Optimal retirement benefit guarantees

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Abstract

The majority of countries that switched to funded private account retirement systems opted to complement such systems with explicit guarantees to retirees and agents saving for retirement. The motivation was that a social insurance system should provide a minimum standard of living in retirement. This paper studies the optimal design of such guarantees. Particular attention is paid to moral hazard, i.e. the incentive to take more risk once the guarantees are in place. Surprisingly, the simple policy of complementing private accounts with a fixed annuity in retirement is shown to be an optimal policy in the baseline model. It is also shown that the standard practice of pricing retirement benefit guarantees as contingent claims and then choosing the minimum cost guarantee may be a misleading indicator for welfare comparisons between alternative policies.

Keywords: Life Cycle savings and portfolio choice, Optimal design of guarantees, Continuous time methods, Retirement benefits, Ricardian Equivalence, Borrowing constraints

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1 Introduction

In the last few decades several countries around the world have experienced dramatic changes in the ways their citizens prepare for retirement. Many countries with substantially different economic structures and histories like Australia, Chile, Mexico and Sweden have replaced mostly unfunded with funded retirement systems[1]. Even in many countries where such changes have not occurred, there has been intense political debate on proposals to switch from unfunded to funded retirement systems. Under such systems individuals are able to invest their savings for retirement in a portfolio of stocks and bonds of their choosing. The switch from defined benefit to defined contribution plans in the private sector has had similar effects.

These trends imply an increased importance of financial markets and especially of the stock market for retirement savings. A common argument given by proponents of the above mentioned changes is that the rate of return for retirement funds in the stock market can be substantially higher than the rate of return in a pay as you go system -especially in the presence of global aging. Additionally, agents can use the added flexibility to better tailor their portfolio to their own preferences.

A risk that is recognized uniformly by proponents and opponents of the move towards retirement systems that are based on the principles of private choice and full funding is that they are too exposed to market risk: A downturn in the stock market could result in substantial losses amongst retirees, especially if agents do not appropriately take such risks into account. Furthermore, the aggregate nature of a stock market downturn, would likely create pressures to "bailout" the investors who lost their retirement savings, attenuating the incentives to hedge against stock market drops.

A commonly proposed remedy for this problem is to offer explicit government guarantees to current and prospective retirees: Variants of such proposals call for the government to explicitly guarantee a minimum level of retirement income, or a minimum return on retirement assets or returns on a benchmark portfolio. Needless to say, this raises the concern of moral hazard. As Becker (2005) put it in the context of the US discussion on privatizing social security:

> As in Chile and other countries with private retirement accounts, the government would guarantee retirees a minimum income - similar to, but larger than, the present

[1] Mitchell and Lachance (2003) report that more than 20 countries have established individual accounts.

> minimum Social Security guarantee. Unfortunately, such guarantees create a "moral hazard" - that is savers may want to make risky investments that give high payoffs if they succeed because the government partly bails them out. Or they may not save at all. [Becker (2005)]

Despite this concern with moral hazard, a very large number of countries that have made the switch to a fully funded system have accompanied this switch with explicit guarantees to retirees. For instance, in Chile retirees are guaranteed a minimum level of retirement income, irrespective of the level of their funds or their withdrawals[2]. The idea to include such guarantees was also present in the discussion to privatize the U.S. social security system[3]. Presumably, such guarantees are so popular for the same reason that led to the very appearance of social security systems in Europe 130 years ago: namely to provide a minimum standard of living to people who cannot rely on their labor income any longer.

Accepting the (political) necessity of such complements to fully funded systems leads to a host of questions: What form should they take? Should the government guarantee a minimum income, a minimum rate of return, or a minimum level of assets upon entering retirement? If individuals reduce their savings and increase the risk in their portfolios in response to such guarantees, how successful are such guarantees likely to be in achieving the stated goal? How large is the cost of such guarantees likely to be and who will finance them?

The existing literature has addressed mostly issues related to the cost and the financing of such guarantees[4]. This research has increased our understanding of the quantitative magnitude of such guarantees. However, to the best of my knowledge, two important issues have not been addressed yet:

a) First, the existing literature has not developed a normative theory, in order to guide the

[2] See e.g. Pennacchi (1999) for a description of the Chilean and other guarantees in various Latin American countries.

[3] See e.g. Feldstein (2005b), Feldstein (2005a), Feldstein and Ranguelova (2001). See also the study by the Congressional Budget Office on the cost of such guarantees. [Sinclair, Lucas, Rehder-Harris, Simpson, and Topoleski (2006)]

[4] See Feldstein and Ranguelova (2001), Feldstein (2005a), Mitchell and Lachance (2003), Constantinides, Donaldson, and Mehra (2002), Smetters (2001), Pennacchi (1999), Sinclair, Lucas, Rehder-Harris, Simpson, and Topoleski (2006).

choice of one type of guarantee over another. Different countries have taken different approaches by guaranteeing retirement income, returns on assets, including mandatory annuities etc. This of course raises the question of finding an optimal way of achieving a minimum standard of living in retirement. Which magnitude (retirement income, assets, returns) should be guaranteed and to what extent?

b) Second and more importantly, existing papers do not take into account the distortions that would be introduced by government guarantees. Even though the importance of this issue is recognized throughout the literature[5], typically the guarantees are priced assuming that individual behavior is not going to be significantly affected by the presence of the guarantee. However, as a matter of theory, it is not clear why the presence of guarantees would have a negligible effect on individual behavior[6].

In this paper the goal is to address these issues and develop an optimality theory of guarantees in the presence of moral hazard.

The paper takes as a starting point that the society in the model has decided to switch to a fully funded system and avoid all pay as you go features in social security - presumably because pay as you go features are viewed as too distortionary. Understanding if that choice is optimal or not and how the transition between the two systems is going to take place is the subject of a voluminous literature and is not addressed here[7].

To motivate the need for guarantees in such a fully funded system, I assume a discrepancy between a social planner's and an individual agent's objectives. The social planner would like to see retired agents provide themselves with a certain minimum standard of living in retirement, but the agents will not necessarily choose to do so. There are a multitude of reasons why that may be the case: On behavioral grounds one could argue that agents may have time inconsistent preferences,

[5] As Feldstein (2005b) points out in his presidential address "Social Insurance programs generally involve a tradeoff of preotection and distortion".

[6] Actually one might expect the opposite: The seminal paper by Bodie, Merton, and Samuelson (1992) illustrates that if an agent can expect to receive some income in the future, the agent's portfolio will contain a component that will perfectly offset variations in the net present value of that income. Viewing guarantees as an anticipated source of income and taking the Bodie, Merton, and Samuelson (1992) argument to its logical conclusion, should lead one to conjecture that such guarantees will be perfectly offset by the agent's portfolio choice.

[7] See e.g. Krueger and Kubler (2006), Ball and Mankiw (2001) for some recent examples on the status of this debate.

so that there is a discrepancy between the choices that they would like to make ex-ante and the actions they choose ex-post. On rational grounds one could argue that retired agents may anticipate government bailouts in the event of a stock market crisis, financed by raising distortionary taxes on the young. This distortion is external to the retirees, thus providing an alternative motivation for the wedge between the central planner's and the agents' objectives.

Whatever the exact reason, the wedge between the central planner's and the agent's objective leads to the same dynamic moral hazard formulation for the problem: In a first step the central planner chooses (optimally) a process of transfers for the agent and then the agent chooses savings and portfolio strategies taking the transfers as given. To avoid the perfect "offsetting" of the government transfers by increased risk taking, the model imposes a borrowing constraint, by requiring that the agent's financial wealth always stay non-negative[8]. Given the backdrop of a fully funded system and the normative nature of the analysis, the baseline model considers the case where the guarantees are fully funded by raising appropriate taxes on the agent while she is working[9].

The first result of the paper is the derivation of an upper bound on the welfare that any set of transfers can attain. The second result is to illustrate that there exist multiple government policies that are optimal. Interestingly, the simplest conceivable policy of just transferring a constant income stream to the agent in retirement is optimal. However, an appropriate type of portfolio insurance policy that guarantees a minimum return on the agent's retirement portfolio -after some cumulated losses- is also optimal. A noteworthy implication of the analysis is that these two policies are equivalent from a welfare perspective, even though they are associated with substantially different actuarial costs. This surprising finding implies that just focusing on the cost of guarantees, as is routinely done in the literature and policy discussions may be misleading for welfare comparisons.

An additional outcome of the analysis, which is of practical importance, is that it derives *explicitly* a minimum level of funds that need to be available when entering retirement, if there is to exist *any* post-retirement set of transfers that will "keep" the agent's post-retirement consumption above the required minimum level. This helps one compute the income taxes that would have to be levied on the agent while she is working in order to ensure that such guarantees can be prefunded. It is also shown that the presence of moral hazard will raise the magnitude of these minimum assets

[8] It is also shown that this constraint arises endogenously as long as the government outlaws securitization of future government transfers.

[9] This is in contrast to e.g. Smetters (2001) where guarantees are unfunded, i.e. pay when needed.

and the associated taxes.

Simple closed form solutions are given for all quantities. An interesting result is that the presence of moral hazard will tend to substantially magnify the amount of transfers that are required to ensure a minimum standard of living.

In summary, this paper lends support to the view expressed by Feldstein (2005b) that by combining elements of a fully funded defined contribution system with some explicit guarantees can achieve the goal of making the retirement system robust to market downturns, even when one *takes account of the "moral hazard" effects.* The model presented here suggests that there may be many equivalent ways to achieve the goal of a minimum standard of living in retirement and just looking at their costs can be misleading. Under certain assumptions, it also suggests that a particularly simple and optimal way of achieving the stated goal is to introduce a minimum constant "fixed annuity" feature next to a purely privatized "defined contribution" system.

The paper is structured as follows. Section 2 sets up the model and briefly lays out the reasons for government intervention. Section 3 introduces a government with the task of keeping the agent's consumption above a minimum level by usage of appropriate taxes and transfers. Section 4 considers the agent's reaction to the presence of such intervention. Section 5 derives an upper bound to welfare (which coincides with the government objective function) no matter which set of admissible taxes/transfers is utilized. Section 6 illustrates two distinct ways of attaining that upper bound, which are hence optimal. Section 7 discusses the cost involved in these transfer schemes and identifies the lowest amount of funds that need to be available in order to achieve the stated goal of guaranteeing a minimum level of retirement consumption. Section 8 discusses extensions to heterogenous incomes and arbitrary stochastic discount factors. Section 9 concludes.

2 The model

2.1 Agents, preferences, and endowments

The model is very similar to the small open economy version of Blanchard (1985). There is a continuum of identical agents with mass 1. At each point an agent faces a constant probability of death q per unit of time dt, and newly born agents also arrive at the same rate. To focus on the moral hazard aspects of guarantees and simplify the setup, I will assume away all heterogeneity in preferences and incomes. In particular, all agents have constant relative risk aversion γ, and a constant discount rate ρ, so that each agent who is born at time T_1 aims to maximize

$$F_{T_1} = E_{T_1} \int_{T_1}^{\infty} e^{-(\rho+q)(s-T_1)} \frac{(c_s)^{1-\gamma}}{1-\gamma} ds \tag{1}$$

Purely for empirical relevance, I will focus on the case $\gamma \geq 1$. It is only a matter of making a few additional technical assumptions to extend the results to $\gamma < 1$. Once born, agents are endowed with a non-tradeable constant income stream of Y per unit of time dt. That income stops after a duration of time equal to T. After this point of time the agent can only rely on her assets to sustain herself. Agents will be referred to as "workers" while they receive labor income and will be referred to as "retirees" once they cannot expect to receive any further income.

2.2 Investment opportunity set

Agents can invest in a riskless and a risky asset. The extension to multiple risky assets is straightforward, and is left out. As is quite standard in the literature that studies incentive problems between a central planner and agents, I will fix the rates of return that agents can earn when accessing financial markets. Alternatively put, I will consider a small open economy and accordingly fix the stochastic discount factor. Section 8.2 shows how the results can be extended to setups where the stochastic discount factor is arbitrary.

In particular, I will assume that agents can invest in the money market, where they receive a constant strictly positive interest rate $r > 0$. In addition they can invest in a risky security with a price per share that evolves as

$$\frac{dP_t}{P_t} = \mu dt + \sigma dB_t$$

where $\mu > r$ and $\sigma > 0$ are given constants and B_t is a one-dimensional Brownian motion on a complete probability space (Ω, F, P).[10] The realization of this Brownian motion will be the only source of uncertainty in this economy.

As is well understood, dynamic trading in the stock and the bond leads to a dynamically complete market. (See e.g. Duffie (2001) or Karatzas and Shreve (1998)). As Karatzas and Shreve

[10] I shall denote by $F = \{F_t\}$ the P-augmentation of the filtration generated by B.

(1998) show, the assumptions of a constant interest rate and risk premium imply the existence of a unique stochastic discount factor (or state price density) which is given by:

$$H(t) = \exp\left\{-\int_0^t \kappa dB_s - rt - \frac{1}{2}\kappa^2 t\right\}, \quad H(0) = 1 \tag{2}$$

where κ is the Sharpe ratio, defined as

$$\kappa = \frac{\mu - r}{\sigma}$$

Using this stochastic discount factor, the no-arbitrage price of any claim that delivers dividends equal to D_s is given by[11]:

$$E_t \int_t^\infty \frac{H_s}{H_t} D_s ds$$

The agent can also enter into contracts with a competitive life insurance company as is explained in detail in Blanchard (1985). As Blanchard (1985), I shall assume that the agent's hazard rate of death is a constant q, so that the insurance company can offer the agent contracts that promise an income stream of q per unit of time dt, in exchange for receiving one dollar if the agent dies over the next interval dt.

2.3 Portfolio and wealth processes

An agent chooses a portfolio process π_t and a consumption process c_t. The portfolio process π_t is the *dollar amount* invested in the risky asset (the "stock market") at time t. The rest, $W_t - \pi_t$, is invested in the money market. The agent has no bequest motives. As Blanchard (1985) shows, it is optimal in this case for the agent to enter an annuity contract: The agent receives from the insurance company an income stream of qW_t per unit of time dt while she is alive. In exchange, the entire remaining wealth of the agent gets transferred to the insurance company when the agent dies. Accordingly, the wealth process of a retired agent evolves as

$$dW_t = qW_t dt + \pi_t \{\mu dt + \sigma dB_t\} + \{W_t - \pi_t\} r dt - c_t dt \tag{3}$$

[11] From a macroeconomic perspective one can also think of H_t as the marginal utility of consumption of the world-representative agent.

and the wealth process of a working agent is given by:

$$dW_t = qW_t dt + \pi_t \{\mu dt + \sigma dB_t\} + \{W_t - \pi_t\} r dt + Y dt - c_t dt$$

An additional requirement is that wealth must remain non-negative throughout:

$$W_t \geq 0 \text{ for all } t \tag{4}$$

This constraint excludes uncollateralized borrowing. Section 3 discusses how this constraint arises endogenously when the government can prohibit financial intermediaries from accepting governmental transfers as collateral.

2.4 A role for government guarantees

As already mentioned in the introductory section, there are many ways to justify the observed popularity of retirement benefit guarantees. From a rational perspective, retirees may have little incentive to hedge the downside of their investments, because they can induce the government to bail them out if their level of consumption were to fall below some lower bound ξ in retirement. Assuming that such bailouts are associated with sufficiently large taxes and distortions on the young (which are external to the old), the central planner will generally have an incentive to alter the lifetime consumption choices of the representative agent, so as to ensure that her consumption in retirement stays above ξ, and as a result bailout demands will not be triggered.

Alternatively, there is a behavioral way to justify retirement benefit guarantees that abstracts from political economy motivations. When the agent evaluates her expected lifetime utility at birth, she may correctly anticipate some inelastic consumption expenditure ξ in retirement (say due to medical or nursing expenditures) so that she might have a preference for considering only paths that involve consumption choices above ξ in retirement. However, future "selves" might ignore this requirement and as a result choose consumption paths that will violate that requirement.

Finally, societies might exhibit altruism towards aging members of the society, so that the utility of the younger members of society would be lower if the consumption of retirees were to fall below a given level ξ. This creates a consumption externality between members of society.

Fortunately, the exact reason why societies opt for retirement benefit guarantees is irrelevant for the purposes of this paper. Section 8 shows formally that either of the above justifications will have a common feature: Namely either approach will introduce a wedge between the central planner's objective and the agent's objective. Specifically, section 8 shows that either of the three justifications will imply that a benevolent central planner will view choices of an agent that involve consumption below ξ in retirement as having *external effects* either on the young (rational, bailout-prevention motivation or altruism) or on the objective of the agent's "self" when she is born (behavioral motivation).

However for all of the paper's results, the exact nature of the external effects is immaterial. All that matters is that under either motivation the objective of the central planner can be expressed as

$$\widehat{F}_{T_1} = \begin{cases} F_{T_1} - b, & \text{if } \Pr\{c_t < \xi \text{ for all } t > T_1 + T\} > 0 \\ F_{T_1} & \text{otherwise} \end{cases} \tag{5}$$

where F_{T_1} is the objective of the agent (given in equation [1]) and $b > 0$ captures the "social cost" associated with the external effects of retiree consumption dropping below ξ. As can be seen by comparing (1) and (5), the difference between the objective of the agent and the central planner is due to the fact that the central planner internalizes the external effects. Assuming that b is large enough, it is reasonable to conjecture that the central planner will want to maximize the agent's lifetime utility (1) subject to the additional constraint:

$$c_t \geq \xi \text{ for all } t \geq T_1 + T \tag{6}$$

To expedite the exposition of the paper's key results, I will assume from this point on that the central planner's objective is to maximize (1) subject to (6). After solving for the optimal retirement benefit guarantees that will maximize (1) subject to (6), I will revisit the nature of the social costs and show that there exists b large enough such that the central planner will indeed find it optimal to enforce the constraint (6) on the agent's choices.

3 Introducing a role for the government

The central planner understands that the agent will not automatically impose on herself the constraint (6). (For brevity I will refer to the central planner as the government). Indeed the choices of the agent will be made so as to maximize (1) without regard to (6). To achieve the goal of imposing constraint (6) on the agent's choices, the government can provide agents with an optimally chosen post retirement transfer process (i.e. a guarantee). Determining such optimal transfers is the goal for the rest of the analysis.

It is most useful to split the problem in two parts: The first step is to solve for the optimal transfer process (i.e. guarantee) that will maximize the agent's retirement utility subject to (6), and the appropriate incentive compatibility constraints, assuming that the guarantee is financed with a lump sum tax upon entering retirement and that the agent has enough assets to pay this tax. This is done in sections 3.1 - 7. The second step is to use the solution of this problem to construct the optimal process of taxes and transfers over the life cycle. This is done in section 7.2.

3.1 Admissible transfers

Given that the time of retirement is central to the analysis, the paper will adopt the following timing convention. Since the setup is time-invariant, time 0 will be the time at which the representative agent retires. Hence the representative agent will be assumed to be born at time $T_1 = -T$. Moreover, since all quantities depend on ratios of the stochastic discount factor between two points in time, I will take the value of the stochastic discount factor at time 0 to be equal to 1. Both of these conventions involve no loss in generality

As already mentioned, post retirement the government can make transfers to the retiree. It is reasonable to follow a common approach in the literature[12] and assume that the government is subject to severe informational constraints compared to the agent.

To capture this, I will assume that governmental transfers can be made contingent on quantities that are "exogenous" to the agent (i.e. the returns in the stock market). However, the government cannot directly observe (or at least verify) an individual agent's consumption, portfolio, or wealth process. Therefore, transfers cannot depend on these quantities.[13] This will form the source of the moral hazard problem, since the agent will need to be induced to choose a consumption process that satisfies (6) given the government's transfers. One exception is that the government will be

[12] See e.g. Cole and Kocherlakota (2001).

[13] There is an analogy to standard principal agent models here. Just as in the standard principal agent model the principal cannot write contracts that are contingent on the agent's effort choice, here it is impossible to write contracts that depend directly on agent's consumption, portfolio choices or assets.

assumed to know an agent's wealth upon entering retirement, namely W_0. This assumption will be relaxed later in the text.

The following definition formalizes the informational requirements:

Definition 1 *Let $\widetilde{\mathcal{F}}_t$ be the filtration generated by the Brownian motion B_t and knowledge of the retiree's assets at the time that she enters retirement (W_0). An admissible cumulative transfer process G_t is a non-decreasing, progressively measurable (with respect to $\widetilde{\mathcal{F}}_t$) process starting at $G_0 = 0$ and satisfying:*

$$E\int_0^\infty e^{-qs}H_s dG_s < \infty$$

With some abuse of mathematical precision, the non-negative increments of the process G_t, namely $dG_s \geq 0$, will be referred to as the "transfers" to the agent.

It is useful to discuss the requirements of Definition 1. The requirement that the process be non-decreasing and start at 0 captures the fact that G_t progressively adds all the positive transfers to the agent.

Progressive measurability with respect to $\widetilde{\mathcal{F}}_t$ is the requirement that captures "exogenous" information and knowledge of W_0. The government is assumed to observe the Brownian path, and hence all the quantities that can depend on that Brownian path (for instance the stock market). However, it cannot condition its transfers directly on an agent's consumption choice, portfolio choice or wealth (except at time 0). It can at most *infer* these choices from its knowledge of the brownian path, along with its knowledge of an agent's optimizing behavior.

Given the assumption that G_t is progressively measurable with respect to $\mathcal{F}_t$, the fair value of a claim delivering the cumulative transfer process G_t is given by[14]:

$$E_t\int_t^\infty e^{-q(s-t)}\frac{H_s}{H_t}dG_s \tag{7}$$

Before formalizing the government's problem, it remains to discuss how these transfers will be financed. Given the "backdrop" of the normative analysis, which is a fully funded system, I will assume until section 7.2 that transfers are funded by levying a lump sum tax D_0 on the agent at

[14]This is a consequence of the martingale representation theorem.

the time of retirement (time 0). The arbitrage free value of the claim that delivers the transfers G_t is:

$$D_0 = E_0 \int_0^\infty e^{-qs} H_s dG_s.$$

An alternative interpretation of this setup is that the government passes a law that requires the agent to buy a contract that offers the payoffs dG_t from competitive financial firms. The competitive financiers will charge D_0 for such a contract. Clearly, raising such a tax will only be feasible if the agent has accumulated a minimum amount of assets by the time she retires. Therefore I will assume that:

$$W_0 \geq \frac{\frac{1}{\gamma} + \phi - 1}{\phi - 1} K \xi \tag{8}$$

where ϕ is a constant that is defined as:

$$\phi = \frac{-\left(\rho - r - \frac{\kappa^2}{2}\right) + \sqrt{\left(\rho - r - \frac{\kappa^2}{2}\right)^2 + 2\left(\rho + q\right)\kappa^2}}{\kappa^2} \tag{9}$$

with the property[15] $\phi > 1$ and K is another constant defined as:

$$K = \frac{\gamma}{\frac{\gamma-1}{\gamma}\frac{\kappa^2}{2} + \gamma\left(r + q\right) + \left(\rho - r\right)} \tag{10}$$

A full discussion of condition (8) will wait until section 7. For now I just remark, that without this condition there would be no combination of D_0, G_t that will safeguard $c_t \geq \xi$. Section 7.2 shows how to use forced savings pre-retirement to ensure that condition (8) is satisfied.

It is now possible to formulate the government's objective.

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[15] To see why $\phi > 1$ notice that ϕ solves the quadratic equation

$$\frac{\kappa^2}{2}\phi^2 + \left(\rho - r - \frac{\kappa^2}{2}\right)\phi - \left(\rho + q\right) = 0$$

Evaluating the left hand side of this equation at $\phi = 1$ gives:

$$-(r + q) < 0$$

Hence the larger of the two roots of the quadratic equation is larger than 1.

Problem 1 *Assuming (8), the government's objective is to determine an admissible cumulative transfer process G_t and an initial tax D_0 so as to maximize:*

$$V = \max_{G_t, D_0} E_0 \int_0^{\infty} e^{-(\rho+q)s} \frac{c_s^{1-\gamma}}{1-\gamma} ds \tag{11}$$

subject to

$$c_t \geq \xi \text{ for all } t > 0 \tag{12}$$

$$D_0 = E_0 \int_0^{\infty} e^{-qt} H_t dG_t \tag{13}$$

and subject to the constraint that c_t solves the agent's optimization problem given G_t

$$c_t = \arg \max_{<c_t, \pi_t>} E_t \int_t^{\infty} e^{-(\rho+q)(s-t)} \frac{c_s^{1-\gamma}}{1-\gamma} ds \tag{14}$$

subject to :

$$dW_t = qW_t dt + \pi_t \{\mu dt + \sigma dB_t\} + \{W_t - \pi_t\} r dt - c_t dt + dG_t \tag{15}$$

$$W_{0^+} = W_0 - D_0 \tag{16}$$

$$W_t \geq 0 \text{ for all } t > 0 \tag{17}$$

There are several remarks on the above setup: As already mentioned, in this section the central planner focuses on maximizing the agent's post retirement welfare, as can be seen from equation (11).

Equation (12) captures the requirement that transfers should induce a consumption process that keeps consumption above the level ξ. Equation (13) requires that the guarantee given to the agent should be financed by the tax raised upon entering retirement.

Equations (14)-(17) capture the incentive compatibility requirement, namely that the consumption process be optimal from the perspective of an agent taking *the governmental taxes and transfers as given.* Equation (15) presents the wealth evolution equation, taking into account the presence of transfers. Equation (16) states that the consumer's financial assets W_0 will be reduced by the tax D_0, so that the agent's post tax assets are given by W_{0^+}.

Equation (17) requires that assets be non-negative at all times. I shall refer to this constraint as the borrowing constraint and it will play a key role in this paper. In practical terms, this constraint implies that the agent has no ability to borrow against future transfers, by -say- securitizing them[16].

[16] This seems plausible, as long as the government can outlaw such securitization. But if the government outlaws

For the purposes of this paper, equation (17) is the key constraint of the analysis. Without this constraint, it would be impossible for the government to find a set of taxes and transfers that would induce the agent to choose a consumption path that satisfies (12). The reason is that the magnitude of the tax D_0 raised at time $t = 0$ is exactly equal to the expected net present value of the government's transfers to the agent. If the agent was unconstrained in her ability to transfer resources between dates and states, she could completely "undo" the effects of the lump sum tax and the transfers by appropriate saving and trading strategies. This result is a manifestation of the well understood principle of Ricardian Equivalence[17]. As long as the agent who is taxed is the same agent that receives the future transfers *and markets are dynamically complete*, Ricardian Equivalence asserts that government intervention will have no effects.

The presence of a borrowing constraint such as (17), however, makes taxes and transfers non-neutral. The reason is that a borrowing constraint implies stronger restrictions than a simple intertemporal budget constraint on the agent's feasible consumption choices. Hence, by a judicious choice of an initial tax and subsequent transfers, the government can affect the agent's consumption.

Sections 4-7 are devoted to the study of problem 1.

4 The agent's consumption choices in the presence of government intervention and borrowing constraints

To solve problem 1 it is instructive to take an intermediate step: This section examines how different forms of transfers will affect the agent's optimal consumption choices.

Specifically, suppose that at the time that the agent enters retirement (time 0) the government taxes her by an amount D_0 and then promises an admissible cumulative transfer process G_t. It is now natural to ask how the agent's consumption choices will be affected by this intervention in the presence of the constraint (17). The following result shows how to obtain the optimal consumption process in this case and is due to He and Pages (1993):

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such securitization, then the only way the agent could borrow against this future income would be by reputation (unsecured lending). However, as is well known from the seminal Bulow and Rogoff (1989) paper, unsecured lending based on reputation cannot be supported. To conclude, as long as the government can outlaw securitization of these transfers, the constraint (17) results naturally from the results in Bulow and Rogoff (1989).

[17] See e.g. Barro (1974),Abel (2003).

Proposition 1 *Let $\mathcal{D}$ be the set of non-increasing, non-negative and progressively measurable processes that start at $X(0) = 1$. Then, the value function $V(W_0)$ of an agent can be expressed as:*

$$V(W_0) = \min_{\lambda>0,\ X_s\in\mathcal{D}} \left[\begin{array}{c} E\left(\int_0^\infty e^{-(\rho+q)s} \max_{c_s}\left(\frac{c_s^{1-\gamma}}{1-\gamma} - \lambda e^{\rho s} H_s X_s c_s\right) ds + \lambda \int_0^\infty e^{-qs} H_s X_s dG_s\right) \\ +\lambda\left(W_0 - D_0\right) \end{array} \right] \quad (18)$$

Let X_t^*, λ^* denote the process X_t and the constant λ that minimize the above expression. Then the optimal consumption process c_t^* for a consumer faced with the borrowing constraint (17) is:

$$c_t^* = \left(\lambda^* e^{\rho t} H_t X_t^*\right)^{-\frac{1}{\gamma}} \text{ for all } t > 0 \quad (19)$$

Moreover, the process X_t^* decreases only when the associated wealth process (W_t) falls to zero and is otherwise constant, i.e.:

$$\int_0^\infty W_t dX_t^* = 0 \quad (20)$$

Finally, the resulting wealth process for any $t > 0$ satisfies:

$$W_t = \frac{E_t\left(\int_t^\infty e^{-q(s-t)} X_s^* H_s c_s^* ds\right)}{X_t^* H_t} - \frac{E_t\left(\int_t^\infty e^{-q(s-t)} X_s^* H_s dG_s\right)}{X_t^* H_t} \quad (21)$$

The simplest way to gain some intuition as to the proposition's statements is to focus first on equation (19). Note that this equation can be rewritten as:

$$e^{-\rho t}\left(c_t^*\right)^{-\gamma} = \lambda^* H_t X_t^* \quad (22)$$

If X_t^* is set to 1 in equation (22), the equation reduces to the standard prediction that one obtains in a dynamically complete market, namely that the marginal utility of consumption is proportional to the stochastic discount factor. In the presence of borrowing constraints, however, the markets will not be dynamically complete and this will make X_t^* different than 1. In particular X_t^* will change when and only when $W_t = 0$ as revealed by (20), i.e. when the borrowing constraint is binding. Hence one can interpret X_t^* as the Lagrange multiplier process associated with the borrowing constraint $W_t \geq 0$: By multiplying the state price density H_t by X_t^* one can "incorporate" the shadow value of the borrowing constraint into the state price density. This means that the solution to the problem of an agent who is constrained by $W_t \geq 0$ and faces the state price density H_t will be identical to the problem of an unconstrained agent who faces the state price density $H_t X_t^*$.

5 Government transfers and their welfare effects: an upper bound

Turning to problem 1, Proposition 1 gives an intuitive way to summarize the effects of the incentive compatibility requirement (equations [14]-[17]).

It asserts that every government transfer process G_t will be associated with a constant λ^G and a Lagrange multiplier process X^G resulting from the minimization problem in equation (18). Given this duality between a choice of G_t and the resulting pair $\left(\lambda^G, X_t^G\right)$, there is a straightforward way to obtain an upper bound to problem 1. In particular consider the following problem:

Problem 2 *Maximize:*

$$J\left(W_0\right) = \max_{c_t, X_t \in \mathcal{D}, \lambda > 0} E_0 \int_0^\infty e^{-(\rho+q)s} \frac{c_s^{1-\gamma}}{1-\gamma} ds \tag{23}$$

subject to:

$$E_0 \left(\int_0^\infty e^{-qs} H_s c_s ds \right) \leq W_0 \tag{24}$$

$$c_t \geq \xi \tag{25}$$

$$c_t = \left(\lambda e^{\rho t} H_t X_t\right)^{-\frac{1}{\gamma}} \tag{26}$$

Problem 2 is the problem of a central planner who can choose directly the consumption of the agent, subject to an intertemporal budget constraint, a constraint on the minimum consumption level (equation [25]) and the additional requirement that any chosen consumption process should have a representation in the form of equation (26).

In effect, problem 2 allows the central planner to choose freely the pair (λ, X_t) without being concerned whether there exist any optimal transfer process G_t that will make (λ, X_t) the optimal solution of the minimization problem in (18).

It is reasonable to conjecture as a result, that problem 2 is a "relaxed" version of the optimization problem 1 and hence the value function of 2 dominates the value function of problem 1. The next proposition proves this assertion and determines the solution of problem 2:

Proposition 2 *For any $\lambda > 0$, let the process $X_t^*(\lambda)$ be defined as*

$$X_t^* = \min\left[1, \frac{\xi^{-\gamma}/\lambda}{\max_{0\leq s\leq t}\left(e^{\rho s} H_s\right)}\right] \tag{27}$$

[To simplify notation,, X_t^* will be used as a shorthand for $X_t^*(\lambda)$]. Assuming (8), the value function of problem (2) is given by:

$$J(W_0) =$$
$$= \min_{\lambda \geq 0} \left[E\left(\int_0^\infty e^{-(\rho+q)s} \frac{(\lambda e^{\rho s} H_s X_s^*)^{1-\frac{1}{\gamma}}}{1-\gamma} ds - \lambda \int_0^\infty e^{-qs} H_s (\lambda e^{\rho s} H_s X_s^*)^{-\frac{1}{\gamma}} ds + \lambda W_0 \right) \right] \quad (28)$$
$$= \min_{\lambda \geq 0} \left[-\frac{K\xi^{1-\gamma}}{\gamma\phi(\phi-1)} \left(\frac{\lambda}{\xi^{-\gamma}} \right)^{\phi} + K\frac{\gamma}{1-\gamma}\lambda^{1-\frac{1}{\gamma}} + \lambda W_0 \right] \quad (29)$$

where K is given in (10) and ϕ is given in (9). Letting λ^* be the scalar that minimizes (29), the optimal triplet that solves problem (2) is given by λ^*, $X_t^*(\lambda^*)$,and the c_t that is implied by equation (26) for $\lambda^*, X_t^*(\lambda^*)$.

Finally, let $\mathcal{G}$ be the class of all admissible transfer processes that lead to a consumption process such that (12) is satisfied and let $V(W_0)$ be given as in equation (18). Then the following upper bound characterizes the value function of problem 1:

$$\max_{G_t \in \mathcal{G}} V(W_0) \leq J(W_0) \quad (30)$$

Proposition 2 illustrates that there exists an upper bound to the value function of the original problem 1 against which one can measure any government policy. At a practical level, the usefulness of Proposition 2 is to provide a very simple test for the optimality of an admissible transfer process G_t: As long as an admissible process G_t attains the upper bound given by proposition 2, such a process must be optimal.

To gain some intuition on Proposition 2, it is useful to ask why the process X_t^* is optimal for problem 2. Taking logs in equation (26) and subtracting $\log \xi$ on both sides gives:

$$\log c_t - \log \xi = -\frac{1}{\gamma}[\log \lambda + \rho t + \log H_t + \log X_t] - \log \xi \quad (31)$$

The above equation implies that $c_t \geq \xi$ if and only if $\log c_t - \log \xi \geq 0$. Hence the constraint (12) will be satisfied if and only if there exists a process $X_t \in \mathcal{D}$ that will safeguard that the right hand side of (31) will always be non-negative. The process of equation (27) does have this property[18].

[18] To see this, consider two cases. The first case is $\lambda \max_{0 \leq s \leq t} e^{\rho s} H_s \leq \xi^{-\gamma}$. In that case $X_t^* = 1$ and accordingly

$$\lambda e^{\rho t} H_t X_t^* = \lambda e^{\rho t} H_t \leq \lambda \max_{0 \leq s \leq t} e^{\rho s} H_s \leq \xi^{-\gamma}$$

Moreover, this process has an additional property. It is the *largest* X_t^* that will satisfy (31).[19]

In practical terms this means that among all decreasing processes that will enforce the requirement (25), X_t^* is the process that will minimize the difference between the consumer's optimal consumption in the absence of government intervention and in its presence. Hence it achieves the goal of having $c_t \geq \xi$ while making the borrowing constraint bind as little as possible, and hence distorting the consumer's consumption choices in a minimal fashion.

6 Optimal Transfer Processes

This section illustrates two distinct policies that can attain the upper bound of (29).

6.1 A constant income stream

The simplest form of government transfer process is a constant income stream: The government collects a lump sum tax of $D_0 = \frac{y_0}{r+q}$ and in exchange it delivers a constant stream of y_0 in annuity. Such a simple policy turns out to be optimal as long as y_0 is chosen appropriately.

The following Proposition illustrates this fact:

Proposition 3 *Let y_0 be given as:*

$$y_0 = (r+q)\, K\xi \left(\frac{\frac{1}{\gamma} + \phi - 1}{\phi - 1} \right) \tag{32}$$

where K is given in (10) and ϕ is given in (9). Assume that the government raises an initial tax of $D_0 = \frac{y_0}{r+q}$ and promises a constant stream of payments equal to y_0 in annuity. Then, the agent will choose a consumption path that satisfies $c_t \geq \xi$ and the value function $V(W_0)$ [as defined in (18)] will attain the upper bound given in Proposition 2.

The second case is $\lambda \max_{0\leq s\leq t} e^{\rho s} H_s \geq \xi^{-\gamma}$. Then $X_t^* = \frac{\xi^{-\gamma}/\lambda}{\max_{0\leq s\leq t}(e^{\rho s}H_s)}$ and accordingly

$$\lambda e^{\rho t} H_t X_t^* = \lambda e^{\rho t} H_t \frac{\xi^{-\gamma}/\lambda}{\max_{0\leq s\leq t}(e^{\rho s}H_s)} = \xi^{-\gamma} \frac{e^{\rho t}H_t}{\max_{0\leq s\leq t}(e^{\rho s}H_s)} \leq \xi^{-\gamma}$$

Hence $\lambda e^{\rho t} H_t X_t^* \leq \xi^{-\gamma}$ as asserted.

[19] This is a consequence of the Skorohod Equation. For a reference on the Skorohod equation see e.g. Karatzas and Shreve (1991), pp. 210-211.

This is a somewhat surprising result. It asserts that by simply promising the agent a constant benefit forever, one can attain the upper bound of Proposition 2. Moreover, there is a simple closed form solution for y_0 depending solely on the parameters.

One can decompose $\frac{y_0}{\xi}$ into two components:

$$\frac{y_0}{\xi} = \underbrace{(r+q)\,K}_{\text{cost with exclusion}} \underbrace{\left(\frac{\frac{1}{\gamma}+\phi-1}{\phi-1}\right)}_{\text{Cost of Moral Hazard}} \tag{33}$$

To understand the first term, it is useful to examine the constant K first. This constant is (up to an adjustment for the probability of death q) the wealth to consumption ratio in the Merton (1971) model. A thought experiment is useful in order to further explore this term: Suppose that the government promises an income stream of $\overline{y}$ to an agent who is *not subject* to the borrowing constraint (17). Assuming that this agent has $W_t = 0$ financial wealth at time t, her total resources are equal to $W_t + \frac{\overline{y}}{r+q} = \frac{\overline{y}}{r+q}$ and hence her consumption will be:

$$c_t = \frac{1}{K}\frac{\overline{y}}{r+q} \tag{34}$$

The term $\frac{\overline{y}}{r+q}$ is the net present value of the promised income stream and $\frac{1}{K}$ is the consumption to wealth ratio in a Merton model. Solving (34) for $\frac{\overline{y}}{c_t}$ gives $\frac{\overline{y}}{c_t} = (r+q)\,K$. This is the first term in equation (33). This term can be interpreted as follows: If the government wanted to keep an agent's consumption at ξ (once $W_t = 0$) by promising her an income stream $\overline{y}$ and then could *unexpectedly and permanently* exclude the agent from financial markets once $W_t = 0$ without making them experience a consumption drop, then the income stream that needs to promised is given by $\overline{y} = (r+q)\,K\xi$. This motivates the term "cost with exclusion".

The second term in (33) is due to the fact that the government cannot undertake such sudden exclusions from financial markets, and instead has to cope with the fact that agents will take more risks due to the guarantee. There are several interesting remarks about the second term. First:

$$\frac{\frac{1}{\gamma}+\phi-1}{\phi-1} > 1$$

since $\phi > 1$. Second, this ratio has intuitive properties: For instance, increased risk aversion (γ) will reduce the amount of y_0 that needs to be promised to the agent in order to make sure that $c_t \geq \xi$. This is intuitive: The larger is the risk aversion of the agent, the less risks she will take in the stock market, and the cheaper it is to insure her against adverse consumption variation. In the limit, as $\gamma \to \infty$, the second term converges to 1, and hence the effect of moral hazard vanishes. It can also be shown that the second term increases as the Sharpe ratio in the market increases. This is equally intuitive: A higher Sharpe ratio will incentivize the agent to invest in the stock market and it will become more expensive to insure a minimum consumption level.

Figure 1 gives a quantitative assessment of the components that enter the ratio of guaranteed income to minimum consumption (y_0/ξ). The figure shows that if the government wants to ensure a minimum consumption of one dollar, it needs to deliver more than one dollar in guaranteed income. What drives this ratio above one is mostly the cost of moral hazard, as can be seen by the decomposition of y_0/ξ into the two components given in (33).

Summarizing, the fact that a constant income policy can attain the upper bound of Proposition 2 is both reassuring and surprising: The constant income policy has very low informational requirements. The government can implement this policy without even knowing the realization of the brownian paths, or the exact magnitude of the agents' assets at time 0. Hence, even though the model setup gives the government the ability to observe the stochastic discount factor and the agent's assets at time 0, this simple "constant income" policy is optimal despite the fact that it doesn't exploit this information. From a practical perspective, the policy has the additional advantage that it is very simple.

The constant income policy is not the unique optimal policy however. The next section presents an alternative approach to achieving the upper bound in (29).

6.2 Portfolio Insurance

Providing agents with a constant income is not the unique optimal way to attain the upper bound in Proposition 2. The approach presented in this section also succeeds in attaining the same upper bound. To describe this approach, let λ^* be the scalar that minimizes (29). Then define the government's transfer process as:

$$dG_t = -\left(\frac{1}{\gamma} + \phi - 1\right) K\xi \frac{dX_t^*}{X_t^*} \tag{35}$$

where $X_t^*(\lambda^*)$ is the process defined in (27).

This section shows the following results:

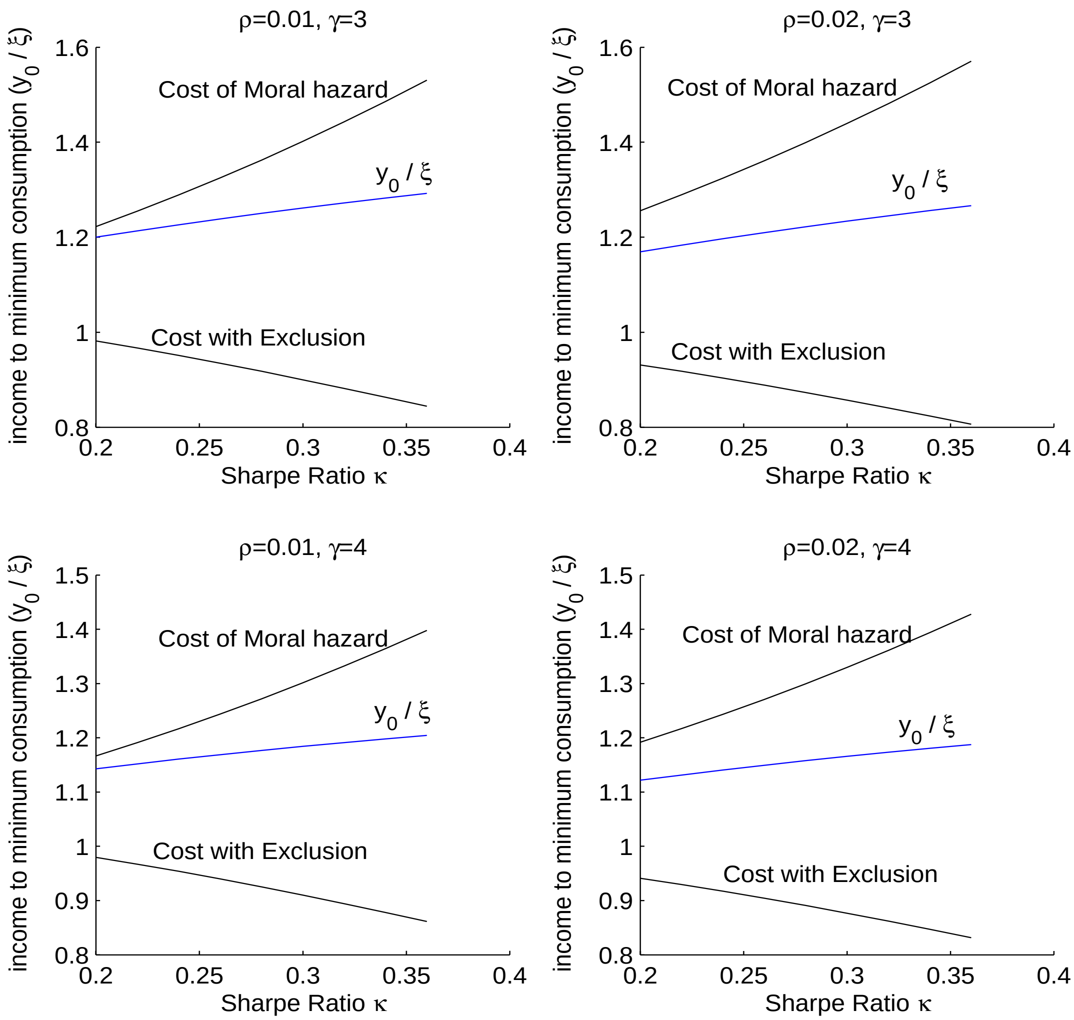

FIGURE 1: The ratio of guaranteed income to the minimum level of guaranteed consumption $\left(\frac{y_0}{\xi}\right)$ and its components. The product of the lines "Cost of Moral hazard" and "Cost with Exclusion" are equal to $\frac{y_0}{\xi}$. The Sharpe ratio used in the calibrations is reported on the x-axis. The preference parameters are given at the top of each figure. The probability of death q is fixed at 0.04 in all figures and the interest rate is equal to 0.02.

a) The process (35) attains the upper bound of Proposition 2.

b) The process (35) has a very intuitive economic interpretation. In particular, the process (35) represents a type of minimum return guarantee (portfolio insurance) on the agent's optimal portfolio of stocks and bonds.

The following proposition formalizes the first claim:

Proposition 4 *Let λ^* be the scalar that minimizes (29) and X_t^* be the process that is given in (27). Consider an agent who anticipates transfers given by (35) and is faced with an initial tax of D_0, where D_0 satisfies (13). Then*

a) her value function will coincide with the upper bound given in (29)

b) That agent will invest

$$\pi_t = \frac{\kappa}{\sigma} K\xi \left[(\phi - 1) \left(\frac{Z_t}{\xi^{-\gamma}} \right)^{\phi-1} + \frac{1}{\gamma} \left(\frac{Z_t}{\xi^{-\gamma}} \right)^{-\frac{1}{\gamma}} \right] \tag{36}$$

dollars *in the stock market and consume*

$$c_t = Z_t^{-\frac{1}{\gamma}} \tag{37}$$

where:

$$Z_t = \lambda^* e^{\rho s} H_s X_s^* \tag{38}$$

The agent's optimal wealth process W_t will be given by:

$$W_t = -K \left(\xi^{-\gamma} \right)^{-\frac{1}{\gamma}} \left(\frac{Z_t}{\xi^{-\gamma}} \right)^{\phi-1} + K Z_t^{-\frac{1}{\gamma}} \tag{39}$$

c) The initial tax D_0 associated with (35) is given by:

$$D_0 = K\xi \frac{\frac{1}{\gamma} + \phi - 1}{\phi - 1} \left(\frac{\lambda^*}{\xi^{-\gamma}} \right)^{\phi-1}$$

The portfolio policy (36) will aid in the interpretation of (35) as a form of portfolio insurance. To obtain some intuition on the nature of (35), consider first the following puzzling feature of (36): As $Z_t \to \xi^{-\gamma}$, equation (39) implies that $W_t \to 0$ whereas the portfolio of the agent becomes:

$$\lim_{Z_t \to \xi^{-\gamma}} \pi_t = \left(\frac{1}{\gamma} + \phi - 1 \right) K\xi \frac{\kappa}{\sigma} > 0 \tag{40}$$

This result may seem surprising at first. How is it possible that the agent's dollar holdings of stock do not go to zero as $W_t \to 0$? To understand the nature of the puzzle, note first that if the agent holds a positive amount of stocks $\pi_t > 0$ when her financial wealth is $W_t = 0$ that means that she must also be holding $W_t - \pi_t = -\pi_t < 0$ in bonds. But then if the stock market experiences a realization of a return that is less than $-r$ over the next interval dt, wouldn't that make the financial wealth of the investor negative?

The resolution of the puzzle is in the nature of the transfers that the agent receives once $W_t = 0$. First, from the definition of Z_t and X_t^* in equations (38) and (27) it follows that X_t^* decreases when and only when $Z_t = \xi$. Accordingly G_t increases when and only when $Z_t = \xi^{-\gamma}$, that is when[20] $c_t = \xi$ and $W_t = 0$. Simply put, the agent starts receiving transfers from the government when (and only when) her wealth becomes 0 and the stock market experiences further negative returns[21]. Because of these transfers, the agent becomes hedged against negative returns when her wealth is equal to zero and hence can afford to hold stock. This motivates the name "portfolio insurance" for the transfer process (35).

An alternative way of thinking about the transfer process G_t in (35) is that the government makes a recommendation to the agent on how she should invest and consume. That recommendation is given by (37) and (36). Based on this recommendation and its observation of stock market realizations, the government can compute the agent's wealth and make "just enough" transfers to the agent when needed, so as to keep her inferred wealth above 0. Proposition 4 asserts that given this transfer structure, the consumer will indeed find it optimal to follow the government's "recommendation".

6.3 Comparing the two policies

Given that both policies attain the upper bound of equation (29), this means that they imply the same value function for the agent, and hence are equivalent from a welfare perspective[22].

However, the two policies do differ. They make transfers of different magnitudes in different states of the world. The initial taxes that they imply are also different. Indeed, the cost of the

[20] This follows from (37) and (38).

[21] Note that the state price density H_t and the stock market P_t are perfectly negatively correlated.

[22] The derivations in the appendix also show that they imply exactly the same consumption process "path by path".

constant income policy is:

$$D_0^{const.} = \frac{y_0}{r+q} = K\xi \left(\frac{\frac{1}{\gamma} + \phi - 1}{\phi - 1} \right) \tag{41}$$

whereas by proposition 4, the cost of the portfolio insurance policy is:

$$D_0^{p.i.} = K\xi \frac{\frac{1}{\gamma} + \phi - 1}{\phi - 1} \left(\frac{\lambda^*}{\xi^{-\gamma}} \right)^{\phi - 1} \tag{42}$$

All the terms in $D_0^{p.i.}$ are explicit, except for λ^* which is determined implicitly by solving the minimization problem of (29). Equation (19) however implies that:

$$c_0^{-\gamma} = \lambda^* \leq \xi^{-\gamma}$$

since $H_0 = X_0^* = 1$. It can also be shown by applying the implicit function theorem to (29) that λ^* is a declining function of W_0. Dividing (42) by (41) gives:

$$\frac{D_0^{p.i.}}{D_0^{const.}} = \left(\frac{\lambda^*}{\xi^{-\gamma}} \right)^{\phi - 1} = \left(\frac{c_0}{\xi} \right)^{-\gamma(\phi - 1)} \leq 1 \tag{43}$$

since $c_0 \geq \xi$ and $\phi > 1$. Hence the "portfolio insurance" policy has a cost that cannot be larger than the cost of the "constant income" policy.

This may seem puzzling, since the two policies imply the same value function, while keeping $c_t \geq \xi$. To resolve the puzzle, note first that the cost of the two policies coincides when $c_0 = \xi$. Furthermore, one can also show that under both policies $c_0 = \xi$ if and only if $W_0 = D_0^{const.}$, so that post tax wealth is equal to $W_{0+} = 0$. Hence the cost of the two policies differs only when the borrowing constraint is *not* binding, but is identical when the borrowing constraint *does* bind.

This observation is helpful, because it hints to the reason why the two policies have different costs but are equivalent from a welfare perspective: The constant income policy delivers the same transfers in all states of the world, including states of the world where the borrowing constraint doesn't bind. By contrast, the "portfolio insurance" policy delivers payments only when the borrowing constraint binds. This gives intuition on why the latter policy is associated with a lower initial tax in general. However, the very reason why the constant income policy "costs" more is because it delivers more payments (in a net present value sense) in states of the world where the borrowing constraint is not binding. One would intuitively expect that Ricardian Equivalence applies in these states. The agent can "undo" the effects of the increased transfers by borrowing and taking more risk in the stock market. Therefore, as the two policies only differ in states where transfers are subject to the Ricardian Equivalence theorem, they are equivalent from a welfare perspective.

The above discussion illustrates a more fundamental point about the evaluation of government guarantees. Determining the cost of government guarantees, as is routinely done in the literature, can be misleading from a welfare perspective. This section illustrated how two policies can both be optimal and imply the same welfare, whereas a net present value of the transfers that they imply could be different.

7 Minimum level of assets and implications for pre-retirement savings

7.1 Minimum assets

The previous sections illustrated several equivalent ways of giving transfers to retirees that will safeguard a consumption process above the level ξ. A maintained assumption was (8). I will now illustrate that this assumption is not only sufficient, but it is also necessary for the existence of transfer processes that will induce a consumption process that satisfies $c_t \geq \xi$.

Proposition 5 *An admissible transfer process G_t that will induce $c_t \geq \xi$ will exist if and only if condition (8) holds.*

The practical implication of this proposition is that it gives an exact lower bound on the assets that need to be available upon retirement in order to ensure the feasibility of attaining the goal $c_t \geq \xi$. Hence, in contrast to existing literature that typically takes this lower bound as exogenously given, the present analysis goes a step further and provides a link between the level of minimal consumption that can be guaranteed and the amount of assets that need to have been accumulated.

Letting $W^{\min}$ be the minimum amount of assets implied by (8), one can decompose $W^{\min}$ in a manner similar to section 6.1 in two parts:

$$\frac{W^{\min}}{\xi} = \underbrace{K}_{\text{Merton's wealth to consumption ratio}} \underbrace{\frac{\frac{1}{\gamma} + \phi - 1}{\phi - 1}}_{\text{Cost due to moral hazard}}$$

Once again, the minimum wealth $\left(W^{\min}\right)$ to minimum consumption (ξ) ratio is not just equal to the wealth to consumption ratio in the Merton model (K) but is increased by the factor $(1/\gamma + \phi - 1)\,/\,(\phi - 1) > 1$, since agents' portfolio and savings decisions will be distorted by any admissible transfer process G_t that will induce $c_t \geq \xi$.

Figure 2 plots the left hand side of the above equation and the associated value of K for various combinations of the parameters. For reasonable parameter choices the resulting numbers are close to 20. For each dollar of minimum guaranteed consumption, the government needs to be able to raise an initial tax close to 20 dollars.

The above discussion has some clear implications for pre-retirement savings. Namely, the government needs to make sure that the agent arrives at retirement with an amount of savings equal to $W^{\min}$. Assuming that the only policy instrument that the government has at its disposal prior to retirement is a proportional tax χ on Y, then a feasible way to ensure that condition (8) is satisfied upon entering retirement is to set χ equal to

$$\chi = \frac{(r+q)\, e^{-(r+q)T}}{1 - e^{-(r+q)T}} \frac{K\xi}{Y} \left(\frac{\frac{1}{\gamma} + \phi - 1}{\phi - 1} \right) \tag{44}$$

and then place the proceeds into a riskless account. I will assume that the parameters are such that $\chi < 1$. A simple computation yields then:

$$\int_0^T \chi Y e^{-(r+q)t} dt = e^{-(r+q)T} W^{\min}$$

In words, the net present value of the tax proceeds are exactly equal to the net present value of $W^{\min}$. Given the non-negativity constraint on the agent's personal financial wealth, such a policy will imply that the agent's total assets once she enters retirement can be no less than the amount that is accumulated in the riskless account, namely $W^{\min}$. Hence condition (8) will be automatically verified. To gain a quantitative sense, Figure 3 illustrates the resulting tax rates for various levels of $\frac{\xi}{Y}$, Sharpe ratios and preference parameters.

7.2 The pre-retirement problem of the government

Sections 3-6 considered only the post-retirement value function of the agent. The analysis established the upper bound (29) to the value function of the agent and showed that it can be attained, as long as condition (8) is satisfied. The previous subsection also established additionally that

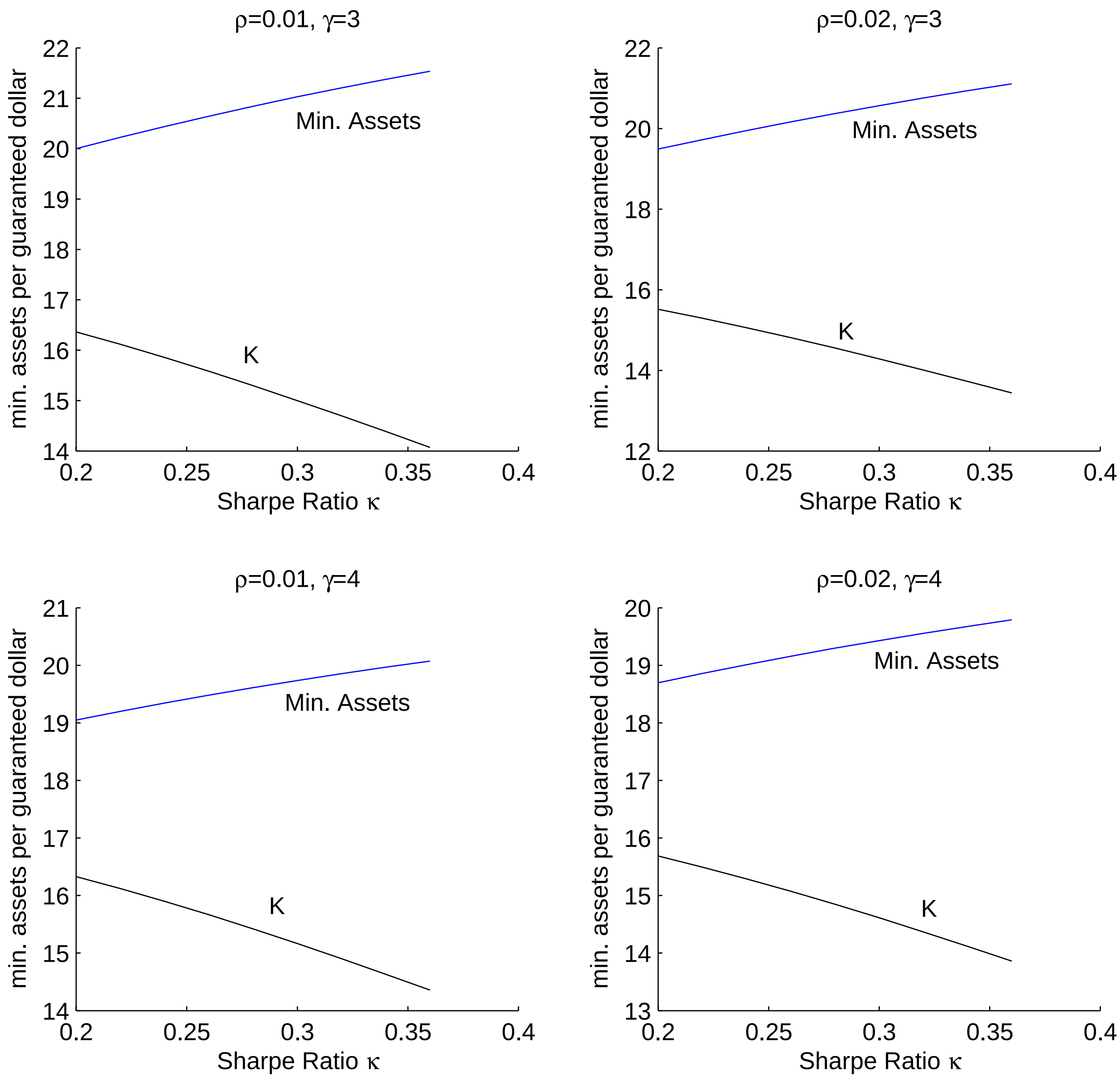

FIGURE 2: The top line in each figure depicts the minimum assets per guaranteed dollar of consumption $\left(\frac{W_0^{\min}}{\xi}\right)$. The bottom line depicts the wealth to consumption ratio in the Merton model (K). The preference parameters are given at the top of each figure. The probability of death q is fixed at 0.04 in all figures and the interest rate is equal to 0.02.

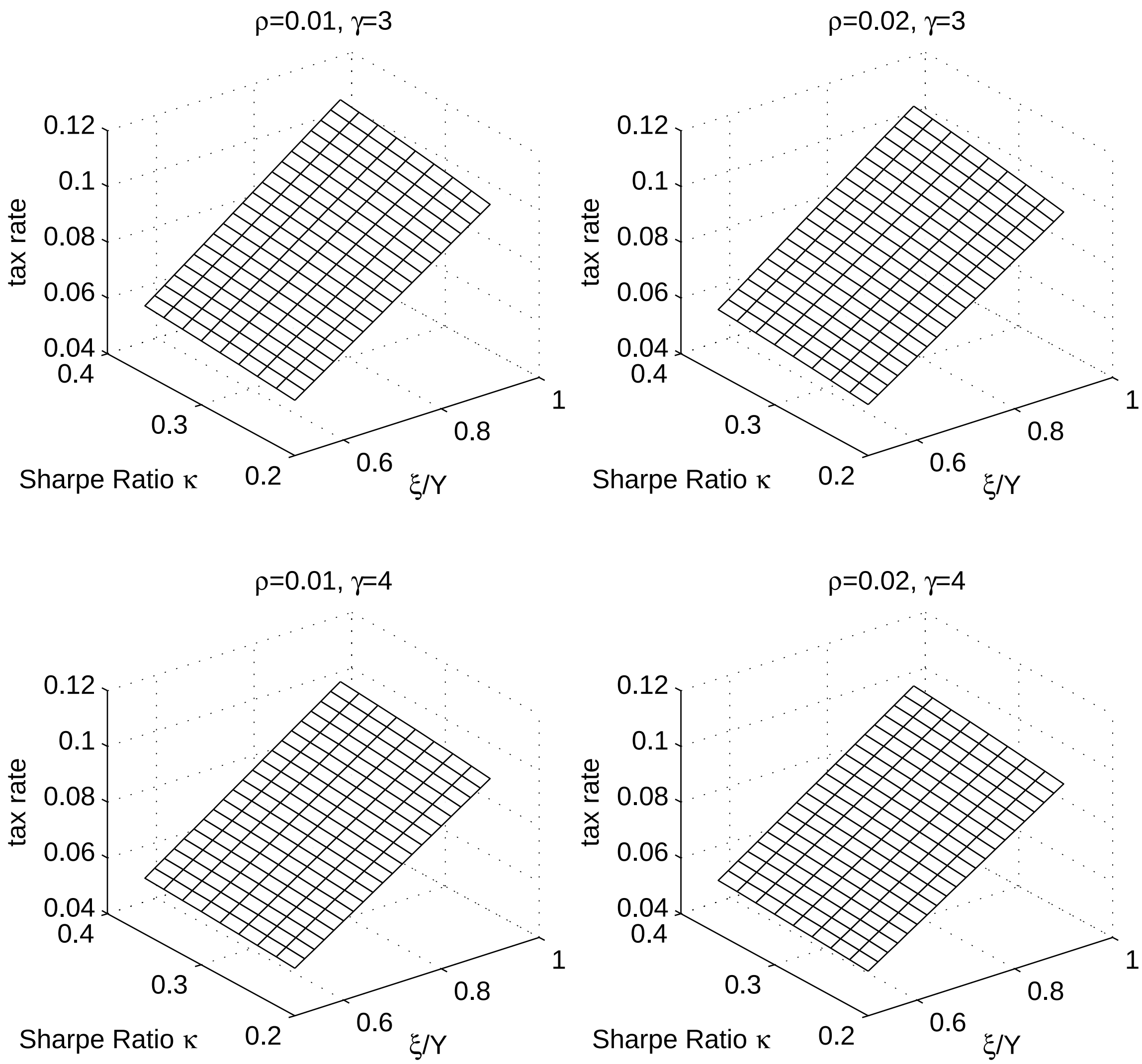

FIGURE 3: The top line in each figure depicts the minimum assets per guaranteed dollar of consumption $\left(\frac{W_0^{\min}}{\xi}\right)$. The bottom line depicts the wealth to consumption ratio in the Merton model (K). The preference parameters are given at the top of each figure. The probability of death q is 0.04. In all figures the interest rate is equal to 0.02.

a feasible policy will *only exist* as long as (8) is satisfied. Assuming that the government's only policy instrument prior to retirement is a constant tax rate on Y, then the unique way of achieving condition (8) is by raising a tax rate that is at least equal to χ in equation (44).

An intuitive argument also shows that it is not optimal to raise the pre-retirement tax rate above that level. Given that the agent is faced with borrowing constraints prior to retirement, any government intervention that taxes income today and returns it in the form of a lump sum payment upon entering retirement will reduce the agent's ability to smooth consumption. The following proposition states this formally:

Proposition 6 *The optimal pre-retirement tax rate that will ensure (8) is given by (44).*

This proposition shows that the optimal policy of the government (pre-retirement) is to set the tax rate on income as low as possible, but subject to the constraint that the agent has accumulated assets equal to $W^{\min}$ at retirement.

To summarize, the optimal government policy over the agent's life cycle is to set taxes equal to (44) prior to the agent's retirement, place the proceeds in riskless assets, and then use the compounded amount to finance either of the two types of guarantees described in section 6.

This describes the optimal government policy, assuming that it is optimal for the government to intervene in the first place. To determine if government intervention is optimal one needs to compare the lifetime expected utility of a new-born agent in the presence of government intervention and in its absence from the perspective of the central planner (equation [5]). Since the distance between the maximized value of F_{T_1} in the presence of government intervention and in its absence is bounded, there always exist a sufficiently high value of b that will justify intervention. Simply put, since the distortions introduced by mandatory savings and transfers will have a bounded effect on the agent's objective F_{T_1}, there will always exist a sufficiently high level of (social) costs associated with violating the constraint $c_t \geq \xi$ in retirement that will justify intervention. Even though the exact nature of these social costs do not form the focus of the analysis, the next section explores in detail possible sources of such costs.

8 Discussion

8.1 Alternative way to justify guarantees

Why have so many countries that switched their retirement systems opted for some form of retirement benefit guarantees? Even though the answer to this question does not affect the nature of the optimal retirement benefit guarantees that were derived previously, it is still interesting to have some simple models to justify the wedge between the central planner's objective and the agent's objective.

8.1.1 Behavioral reasons

Perhaps the simplest justification behind such a wedge is behavioral. If one were to interpret ξ as inelastic expenditures associated with aging (say medical costs) then it is plausible to believe that a newly born agent is endowed with preferences given by (5). (If ξ is literally a subsistence level in retirement, then $b = \infty$). If, however, all future "selves"[23] of the agent ignore the necessity to make consumption choices that will provide her with the appropriate subsistence level ξ in retirement they will maximize (1). Assuming that the agent cannot restrain the choices of her future selves, then the central planner will be led to solve problem 1.

8.1.2 Rational reasons

The behavioral approach is only one of the ways to justify retirement benefit guarantees. Here, I give a rational motivation, which is based on the idea that retirees can force distortionary transfers from the young agents to the retired agents.

To capture this idea, I will assume that each cohort of agents that are born at time T_1 and retire at time $T_1 + T$ have simple time consistent CRRA preferences which are given by (1). However, upon entering retirement the entire cohort of agents can costelessly join a "union". The union operates a linear "transfer" technology: Its members that decide to exert a (flow) effort n protesting, can receive a flow of n in transfers from the young[24]. Exerting however an effort of n creates a disutility of $\xi^{-\gamma} n$. Mathematically, this implies that the utility of a representative retired agent can

[23]For economic applications of the concept of "multiple selves", see e.g. Harris and Laibson (2001),Amador, Werning, and Angeletos (2006).

[24]This implicitly assumes that the union members who did not join in the protest do not receive transfers. This assumption is made in order to exclude free rider problems.

be expressed as:

$$\max_{c_t,\pi_t,n_{t,T_1+T}} E_{T_1+T}\int_{T_1+T}^{\infty} e^{-(\rho+q)(s-T_1-T)}\frac{c_s^{1-\gamma}}{1-\gamma}ds - E_{T_1+T}\int_{T_1+T}^{\infty} e^{-(\rho+q)(s-T_1-T)}\xi^{-\gamma}n_{s,T_1+T}ds \quad (45)$$

where the notation $n_{s,T_{1+T}}$ denotes the effort at time s of an agent who retired at time T_1+T. The dynamic budget constraint after time T_1+T now becomes:

$$dW_t = qW_t dt + \pi_t\{\mu dt + \sigma dB_t\} + \{W_t - \pi_t\} r dt + dt - c_t dt + n_{t,T_1+T}dt$$

for a retired agent. Naturally, the transfers to the retired agents need to be subtracted from the budget constraint of the young agents. Letting Φ_t denote the total transfers to the old one obtains:

$$\Phi_t = qe^{-qT}\int_0^{\infty} e^{-qs}n_{t,t-s}ds$$

which means that the representative non-retired agent pays needs to be taxed by $\frac{\Phi_t}{1-e^{-qT}}$. A simple way to capture the idea that such taxation is likely to be distortionary, is to assume that for each dollar raised in taxes, x dollars get "destroyed" in the process, so that the representative young agent's budget is given by:

$$\begin{aligned} dW_t &= qW_t dt + \pi_t\{\mu dt + \sigma dB_t\} + \{W_t - \pi_t\} r dt + dt - c_t dt + \\ &\quad + \left[Y - (1+x)\frac{\Phi_t}{1-e^{-qT}}\right] dt \end{aligned}$$

Finally, I will make the assumption that $n_{t,s}$ can only be chosen to be between 0 and $\overline{n}$, where $\overline{n}$ is a constant that satisfies

$$\overline{n} < \frac{\left(1-e^{-qT}\right)Y}{(1+x)e^{-qT}} \quad (46)$$

This condition will safeguard that even if all the vintages of retirees decided to protest at the same time, the representative young agent would still be able to pay the associated taxes.

With this setup in hand, it is now possible to investigate the incentives of retired agents to protest. Given the linearity of effort disutility in equation (45) leads to the following simple cutoff rule for the optimal n_{s,T_1+T} :

$$n_{t,T_1+T} = \begin{cases} 0 \text{ if } c_t \geq \xi \\ \overline{n} \text{ if } c_t < \xi \end{cases} \quad (47)$$

The proof of this fact is straightforward and hence I give only a brief sketch. Writing down the Hamilton Jacobi Bellman equation for the value function J of a retired agent and keeping only terms that involve n_{t,T_1+T} shows that n_{t,T_1+T} is determined as the solution to the following maximization problem:

$$\max_{0 \leq n_{t,T_1+T} \leq \overline{n}} n_{t,T_1+T} \left(J_W - \xi^{-\gamma} \right)$$

where J_W is the derivative of the value function with respect to wealth. The solution to this linear problem is trivial. If $J_W \geq \xi^{-\gamma}$ then the agent will choose $n_{t,T_1+T} = \overline{n}$, else $n_{t,T_1+T} = 0$. Combining this observation with the envelope condition[25] $c^{-\gamma} = J_W$ leads to (47).

This shows that a central planner who would like to prevent retired agents from protesting should ensure that their consumption stays above ξ in retirement, so that they have no incentive to protest. But this is precisely the requirement in (6).

With this observation it is possible to show that if the central planner's objective is to maximize the expected utility of a newly born agent in steady state, and if the distortion x associated with taxes is large enough, then the central planner would find it optimal to provide all newly born agents with a pattern of taxes and transfers over their lifetime that will ensure that their consumption is above ξ in retirement. Searching for the optimal such transfer process is precisely the goal of problem 1.

Summarizing, if bailouts introduce distortions (x) that are larger than the distortions associated with a prefunded optimal guarantee, then the central planner will find it optimal to induce agents to choose a consumption process in retirement that will prevent such bailouts. Choosing the optimal such guarantee leads to problem 1.

8.2 Arbitrary stochastic discount factors and multiple assets and sources of uncertainty

The exogeneity of the stochastic discount factor and the assumption of a single source of risk and a single asset are not as restrictive as they may seem at first. Even if the stochastic discount factor was driven by multiple sources of uncertainty and the risk premia and interest rates were time

[25] See e.g. Merton (1971).

varying, most of the results of the paper would survive. The only indispensable assumption is that markets be dynamically complete from the perspective of a new-born agent in the absence of the borrowing constraint and the stochastic discount factor be a continuous function of time.

A close examination of the proof that (28) provides an upper bound to problem 1 reveals that none of the steps depend on the functional form of H_t. The proof goes through for any continuous stochastic discount factor H_t.

It is also possible to show that there always exist variants of the "portfolio insurance" policy that will attain the upper bound of proposition 2 for any stochastic discount factor. The result that seems however to not be true in general is that the "constant income" policy also attains the upper bound of proposition 2.

These observations imply that many of the results of the paper would survive, even if one closed the model in general equilibrium[26]. In that case, the prices of the guarantees and all the parametric formulas would be altered. However, qualitatively the characterization of the upper bound and the existence of at least a policy that attains it, would remain unchanged.

9 Conclusion

This paper presented an optimality theory on how to design a retirement system with the ability to guarantee a minimum standard of living to retirees.

The key results of the paper can be summarized as follows:

First, there can be multiple optimal solutions to ensuring a minimum standard of living in retirement. Two such solutions are a constant income policy and a portfolio insurance policy.

Second, the cost of a policy can be a misleading indicator of its implications for welfare. The two solutions that were discussed in this paper have identical implications for welfare, yet their costs are different in general.

Third, the paper showed that the presence of moral hazard will tend to increase the cost of guarantees. The simple policy that delivers a constant income in retirement illustrates this best:

[26] Of course in general equilibrium care should be taken to make sure that it is feasible to keep retiree consumption above any lower bound. If the aggregate endowment followed a lognormal process, this would only be possible if one reformulated the constraint $c_t \geq \xi$ so as to make ξ proportional to aggregate consumption.

In order to safeguard that an agent's consumption will not fall below a minimum amount (say a dollar), more than one dollar needs to be given in retirement income.

Fourth, the model derives explicitly a minimum amount of assets that need to be available in retirement so as to safeguard that consumption will not drop below a minimum level. Calibration exercises indicate that this minimum level of assets is about 20 times the guaranteed consumption level.

Several issues are unexplored by the present paper. A first question concerns unobserved preference heterogeneity. If agents have different risk aversions, or discount factors, then the government needs to offer menus of contracts in the spirit of discriminatory pricing. It appears straightforward to extend the analysis to allow for this possibility. A particularly interesting question that would emerge in such a setting is whether the need to enforce sorting into different types of contracts would affect the optimal security design or not.

A second question concerns the implications of such guarantees for asset prices. Even though the results of the paper go through for arbitrary stochastic discount factors, it is certain that extensive coverage of retirees by these guarantees would affect the stochastic discount factor in general equilibrium. Studying these two questions is left for future research.

A Appendix

A.1 Proof of proposition 1

Proof. Subject to minor modifications, the proof of this proposition is identical to the first theorem of He and Pages (1993) and is therefore omitted. However, it is possible to give a sketch of the main argument behind Proposition 1.

The consumer needs to choose her optimal consumption/portfolio path taking the process for transfers and the initial tax as given, while satisfying the constraint $W_t \geq 0$. The first step is to note that *for any* consumption / portfolio policy that satisfies (15)-(17), one can apply Ito's Lemma to $e^{-qt}H_tW_t$ and then integrate to obtain:

$$\int_0^t e^{-qs}H_sc_sds + e^{-qt}H_tW_t = W_0 - D_0 + \int_0^t e^{-qs}H_sdG_s + \int_0^t \psi_s dB_s$$

for an appropriate process[27] ψ_s that will depend on the agent's portfolio choice. Intuitively, this is just the dynamic budget constraint "integrated forward". Since $W_t \geq 0$, the above equation implies that:

$$\int_0^t e^{-qs}H_sc_sds - \left(W_0 - D_0 + \int_0^t e^{-qs}H_sdG_s + \int_0^t \psi_s dB_s\right) \leq 0 \text{ for all } t \geq 0 \tag{48}$$

For any non-increasing and positive process X_t (starting at $X_0 = 1$) and any positive constant λ, equation (48) implies:

$$\lambda \int_0^\infty \left[\int_0^t e^{-qs}H_sc_sds - \left(W_0 - D_0 + \int_0^t e^{-qs}H_sdG_s + \int_0^t \psi_s dB_s\right)\right] dX_t \geq 0$$

since dX_t is (weakly) decreasing.

Hence, for any consumption policy that satisfies the (15)-(17) it follows that:

$$\begin{aligned} E_0 \int_0^\infty e^{-(\rho+q)s}\frac{c_s^{1-\gamma}}{1-\gamma}ds \quad &\leq \quad E_0 \int_0^\infty e^{-(\rho+q)s}\frac{c_s^{1-\gamma}}{1-\gamma}ds \\ &+\lambda E_0 \int_0^\infty dX_t \int_0^t e^{-qs}H_sc_sds \\ &-\lambda E_0 \int_0^\infty dX_t \left(W_0 - D_0 + \int_0^t e^{-qs}H_sdG_s + \int_0^t \psi_s dB_s\right) \end{aligned} \tag{49}$$

Equation (49) suggests a natural interpretation for X_t as a process of (cumulative) Lagrange multipliers associated with the requirement (48), which follows from[28] $W_t \geq 0$. For short, from now I will refer to X_t as the process of Lagrange multipliers.

[27] This is an implication of the martingale representation theorem. See e.g. He and Pages (1993) for details.

[28] As one might expect, the inequality in (49) can only become an equality if (20) holds, i.e. if X_t decreases only when $W_t = 0$ and remains otherwise constant.

Applying integration by parts to the second and third line of equation (49) and noting that $X(0)=1$, $\lim_{t\to\infty} X(t) \geq 0$ and using (48) gives:

$$
\begin{aligned}
&E_0 \int_0^\infty e^{-(\rho+q)s} \frac{c_s^{1-\gamma}}{1-\gamma} ds \leq \\
&\leq \;\; E_0 \left(\int_0^\infty e^{-(\rho+q)s} \frac{c_s^{1-\gamma}}{1-\gamma} ds - \lambda \int_0^\infty e^{-qs} X_s H_s c_s ds + \lambda \int_0^\infty e^{-qs} H_s X_s dG_s \right) \\
&\quad\;\; + \lambda \left(W_0 - D_0 \right) \\
&\leq \;\; E_0 \left(\int_0^\infty e^{-(\rho+q)s} \max_c \left(\frac{c_s^{1-\gamma}}{1-\gamma} - \lambda e^{\rho s} H_s X_s c_s \right) ds + \lambda \int_0^\infty e^{-qs} H_s X_s dG_s \right) \\
&\quad\;\; + \lambda \left(W_0 - D_0 \right)
\end{aligned} \tag{50}
$$

Since this inequality holds for any decreasing, positive progressively measurable process X_t and any positive constant λ, it must also hold for the process X_t that minimizes the right hand side of (50). By a similar argument, since the above inequality holds for any choice of c_t, π_t that satisfies (15)-(17) it must also hold for the optimal c_t, π_t. Hence:

$$
\begin{aligned}
&V(W_0) = \max_{c_s, \pi_s} E_0 \int_0^\infty e^{-(\rho+q)s} \frac{c_s^{1-\gamma}}{1-\gamma} ds \\
&\leq \min_{\lambda>0,\ X_s \in \mathcal{D}} \left[\begin{array}{c} E \int_0^\infty e^{-(\rho+q)s} \max_{c_s} \left(\frac{c_s^{1-\gamma}}{1-\gamma} - \lambda e^{\rho s} H_s X_s c_s \right) ds \\ + \lambda E \int_0^\infty e^{-qs} H_s X_s dG_s \\ + \lambda \left(W_0 - D_0 \right) \end{array} \right]
\end{aligned}
$$

These arguments establish that the right hand side of (18) provides an upper bound to the value function of the consumer. Establishing that it also provides a lower bound is achieved by showing that there exists a consumption / portfolio policy that satisfies (15)-(17) and attains this upper bound. This part of the proof is contained in He and Pages (1993) and the reader is referred to that paper for details. ■

A.2 Proof of Proposition 2

The proof of Proposition 2 is established in steps. The following Lemma will prove useful in establishing the last part of the proposition.

Lemma 1 *Take any $\lambda \in (0, \xi^{-\gamma}]$ and any process G_t and define*

$$
\widehat{X}_t = \arg \min_{X_t \in \mathcal{D}} E \left(\int_0^\infty e^{-(\rho+q)s} \max_{c_s} \left(\frac{c_s^{1-\gamma}}{1-\gamma} - \lambda e^{\rho s} H_s X_s c_s \right) ds + \lambda \int_0^\infty e^{-qs} H_s \left(X_s - 1 \right) dG_s \right) \tag{51}
$$

Then:

$$
\lambda E \left(\int_0^\infty e^{-qs} H_s \left(\widehat{X}_s - 1 \right) dG_s \right) = E \int_0^\infty e^{-(\rho+q)s} \left(e^{\rho s} \lambda H_s \widehat{X}_s \right)^{1-\frac{1}{\gamma}} \left(1 - \frac{1}{\widehat{X}_s} \right) ds \tag{52}
$$

Proof of Lemma 1. Let:

$$\Lambda_t \equiv 1 - \frac{1}{\widehat{X}_t} \tag{53}$$

Applying Ito's Lemma to Λ_t we obtain:

$$d\Lambda_t \equiv \frac{d\widehat{X}_t}{\left(\widehat{X}_t\right)^2} \tag{54}$$

Hence Λ_t changes when and only $\widehat{X}_t$ changes. By Theorem 1 of He and Pages (1993):

$$\int_0^\infty \left[E_t\left(\int_t^\infty \widehat{X}_s e^{-qs} H_s dG_s\right) - E_t\left(\int_t^\infty \widehat{X}_s e^{-qs} H_s c_s ds\right)\right] d\widehat{X}_t = 0 \tag{55}$$

where c_s is given explicitly by:

$$c_s = \left(e^{\rho s}\lambda H_s X_s\right)^{-\frac{1}{\gamma}} \tag{56}$$

Plugging (56) into (55) and then using (54) and observing that Λ_t changes when and only $\widehat{X}_t$ changes implies that:

$$\int_0^\infty \left(E_t \int_t^\infty \widehat{X}_s e^{-qs} H_s dG_s - E_t \int_t^\infty \widehat{X}_s e^{-qs} H_s \left(e^{\rho s}\lambda H_s \widehat{X}_s\right)^{-\frac{1}{\gamma}} ds\right) d\Lambda_t = 0$$

Then, for any admissible G_t and $\widehat{X}_t$ given by (51):

$$\begin{aligned} &\lambda E\left(\int_0^\infty e^{-qs} H_s \left(\widehat{X}_s - 1\right) dG_s\right) = \\ &\quad \lambda E\left[\int_0^\infty e^{-qs} H_s \left(\widehat{X}_s - 1\right) dG_s - \int_0^\infty \left(E_t \int_t^\infty \widehat{X}_s e^{-qs} H_s dG_s\right) d\Lambda_t\right] \\ &\quad + \lambda E\left\{\int_0^\infty E_t\left[\int_t^\infty \widehat{X}_s e^{-qs} H_s \left(e^{\rho s}\lambda H_s \widehat{X}_s\right)^{-\frac{1}{\gamma}} ds\right] d\Lambda_t\right\} \end{aligned} \tag{57}$$

Next consider the martingale:

$$\mathcal{M}_t = E_t \int_0^\infty \widehat{X}_s e^{-qs} H_s dG_s = \int_0^t \widehat{X}_s e^{-qs} H_s dG_s + E_t \int_t^\infty \widehat{X}_s e^{-qs} H_s dG_s \tag{58}$$

According to the martingale representation theorem, there exists a square integrable $\widetilde{\psi}_s$ such that:

$$\mathcal{M}_t = \mathcal{M}_0 + \int_0^t \widetilde{\psi}_s dB_s \tag{59}$$

Combining (58) and (59) gives:

$$\begin{aligned} d\left(E_t \int_t^\infty \widehat{X}_s e^{-qs} H_s dG_s\right) &= d\mathcal{M}_t - \widehat{X}_t e^{-qt} H_t dG_t \\ &= \widetilde{\psi}_t dB_t - \widehat{X}_t e^{-qt} H_t dG_t \end{aligned}$$

Now, fixing an arbitrary $\varepsilon > 0$, letting τ^ε be the first time t such that $|\Lambda_t| \geq \frac{1}{\varepsilon}$, applying integration by parts and using the fact that $\Lambda_0 = 0$, gives:

$$\begin{aligned} -E\int_0^{T\wedge\tau^\varepsilon}\left(E_t\int_t^\infty \widehat{X}_s e^{-qs}H_s dG_s\right)d\Lambda_t &= -E\int_0^{T\wedge\tau^\varepsilon}\Lambda_s\widehat{X}_s e^{-qs}H_s dG_s \\ &\quad +E\int_0^{T\wedge\tau^\varepsilon}\Lambda_s\widetilde{\psi}_s dB_s \\ &\quad -E\left[\Lambda_{T\wedge\tau^\varepsilon}\left(E_{T\wedge\tau^\varepsilon}\int_{T\wedge\tau^\varepsilon}^\infty \widehat{X}_s e^{-qs}H_s dG_s\right)\right] \end{aligned}$$

Since ψ_s is square integrable and $|\Lambda_s|$ is bounded in $\left[0, \frac{1}{\varepsilon}\right]$ the second term on the right hand side of the above expression is 0. We also note that:

$$-E\left[\Lambda_{T\wedge\tau^\varepsilon}\left(E_{T\wedge\tau^\varepsilon}\int_{T\wedge\tau^\varepsilon}^\infty \widehat{X}_s e^{-qs}H_s dG_s\right)\right] = -E\left[\widehat{X}_{T\wedge\tau^\varepsilon}\Lambda_{T\wedge\tau^\varepsilon}J\right] \tag{60}$$

where

$$J = \left(E_{T\wedge\tau^\varepsilon}\int_{T\wedge\tau^\varepsilon}^\infty \frac{\widehat{X}_s}{\widehat{X}_{T\wedge\tau^\varepsilon}}e^{-qs}H_s dG_s\right) \leq E_{T\wedge\tau^\varepsilon}\int_{T\wedge\tau^\varepsilon}^\infty e^{-qs}H_s dG_s \tag{61}$$

since $\widehat{X}_t$ is non-increasing. Combining (61) with (60) and noting that $0 < \widehat{X}_t \leq 1$

$$-E\left[\widehat{X}_{T\wedge\tau^\varepsilon}\Lambda_{T\wedge\tau^\varepsilon}J\right] = E\left[\left(1-\widehat{X}_{T\wedge\tau^\varepsilon}\right)J\right] \leq E_{T\wedge\tau^\varepsilon}\int_{T\wedge\tau^\varepsilon}^\infty e^{-qs}H_s dG_s \tag{62}$$

Given that:

$$E\int_0^\infty e^{-qs}H_s dG_s < \infty$$

it follows that:

$$E_{T\wedge\tau^\varepsilon}\int_{T\wedge\tau^\varepsilon}^\infty e^{-qs}H_s dG_s \to 0 \tag{63}$$

as $\varepsilon \to 0, T \to \infty$. This leads to the inequalities:

$$\begin{aligned} -E\int_0^\infty\left(E_t\int_t^\infty \widehat{X}_s e^{-qs}H_s dG_s\right)d\Lambda_t &\geq -E\int_0^{T\wedge\tau^\varepsilon}\left(E_t\int_t^\infty \widehat{X}_s e^{-qs}H_s dG_s\right)d\Lambda_t \\ &\geq -E\int_0^{T\wedge\tau^\varepsilon}\Lambda_s\widehat{X}_s e^{-qs}H_s dG_s \end{aligned}$$

Letting $\varepsilon \to 0, T \to \infty$, using the monotone convergence theorem, and using (62) and (63), gives

$$-\int_0^\infty\left(E_t\int_t^\infty \widehat{X}_s e^{-qs}H_s dG_s\right)d\Lambda_t = -E\int_0^\infty \Lambda_s\widehat{X}_s e^{-qs}H_s dG_s \tag{64}$$

Using (64) and the definition of Λ_t gives:

$$\lambda E\left[\int_0^\infty e^{-qs}H_s\left(\widehat{X}_s - 1\right)dG_s - \int_0^\infty\left(E_t\int_t^\infty \widehat{X}_s e^{-qs}H_s dG_s\right)d\Lambda_t\right] =$$

$$= E\left[\lambda\int_0^\infty e^{-qs}H_s\left(\widehat{X}_s-1\right)dG_s-\lambda\int_0^\infty e^{-qs}H_s\widehat{X}_s\Lambda_s dG_s\right]=0$$

Returning now to (57) and using the above equation yields:

$$\lambda E\left(\int_0^\infty e^{-qs}H_s\left(\widehat{X}_s-1\right)dG_s\right) = \lambda E\left\{\int_0^\infty E_t\left[\int_t^\infty \widehat{X}_s e^{-qs}H_s\left(e^{\rho s}\lambda H_s\widehat{X}_s\right)^{-\frac{1}{\gamma}}ds\right]d\Lambda_t\right\} \tag{65}$$

$$= E\left[\int_0^\infty e^{-(\rho+q)t}\left(e^{\rho t}\lambda H_t\widehat{X}_s\right)^{1-\frac{1}{\gamma}}\Lambda_t dt\right] \tag{66}$$

where (66) follows from a similar integration by parts argument as the one in equations (58)-(64). ■

The next Lemma uses Lemma 1 to prove (30).

Lemma 2 *For all admissible processes $G_t \in \mathcal{G}$:*

$$\max_{G_t\in\mathcal{G}} V(W_0) \le \min_{\lambda\in(0,\xi^{-\gamma}]}\left[E\left(\int_0^\infty e^{-(\rho+q)s}\frac{\left(\lambda e^{\rho s}H_sX_s^*\right)^{1-\frac{1}{\gamma}}}{1-\gamma}ds-\lambda\int_0^\infty e^{-qs}H_s\left(\lambda e^{\rho s}H_sX_s^*\right)^{-\frac{1}{\gamma}}ds+\lambda W_0\right)\right] \tag{67}$$

Proof. Proposition 1 along with Lemma 1 implies that for any admissible process G_t there exists a $\lambda^G > 0$ and a decreasing process $X_t^G \in \mathcal{D}$ that minimize (18) such that:

$$V(W_0) = E\left(\int_0^\infty e^{-(\rho+q)s}\max_{c_s}\left(\frac{c_s^{1-\gamma}}{1-\gamma}-\lambda^G e^{\rho s}H_sX_s^Gc_s\right)ds+\lambda^G\int_0^\infty e^{-qs}H_s\left(X_s^G-1\right)dG_s\right)+\lambda^G W_0$$

$$= E\int_0^\infty e^{-(\rho+q)s}\left(\frac{\left(e^{\rho s}\lambda^G H_sX_s^G\right)^{1-\frac{1}{\gamma}}}{1-\gamma}-\lambda^G e^{\rho s}H_s\left(e^{\rho s}\lambda^G H_sX_s^G\right)^{-\frac{1}{\gamma}}\right)ds+\lambda^G W_0 \tag{68}$$

Moreover, since the process G_t enforces $c_t \ge \xi$, equation (19) implies that $\lambda^G \le \xi^{-\gamma}$. Next take an arbitrary $\lambda > 0$. Since

$$c_t = \left(e^{\rho t}\lambda^G H_tX_t^G\right)^{-\frac{1}{\gamma}}$$

is an optimal consumption process, it exhausts the "budget constraint" of the consumer so that:

$$E\int_0^\infty e^{-(\rho+q)s}e^{\rho s}H_s\left(e^{\rho s}\lambda^G H_sX_s^G\right)^{-\frac{1}{\gamma}}ds = W_0-D_0+E\int_0^\infty e^{-qs}H_s dG_s$$

Using (13), this implies that:

$$E\int_0^\infty e^{-(\rho+q)s}e^{\rho s}H_s\left(e^{\rho s}\lambda^G H_sX_s^G\right)^{-\frac{1}{\gamma}} = W_0$$

This furthermore implies that (68) can be rewritten as:

$$V(W_0) = E\int_0^\infty e^{-(\rho+q)s}\left(\frac{\left(e^{\rho s}\lambda^G H_sX_s^G\right)^{1-\frac{1}{\gamma}}}{1-\gamma}-\lambda e^{\rho s}H_s\left(e^{\rho s}\lambda^G H_sX_s^G\right)^{-\frac{1}{\gamma}}\right)ds+\lambda W_0 \tag{69}$$

Next define X_t^* as in equation (27), and let the process N_t be given as:

$$N_t = \frac{\lambda^G}{\lambda}\frac{X_t^G}{X_t^*}$$

Using N_t one can rewrite equation (69) as

$$V(W_0) = E\int_0^{\infty} e^{-(\rho+q)s}\left(\frac{\left(e^{\rho s}\lambda H_s X_s^* N_s\right)^{1-\frac{1}{\gamma}}}{1-\gamma} - \lambda e^{\rho s} H_s \left(e^{\rho s}\lambda H_s X_s^* N_s\right)^{-\frac{1}{\gamma}}\right) ds + \lambda W_0 \tag{70}$$

Since $\lambda^G X_t^G$ is a decreasing process that starts at λ^G and always stays below $\xi^{-\gamma}$, the Skorohod equation[29] implies that there exists another decreasing process $\lambda^G X_t^{*G}$ that also starts at λ^G and stays below $\xi^{-\gamma}$, with the property

$$\lambda^G X_t^G \leq \lambda^G X_t^{*G} \tag{71}$$

This process is given by:

$$X_t^{*G} = \min\left[1, \frac{\xi^{-\gamma}/\lambda^G}{\max_{0\leq s\leq t}\left(e^{\rho s} H_s\right)}\right]$$

Note that X_t^{*G} is identical to X_t^* with the exception that λ replaces λ^G. Using (71) and the definition of N_t yields:

$$N_t = \frac{\lambda^G}{\lambda}\frac{X_t^G}{X_t^*} \leq \frac{\lambda^G}{\lambda}\frac{X_t^{*G}}{X_t^*} \tag{72}$$

Using (72) and (70) leads to:

$$V(W_0) \leq E\int_0^{\infty} e^{-(\rho+q)s} A(s) ds + \lambda W_0 \tag{73}$$

where:

$$A(s) = \max_{N_s \leq Q_s}\left(\widetilde{A}(s)\right) \tag{74}$$

and $\widetilde{A}(s)$ is defined as

$$\widetilde{A}(s) = \frac{\left(e^{\rho s}\lambda H_s X_s^* N_s\right)^{1-\frac{1}{\gamma}}}{1-\gamma} - \lambda e^{\rho s} H_s \left(e^{\rho s}\lambda H_s X_s^* N_s\right)^{-\frac{1}{\gamma}}$$

and

$$Q_s = \max\left[1, \frac{\lambda^G}{\lambda}\frac{X_s^{*G}}{X_s^*}\right]$$

To study the maximization problem of equation (74) it is useful to compute the derivative of $\widetilde{A}_s$ with respect to N_s. Performing this computation and combining terms gives:

$$\frac{\partial \widetilde{A}_s}{\partial N_s} = -\frac{1}{\gamma}\left(e^{\rho s}\lambda H_s X_s^* N_s\right)^{1-\frac{1}{\gamma}} N_s^{-1}\left(1 - \frac{1}{N_s X_s^*}\right) \tag{75}$$

[29] For the Skorohod equation see Karatzas and Shreve (1991).

At this stage it is useful to consider two cases separately. The first case is $\lambda > \lambda^G$. In this case, it is straightforward to show that:

$$Q_s = 1$$

Hence in maximizing $\widetilde{A}(s)$, one can constrain attention to values of $N_s \leq 1$. An examination of (75) reveals that $\frac{\partial \widetilde{A}(s)}{\partial N_s} \geq 0$ for all $N_s \leq 1$ and all X_s^*, since $X_s^* \leq 1$. Hence the solution to (74) is $N_s = 1$ when $\lambda > \lambda^G$.

In the case where $\lambda < \lambda^G$ it is also true that the optimal N_s in (74) is equal to one. To see this, observe that:

$$Q_s = \begin{cases} \frac{\lambda^G}{\lambda} \frac{X_s^{*G}}{X_s^*} & \text{when } X_s^* = 1 \\ 1 & \text{when } X_s^* < 1 \end{cases}$$

Using this observation in (75) reveals that the optimal choice for N_s is always equal to 1.[30]

The above reasoning shows that the optimal solution of (74) is given by $N_s = 1$. Returning to (73), this implies that:

$$V(W_0) \leq E \int_0^\infty e^{-(\rho+q)s} \left(\frac{(e^{\rho s} \lambda H_s X_s^*)^{1-\frac{1}{\gamma}}}{1-\gamma} ds - \lambda e^{\rho s} H_s \left(e^{\rho s} \lambda H_s X_s^*\right)^{-\frac{1}{\gamma}} ds \right) + \lambda W_0$$

Since this bound holds for arbitrary $\lambda \in (0, \xi^{-\gamma}]$ and arbitrary $G_t \in \mathcal{G}$, it also holds for the $\lambda \in (0, \xi^{-\gamma}]$ that minimizes the right hand side of the above equation and the $G_t \in \mathcal{G}$ that maximizes the right hand side. Hence (67) follows. ■

The next part of the proof is to show that equation (28) holds. A first step is to show that (28) provides an upper bound to $J(W_0)$:

Lemma 3 *The value function of problem 2 is bounded above by:*

$$J(W_0) \leq$$
$$\min_{\lambda \in (0, \xi^{-\gamma}]} \left[E \left(\int_0^\infty e^{-(\rho+q)s} \frac{(\lambda e^{\rho s} H_s X_s^*)^{1-\frac{1}{\gamma}}}{1-\gamma} ds - \lambda \int_0^\infty e^{-qs} H_s \left(\lambda e^{\rho s} H_s X_s^*\right)^{-\frac{1}{\gamma}} ds + \lambda W_0 \right) \right] \tag{76}$$

Proof. The proof of this Lemma follows identical steps to the proof of the previous Lemma. To see this, take an arbitrary triplet $< \widehat{\lambda}, X_t, c_t >$ that satisfies equations (24)-(26) of Problem 2. Then for any $\lambda > 0$ one obtains:

$$J(W_0) \leq E \left(\int_0^\infty e^{-(\rho+q)s} \frac{\left(\widehat{\lambda} e^{\rho s} H_s X_s\right)^{1-\frac{1}{\gamma}}}{1-\gamma} - \lambda \int_0^\infty e^{-qs} H_s \left(\widehat{\lambda} e^{\rho s} H_s X_s\right)^{-\frac{1}{\gamma}} + \lambda W_0 \right)$$

[30] To see this distinguish cases. When $X_s^* = 1$, then solving $\frac{\partial \widetilde{A}(s)}{\partial N_s} = 0$ gives $N_s = 1 \leq Q_s$. Hence N_s is the unique interior solution. When $X_s^* < 1$, then $\frac{\partial \widetilde{A}(s)}{\partial N_s} > 0$ for all $N_s \leq Q_s = 1$. Hence the solution is given by the corner $N_s = Q_s = 1$.

Notice that this equation is identical to equation (69), with the exception that λ^G is replaced by $\widehat{\lambda}$ and X_t^G is replaced by X_t. Since the equations following (69) hold for any λ^G, X_t^G they also hold for $\widehat{\lambda}, X_t$. Accordingly, by repeating the same steps, one can arrive at (76). ■

The next step in the proof of the proposition is to show that the inequality in (76) holds with equality for the optimal policy. The following Lemma presents a step in this direction:

Lemma 4 *Let $F(\lambda)$ be given by:*

$$F(\lambda) = E\left(\int_0^\infty e^{-(\rho+q)s}\frac{(\lambda e^{\rho s}H_s X_s^*)^{1-\frac{1}{\gamma}}}{1-\gamma}ds - \lambda\int_0^\infty e^{-qs}H_s\left(\lambda e^{\rho s}H_s X_s^*\right)^{-\frac{1}{\gamma}}ds\right) \tag{77}$$

Then

$$F(\lambda) = -\frac{K\xi^{1-\gamma}}{\gamma\phi(\phi-1)}\left(\frac{\lambda}{\xi^{-\gamma}}\right)^{\phi} + K\frac{\gamma}{1-\gamma}\lambda^{1-\frac{1}{\gamma}} \tag{78}$$

Assume moreover that (8) is met. Then

$$\min_{\lambda\in(0,\xi^{-\gamma}]}\left[F(\lambda)+\lambda W_0\right] = \min_{\lambda>0}\left[F(\lambda)+\lambda W_0\right] \tag{79}$$

and (76) can be rewritten as:

$$J(W_0) \leq \min_{\lambda>0}\left[F(\lambda)+\lambda W_0\right]$$

Moreover, letting λ^* be given as:

$$\lambda^* = \arg\min_{\lambda>0}\left[F(\lambda)+\lambda W_0\right]$$

implies that:

$$E_0\left[\int_0^\infty e^{-qs}H_s\left(\lambda^* e^{\rho s}H_s X_s^*\right)^{-\frac{1}{\gamma}}\right] = W_0$$

and accordingly $c_s^* = (\lambda^* e^{\rho s}H_s X_s^*)^{-\frac{1}{\gamma}}$ is a feasible consumption to the central planner of problem 2.

Proof. To save notation, let

$$Z_t = \lambda e^{\rho t}H_t X_t^* \tag{80}$$

and note that $Z_0 = \lambda$, and that $Z_t \in (0, \xi^{-\gamma}]$ by the definition of X_t^* in equation (27). Equation (77) can now be rewritten as:

$$F(\lambda) = E\left[\int_0^\infty e^{-(\rho+q)s}\frac{1}{1-\gamma}(Z_s)^{1-\frac{1}{\gamma}}ds - \int_0^\infty e^{-(\rho+q)s}\frac{Z_s^{1-\frac{1}{\gamma}}}{X_s^*}ds\right] \tag{81}$$

It will be convenient to compute the two terms inside equation (81) separately. Define first:

$$G(Z_t) = E\left[\int_t^\infty e^{-(\rho+q)(s-t)} \frac{1}{1-\gamma} (Z_s)^{1-\frac{1}{\gamma}} ds | Z_t\right] \tag{82}$$

To compute $G(Z_t)$, it is easiest to let τ be the first hitting time of Z_t to the level $\varepsilon > 0$:

$$\tau = \inf_{s \geq t} \{Z_s = \varepsilon\}$$

and then compute the expression:

$$G^\varepsilon(Z_t) = E\left[\int_t^\tau e^{-(\rho+q)s} \frac{1}{1-\gamma} (Z_s)^{1-\frac{1}{\gamma}} ds | Z_t\right] \tag{83}$$

To compute (83) apply first Ito's Lemma to (80) to obtain:

$$\frac{dZ_t}{Z_t} = (\rho - r) dt - \kappa dB_t + \frac{dX_t^*}{X_t^*}$$

Next, I shall construct a function $G^\varepsilon(Z)$ that satisfies the ODE:

$$\frac{\kappa^2}{2} G_{ZZ}^\varepsilon Z^2 + G_Z^\varepsilon Z(\rho - r) - (\rho + q) G^\varepsilon + \frac{1}{1-\gamma} (Z)^{1-\frac{1}{\gamma}} = 0 \tag{84}$$

subject to the boundary conditions:

$$G_Z^\varepsilon\left(\xi^{-\gamma}\right) = 0 \tag{85}$$

$$G^\varepsilon(\varepsilon) = 0 \tag{86}$$

(84) is a linear ordinary differential equation with general solution:

$$G^\varepsilon(Z) = C_1 Z^\chi + C_2 Z^\phi + K \frac{1}{1-\gamma} Z^{1-\frac{1}{\gamma}}$$

where C_1, C_2 are arbitrary constants, K is given in equation (10), $\phi > 0$ in (9), and χ is given by:

$$\chi = \frac{-\left(\rho - r - \frac{\kappa^2}{2}\right) - \sqrt{\left(\rho - r - \frac{\kappa^2}{2}\right)^2 + 2(\rho + q)\kappa^2}}{\kappa^2} < 0 \tag{87}$$

To satisfy (85), (86) C_1 and C_2 must be chosen so that:

$$\begin{aligned} \chi C_1 \left(\xi^{-\gamma}\right)^\chi + \phi C_2 \left(\xi^{-\gamma}\right)^\phi - \frac{1}{\gamma} K \left(\xi^{-\gamma}\right)^{1-\frac{1}{\gamma}} &= 0 \\ C_1 \varepsilon^\chi + C_2 \varepsilon^\phi + K \frac{1}{1-\gamma} \varepsilon^{1-\frac{1}{\gamma}} &= 0 \end{aligned}$$

Solving this system yields:

$$C_2 = \frac{K\left[\frac{1}{\gamma\chi}\left(\xi^{-\gamma}\right)^{1-\frac{1}{\gamma}-\chi} \varepsilon^\chi + \frac{1}{1-\gamma} \varepsilon^{1-\frac{1}{\gamma}}\right]}{\frac{\phi}{\chi}\left(\xi^{-\gamma}\right)^{\phi-\chi} \varepsilon^\chi - \varepsilon^\phi}$$

and

$$C_1 = -C_2 \varepsilon^{\phi-\chi} - K \frac{1}{1-\gamma} \varepsilon^{1-\frac{1}{\gamma}-\chi}$$

It remains now to verify that $G^{\varepsilon}(Z_t)$ satisfies (83). To this end, apply Ito's Lemma to $e^{-(\rho+q)t} G^{\varepsilon}(Z_t)$ to obtain for any time $T \wedge \tau$:

$$\begin{aligned} e^{-(\rho+q)T} G^{\varepsilon}(Z_{T\wedge\tau}) - e^{-(\rho+q)t} G^{\varepsilon}(Z_t) &= \int_t^{T\wedge\tau} \left(\frac{\kappa^2}{2} G^{\varepsilon}_{ZZ} Z_s^2 + G^{\varepsilon}_Z Z_s (\rho - r) - (\rho+q) G^{\varepsilon} \right) e^{-(\rho+q)s} ds \\ &\quad - \int_t^{T\wedge\tau} e^{-(\rho+q)s} \kappa G^{\varepsilon}_Z Z_s dB_s \\ &\quad + \int_t^{T\wedge\tau} e^{-(\rho+q)s} G^{\varepsilon}_Z \left(\xi^{-\gamma}\right) \xi^{-\gamma} \frac{dX_s^*}{X_s^*} \end{aligned}$$

Using (84) in the first line of the above equation along with (85) in the third line, letting $T \to \infty$ along with (86) and using the monotone convergence theorem gives:

$$G^{\varepsilon}(Z_t) = E_t \left[\int_t^{\tau} e^{-(\rho+q)(s-t)} \frac{1}{1-\gamma} (Z_s)^{1-\frac{1}{\gamma}} ds + \int_t^{\tau} e^{-(\rho+q)(s-t)} \kappa G^{\varepsilon}_Z Z_s dB_s \right] \tag{88}$$

Since $G^{\varepsilon}_Z Z$ is bounded between t and τ, the second term in the above expression is a martingale and hence obtain (110). Next, letting $\varepsilon \to 0$, it is straightforward to show that:

$$C_2 = \frac{K\left[\frac{1}{\gamma\chi}\left(\xi^{-\gamma}\right)^{1-\frac{1}{\gamma}-\chi} - \frac{1}{1-\gamma}\varepsilon^{1-\frac{1}{\gamma}-\chi}\right]}{\frac{\phi}{\chi}\left(\xi^{-\gamma}\right)^{\phi-\chi} - \varepsilon^{\phi-\chi}} \to K\frac{1}{\gamma\phi}\left(\xi^{-\gamma}\right)^{1-\frac{1}{\gamma}-\phi}$$

since $\varepsilon^{\phi-\chi} \to 0$ and $\varepsilon^{1-\frac{1}{\gamma}-\chi} \to 0$. By a similar argument it is easy to show that $C_1 \to 0$ and hence:

$$\lim_{\varepsilon\to 0} G^{\varepsilon}(Z) = G(Z) = \frac{1}{\phi}\frac{1}{\gamma} K \xi^{1-\gamma} \left(\frac{Z}{\xi^{-\gamma}}\right)^{\phi} + K \frac{1}{1-\gamma} Z^{1-\frac{1}{\gamma}} \tag{89}$$

Now (82) is a consequence of the monotone convergence theorem.

It remains to compute the expression

$$N(Z_t, X_t^*) = E_t \left(\int_t^{\infty} e^{-(\rho+q)(s-t)} \frac{Z_s^{1-\frac{1}{\gamma}}}{X_s^*} ds \right)$$

Using a similar logic as above, the next step is to search for a function N that satisfies:

$$\begin{aligned} \frac{\kappa^2}{2} N_{ZZ} Z^2 + N_Z Z (\rho - r) - (\rho+q) N + \frac{(Z)^{1-\frac{1}{\gamma}}}{X^*} &= 0 \\ N_Z\left(\xi^{-\gamma}, X^*\right) \frac{\xi^{-\gamma}}{X^*} + N_X\left(\xi^{-\gamma}, X^*\right) &= 0 \end{aligned}$$

One can check that such a function exists and is given by:

$$N(Z, X^*) = \frac{1}{(\phi-1)} \frac{1}{\gamma} \frac{K\left(\xi^{-\gamma}\right)^{1-\frac{1}{\gamma}}}{X^*} \left(\frac{Z}{\xi^{-\gamma}}\right)^{\phi} + K \frac{Z^{1-\frac{1}{\gamma}}}{X^*} \tag{90}$$

One can now use the same steps as for the function $G(Z_t)$ to verify that:

$$N\left(Z_t, X_t^*\right) = E_t\left(\int_t^\infty e^{-(\rho+q)(s-t)} \frac{Z_s^{1-\frac{1}{\gamma}}}{X_s^*} ds\right) \tag{91}$$

It is now possible to compute $F(\lambda)$ which is given by:

$$\begin{aligned} F(\lambda) &= G(\lambda) - N(\lambda, 1) = \\ &= -\frac{K\xi^{1-\gamma}}{\gamma\phi(\phi-1)}\left(\frac{\lambda}{\xi^{-\gamma}}\right)^{\phi} + K\frac{\gamma}{1-\gamma}\lambda^{1-\frac{1}{\gamma}} \end{aligned} \tag{92}$$

To show the second part of the proposition, observe that (91), (80) and (90) imply that

$$\begin{aligned} \frac{N(\lambda, 1)}{\lambda} &= \frac{1}{\lambda}E_0\left(\int_0^\infty e^{-(\rho+q)s} \frac{Z_s^{1-\frac{1}{\gamma}}}{X_s^*} ds\right) = E_0\left(\int_0^\infty e^{-qs} H_s\left(\lambda e^{\rho s} H_s X_s^*\right)^{-\frac{1}{\gamma}} ds\right) = \\ &= \frac{K\xi^{1-\gamma}}{(\phi-1)}\frac{1}{\gamma}\left(\frac{\lambda}{\xi^{-\gamma}}\right)^{\phi}\frac{1}{\lambda} + K\lambda^{-\frac{1}{\gamma}} \end{aligned} \tag{93}$$

Moreover, computing $F'(\lambda)$ in (92) yields:

$$F'(\lambda) = -\frac{K\xi^{1-\gamma}}{(\phi-1)}\frac{1}{\gamma}\left(\frac{\lambda}{\xi^{-\gamma}}\right)^{\phi}\frac{1}{\lambda} - K\lambda^{-\frac{1}{\gamma}} \tag{94}$$

Combining (93) and (94) yields:

$$\begin{aligned} F'(\lambda) &= -\frac{N(\lambda, 1)}{\lambda} = \\ &= -E_0\left(\int_0^\infty e^{-qs} H_s\left(\lambda e^{\rho s} H_s X_s^*\right)^{-\frac{1}{\gamma}} ds\right) \end{aligned} \tag{95}$$

Using the formula for $F(\lambda)$, equation (76) can be expressed as:

$$\min_{\lambda\in(0,\xi^{-\gamma}]} \left\{F(\lambda) + \lambda W_0\right\}$$

which leads to the first order condition for the minimizing λ^* :

$$F'\left(\lambda^*\right) = -W_0 \tag{96}$$

Using (95) leads to:

$$W_0 = E_0\left(\int_0^\infty e^{-qs} H_s\left(\lambda^* e^{\rho s} H_s X_s^*\right)^{-\frac{1}{\gamma}} ds\right) = E_0\left(\int_0^\infty e^{-qs} H_s c_s^* ds\right)$$

This last equation implies that λ^*, X_t^* and the associated consumption process $c_t^* = \left(\lambda^* e^{\rho t} H_t X_t^*\right)^{-\frac{1}{\gamma}}$ satisfy (24) and (26). To show that the choice $\langle\lambda^*, X_t^*, c_t^*\rangle$ constitutes a feasible triplet, it remains to show that it also satisfies (25). By construction of X_t^* this will be the case as long as $\lambda^* < \xi^{-\gamma}$. This will indeed be the case as long as W_0 satisfies (8). To see this, note that $\xi^{-\gamma}$ is the unique solution of (96) when W_0 is given by

$$W_0 = \frac{\frac{1}{\gamma} + \phi - 1}{\phi - 1} K\xi$$

Moreover, equation (94) implies that:

$$\begin{aligned} F''(\lambda) &= -K\left(\xi^{-\gamma}\right)^{1-\frac{1}{\gamma}} \frac{1}{\gamma}\left(\frac{1}{\xi^{-\gamma}}\right)^{\phi} \lambda^{\phi-2} + \frac{1}{\gamma} K \lambda^{-\frac{1}{\gamma}-1} \\ &= \frac{1}{\gamma} K \lambda^{-\frac{1}{\gamma}-1}\left[1 - \left(\frac{\lambda}{\xi^{-\gamma}}\right)^{\phi+\frac{1}{\gamma}-1}\right] > 0 \end{aligned} \tag{97}$$

The above equation shows that $F^{'}(\lambda)$ is an increasing function of λ for $0 < \lambda < \xi^{-\gamma}$ and hence the solution λ^* of equation (96) is a decreasing function of W_0. Hence, as long as W_0 satisfies (8), then $\lambda^* < \xi^{-\gamma}$. Since the interior solution λ^* is smaller than $\xi^{-\gamma}$, equation (79) follows. ■

Combining the above Lemma with (76) implies that:

$$\begin{aligned} J(W_0) &\leq \min_{\lambda>0}\left[F(\lambda) + \lambda W_0\right] = F(\lambda^*) + \lambda^* W_0 = \\ &= E\left(\int_0^{\infty} e^{-(\rho+q)s} \frac{\left((\lambda^* e^{\rho s} H_s X_s^*)^{-\frac{1}{\gamma}}\right)^{1-\gamma}}{1-\gamma} ds\right) \\ &= E\left(\int_0^{\infty} e^{-(\rho+q)s} \frac{(c_s^*)^{1-\gamma}}{1-\gamma} ds\right) \leq J\left(W_0\right) \end{aligned}$$

The last inequality follows because $c_s^* = (\lambda^* e^{\rho s} H_s X_s^*)^{-\frac{1}{\gamma}}$ is a feasible consumption process for problem for problem 2 and $J\left(W_0\right)$ is the value function of the problem. The above three lines imply that equation (76) holds with equality as long as one chooses the optimal solution in the statement of the proposition. This concludes the proof of Proposition 2.

B Remaining Proofs

Proof of Proposition 3. The proof of this Proposition is just a special case of Section 6 in He and Pages (1993) and hence I give only a sketch and refer the reader to He and Pages (1993) for details.

To start define:

$$\widetilde{V}(\lambda) = \min_{X_s \in \mathcal{D}} E\left[\int_0^{\infty} e^{-(\rho+q)s} \max_{c_s}\left(\frac{c_s^{1-\gamma}}{1-\gamma} - \lambda e^{\rho s} H_s X_s c_s\right) ds + \lambda \int_0^{\infty} e^{-qs} H_s X_s y_0 ds\right] \tag{98}$$

By equation (13) and equation (18) of Proposition 1

$$V(W_0) = \min_{\lambda>0}\left[\widetilde{V}(\lambda) + \lambda\left(W_0 - \frac{y_0}{r+q}\right)\right] \tag{99}$$

since

$$y_0 E \int_0^\infty H_s ds = \frac{y_0}{r}$$

Next, for an arbitrary decreasing process X_t let Z_t be defined as:

$$Z_t = \lambda e^{\rho s} H_s X_s$$

Note that $Z_0 = \lambda$. Applying Ito's Lemma to Z_t gives:

$$\frac{dZ_t}{Z_t} = (\rho - r)\, dt - \kappa dB_t + \frac{dX_t}{X_t} \tag{100}$$

With this definition of Z_t one can solve the maximization problem inside (98) and rewrite $\widetilde{V}(\lambda)$ as

$$\widetilde{V}(Z_0) = \min_{X_s \in \mathcal{D}} E\left[\int_0^\infty e^{-(\rho+q)s}\left(\frac{\gamma}{1-\gamma} Z_s^{1-\frac{1}{\gamma}} + y_0 Z_s\right) ds\right] \tag{101}$$

From this point on, one can use similar arguments to He and Pages (1993) and treat (101) as a singular stochastic control problem over the set of decreasing processes X_t. As He and Pages (1993) show, the optimal solution is to always decrease X_t appropriately, so as to keep Z_t in the interval $(0, \overline{Z}]$. $\overline{Z}$ is a free boundary that is determined next.

Using this conjecture for the optimal policy one can now proceed as He and Pages (1993) to establish that $\widetilde{V}(Z)$ satisfies the ordinary differential equation:

$$\frac{\kappa^2}{2}\widetilde{V}_{ZZ} Z^2 + (\rho - r)\, \widetilde{V}_Z Z - (\rho + q)\widetilde{V} + \frac{\gamma}{1-\gamma} Z^{1-\frac{1}{\gamma}} + y_0 Z = 0 \text{ for all } Z \in (0, \overline{Z}]$$

The general solution to this equation is:

$$\widetilde{V}(Z) = C_1 Z^\phi + C_2 Z^\chi + K\frac{\gamma}{1-\gamma} Z^{1-\frac{1}{\gamma}} + \frac{y_0}{r+q} Z \tag{102}$$

where K is given in (10), ϕ in (9) and χ in (87) and C_1, C_2 are arbitrary constants. As in He and Pages (1993) one can set $C_2 = 0$ (since $\chi < 0$). Hence it remains to determine C_1 and the free boundary $\overline{Z}$. As most singular stochastic control problems, one can employ a "smooth pasting" and "high contact" principle, namely by determining C_1 and $\overline{Z}$ so that:

$$\widetilde{V}_Z\left(\overline{Z}\right) = 0 \tag{103}$$

$$\widetilde{V}_{ZZ}\left(\overline{Z}\right) = 0 \tag{104}$$

Using the general solution in (102) along with $C_2 = 0$ and plugging into the equations (103) and (104) gives the system of equations

$$\begin{aligned} \phi C_1 \overline{Z}^{\phi-1} - K\overline{Z}^{-\frac{1}{\gamma}} + \frac{y_0}{r+q} &= 0 \\ \phi(\phi-1) C_1 \overline{Z}^{\phi-2} + \frac{1}{\gamma} K \overline{Z}^{-\frac{1}{\gamma}-1} &= 0 \end{aligned}$$

Solving this system for C_1 and $\overline{Z}$ gives:

$$\overline{Z}^{-\frac{1}{\gamma}} = \frac{1}{K}\frac{y_0}{r+q}\left(\frac{\phi-1}{\frac{1}{\gamma}+\phi-1}\right) \tag{105}$$

$$C_1 = -\frac{\frac{1}{\gamma}\frac{y_0}{r+q}}{\phi\overline{Z}^{\phi-1}\left[\frac{1}{\gamma}+\phi-1\right]} \tag{106}$$

The next steps to verify that the conjectured policy is indeed optimal are identical to He and Pages (1993) and are left out.

To conclude the proof , note that sofar the calculations were true for an arbitrary y_0. To determine the y_0 that will safeguard that $c_t \geq \xi$ observe that:

$$c_t = Z^{-\frac{1}{\gamma}}$$

by equation (19). Since the optimal policy is to control X_t so as to "keep" Z_t in the interval $(0, \overline{Z}]$ it follows that the minimum level of consumption is given by $\overline{Z}^{-\frac{1}{\gamma}}$. Hence, in order to guarantee condition $c_t \geq \xi$ it suffices to determine y_0 so that:

$$\xi = \overline{Z}^{-\frac{1}{\gamma}} = \frac{1}{K}\frac{y_0}{r+q}\left(\frac{\phi-1}{\frac{1}{\gamma}+\phi-1}\right)$$

Solving for y_0 gives:

$$y_0 = \xi(r+q)K\frac{\frac{1}{\gamma}+\phi-1}{\phi-1}$$

One can now substitute that level of y_0 into (106), (105) and use the resulting expressions to obtain from (102) the following expression for $\widetilde{V}(Z)$:

$$\widetilde{V}(Z) = -\frac{K\xi^{1-\gamma}}{\gamma\phi\left(\phi-1\right)}\left(\frac{Z}{\xi^{-\gamma}}\right)^{\phi} + K\frac{\gamma}{1-\gamma}Z^{1-\frac{1}{\gamma}} + \frac{y_0}{r+q}Z$$

Evaluating this expression at $Z_0 = \lambda$ and using equation (99) gives the value function of the agent as

$$\begin{aligned} V(W_0) &= \min_{\lambda>0}\left[\widetilde{V}(\lambda) + \lambda\left(W_0 - \frac{y_0}{r+q}\right)\right] = \\ &= \min_{\lambda>0}\left[-\frac{K\xi^{1-\gamma}}{\gamma\phi\left(\phi-1\right)}\left(\frac{\lambda}{\xi^{-\gamma}}\right)^{\phi} + K\frac{\gamma}{1-\gamma}\lambda^{1-\frac{1}{\gamma}} + \lambda W_0\right] \end{aligned}$$

This last equation is precisely equation (29) of proposition 2, which shows that the "constant income" policy of the current proposition attains the upper bound of proposition 2. ■

Proof of Proposition 4. The proof of this proposition proceeds in steps. The first two Lemmas establish that the proposed transfer policy will make it possible for an agent who follows the optimal consumption process of proposition 4 to satisfy the intertemporal budget constraint. The proof then continues to show that the wealth process associated with the optimal consumption process of proposition 4 along with the portfolio process (36) will lead to non-negative levels of wealth at all times. Finally, it is shown that the consumption policy of proposition 4 along with the portfolio choice (36) are optimal for an agent who is faced with transfers given by (35) and attain the upper bound of proposition 2.

Lemma 5 *Let K and ϕ be given by (10) and (9) and for any $0 < \lambda < \xi^{-\gamma}$ let:*

$$Z_t = \lambda e^{\rho s} H_s X_s^*$$

Then:

$$\int_0^\infty E_t \left(\int_t^\infty e^{-q(s-t)} H_s X_s^* dG_s - \int_t^\infty e^{-q(s-t)} H_s X_s^* Z_s^{-\frac{1}{\gamma}} ds \right) dX_t^* = 0 \tag{107}$$

Proof of Lemma 5. It will simplify notation to let:

$$\eta = -K\xi \left(\phi - 1 + \frac{1}{\gamma} \right) \tag{108}$$

The first step is to compute

$$\frac{E_t \int_t^\infty e^{-qs} H_s X_s^* dG_s}{e^{-qt} H_t X_t^*} = \eta \frac{E_t \int_t^\infty e^{-qs} H_s dX_s^*}{e^{-qt} H_t X_t^*} \tag{109}$$

Applying integration by parts and using the definition of Z_t gives:

$$E_t \left(\int_t^\infty e^{-qs} H_s dX_s^* \right) = \frac{1}{\lambda} \left[-e^{-(\rho+q)t} Z_t + E_t \left(\int_t^\infty (r+q) e^{-(\rho+q)s} Z_s ds \right) \right] \tag{110}$$

Using (110) in equation (109) gives:

$$\frac{E_t \int_t^\infty e^{-qs} H_s X_s^* dG_s}{e^{-qt} H_t X_t^*} = \eta \left[(r+q) \frac{E_t \left(\int_t^\infty e^{-(\rho+q)(s-t)} Z_s ds \right)}{Z_t} - 1 \right] \tag{111}$$

By using a logic similar to equations (84)-(88) it can be shown that:

$$E_t \left(\int_t^\infty e^{-(\rho+q)(s-t)} Z_s ds \right) = -\frac{1}{\phi} \frac{\xi^{-\gamma}}{r+q} \left(\frac{Z_t}{\xi^{-\gamma}} \right)^\phi + \frac{1}{r+q} Z_t \tag{112}$$

where ϕ is defined in equation (9). Plugging back (112) into (111) gives:

$$\frac{E_t \int_t^\infty e^{-qs} H_s X_s^* dG_s}{e^{-qt} H_t X_t^*} = -\frac{\eta}{\phi} \left(\frac{Z_t}{\xi^{-\gamma}} \right)^{\phi-1} \tag{113}$$

To conclude the proof, note that equations (82) and (89) imply that:

$$\frac{E_t \left(\int_t^\infty e^{-qs} H_s X_s^* Z_s^{-\frac{1}{\gamma}} ds \right)}{e^{-qt} H_t X_t^*} = \frac{E_t \left(\int_t^\infty e^{-(\rho+q)(s-t)} Z_s^{1-\frac{1}{\gamma}} ds \right)}{Z_t} = \frac{\frac{1}{\phi} \frac{1-\gamma}{\gamma} K \xi^{1-\gamma} \left(\frac{Z_t}{\xi^{-\gamma}} \right)^\phi + K Z_t^{1-\frac{1}{\gamma}}}{Z_t} \tag{114}$$

Combining (114) with (113) gives:

$$\frac{E_t\left(\int_t^\infty e^{-qs}H_sX_s^*dG_s - \int_t^\infty e^{-qs}H_sX_s^*Z_s^{-\frac{1}{\gamma}}ds\right)}{e^{-qt}H_tX_t^*} =$$

$$= -\frac{\eta}{\phi}\left(\frac{Z_t}{\xi^{-\gamma}}\right)^{\phi-1} - \frac{\frac{1}{\phi}\frac{1-\gamma}{\gamma}K\xi^{1-\gamma}\left(\frac{Z_t}{\xi^{-\gamma}}\right)^{\phi} + KZ_t^{1-\frac{1}{\gamma}}}{Z_t}$$

Since $dX_t^* \neq 0$ when and only when $Z_t = \xi^{-\gamma}$, equation (107) amounts to checking that:

$$-\frac{\eta}{\phi} - \left(\frac{1}{\phi}\frac{1-\gamma}{\gamma} + 1\right)K\xi = 0$$

which follows easily from the definition of η. ■

Lemma 6 *Let Z_s be as in the statement of the proposition 4 and let G_t be as in (35). Then the consumption policy:*

$$c_s^* = (Z_s)^{-\frac{1}{\gamma}} \tag{115}$$

satisfies:

$$E\int_0^\infty e^{-qs}H_sX_s^*c_s^*ds = W_0 + \int_0^\infty e^{-qs}H_s\left(X_s^* - 1\right)dG_s \tag{116}$$

Proof of Lemma 6. Taking any $\lambda \in (\ 0, \xi^{-\gamma}]$, using the definition of X_t^*, and equation (107), the same reasoning behind (57) leads to:

$$E\left(\int_0^\infty e^{-(\rho+q)s}\max_{c_s}\left(\frac{c_s^{1-\gamma}}{1-\gamma} - \lambda e^{\rho s}H_sX_s^*c_s\right)ds + \lambda\int_0^\infty e^{-qs}H_s\left(X_s^*-1\right)dG_s\right) + \lambda W_0 = \tag{117}$$

$$= E\left[\int_0^\infty e^{-(\rho+q)s}\frac{\gamma}{1-\gamma}\left(e^{\rho s}\lambda H_sX_s^*\right)^{\frac{\gamma-1}{\gamma}}ds + \int_0^\infty e^{-(\rho+q)s}\left(e^{\rho s}\lambda H_sX_s^*\right)^{1-\frac{1}{\gamma}}\left(1-\frac{1}{X_s^*}\right)ds\right] + \lambda W_0 \tag{118}$$

Hence the λ^* that minimizes (29) (and hence minimizes [118]) also minimizes (117). But since λ minimizes (117), the same argument as in He and Pages (1993) (Proof of Theorem 1) leads to (116). ■

Proof of Proposition 4 continued. Lemma 6 has demonstrated that the consumption policy (115) satisfies the intertemporal budget constraint (116). It remains to show that this consumption policy along with the portfolio policy (36) will lead to a process for financial wealth that satisfies $W_t \geq 0$. To that end let η be given as in (108) and define:

$$W^*\left(Z_t\right) = -K\left(\xi^{-\gamma}\right)^{-\frac{1}{\gamma}}\left(\frac{Z_t}{\xi^{-\gamma}}\right)^{\phi-1} + KZ_t^{-\frac{1}{\gamma}} \tag{119}$$

It is straightforward to verify the following facts about $W^*\left(Z_t\right)$:

$$\frac{\kappa^2}{2}Z^2W_{ZZ}^* + \left(\rho - r + \kappa^2\right)ZW_Z^* - (r+q)W + (Z)^{-\frac{1}{\gamma}} = 0 \tag{120}$$

$$W^*\left(\xi^{-\gamma}\right)=0, W^*\left(Z\right)\geq 0 \text{ for all } Z\in\left(0,\xi^{-\gamma}\right] \tag{121}$$

$$W_Z^*\left(\xi^{-\gamma}\right)=-K\xi\left(\phi-1+\frac{1}{\gamma}\right)\left(\xi^{-\gamma}\right)^{-1}=\frac{\eta}{\xi^{-\gamma}} \tag{122}$$

The next step is to verify that $W^*\left(Z_t\right)$ is the stochastic process for the financial wealth of the agent. To see this, use the definition of c_s^* (equation [115]) along with the definitions of dG_t, W_t^* (equations [35] and [119] respectively) and apply Ito's Lemma to obtain:

$$d\left(\int_0^t c_s^* ds-\int_0^t dG_s+W_t^*\right)=$$

$$\begin{aligned}
&= c_t^* dt-\eta\frac{dX_t^*}{X_t^*}+W_Z^* dZ_t+\frac{\kappa^2}{2}W_{ZZ}^* Z_t^2 dt\\
&= \left(c_t^*-Z_t^{-\frac{1}{\gamma}}\right)dt+\left[W_Z^*\left(\xi^{-\gamma}\right)\xi^{-\gamma}-\eta\right]\frac{dX_t^*}{X_t^*}+(r+q)W_t^* dt-\kappa^2 Z_t W_Z^* dt-\kappa W_Z^* Z_t dB_t=\\
&= (r+q)W_t^* dt-\kappa^2 Z_t W_Z^* dt-\frac{\kappa}{\sigma}W_Z^* Z_t\left(\frac{dP_t}{P_t}-\mu dt\right)\\
&= (r+q)W_t^* dt-\kappa^2 Z_t W_Z^* dt-\frac{\kappa}{\sigma}W_Z^* Z_t\left(\frac{dP_t}{P_t}-\left(\mu-r\right)dt-rdt\right)=\\
&= qW_t^* dt+r\left(W_t^*+\frac{\kappa}{\sigma}W_Z^* Z_t\right)dt-\frac{\kappa}{\sigma}W_Z^* Z_t\frac{dP_t}{P_t}=\\
&= qW_t^* dt+r\left(W_t^*-\pi_t^*\right)dt+\pi_t^*\frac{dP_t}{P_t}
\end{aligned}$$

Integrating gives

$$\int_0^t c_s^* ds+W_t^*=W_0-D_0+\int_0^t dG_s+\int_0^t qW_s^* dt+\int_0^t r\left(W_t^*-\pi_t^*\right)dt+\int_0^t \pi_t^*\frac{dP_t}{P_t}$$

Hence the process W_t^* satisfies the equation (15) for an agent who chooses a consumption policy given by (115) and a portfolio policy given by (36). Accordingly, it is the financial wealth process that is associated with that policy pair. Moreover, by equation (121) the financial wealth process is non-negative. Accordingly, the policies given by (115) and (36) are feasible for an agent who is faced with the transfer process (35).

Verifying the optimality of the stated policy pair is straightforward. According to proposition 1:

$$V(W_0)=\min_{\lambda>0,\ X_s\in\mathcal{D}}\left[\begin{array}{c} E\left(\int_0^\infty e^{-(\rho+q)s}\max_{c_s}\left(\frac{c_s^{1-\gamma}}{1-\gamma}-\lambda e^{\rho s}H_s X_s c_s\right)ds+\lambda\int_0^\infty e^{-qs}H_s X_s dG_s\right)\\ +\lambda\left(W_0-D_0\right)\end{array}\right]\leq Q(W_0)$$

where:

$$Q(W_0)=\min_{\lambda>0}\left[\begin{array}{c} E\left(\int_0^\infty e^{-(\rho+q)s}\max_{c_s}\left(\frac{c_s^{1-\gamma}}{1-\gamma}-\lambda e^{\rho s}H_s X_s^* c_s\right)ds+\lambda\int_0^\infty e^{-qs}H_s X_s^* dG_s\right)\\ +\lambda\left(W_0-D_0\right)\end{array}\right]$$

One can use now Lemma 6 to illustrate that the consumption policy (115) leads to a payoff for the agent equal to $Q(W_0)$ which is an upper bound to the value function of the agent $V(W_0)$. Since the consumption policy (115) is also feasible, the payoff associated with that policy also provides a lower bound to the value function $V(W_0)$. Hence this policy must be optimal, since the payoff associated with it is equal to the value function.

Finally, the easiest way to show that

$$D_0 = K\xi \frac{\frac{1}{\gamma}+\phi-1}{\phi-1}\left(\frac{\lambda^*}{\xi^{-\gamma}}\right)^{\phi-1}$$

is to observe that the intertemporal budget constraint implies that:

$$E_{\tau_0}\left(\int_{\tau_0}^{\infty} e^{-q(s-\tau_0)}\frac{H_s}{H_{\tau_0}}c_s^* ds\right) = E_{\tau_0}\left(\int_{\tau_0}^{\infty} e^{-q(s-\tau_0)}\frac{H_s}{H_{\tau_0}}dG_s\right)$$

where τ_0 is the first time that $X_{\tau_0} \geq 1$ (or equivalently the first time that $W_{\tau_0} = 0$ and $\lambda^* e^{\rho\tau_0}H_{\tau_0} = \xi^{-\gamma}$) . A few manipulations can be used to show that

$$E_{\tau_0}\left(\int_{\tau_0}^{\infty} e^{-q(s-\tau_0)}\frac{H_s}{H_{\tau_0}}c_s^* ds\right) = \frac{N\left(\xi^{-\gamma},1\right)}{\xi^{-\gamma}} = K\xi\frac{\frac{1}{\gamma}+\phi-1}{\phi-1}$$

where N is defined and computed in (90) and (91). Finally, since there are no transfers between 0 and τ_0 :

$$\begin{aligned} D_0 &= E\left(e^{-q\tau_0}H_{\tau_0}\right)K\xi\frac{\frac{1}{\gamma}+\phi-1}{\phi-1} = \frac{1}{\lambda^*}E\left(e^{-(\rho+q)\tau_0}\lambda^* e^{\rho\tau_0}H_{\tau_0}\right)K\xi\frac{\frac{1}{\gamma}+\phi-1}{\phi-1} = \\ &= \frac{\xi^{-\gamma}}{\lambda^*}E\left(e^{-(\rho+q)\tau_0}\right)K\xi\frac{\frac{1}{\gamma}+\phi-1}{\phi-1} = \left(\frac{\lambda^*}{\xi^{-\gamma}}\right)^{\phi-1}K\xi\frac{\frac{1}{\gamma}+\phi-1}{\phi-1} \end{aligned}$$

where the proof of $E\left(e^{-(\rho+q)\tau_0}\right) = \left(\frac{\lambda^*}{\xi^{-\gamma}}\right)^{\phi}$ is identical to the one given in Oksendal (1998), Chapter 10. ■

Proof of Proposition 5. . Take any transfer process G_t such that the resulting consumption process of the agent satisfies $c_t \geq \xi$. Proposition 1 implies then that there exists a cumulative multiplier process X_t^G and a constant λ^G such that:

$$c_t = \left(\lambda^G e^{\rho t}H_t X_t^G\right)^{-\frac{1}{\gamma}} \geq \xi$$

Letting:

$$X_t^* = \min\left[1, \frac{\xi^{-\gamma}/\lambda^G}{\max_{0\leq s\leq t}\left(e^{\rho s}H_s\right)}\right]$$

and:

$$P = E\left(\int_0^{\infty} e^{-qs}H_s c_s ds\right)$$

gives:

$$P = E\left(\int_0^{\infty} e^{-qs}H_s\left(\lambda^G e^{\rho s}H_s X_s^G\right)^{-\frac{1}{\gamma}} ds\right) \geq E\left(\int_0^{\infty} e^{-qs}H_s\left(\lambda^G e^{\rho s}H_s X_s^*\right)^{-\frac{1}{\gamma}} ds\right) \tag{123}$$

since[31] $X_s^*\left(\lambda^G\right) \geq X_s^G$. Equation (93) implies that:

$$E\left(\int_0^\infty e^{-qs}H_s\left(\lambda^G e^{\rho s}H_s X_s^*\right)^{-\frac{1}{\gamma}}ds\right) = \frac{K\xi^{1-\gamma}}{(\phi-1)}\frac{1}{\gamma}\left(\frac{\lambda^G}{\xi^{-\gamma}}\right)^{\phi}\frac{1}{\lambda^G} + K\left(\lambda^G\right)^{-\frac{1}{\gamma}}$$

Combining (95) and (97) implies that the right hand side of the above equation is decreasing in λ^G whenever $\lambda^G \leq \xi^{-\gamma}$. Since $c_0 = \left(\lambda^G\right)^{-\frac{1}{\gamma}} \geq \xi$ this implies furthermore:

$$\begin{aligned} E\left(\int_0^\infty e^{-qs}H_s\left(\lambda^G e^{\rho s}H_s X_s^*\right)^{-\frac{1}{\gamma}}ds\right) &\geq \frac{K\xi^{1-\gamma}}{(\phi-1)}\frac{1}{\gamma}\frac{1}{\xi^{-\gamma}} + K\xi = K\xi\left(1+\frac{1}{\phi-1}\frac{1}{\gamma}\right) \\ &= K\xi\left(\frac{\frac{1}{\gamma}+\phi-1}{\phi-1}\right) \end{aligned} \tag{124}$$

Combining (123) and (124) yields the conclusion of the theorem. ■

Proof of Proposition 6. First note that a marginal increase in the tax rate χ in each period prior to retirement can raise the agents' minimum assets by:

$$\omega = Y\int_{-T}^0 e^{-(r+q)s}ds = Y\frac{e^{(r+q)T}-1}{r+q}$$

By an argument similar to Proposition 1, the agent's value function at birth (time $-T$) can be rewritten as:

$$F = \min_{\widetilde{X}_s,\lambda>0} E_{(-T)}\left[\begin{array}{c} \int_{-T}^0 e^{-(\rho+q)(s+T)}\max_{c_s}\left(\frac{c_s^{1-\gamma}}{1-\gamma} - \lambda e^{\rho(s+T)}\widetilde{X}_s\frac{H_s}{H_{(-T)}}c_s\right)ds \\ +\lambda(1-\chi)Y\int_{-T}^0 e^{-q(s+T)}\frac{H_s}{H_{(-T)}}\widetilde{X}_s ds + \max_{W_{0^+}\geq 0}\left(e^{-(\rho+q)T}J(W_{0^+}+\chi\omega) - \lambda\widetilde{X}_0 e^{-qT}\frac{H_0}{H_{(-T)}}W_{0^+}\right)\end{array}\right] \tag{125}$$

where $J\left(W_{0^+}+\chi\omega\right)$ is given in proposition 2 and $\widetilde{X}_s$ is a decreasing process starting at $\widetilde{X}_{(-T)} = 1$. Let the expected value of the expression inside the square brackets be denoted as $U(\widetilde{X}_s, \lambda)$, so that:

$$F^{(\chi)} = \min_{\widetilde{X}_s,\lambda} U(\widetilde{X}_s, \lambda; \chi)$$

Differentiating $U(\widetilde{X}_s, \lambda; \chi)$ with respect to χ gives:

$$U_\chi = E_{(-T)}\left[1_{\left\{\lambda\widetilde{X}_0\frac{H_0}{H_{(-T)}}<\xi^{-\gamma}\right\}}e^{-(\rho+q)T}J'(W_{0^+}+\chi\omega)\omega - \lambda Y\int_{-T}^0 e^{-q(s+T)}\frac{H_s}{H_{(-T)}}\widetilde{X}_s ds\right] \tag{126}$$

Whenever $\lambda\widetilde{X}_0\frac{H_0}{H_{(-T)}} \geq \xi^{-\gamma}$, so that the constraint $W_{0^+} \geq 0$ does not bind, one can use the first order condition from the second maximization problem inside the square brackets of (125) to obtain

$$J'(W_{0^+}+\chi\omega) = \lambda\widetilde{X}_0 e^{\rho T}\frac{H_0}{H_{(-T)}}$$

[31] This is an implication of the Skorohod equation. See Karatzas and Shreve (1991).

This allows one to rewrite expression (126) as

$$
\begin{aligned}
U_\chi &= E_{(-T)}\left[\left(1_{\left\{\lambda \widetilde{X}_0 \frac{H_0}{H_{(-T)}}<\xi^{-\gamma}\right\}}\lambda \widetilde{X}_0 e^{-qT}\frac{H_0}{H_{(-T)}}\omega - \lambda Y\int_{-T}^{0} e^{-q(s+T)}\frac{H_s}{H_{(-T)}}\widetilde{X}_s ds\right)\right] \\
&\leq E_{(-T)}\left[\left(\lambda \widetilde{X}_0 e^{-qT}\frac{H_0}{H_{(-T)}}\omega - \lambda Y\int_{-T}^{0} e^{-q(s+T)}\frac{H_s}{H_{(-T)}}\widetilde{X}_s ds\right)\right] \\
&= \lambda\delta e^{-qT}\omega - \lambda\delta E_{(-T)}\left(\int_{-T}^{0} e^{-q(s+T)} Y\frac{H_s}{H_{(-T)}}\frac{\widetilde{X}_s}{\delta}ds\right) \qquad (127)
\end{aligned}
$$

where:

$$\delta = E_{(-T)}\left(\widetilde{X}_0\frac{H_0}{H_{(-T)}}\right)$$

Furthermore,

$$
\begin{aligned}
E_{(-T)}\left(\int_{-T}^{0} e^{-q(s+T)} Y\frac{H_s}{H_{(-T)}}\frac{\widetilde{X}_s}{\delta}ds\right) &= \int_{-T}^{0} e^{-q(s+T)} Y\frac{E_{(-T)}\left(H_s\widetilde{X}_s\right)}{E_{(-T)}\left(H_0\widetilde{X}_0\right)}ds = \\
&= e^{rT}\int_{-T}^{0} e^{-(r+q)(s+T)} Y\frac{E_{(-T)}\left(e^{r(s+T)}\frac{H_s}{H_{(-T)}}\widetilde{X}_s\right)}{E_{(-T)}\left(e^{rT}\frac{H_0}{H_{(-T)}}\widetilde{X}_0\right)}ds \\
&\geq Ye^{rT}\int_{-T}^{0} e^{-(r+q)(s+T)}ds = \omega e^{-qT} \qquad (128)
\end{aligned}
$$

where the inequality follows from the fact that $e^{rs}H_s$ is a martingale while $\widetilde{X}_s$ is a decreasing process, so that $\widetilde{X}_s \geq \widetilde{X}_0$ for all $s \in [-T, 0]$. Combining (127) and (128) leads to $U_\chi \leq 0$.

Hence, letting $\chi^{\min}$ denote the minimum tax rate that will satisfy (8) as given by (44), it follows that $U(\widetilde{X}_s, \lambda; \chi^{\min}) > U(\widetilde{X}_s, \lambda; \chi)$ for all $\chi \in \left(\chi^{\min}, 1\right)$. This furthermore implies that:

$$F^{(\chi^{\min})} = U(\widetilde{X}_s^{\chi^{\min}}, \lambda^{\chi^{\min}}; \chi^{\min}) \geq U(\widetilde{X}_s^{\chi^{\min}}, \lambda^{\chi^{\min}}; \chi) \geq U(\widetilde{X}_s^{\chi}, \lambda^{\chi}; \chi) = F^{(\chi)}$$

where $\widetilde{X}_s^{\chi}, \lambda^{\chi}$ denote the minimizers of U given χ and similar for $\widetilde{X}_s^{\chi^{\min}}, \lambda^{\chi^{\min}}$. Hence it is never optimal to set the tax rate above $\chi^{\min}$. ■

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