one-period interest rates are constant.[8] The following proposition summarizes these derivations.

Proposition 3 *If one-period interest rates are constant in every period, without technology shocks, the returns to the firm (and the investment returns) are equal to the one-period interest rate.*

[8] As shown in the appendix, the continuous-time version of the model admits a more immediate proof for this property.

A consequence of this result is that, without technology shocks, if investment returns for one capital stock are state-contingent, then one-period interest rates cannot be constant. The importance of this result is that one cannot work with a "nice" benchmark environment with constant interest rates, in general.[9] On the other hand, as shown in the quantitative applications below, interest rate volatility doesn't have to be excessively high, even when investment returns are quite volatile.

5 Calibration

Parameter values are assigned based on 3 types of criteria. First, a set of parameter values are picked to match direct empirical counterparts. Second, some parameters are chosen to yield the best implications for key asset pricing moments. Third, some parameters are chosen to make sure the derived state-prices are admissible. I first present a short summary of the baseline calibration. The details and the specification with shocks to the investment technology are given thereafter.

5.1 Summary

Table 1 lists the main parameters chosen for the baseline case,

Table 1: Parameter values

ρ	$=$	$(0.2, 0)$
$\lambda^I(\mathfrak{s}_1), \lambda^I(\mathfrak{s}_2)$	$=$	$0.9587, 1.1078$
δ_E, δ_S	$=$	$0.112, 0.031$
$\overline{(K_E/Z_E)/(K_S/Z_S)}$	$=$	0.6
b_E, b_S, c_E, c_S so that $\overline{qZ}$	$=$	1.5
ν_E, ν_S	$=$	$2.115, 3.854$
A_E, A_S so that $\bar{R}_E, \bar{R}_S$	$=$	$1.04644, 1.08026$

where ρ stands for the first-order serial correlation of investment growth. A set of parameters is chosen based on direct empirical counterparts; namely, ρ, $\lambda^I(\mathfrak{s}_1), \lambda^I(\mathfrak{s}_2), (\delta_E, \delta_S)$, and $\overline{(K_E/Z_E)/(K_S/Z_S)}$. In order to replicate steady-state values for qZ, (b_E, b_S) are selected; (c_E, c_S) are then determined to generate the lowest possible total adjustment cost. The curvature parameters, ν_E and ν_S, and the steady state returns $\bar{R}_E$ and $\bar{R}_S$, (implicitly A_E and A_S), are chosen to match historical first and second moments of the market return and the risk free rate.

[9] Note, setting $v = 1 = b$, and $c = 0$ for one of the firms in our analysis would seem to imply constant interest rates. However, this is not an admissible specification, because the first-order conditions do not describe optimal firm behavior in general, as this problem is linear.

5.2 Details of calibration

This section provides additional information about parameter choices and data sources.

5.2.1 Investment and productivity processes

I consider the Bureau of Economic Analysis' (BEA) quantity indexes of investment for equipment & software as well as for structures as the empirical counterparts to investment in units of capital goods, IZ. Because Z measures the number of units of capital goods that can be produced from one unit of the final good, ruling out arbitrage implies that $1/Z$ is the price of the capital good in terms of the final good. Equivalently, $1/Z$ is the replacement cost for capital (not including adjustment cost), or the book value of capital. For both types of capital, Z is computed as the deflator for nondurable consumption and services divided by the deflator of the investment good. Investment expenditure, I, can then be obtained by combining the series for IZ and Z. Based on annual data covering 1947-2003, the properties of the growth rates of these series are shown in Table 2.

Table 2: U.S. Investment 1947-2003 (Growth rates)

		Mean	Standard Deviation	1^{st} Autocorrelation
Investment expenditure	I_E	3.81%	6.98%	.08
	I_S	2.85%	7.94%	.27
Investment	IZ_E	5.71%	7.81%	.13
	IZ_S	2.29%	6.86%	.28
Investment technology	Z_E	1.82%	2.56%	.66
	Z_S	$-.44\%$	2.35%	.31

As is well known, the price of equipment & software has been decreasing over time. The 1.82% annual increase in Z shows that in Table 2. Table 2 also shows that the volatilities of investment, and investment expenditure, are very similar for both types of capital.

The calibration of the investment growth process proceeds in two steps. First, the probability matrix is determined to match the serial correlation and the frequency of low and high growth states. These two moments do not depend on the shock values themselves but only on the probabilities. Specifically, the two diagonal elements of the probability matrix are given as

$$\pi_{11} = \frac{\rho + fr}{1 + fr}; \ \pi_{22} = \frac{1 + fr \cdot \rho}{1 + fr},$$

where fr is the relative frequency of state 1, the recession state. The numbers of realizations of investment growth rates above and below the mean are almost the same; thus I set $fr = 1$.

As shown in Table 2, the first-order serial correlations of the growth rates of investment are 0.13 and 0.28, respectively, and 0.08 and 0.27 for investment expenditure. The common ρ is set at the average for investment expenditure of 0.2; the natural benchmark case where $\rho = 0$ is also considered.

For the baseline calibration, I abstract from shocks to the investment technology, Z. Due to the balanced growth requirement, the growth rates of investment expenditures are equalized across sectors. The mean of $\lambda^I - 1$ is set at 3.33% per year, which is the average of the historical investment growth rates across the two types of capital. The implied standard deviation is 7.46%, the historic average of the standard deviations across the two types of capital. Note that the perfect positive correlation of the investment growth rates in the model is not that far from the historical reality. Indeed, the historical sample correlations for investment across the two sectors are 0.61 and 0.64, for investment and investment expenditure, respectively.

To help the model produce admissible outcomes, that is, positive state prices based on finite firm values, I bound the domain of the investment capital ratios. Specifically, an upper and a lower bound for the investment capital ratio for equipment & software, IZ/K_E, are set. The upper bound corresponds to IZ/K_E after 7 high (positive) investment growth rates starting from the steady state value; for the lower bound it is 7 low (negative) growth rates. The presented quantitative results are not significantly affected by the values of these bounds. However, without the bounds, the requirement of finite firm values in particular cannot necessarily be satisfied within the presented model specification. Mechanically, the bound is enforced by replacing IZ/K_E values beyond a given bound with the value of the bound. The implied investment growth rate λ^I is then also applied to the other type of capital to ensure balanced growth. This procedure also implicitly bounds IZ/K_S.

For the case where the investment specific technology Z is allowed to vary in both sectors, the 6 values for the realized growth rates of investment expenditure (2) and the sector specific investment technologies (4) are set so as to match as closely as possible the 8 means and standard deviations (equally weighted) of the growth rates of IZ_E, IZ_S, Z_E and Z_S. This objective can be achieved quite well. The empirical correlation of investment with its technological growth are 0.43 and -0.32, for the two types of capital respectively, while the correlation of the technological growth across types is 0.3. Clearly, due to limited degrees of freedom, the two-state process cannot match all these correlations. As shown below, for most quantities of interest, the Z shocks don't turn out to matter that much.

5.2.2 Depreciation rates

The depreciation rates for equipment & software as well as for structures, (δ_E, δ_S), are based on time series averages of the depreciation rates reported in the Fixed Assets tables form the BEA. These are 13.06% and 2.7%, respectively, for the period 1947-2002. Because the BEA's depreciation includes physical wear as well as economic obsolescence, the data is adjusted to take into account that depreciation in the model covers only physical depreciation. To do this the price increase in the capital good is added, so that

$$\delta_t = \frac{D_t}{K_t} + (Z_{t-1}/Z_t - 1),$$

with D_t depreciation according to the BEA. This adjustment decreases depreciation by 1.82% for equipment and -0.44% for structures, so that $(\delta_E, \delta_S) = (.112, .031)$.

5.2.3 Relative size of capital stocks

The capital stock ratio, $(K_{E,t}/Z_{E,t}) / (K_{S,t}/Z_{S,t})$, is needed only for computing aggregate returns, which, as shown earlier, are value-weighted averages of the two capital returns. Based on the Current-Cost Net Stocks of Fixed Assets from the BEA, for the period 1947-2002, the average of $(K_{E,t}/Z_{E,t}) / (K_{S,t}/Z_{S,t})$ is 0.6. We set the steady state ratio in the model equal to this value. In the model, the ratio of the physical capital stocks $K_{E,t}/K_{S,t}$ is nonstationary, while–given the balanced growth requirements–the ratio of the book values of the capital stocks $(K_{E,t}/Z_{E,t}) / (K_{S,t}/Z_{S,t})$, is stationary. This seems consistent with the behavior of the empirical counterparts.

5.2.4 Adjustment costs and marginal products

Given the limited direct evidence on the values of ν_E and ν_S as well as $\bar{R}_E$ and $\bar{R}_S$, these parameters are chosen with the objective to get the best possible model fit for the first and second moments of the aggregate return and the risk free rate, assuming that $\nu_S > \nu_E$. As shown below, for the considered empirical counterparts, the four moments can be perfectly matched with the values $(\nu_E, \nu_S, \bar{R}_E, \bar{R}_S) = (2.11, 3.875, 1.04622, 1.08108)$; with the implied marginal product terms $(A_E, A_S) = (0.1762, 0.1384)$.[10] Mechanically, I draw a sample for the exogenous state s_t of 100'000 periods and search in the 4 dimensional parameter space to match the 4 moments.

Each of the four parameters affect all four moments, but there are differences in sensitivities. In particular, in line with equation 11, the average of the curvature parameters affects the volatility

[10] Return data is from Ibbotson Associates (2004). Arguably, the model could be compared to an unlevered return to capital. For comparability with the literature, this isn't done here.

of the aggregate return most strongly. The level of the $\bar{R}_j' s$ has a strong effect on the mean risk free rate, as suggest by equation 14. Consistent with expression 13, the difference between $\bar{R}_E$ and $\bar{R}_S$ strongly affects the Sharpe ratio. Finally, a smaller difference between ν_E and ν_S has a positive effect on the volatility of the risk free rate.

Most readers would probably find the assumption that $\nu_S > \nu_E$ as a priori reasonable. There is also more direct evidence that suggests that the adjustment cost curvature should be larger for structures than for equipment & software. For example, as shown in Table 2, the fact that the first order serial correlation of the growth rates is somewhat higher for structures than for equipment can be interpreted as an expression of the desire to smooth investment over time due to the relatively higher adjustment cost. As another example, Guiso and Parigi (1999) examine investment behavior for equipment and structures with Italian data on investment and sales, but no asset price data. Their findings are also consistent with the notion that structures are more costly to adjust than equipment.

One way to gauge whether the adjustment cost parameters are reasonable is to consider the amount of resources lost due to the adjustment process. For the baseline calibration, the mean average adjustment cost (from the simulated model) is 8.1% and 11.6% of investment for equipment & software and structures, respectively. These values depend primarily on the target value for qZ, which itself does not affect much the model's asset pricing implications. When compared to the extreme risk aversion required to make consumption data consistent with the equity premium, the adjustment cost curvatures required here are much smaller. A prime reason for this is that investment growth is substantially more volatile than consumption growth.

There is a large literature estimating adjustment costs at the microeconomic level, see for instance the survey by Hamermesh and Pfann (1996) or more recently Hall (2004). From these, there doesn't emerge much agreement about the importance of adjustment cost. One difficulty in linking the results of such studies to mine is that it is typically assumed that adjustment cost functions are quadratic. Another difficulty is that at a disaggregated level fixed costs are likely to play an important role.

As shown in equation 13, if $\nu_S > \nu_E$, it isn't surprising that $\bar{R}_S > \bar{R}_E$, as both terms in the equation are then positively contributing to the model's Sharpe ratio and the equity premium. While direct empirical evidence on the $\overline{R}_j' s$ seems elusive, indirect evidence supports the model implication that structures have higher expected returns than equipment. Indeed, Tuzel (2007) considers the returns of portfolios of firm returns sorted on real estate holdings. She finds that the returns of firms in the quintile with the highest shares of real estate capital exceed that of firms in the quintile with the lowest shares of real estate by 3-6% annually. While her firm groupings display considerable variation in the ratios of real estate capital, none of the extreme quintiles represents returns to either structures or equipment alone. The implied differences in expected returns in Table 3 do therefore not appear excessive in light of this evidence.

The values for b_j are picked to replicate steady values for Tobin's Q, $\overline{qZ}$ of 1.5 for both types of capital. The c_j's are then picked to minimize the overall amount of output lost due to adjustment cost. These parameters have very limited influence on the model's return implications.

There are many examples of studies that estimate qZ. Lindenberg and Ross (1981) report averages for two-digit sectors for the period 1960-77 between .85 and 3.08. Lewellen and Badrinath (1997) report an average of 1.4 across all sectors for the period 1975-91. Gomes (1999) reports an average of 1.56. Based on this, I use a steady-state target value for qZ, $\overline{qZ}$, of 1.5 for both sectors. One problem with using empirical studies to infer the required heterogeneity of costs across types is that most studies consider adjustment costs by sector of activity. For the analysis here, I would need information about the adjustment costs by type of capital.

The marginal product terms $A_j\left(s^{t+1}\right)$ are made state-contingent so as to guarantee that the implied state prices are always positive. I choose to do this by introducing state-contingency only when needed and then in a very limited way. In particular, $A_E\left(s^{t+1}\right)$ is kept constant at A_E throughout. $A_S\left(s^{t+1}\right)$ is constant at A_S except if the state price were to be negative, which is the case for low values of IZ/K_S. In this case $A_S\left(s^t, s_{t+1}\right) = A_S\left(1 \pm x\left(s^t\right)\right)$, with $x\left(s^t\right)$ set to obtain a state price in state 2 equal to 0. For the benchmark calibration, the shock is turned on 19.3% of the time. In 83% of these cases, $x\left(s^t\right)$ is smaller than 0.05, 0.5% of the time it is larger than 0.5., and no realizations are larger than 0.6. While these shocks are useful in insuring that the implied state-prices are admissible, they have only second-order effects on key asset pricing moments. This is because the marginal product components A_j represents a small part of the overall return. Note also that the implied correlation between productivity shocks and investment is positive, which seems reasonable.

6 Quantitative properties

Table 3 presents model implications for the baseline calibration as well as empirical counterparts for a set of moments. Model results are based on a sample of 100'000 yearly periods starting from steady state. For unconditional moments, the key finding is that the model is able to match the historical mean equity premium and risk free rate, by also matching return volatilities for the aggregate return and the risk free rate. In Table 4, the model with IID investment growth rates, but otherwise unchanged, implies essentially the same unconditional moments, with the risk free rates being slightly less volatile.

Of particular interest is the model's ability to generate substantial time variation in expected excess returns and in Sharpe ratios. Indeed, the standard deviation of the one-period ahead conditional equity premium is 6.32% and 5.42% for the baseline calibration with and without serially correlated investment growth rates, respectively. It is worth emphasizing that despite the high volatility in risk premiums, the volatility of the risk free rate is not excessive, with a standard deviation of 2.07% and 1.76%, respectively. A number of empirical studies measure excess return predictability. For example, Campbell and Cochrane (1999) report $R^{2\prime}s$ of 0.18 and 0.04 for regressions of excess returns on lagged price-dividend ratios at a one-year horizon for the periods $1947-95$ and $1871-1993$, respectively. Combining the R^2 with the volatility of the excess returns, $\sqrt{R^2} std\left(R-R^f\right)$ provides an estimate of the volatility of the conditional equity premium. Setting $R^2=0.1$ this would be $\sqrt{0.1}\times 0.17 = 5.27\%$. Thus, the model's values of 6.32% and 5.42% are close.

In the model, the high volatility of the (conditional) equity premium can be understood as the combination of volatile expected investment returns for both types of capital and a relatively stable risk free rate. The main driver of the expected return of a given type of capital is its investment capital ratio, as is clearly shown in the return equations 4 and 11. In the calibrated model, investment-capital ratios are negatively related to expected returns. Figure 1 illustrates this relationship by plotting the (simulated) expected investment returns for each type of capital against its own investment-capital ratio. In this case, the state of the economy consists of the two investment-capital ratios and the realized investment growth rate. The realized investment growth rate matters, because with serially correlated growth rates, it affects the forecast of next period's growth rate. Higher expected growth rates increase expected returns, as can clearly be seen in the return equation 11. Thus, in Figure 1, the upper line (or set of points) in each panel corresponds to the high growth rate, and for the IID case (not shown) there would be only one line in each graph. In addition to the investment growth rate, for equipment, the expected return depends only on its own investment-capital ratio. For structures, with extreme investment-capital ratios, the investment-capital ratio of equipment matters too because of the shocks to the marginal product terms (in the lower range only) and because of the bounds on the investment-capital processes. Intuitively, the main mechanism at work is that when an investment-capital ratio is high, the current cost of adding capital (that is Tobin's Q, $b\left(IZ/K\right)^{\nu-1}$) is high, and thus the expected return going forward is low. Given the considerable volatility of expected returns illustrated in Figure 1, and given the relatively stable risk free rates, expected excess returns (and thus the equity premium) inherit most of the dynamic properties of the expected returns. Given that investment-capital ratios are strongly pro-cyclical (positively correlated with GDP), a model with

IID investment growth rate predicts a counter-cyclical equity premium.

From the more general perspective of merely assuming the absence of arbitrage, the conditional equity premium can be written as

$$E_t\left(R_{t+1}-R_t^f\right)=-\frac{\sigma_t\left(m_{t+1}\right)}{E_t m_{t+1}}\sigma_t\left(R_{t+1}\right)\rho_t\left(m_{t+1},R_{t+1}\right).$$

In the model, the conditional return volatility $\sigma_t\left(R_{t+1}\right)$ doesn't move very much, with standard deviations of 1.11% and 0.87% for the benchmark cases with and without serial correlation displayed in Table 3 and 4. For the continuous-time approximation, as shown in equation 11, with homoscedastic investment growth rates, instantaneous returns are also homoscedastic. In this case, the conditional standard deviation of aggregate returns moves only through shifts in the relative value-weights of the two capital stocks. In the simulated model, the shocks to the marginal product terms and the limits on the range of the investment-capital process also create some heteroscedasticity. Given the relatively stable conditional return volatility, the Sharpe ratio implied by the aggregate market, $E_t\left(R_{t+1}-R_t^f\right)/\sigma_t\left(R_{t+1}\right)$, inherits the dynamic properties of the conditional equity premium. A number of recent studies provide empirical support for volatile and countercyclical Sharpe ratios, see for instance Brandt and Kang (2004) and Ludvigson and Ng (2007). The model with IID investment growth is consistent with these findings. In the model, the Sharpe ratio is mainly driven by time variation in the market price of risk, $\sigma_t\left(m_{t+1}\right)/E_t m_{t+1}$. However, the correlation between the stochastic discount factor and the market return is also time-varying. Of course, this being a two-state model, conditional correlations are either 1 or -1. While the correlation is typically equal to -1, it changes sign at times when the investment-capital ratios are very high, that is, when Sharpe ratios are very low. The slightly higher volatility of the Sharpe ratio compared to the Market price of risk, as displayed in Table 3 and 4, is a reflection of this.

To further illustrate model properties, I consider the implications from feeding the investment realizations for the U.S. for the period 1947-2003 through the model.[11] Given that investment growth in the model follows a two-state distribution, the fit of the driving process is not perfect. Nevertheless, as shown in Figure 2, the fit can be very good, with correlations between the model and the data of 0.78 and 0.71 for equipment and structures, respectively. Figure 3 shows that the model-generated returns are indeed related to actually realized stock returns, with a correlation of 0.48. Figure 4, a and b, show conditional moments. In Figure 4a, the high frequency movements in expected returns as well as Sharpe ratios are driven by the forecastable component of the investment growth rates; the low frequency movements are driven by the investment-capital ratios.

[11] In particular, if the average of the deviations from the unconditional means for the two types of capital is positive, the common investment growth realization is set to the high rate and vice versa.

For the IID case displayed in Figure 4b, investment-capital ratios are the only drivers of time-varying asset returns. It is interesting to consider the 1990s. As shown in Figure 2, the decade produced a series of eight high investment growth realizations in a row. Through that sequence, investment-capital ratios are continuously increasing. As shown in Figure 4b, at the end of this sequence, the expected equity premium becomes negative, and thus the conditional correlation between the stochastic discount factor and realized returns has switched sign. From the perspective of the firms making investment decisions, the story told by the model is that throughout the 90's firms continued to invest heavily, despite declining expected returns, because investment returns were considered less and less risky.

6.1 Sensitivity and discussion

I consider here the effects of the investment specific technology shocks and the shocks to the marginal product terms. The quantitative content of the continuous-time approximations is also examined.

Tables 5 and 6 show results for the calibrations with investment specific technology shocks Z. In Table 5 the correlation of Z with the investment growth of the same type equals, 1; in Table 6, it is -1. While there are some quantitative differences compared to the baseline case, and between the two cases considered here, none of the main conclusions are affected.

Table 7 illustrates the effect of the shocks to the marginal product terms. In this case, the shocks to the marginal product term A_S are always turned on at $\pm 30\%$, and sometimes higher if needed to make prices stationary. Comparing this to the benchmark case in Table 3 without the shocks (except if needed to make prices stationary), there is little difference. Having the shocks on all the time, increases the risk free rate by 81 basis points and reduces the equity premium roughly by the same amount. Return volatilities are essentially the same in the two cases.

Finally, I reconsider the closed form expressions derived for the continuous-time model at steady-state for the Sharpe ratio and the risk free rate. This allows us to compare the continuous-time setup to the more fully specified simulated discrete-time model, as well as to appreciate the difference between steady state values and unconditional averages.

As shown in equation 13 and 14 the Sharpe ratio and the risk free rate at steady-state in the continuous-time model are function of $\left(\nu_E, \nu_S, \bar{R}_E, \bar{R}_S\right)$ and σ_I only. Based on the values of these parameters used for the baseline calibration, the Sharpe ratio and the risk free rate equal

$$0.3762 \text{ and } 1.54\%,$$

respectively. The discrete-time model with IID shocks evaluated at steady state when the investment growth rate is set equal to the average implies

$$0.3721 \text{ and } 1.62\%,$$

for these two quantities. Thus, in these two dimensions, continuous-time and discrete-time versions are very close. For mean values reported in Table 4 the two are

$$0.51 \text{ and } 1.01\%.$$

In this case, averages are somewhat different from steady-state values. The key feature that makes the average Sharpe ratio relatively larger can be seen in Figure 1. Indeed, for Structures–that have the higher adjustment cost curvature–expected returns are strongly convex in the investment-capital ratio.

7 Conclusions

The paper has examined the implications of producers' first-order conditions for asset prices in a model where convex adjustment cost play a major role. Similarly to Mehra and Prescott's approach of presenting asset prices implied by aggregate consumption through standard preferences, I have considered a mapping from investment in equipment and structures to asset prices. I show analytically the importance of some of the key features of the production technology and the investment process. Quantitatively, the conclusion reached here of are very different from Mehra and Prescott. In particular, there is no equity premium puzzle viewed from the producer side, and risk premiums are strongly time-varying.

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Appendix: Continuous-time model

This appendix presents a continuous-time investment model that replicates the setup of the discrete-time environment. The technology side of the model follows Abel and Eberly (1994) but without shocks. The main difference is that here the firm faces changing state prices, while in their case pricing is risk neutral with constant interest rates. The steps needed to derive the return equation (11) are also presented.

The capital stock evolves as $dK_t = (I_t - \delta K_t)\, dt$, and the investment cost is given by

$$H(I_t, K_t) = \left\{ \frac{b}{\nu} (I_t/K_t)^{\nu} + c \right\} K_t,$$

which is homogenous of degree one in I and K. The gross profit is given as

$$AK_t.$$

Assume that the state-price process is given as

$$d\Lambda_t = -\Lambda_t r(x_t)\, dt + \Lambda_t \sigma(x_t)\, dz_t,$$

where dz_t is a one-dimensional Brownian motion, and

$$dx_t = \mu_x(x_t) dt + \sigma_x(x_t) dz_t.$$

Assume that the functions $\mu_x(x_t), \sigma_x(x_t), r(x_t)$ and $\sigma(x_t)$ satisfy the regular conditions such that there are solutions for the above two SDEs.

The firm maximizes its value

$$V = \max_{\{I_{t+s}\}} E_t \left\{ \int_0^{\infty} [AK_{t+s} - H(I_{t+s}, K_{,t+s})] \frac{\Lambda_{t+s}}{\Lambda_t} ds \right\}.$$

Given the dynamics of Λ_t, it is obvious that the firm's value function V is independent of Λ_t. Following from the Markov property of the state variable x_t, the firm's value function would be a function of (K_t, x_t). The HJB equation is

$$rV = \max_{\{I_t\}} \left\{ [AK_t - H(I_t, K_t)] + (I_t - \delta K_t) V_K + \mu_x V_x + \frac{1}{2}\sigma_x^2 V_{xx} + \sigma \sigma_x V_x \right\}.$$

The first-order condition is

$$H_I(I_t, K_t) = V_K \equiv q_t$$

That is,

$$\begin{aligned} V_K &= b(I_t/K_t)^{\nu - 1} \\ I_t &= \left(\frac{V_K}{b} \right)^{\frac{1}{\nu - 1}} K_t \end{aligned}$$

Because of constant returns to scale in K_t, following Hayashi, it is easy to see that $V(K_t, x_t) = K_t V_K(x_t)$. Thus, it is clear that optimal investment follows an Ito process, $dI_t/I_t = \mu_I(K_t, x_t)\,dt + \sigma_I(K_t, x_t)\,dz_t$.

Define realized returns to the firm as

$$\frac{AK_t - H(I_t, K_{,t})}{V_t}dt + \frac{dV_t}{V_t}.$$

Given Hayashi's result and the first-order conditions

$$\frac{AK_t - H(I_t, K_{,t})}{V_t}dt + \frac{dV_t}{V_t} = \frac{AK_t - H(I_t, K_{,t})}{q_t K_t}dt + \frac{dK_t}{K_t} + \frac{dq_t}{q_t}.$$

Using $q_t = b(I_t/K_t)^{\nu-1}$ and Ito's lemma, the return equation 11 given in the main text can be derived.

Proposition 3 in the text shows that for the model without technology shocks, constant interest rates imply constant investment returns. The continuous time model admits a more compact proof for this property. Indeed, changing to the risk-neutral measure $\mathbb{Q}$, the firm's problem becomes

$$V = \max_{\{I_{t+s}\}} E_t^{\mathbb{Q}}\left\{\int_0^\infty e^{-\int_t^{t+s} r_u du}\left[AK_{t+s} - H(I_{t+s}, K_{,t+s})\right]ds\right\},$$

with

$$dx_t = (\mu_x(x_t) + \sigma(x_t)\sigma_x(x_t))\,dt + \sigma_x(x_t)dz_t^{\mathbb{Q}}$$

and

$$dK_t = (I_t - \delta K_t)\,dt.$$

Written in this form, it is obvious that if the interest rate r_u is constant, the firm faces no uncertainty, and thus, it will not introduce any uncertainty into an optimal investment plan.

Table 3
Asset Pricing Implications: Baseline calibration

	R^M	R^M-R^f	Rf	Market Price of Risk	Sharpe Market
Mean		8.35%	1.09%	0.55	0.52
Std	17.24%		2.07%	0.34	0.38

	R^E	R^E-R^f	R^S	R^S-R^f
Mean		4.15%		12.34%
Std	8.48%		25.00%	

Std[E(R^M-R^f|t)]	6.27%
Std[Std(R^M-R^f|t)]	1.03%

Real returns 1947-2003	R^M	R^M-R^f	Rf
Mean		8.35%	1.09%
Std	17.24%		2.07%

Returns: R^M, market; R^f, risk free; R^E, equipment and software; R^S, structures
(v_E, v_S, R_E, R_S) = (2.11, 3.875, 1.04622, 1.08108)

Table 4
Asset Pricing Implications: IID case, (no serial correlation)

	R^M	R^M-R^f	Rf	Market Price of Risk	Sharpe Market
Mean		8.25%	1.01%	0.52	0.51
Std	17.26%		1.75%	0.31	0.33

	R^E	R^E-R^f	R^S	R^S-R^f
Mean		4.18%		11.89%
Std	8.66%		24.22%	

Std[E(R^M-R^f|t)]	5.36%
Std[Std(R^M-R^f|t)]	0.81%

Real returns 1947-2003	R^M	R^M-R^f	Rf
Mean		8.35%	1.09%
Std	17.24%		2.07%

Returns: R^M, market; R^f, risk free; R^E, equipment and software; R^S, structures
(v_E, v_S, R_E, R_S) = (2.11, 3.875, 1.04622, 1.08108)

Table 5

Asset Pricing Implications: with shocks to investment technology, positive correlation λ^I and λ^Z

	R^M	R^M-R^f	Rf	Market Price of Risk	Sharpe Market
Mean		6.72%	2.34%	0.55	0.52
Std	14.20%		2.52%	0.35	0.40

	R^E	R^E-R^f	R^S	R^S-R^f
Mean		2.78%		10.50%
Std	6.09%		21.75%	

Std[E(R^M-R^f|t)]	5.28%
Std[Std(R^M-R^f|t)]	1.08%

Real returns 1947-2003	R^M	R^M-R^f	Rf
Mean		8.35%	1.09%
Std	17.24%		2.07%

Returns: R^M, market; R^f, risk free; R^E, equipment and software; R^S, structures

(v_E, v_S, R_E, R_S) = (2.11, 3.875, 1.04622, 1.08108)

Table 6
Asset Pricing Implications: with shocks to investment technology, negative correlation λI and λZ

	R^M	R^M-R^f	Rf	Market Price of Risk	Sharpe Market
Mean		10.09%	-0.24%	0.57	0.55
Std	19.28%		2.91%	0.34	0.39

	R^E	R^E-R^f	R^S	R^S-R^f
Mean		5.71%		14.26%
Std	10.77%		27.11%	

Std[E(R^M-R^f|t)]	7.20%
Std[Std(R^M-R^f|t)]	1.17%

Real returns 1947-2003	R^M	R^M-R^f	Rf
Mean		8.35%	1.09%
Std	17.24%		2.07%

Returns: R^M, market; R^f, risk free; R^E, equipment and software; R^S, structures
(v_E, v_S, R_E, R_S) = (2.11, 3.875, 1.04622, 1.08108)

Table 7

Asset Pricing Implications: Baseline calibration with A shocks for structures always on

	R^M	R^M-R^f	Rf	Market Price of Risk	Sharpe Market
Mean		7.52%	1.90%	0.45	0.42
Std	18.83%		1.91%	0.29	0.33

	R^E	R^E-R^f	R^S	R^S-R^f
Mean		3.35%		11.47%
Std	8.48%		27.67%	

Std[E(R^M-R^f|t)]	6.05%
Std[Std(R^M-R^f|t)]	0.63%

Real returns 1947-2003	R^M	R^M-R^f	Rf
Mean		8.35%	1.09%
Std	17.24%		2.07%

Returns: R^M, market; R^f, risk free; R^E, equipment and software; R^S, structures

(v_E, v_S, R_E, R_S) = (2.11, 3.875, 1.04622, 1.08108); A_S shock x=0.3 or larger if needed for positive prices

Figure 1 Expected Investment Returns as a Function of Investment-Capital Ratios

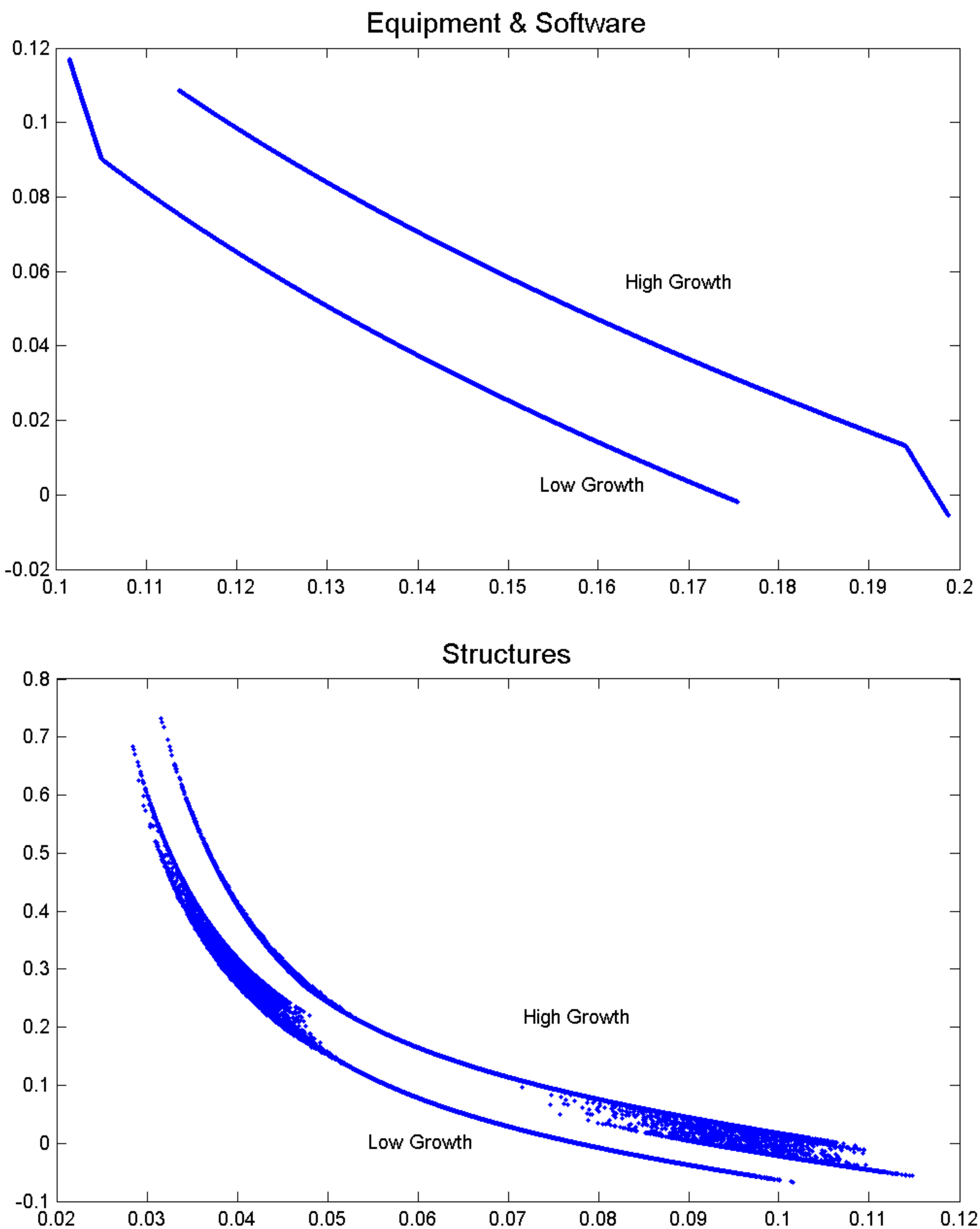


Figure 2

Realized Investment Growth 1948-2003

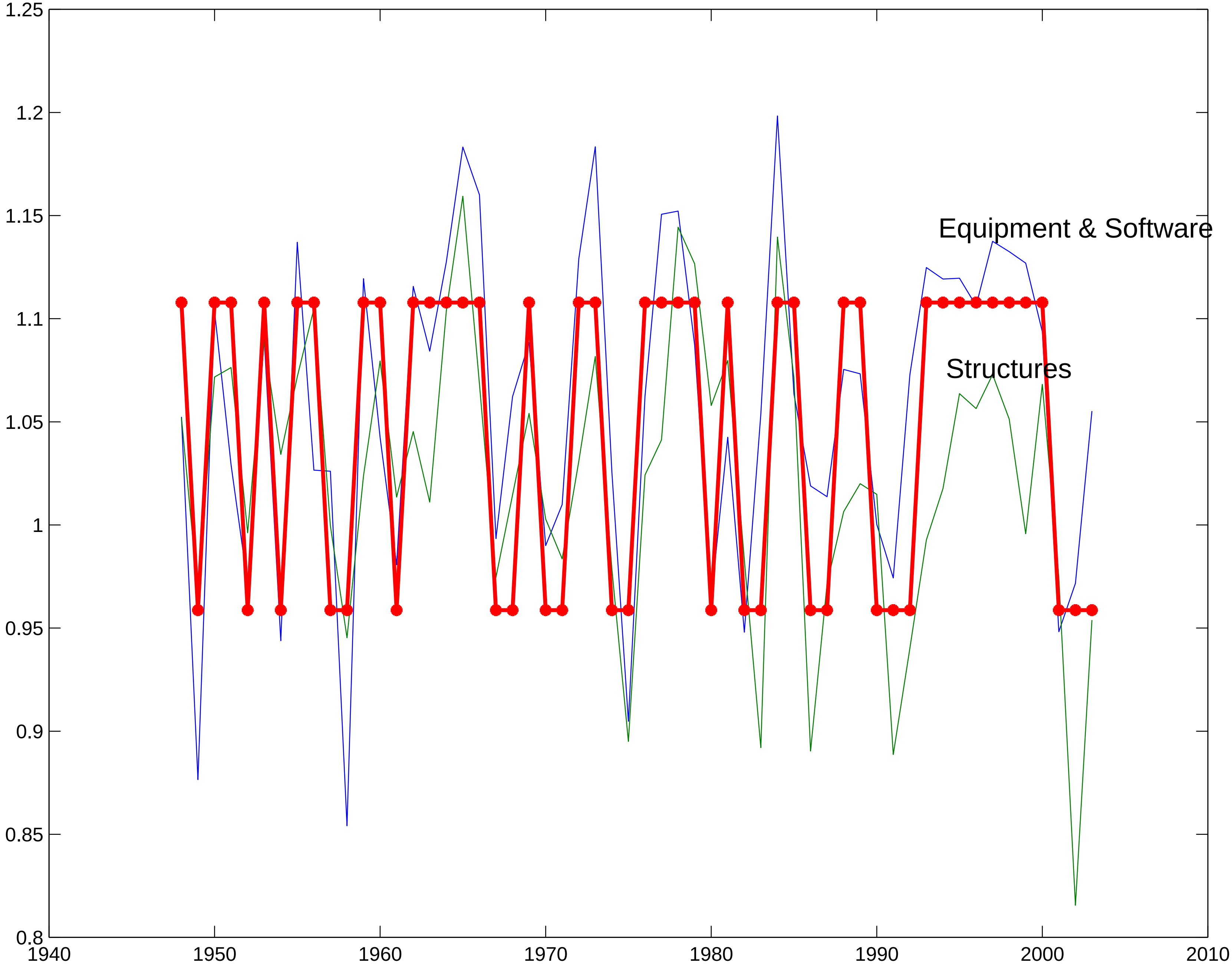
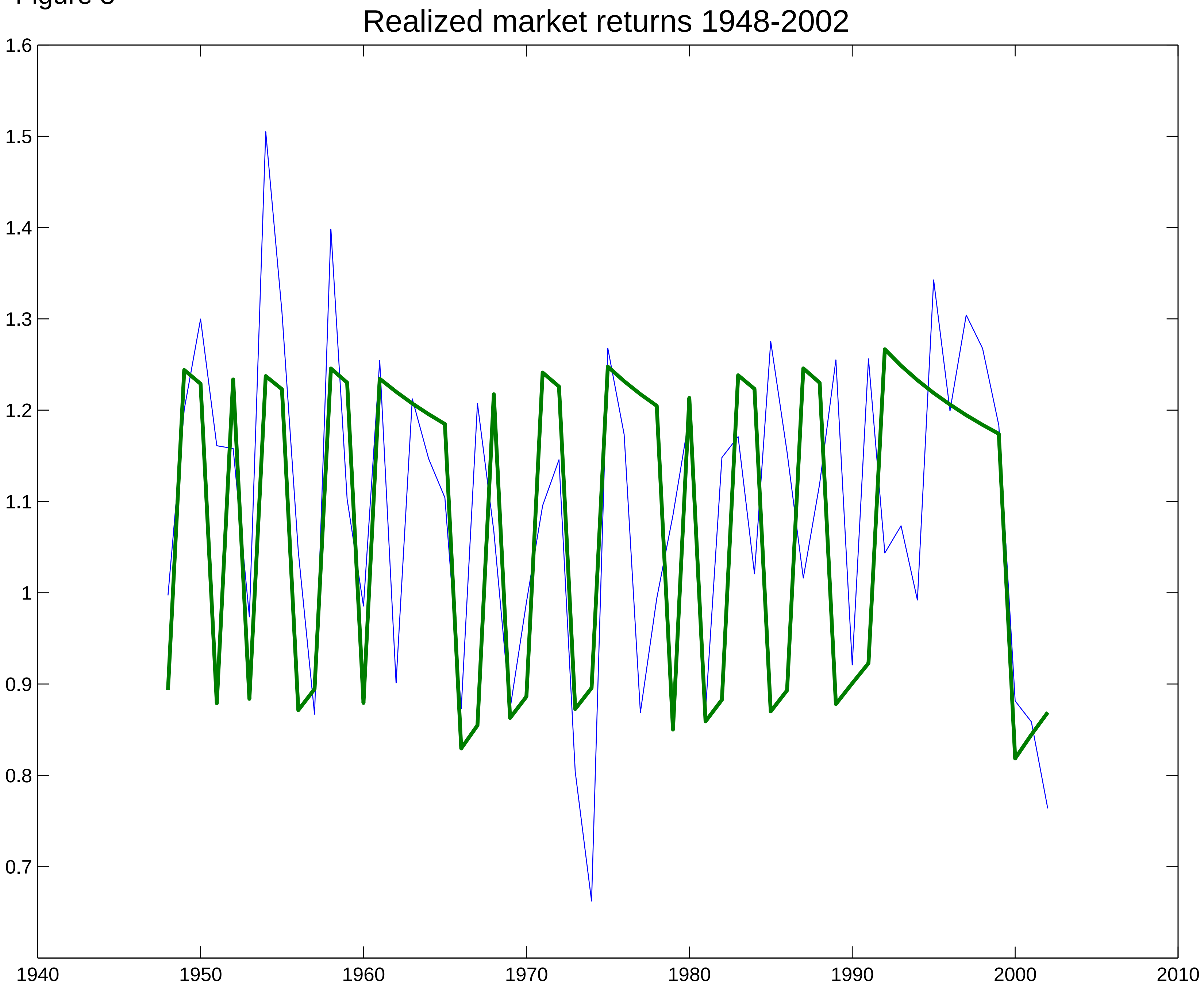


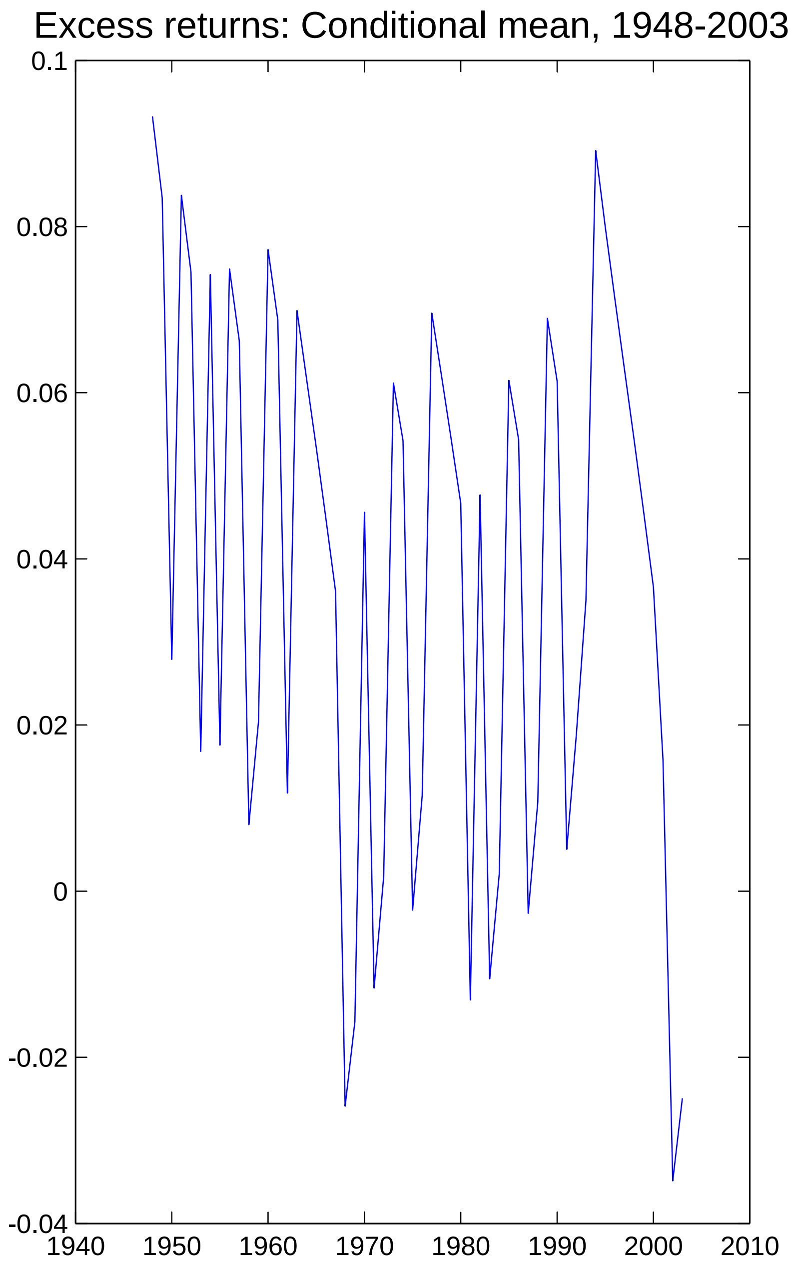
Figure 3



Realized market returns 1948-2002
1.6
1.5
1.4
1.3
1.2
1.1
1
0.9
0.8
0.7
1940
1950
1960
1970
1980
1990
2000
2010

Figure 4a

Baseline Calibration with Serially Correlated Investment Growth Rates



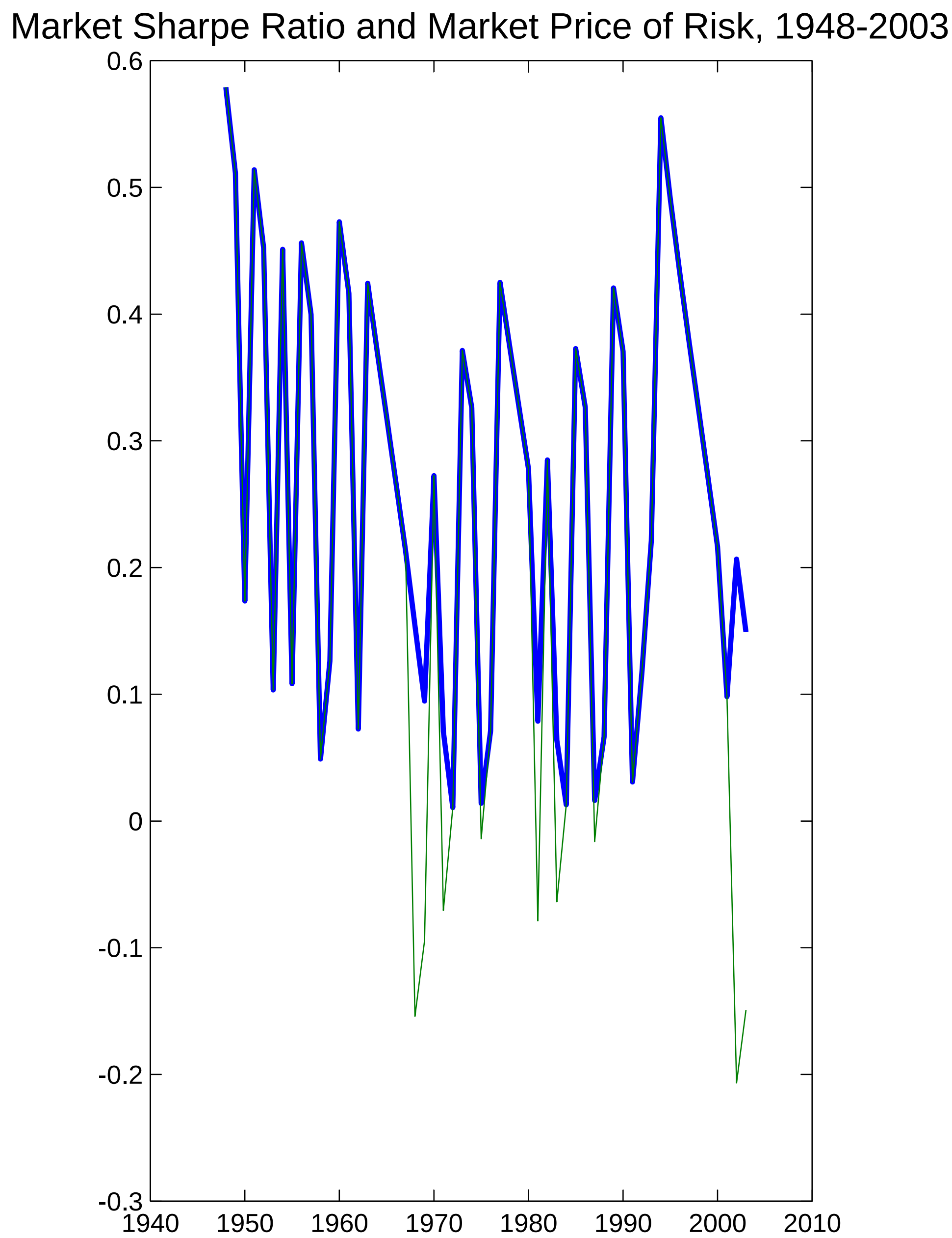
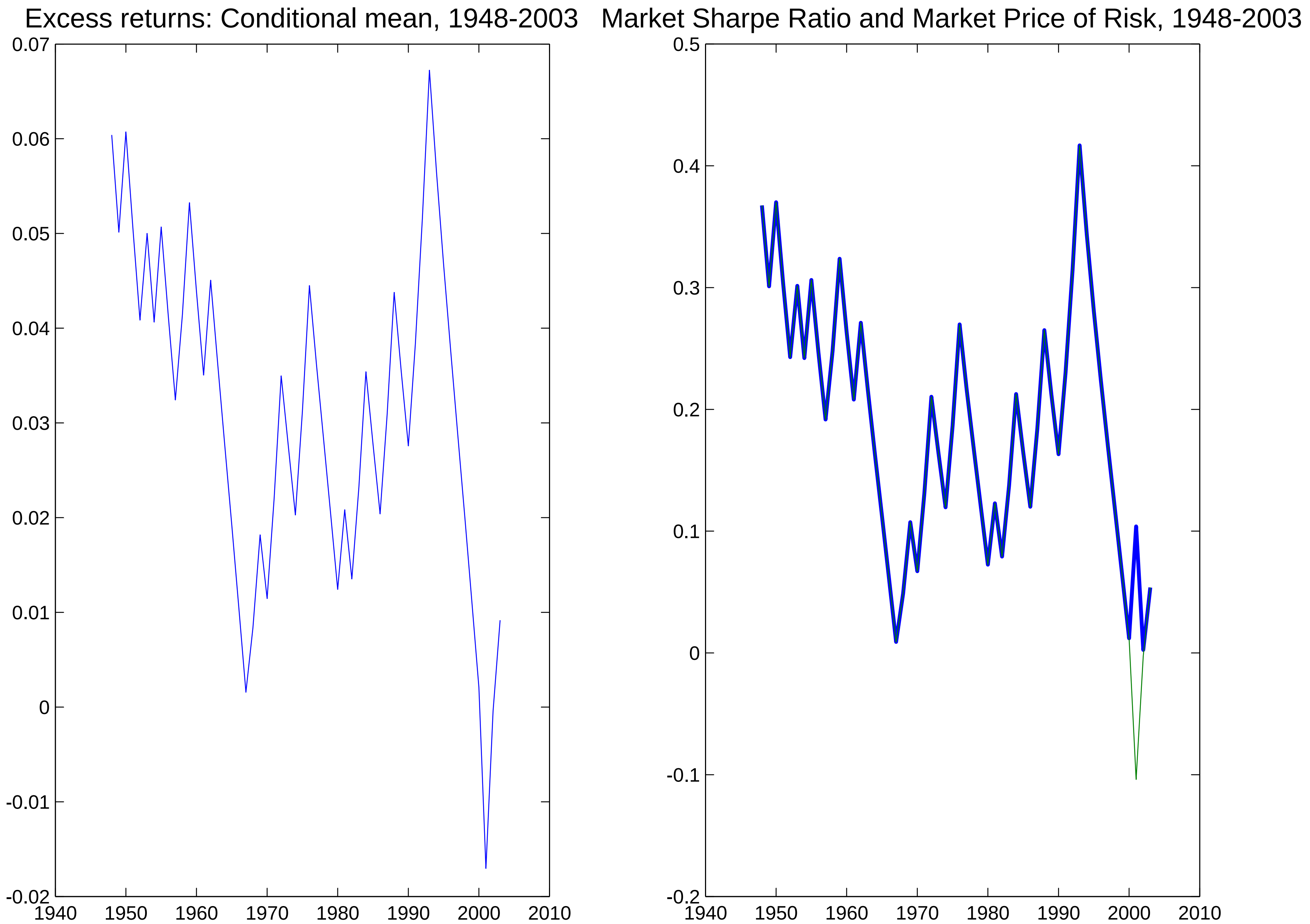


Figure 4b

Baseline Calibration with IID Investment Growth Rates



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