Technological Growth, Asset Pricing, and Consumption Risk

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Abstract

In this paper we develop a theoretical model in order to understand co-movements between asset returns and consumption over short and long horizons. We present an intertemporal general equilibrium model featuring two types of shocks: "small", frequent and disembodied shocks to productivity and "large" technological innovations, which are embodied into new vintages of the capital stock. The latter types of shocks affect the economy with lags, since firms need to invest before they can take advantage of the new technologies. The delayed reaction of consumption to a large technological innovation helps us explain why short run correlations between returns and consumption growth are weaker than their long run counterparts. Because of this effect, the model can shed some light into the economic mechanisms that make consumption based asset pricing more successful at lower frequencies.

Keywords: Production based asset pricing, Continuous time methods, irreversible investment, technology, consumption risk

JEL Classification: G0, G1, E1, E2

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1 Introduction

The process of invention, development and diffusion of new technologies has been widely studied in the economic literature. It is commonly believed that technological progress is the most important factor in determining living standards over the long run. It appears equally plausible that technological advancement is a key determinant of asset price movements during many periods of economic history.

Our goal in this paper is a) to illustrate how adoption of large technological innovations will lead to cycles in output and asset prices and b) to use our model in order to understand the *economic forces* behind the empirical success of recent literature that has re-ignited interest in consumption based asset pricing, by emphasizing the correlation between returns and consumption growth over long horizons.

The key idea behind our theoretical framework is that productivity growth comes in the form of two shocks. The first type are "small", frequent, disembodied shocks, that affect earnings in the entire economy. One should think of them as daily news that appear in the financial press (variations in the supply of raw materials, political decisions that affect production, bad weather etc.). However, these types of shocks do not fundamentally alter the technology used to produce output. The second type of shocks are Poisson arrivals of major technological or organizational innovations, like automobiles, the internet, just in time manufacturing etc.. These shocks will not affect the economy on impact, but only with a lag. The reason is that firms will need to make investments in order to take advantage of these innovations. Given the irreversibility of the investment decisions, and the high relative cost of these new technologies on arrival, there is an endogenous lag between the impact of the second type of shock and its eventual effects on output. Importantly, we show that the process of adoption of new technologies leads to endogenous persistence and cycles, even though all shocks in the model arrive in an unpredictable *pure i.i.d.* fashion.

The link between the macroeconomy and asset pricing in our model revolves around the idea that growth options of firms exhibit a “life cycle” as technologies diffuse. On impact of a major technological shock, growth options emerge in the prices of all securities. We show that these growth options are riskier than assets in place. Hence, in the initial phases of the technological cycle (i.e. when the economy is below its stochastic trend line) expected returns in the stock market are higher, simply because most growth options have not been exercised. As time passes, firms start converting growth options into assets in place, hence reducing the implicit riskiness of their stock. Eventually, the new technology enters the region of diminishing marginal returns at the aggregate, most growth options get exercised and expected returns become low. Therefore, the model produces countercyclical variation in risk premia at the aggregate.

By combining the above two observations we derive a number of new predictions about the correlation between consumption and returns at high and low frequencies. In the model, the effects of a major technological innovation produce consumption gains with a lag, whereas they immediately affect returns. This attenuates the correlation between consumption and returns in the short run and strengthens it over longer horizons.

This simple observation drives the success of consumption based asset pricing at lower frequencies: The model allows us to endogenously obtain the cross sectional distribution of returns, as a function of size and value. We are thus able to use our model as a laboratory in order to examine the consumption CAPM over different horizons. We show that we can replicate certain patterns in the data, namely the success of long horizon versions of the CAPM and the failure of short horizon versions. In the model, it is the covariance between returns and certain permanent shocks to consumption that drives expected returns. Accordingly, lower frequency correlations between consumption and returns are better able to capture the source of these differences for different portfolios.

1.1 Relation to the literature

The literature closest to this paper is the production based asset pricing literature[1]. The papers by Gomes, Kogan, and Zhang [2003] and Carlson, Fisher, and Giammarino [2004] are the most related to ours.

The paper by Carlson, Fisher, and Giammarino [2004] develops the intuition that the exercise of growth options can lead to variation in expected returns in a partial equilibrium setting. In our paper, a similar mechanism is at operation in general equilibrium. By taking the model to

[1] For contributions to this literature, see Cochrane [1996], Jermann [1998], Berk, Green, and Naik [1999], Berk, Green, and Naik [2004], Kogan [2001], Kogan [2004], Gomes, Kogan, and Zhang [2003], Carlson, Fisher, and Giammarino [2004], Zhang [2005], Cooper [2004], Gourio [2004], Gala [2005].

general equilibrium, we are able to obtain consumption, the stochastic discount factor, and asset returns endogenously. Importantly, the general equilibrium framework allows us to extend the intuitions in Carlson, Fisher, and Giammarino [2004] so as to discuss a richer set of implications for asset pricing, such as short and long run correlations between consumption and returns in the time series and the cross section. Gomes, Kogan, and Zhang [2003] also examine the cross section and the time series of returns in a general equilibrium setting, as we do. The two most significant differences between their model and ours is a) the distinction between "embodied" and "disembodied" aggregate technological shocks, and b) the presence of a true timing decision as to the exercise of the growth options. Gomes, Kogan, and Zhang [2003] follow the seminal paper by Berk, Green, and Naik [1999] and assume that options arrive in an i.i.d. fashion across firms, and have a "take it or leave it" nature: The firms must decide "on the spot" if they want to proceed with the investment or not. By contrast in our model, all firms have discretion as to the timing of their investment. This is not a mere technicality. It is the very reason for the simultaneity in the exercise of growth options that leads to our endogenous cycles. Alternatively put, in Gomes, Kogan, and Zhang [2003] cycles emerge out of the assumption of a trend stationary productivity process. In our model, all exogenous shocks follow random walks. Cycles emerge *endogenously* as the result of technological adoption. The practical implication is that the consumption process in our model features both permanent trend shocks and some small endogenously arising cyclical shocks. Our model preserves thus a strong random walk component in consumption, and hence is able to match the variability of consumption over the short and the long run. This appears particularly important when studying consumption based asset pricing using consumption growth over varying horizons.

The present paper also provides a first step towards establishing a bridge between the production based asset pricing literature and the literature on long run risk[2]. Papers in the long run risk literature typically use an Epstein-Zin utility specification that introduces long run risk into the pricing kernel. From that point on, the papers use the empirical fact that correlations between long run consumption growth and cross sectional differences in returns are stronger than their short run counterparts, a result established by Parker and Julliard [2005] and Bansal, Dittmar, and Lundblad

[2] See Bansal and Yaron [2004], Bansal, Dittmar, and Kiku [2005], Bansal, Dittmar, and Lundblad [2004], Hansen, Heaton, and Li [2005], Daniel and Marshall [1999], Parker and Julliard [2005].

[2004][3]. In this paper our goal is different: Instead of taking the difference in correlations in the short run and the long run as given, and deriving implications for average returns, we try to understand economically *why* the correlations are different in the first place. Hence, the analysis is driven by the production side of the model, since our main goal is to understand how the heterogeneity across firms can help us understand correlation patterns in the data. In particular our simulation results replicate the patterns observed by Parker and Julliard [2005].

We believe that in light of the notorious difficulties in measuring long run correlations in the data[4], it appears particularly useful to develop some theoretical intuition on the economic mechanisms behind these correlations. And even though our utility specification in this paper is simple, in order to make the effects of production more transparent, we can reasonably conjecture that richer utility specifications such as Epstein and Zin [1989] will further strengthen the conclusions. The same is true for approaches that make consumption adjust sluggishly by assuming inattentiveness[5].

Our paper also complements the work of Menzly, Santos, and Veronesi [2004]. In that paper the behavior of consumption and dividends are assumed exogenously. Interestingly, our analysis will endogenously produce a process for the dividends of a firm that will resemble Menzly, Santos, and Veronesi [2004], in the sense that the total dividends of an individual firm will be cointegrated with aggregate consumption. However, given that our consumption process has some small predictable components, we can additionally characterize the differences between its short horizon and long horizon covariance with returns.

A recent paper that is related to the present one is Pastor and Veronesi [2005]. In their model, Pastor and Veronesi [2005] connect the arrival of technological growth with the "bubble"-type behavior of asset prices around these events[6]. Our model produces some patterns that are similar. However, the focus of the two papers and the mechanisms are different. Our mechanism uses the endogenous exercise of growth options to produce variations in expected returns, and we focus on providing a link between technological arrivals and correlations between consumption and returns

[3]Bansal, Dittmar, and Lundblad [2004] examine the cointegrating relationship between dividends per share and consumption growth, whereas Parker and Julliard [2005] directly study the covariance between returns and consumption over longer intervals, as we do in simulations of the model.

[4]Hansen, Heaton, and Li [2005]

[5]See e.g. Abel, Eberly, and Panageas [2006].

[6]Other papers that have analyzed the recent upswing in prices include Pastor and Veronesi [2004], Jermann and Quadrini [2002].

in the time series and the cross section.

There is a vast literature in macroeconomics and growth that analyzes innovation, dissemination of new technologies and the impact of the arrival of new capital vintages[7]. Our paper has however a fundamentally different scope than the literature on growth and innovation. In that literature, uncertainty and the pricing of risk are not the focus of the analysis. By contrast, these papers analyze innovation decisions in much greater depth than we do. The trade-off is that they cannot allow for sufficiently rich uncertainty, and an endogenous determination of the stochastic discount factor as is possible in the simpler setup of our paper. This is why most of this literature cannot be readily used for an in-depth asset pricing analysis in the time series and the cross section. Our approach is to simplify the model sufficiently, so as to obtain some of the key predictions of this literature, while being able to obtain tractable closed form solutions in a framework where the pricing of risk is central.

Finally, the model of this paper can also help link the findings of a recent literature in macroeconomics on the delayed reaction of the economy to technological shocks[8] with the findings in the finance literature on the success of consumption based asset pricing at longer horizons.

A technical contribution of our work is that it provides a tractable solution to a general equilibrium model, where the micro-decsions are "lumpy" and exhibit optimal stopping features. The micro decision of the firm has a similar structure to the recent sequence of papers by Abel and Eberly [2003], Abel and Eberly [2002b], Abel and Eberly [2004]. Just as firms in these papers adapt to the technological frontier at an optimally chosen time, firms in our framework decide on the optimal time to plant new trees. Moreover, by assuming cross sectional heterogeneity only at the beginning of an epoch, we can aggregate over firms in a much simpler way than the existing literature[9].

The structure of the paper is as follows: Section 2 presents the model and Section 3 the resulting equilibrium allocations. Section 4 presents the qualitative and quantitative implications of the

[7] This is a truly vast literature and we do not attempt to review it. We just mention the papers by Jovanovic and Rousseau [2004], Jovanovic and MacDonald [1994], Jovanovic and Rousseau [2003], Greenwood and Jovanovic [1999], Atkeson and Kehoe [1999], Atkeson and Kehoe [1993], Helpman [1998] as representative examples.

[8] See Greenwood and Yorukoglu [1997], Basu, Fernald, and Kimball [2002], Vigfusson [2004].

[9] For other analytically tractable approaches to aggregation see Caballero and Engel [1999], Caballero and Engel [1991], Caballero and Pindyck [1996], Novy-Marx [2003].

model. Section 5 concludes. All proofs are given in the appendix.

2 The model

2.1 Trees, Firms and Technological Epochs

There exists a continuum of firms indexed by $j \in [0,1]$. Each firm owns a collection of trees that have been planted in different technological epochs, and its total earnings is just the sum of the earnings produced by the trees it owns. Each tree in turn produces earnings that are the product of three components: a) a vintage specific component that is common across all trees of the same technological epoch, b) a time invariant tree specific component and c) an aggregate productivity shock. To introduce notation, let $Y_{N,i,t}$ denote the earnings stream of tree i at time t, which was planted in the technological epoch $N \in (-\infty..-1,0,1,..+\infty)$. In particular, assume the following functional form for $Y_{N,i,t}$:

$$Y_{N,i,t} = \left(\overline{A}\right)^N \zeta(i)\theta_t \tag{1}$$

$\left(\overline{A}\right)^N$ captures the vintage effect. $\overline{A} > 1$ is a constant. $\zeta(\cdot)$ is a positive strictly decreasing function on $[0,1]$, so that $\zeta(i)$ captures a tree specific effect. θ_t is the common productivity shock and evolves as a geometric Brownian Motion:

$$\frac{d\theta_t}{\theta_t} = \mu dt + \sigma dB_t \tag{2}$$

where $\mu > 0, \sigma > 0$ are constants, and B_t is a standard Brownian Motion.

Technological epochs arrive at the Poisson rate $\lambda > 0$. Once a new epoch arrives, the index N becomes $N+1$, and every firm gains the option to plant a single tree of the new vintage at a time of its choosing. Since $\overline{A} > 1$, and N grows to $N+1$, equation (1) reveals that trees of a later epoch are on average "better" than previous trees. To keep notation compact, we shall use the letter N to refer to the current epoch instead of N.

Firm heterogeneity is introduced as follows: Once epoch N arrives, each firm j draws a random number $i_{j,N}$ from a uniform distribution on $[0,1]$. This number informs the firm of the type of tree that it can plant in the new epoch. In particular a firm that drew the number $i_{j,N}$ can plant a tree with tree specific productivity $\zeta(i_{j,N})$. These numbers are drawn in an i.i.d fashion across epochs: It is possible that firm j draws a low $i_{j,N}$ in epoch N, a high $i_{j,N+1}$ in epoch $N+1$ etc.

To simplify the setup, we shall assume that once an epoch changes, the firm loses the option to plant a tree that corresponds to any previous epoch. It can only plant a tree corresponding to the technology of the current epoch[10].

Let:

$$X_{j,t} = \sum_{n=-\infty..N} \overline{A}^{n} \zeta(i_{j,n}) 1_{\{\widetilde{\chi}_{n,j}=1\}} \tag{3}$$

where N denotes the technological epoch at time t and $1_{\{\widetilde{\chi}_{n,j}=1\}}$ is an indicator function that is 1 if firm j decided to plant a tree in technological epoch n and 0 otherwise. A firm's total earnings are then given by:

$$Y_{j,t} = X_{j,t}\theta_t$$

Any given firm determines the time at which it plants a tree in an optimal manner. Planting a tree at time t requires a fixed cost of q_t. This cost is the same for all trees of a given epoch and represents payments that need to be given to "gardeners" who will plant these trees. To keep with the usual assumptions of a Lucas tree economy, we shall assume that the company finances these fixed payments by issuing new equity in the amount q_t.

Assuming complete markets, the firm's objective is to maximize its share price. Given that options to plant a tree arrive in an i.i.d fashion across epochs, there is no linkage between the decision to plant a tree in this epoch and any future epochs. Thus, the option to plant a tree can be studied in isolation in each epoch.

The optimization problem of firm j in epoch N amounts to choosing the optimal stopping time τ:

$$P^{o}_{N,j,t} \equiv \sup_{\tau} E_t \left\{ 1_{\{\tau<\tau_{N+1}\}} \left[\left(\overline{A}^{N} \zeta(i_{j,N}) \int_{\tau}^{\infty} \frac{H_s}{H_t}\theta_s ds \right) - \frac{H_\tau}{H_t} q_\tau \right] \right\} \tag{4}$$

where H_s is the (endogenously determined) stochastic discount factor, τ_{N+1} is the random time at which the next epoch arrives, while $P^{o}_{N,j,t}$ denotes the (real) option of planting a new tree in epoch N.

Given the setup, a firm's price will consist of three components: a) the value of assets in place, b) the value of the growth option in the current technological epoch and c) the value of the growth

[10] This assumption would result endogenously, as long as it costs the same to plant a "new" tree and an "old" tree, along with an assumption that the least productive tree of the new vintage is more productive than the most productive tree of the previous vintage.

options in all subsequent epochs. To see this, let:

$$P^A_{j,t} \equiv X_{j,t}\left(E_t \int_t^\infty \frac{H_s}{H_t}\theta_s ds\right) \tag{5}$$

denote the value of assets in place (with $X_{j,t}$ as defined in [3]). Then the price of firm j, assuming it has not planted a tree (yet) in technological epoch N is

$$P_{N,j,t} = P^A_{j,t} + P^o_{N,j,t} + P^f_{N,t} \tag{6}$$

where:

$$P^f_{N,t} = E_t\left(\sum_{n=N+1..\infty} \frac{H_{\tau_n}}{H_t} P^o_{n,j,\tau_n}\right)$$

and τ_n denotes the time at which technological epoch $n = N+1..\infty$ arrives. The first term on the right hand side of (6) is the value of assets in place, while the second term is the value of the growth option in the current epoch. The third term is the value of all future growth options. Naturally, for a firm that has planted a tree in the current technological epoch there exists no longer a current epoch option and hence its value is given by:

$$P_{N,j,t} = P^A_{j,t} + E_t\left(\sum_{n=N+1...\infty} \frac{H_{\tau_n}}{H_t} P^o_{n,j,\tau_n}\right)$$

2.2 Aggregation

The total output in the economy at time t is given by

$$Y_t = \int_0^1 Y_t(j)dj = \left(\int_0^1 X_{j,t}dj\right)\theta_t = X_t\theta_t \tag{7}$$

with $X_{j,t}$ defined in (3) and X_t defined as

$$X_t = \int_0^1 X_{j,t}dj \tag{8}$$

It will be particularly useful to introduce one extra piece of notation. Let $K_{N,t} \in [0,1]$ denote the mass of firms that have updated their technology in technological epoch N up to time t. We show formally later that $K_{N,t}$ coincides with the index of the last tree that was planted in epoch N.

Since investment in new trees is irreversible, $K_{N,t}$ (when viewed as a function of time) will be an increasing process. Given the definition of $K_{N,t}$, the aggregate output is given as

$$Y_t = \left[\sum_{n=-\infty..N-1} \overline{A}^{(n-N)} \left(\int_0^{K_{n,\overline{\tau_n}}} \zeta(i)di \right) + \int_0^{K_{N,t}} \zeta(i)di \right] \overline{A}^N \theta_t$$

where $\overline{\tau_n} = \tau_{n+1}$ denotes the time at which epoch n ended (and epoch $n+1$ started). To analyze this decomposition it will be easiest to define

$$F(x) = \int_0^x \zeta(i)di$$

It can easily be verified that, $F_x \geq 0$ (since $\zeta(\cdot) > 0$) and $F_{xx} < 0$, (since $\zeta(\cdot)$ is declining). Hence $F(x)$ has the two key properties of a production function. Using the definition of $F(\cdot)$, Y_t can accordingly be rewritten as

$$Y_t = \left[\sum_{n=-\infty..N-1} \overline{A}^{(n-N)} F\left(K_{n,\overline{\tau_n}}\right) + F(K_{N,t}) \right] \overline{A}^N \theta_t \tag{9}$$

Aggregate output is thus the product of two components: A stationary component (inside the square brackets) and a stochastic trend $\left(\overline{A}^N \theta_t\right)$ which captures the joint effects of technological progress due to the arrival of epochs $\left(\overline{A}^N\right)$ and aggregate productivity growth (θ_t). The term inside the square brackets is a weighted average of the contributions of the different vintages of trees towards the aggregate product. The weight on trees that were planted in previous epochs decays geometrically at the rate $\overline{A}$. In this sense, $\overline{A}$ is simultaneously the rate of technological progress (in terms of new trees) and technological obsolescence (in terms of existing ones).

2.3 Markets

As is typically assumed in "Lucas Tree" models, each firm is fully equity financed and the representative agent holds all its shares. Moreover, claims to the output stream of these firms are the only assets in positive supply, and hence the total value of positive supply assets in the economy is:

$$P_{N,t} = \int_0^1 P_{N,j,t} dj \tag{10}$$

Next to the stock market for shares of each company there exists a (zero net supply) bond market, where agents can trade zero-coupon bonds of arbitrary maturity. We shall assume that markets are complete.[11] Accordingly, the search for equilibrium prices can be reduced to the search for a stochastic discount factor H_t, which will coincide with the marginal utility of consumption for the representative agent. (See Karatzas and Shreve [1998], Chapter 4)

[11] In particular there exist markets where agents can trade securities (in zero net supply) that promise to pay 1 unit

2.4 Consumers, Gardeners, and Preferences

To keep with Lucas's analogy of "trees", we shall assume that trees can only be planted by "gardeners". The economy is populated by a continuum of identical consumers/gardeners that can be aggregated into a single representative agent. The representative agent owns all the firms in the economy, and is also the (competitive) provider of gardening services.

We shall allow the agent's utility to exhibit external habit formation with respect to the running maximum of aggregate consumption for both substantive and technical reasons that will become clear in the next subsection. The representative consumer's preference over consumption streams is characterized by a utility function of the form

$$U(C_t, M_t^C)$$

where:

$$M_t^C = \max_{s \leq t} \{C_s\} \tag{11}$$

denotes the running maximum of aggregate consumption up to time t, and $U\left(C_t, M_t^C\right)$ satisfies $U_C > 0$, $U_{CC} < 0$, $U_{M^C} < 0$, $U_{CM^C} > 0$.

Gardeners have a disutility of effort for planting new trees and need to be compensated accordingly. Planting a tree creates a fixed disutility of $U_C(s)\eta(s)$ per tree planted. Hence, the representative agent's utility function is given by:

$$\max_{C_s, dl_s} E_t \left[\int_t^\infty e^{-\rho(s-t)} U(C_s, M_s^C) ds - \int_t^\infty e^{-\rho(s-t)} U_C(s)\eta(s) dl_s \right] \tag{12}$$

where $\rho > 0$ is the subjective discount factor, and $dl(s) \geq 0$ denotes the increments in the number of trees that the representative consumer / gardener has planted.

This utility specification for the representative agent captures the fact that labor services are sunk in this model, i.e. the effort of planting a tree cannot be reversed. Furthermore, there is no loss in generality from specifying the disutility of labor (per tree planted) as $U_C(s)\eta(s)$, since $\eta(s)$ is an arbitrary process.

of the numeraire when technological round N arrives. These markets will be redundant in general equilibrium, since agents will be able to create dynamic portfolios of stocks and bonds that produce the same payoff as these claims. However, it will be easiest to assume their existence throughout to guarantee ex-ante that markets are complete.

Equation (12) implies that the supply of gardening services at time t is perfectly elastic[12] at the price η_t. To see this, let V_W denote the derivative of the gardener's value function with respect to wealth. A gardener will have an incentive to plant a tree if and only if:

$$q_t V_W \geq \eta_t U_C$$

Imposing the envelope condition[13], we obtain $V_W = U_C$. Furthermore, assuming perfect competition among gardeners reveals that the price for planting trees will be given by:

$$q_t = \eta_t \tag{13}$$

The consumer maximizes (12) over consumption plans in a complete market:

$$\max_{C_s, dl_s} E_t \left[\int_t^\infty e^{-\rho(s-t)} U(C_s, M_s^C) ds - \int_t^\infty e^{-\rho(s-t)} U_C(s) \eta(s) dl_s \right] \tag{14}$$

$$s.t.$$

$$E_t \left(\int_t^\infty \frac{H_s}{H_t} C_s ds \right) \leq \int_0^1 P_{N,j,t} dj + E_t \left(\int_t^\infty \frac{H_s}{H_t} q_s dl_s \right) \tag{15}$$

Note that the representative consumer owns all the trees and receives gardening fees q_t every time a firm plants a tree.

2.5 Functional Forms and Discussion

Before proceeding, we need to make certain assumptions on functional forms, in order to solve the model explicitly. The assumptions that we make are intended either a) to allow for tractability or b) to ensure that the solution of the model satisfies certain desirable properties.

The first assumption on functional form concerns the utility $U\left(C_t, M_t^C\right)$. We shall assume that

$$U\left(C_t, M_t^C\right) = \left(M_t^C\right)^\gamma \frac{C_t^{1-\gamma}}{1-\gamma}, \quad \gamma > 1 \tag{16}$$

[12] Perfectly elastic supply of gardening services will safeguard that the supply of capital is elastic from the perspective of shareholders. This is analogous to the standard assumption in the neoclassical theory of investment (without adjustment costs). The only difference with the standard neoclassical growth model is that planting trees does not "crowd" out current output, because it requires effort that is specific to planting trees. This is a particularly plausible assumption, especially since a new technology requires agents to exert effort in order to learn how to install and use the new vintages of capital.

[13] The envelope condition follows directly from the first order equations associated with the Bellman equation (see e.g. Øksendal [2003], Chapter 11)

It can be easily verified that $U_C > 0, U_{CC} < 0, U_{CM^C} > 0, U_{M^C} < 0$, and hence this utility is closely related to the utilities studied in Abel [1990] and exhibits both "envy" ($U_{M^C} < 0$) and catching up with the Joneses ($U_{CM^C} > 0$) in the terminology of Dupor and Liu [2003]. The main difference is that the habit index is in terms of the past consumption maximum, not some exponential average of past consumption as in Campbell and Cochrane [1999] or Chan and Kogan [2002]. Using the running maximum of consumption M_t^C as the habit index is particularly attractive for our purposes, because of the analytic tractability that it will allow[14].

At a substantive level, this utility specification will serve three purposes: First, it will allow us to match first and second moments of the equity premium and interest rates. Second, it will imply that the growth cycles that will arise in the model will leave interest rates unaffected. To see this, note that

$$U_C = \left(\frac{C_t}{M_t^C}\right)^{-\gamma}$$

In equilibrium, it will turn out that

$$\frac{C_t}{M_t^C} = \frac{\theta_t}{\max_{s<t} \theta_s} \tag{17}$$

The right hand side of (17) is unaffected by the investment decisions of firms and this will in turn be true for the stochastic discount factor and therefore the real interest rate. This is a key property of this specification. Without habit formation most of the effects of technological innovations will work through the real interest rate, which is unattractive[15]. Lastly, these preferences will imply constant relative risk aversion, so that we can attribute all our results to the effect of fluctuations in the relative weight of growth options on expected returns. Clearly, with time varying risk aversion,

[14] As most habit level specifications already proposed in the literature, it has the very attractive property that it is "cointegrated" with aggregate consumption in the sense that the difference between $\log(C_t)$ and $\log(M_t^C)$ will be stationary. Moreover, the ratio between C_t and M_t^C will be bounded between 0 and 1 (as is the surplus in Campbell and Cochrane [1999]).

[15] See Campbell and Cochrane [1999]. Keeping interest rates unaffected will overcome an additional severe problem of simple constant relative risk aversion (CRRA) utilities: Agents with CRRA utilities will try to smooth the future consumption gains that arise at the onset of a technological epoch by dissaving. In general equilibrium, savings must remain at 0, and so real interest rates will have to rise in order to bring savings back to zero. With a coefficient of relative risk aversion above 1, the smoothing motive will be sufficiently strong as to push the interest rate so high, that the market price to earnings ratio will *decline*. This appears to be at odds with the stock market booms that are typically observed in periods of rapid technological innovation as the nineties and the twenties. By leaving interest rates unaffected by growth cycles, our specification avoids this problem.

our results would look even stronger. We would finally like to remark that a specification of the model with standard CRRA utilities would still produce most of the key results of the paper (i.e. correlations between consumption and returns in the long run and the short run) but would miss the first two unconditional moments of excess returns and interest rates.

Our next choice of functional form concerns the specification of the disutility of effort for gardening services η_t. In general, we think of "gardening" services as compensation for the "know how" and the effort that is provided by experts who need to invent, create and install the new capital stock. Our choice for the functional form of these costs is motivated by four main considerations: First, we want the magnitude of this compensation to share the same trend as aggregate output. Second, we want to keep the amount of gardening services provided stationary. Third, we want to keep the gardening fees constant within each epoch, in order to keep the analysis transparent and tractable. Furthermore, this will imply that the correlation between these effort payments and output will be zero in the short run and will strengthen only in the long run, a pattern that is consistent with the behavior of real wages in the data. Fourth, we want to capture the idea that the costs of planting a tree are prohibitively high at the beginning of an epoch, so that even the most productive firm will have an incentive to wait.

To give a specification that satisfies all four objectives simultaneously, define

$$M_t = \max_{s \leq t} \theta_s \tag{18}$$

and let $\eta_t = q\overline{A}^N M_{\tau_N}$ so that equation (13) implies

$$q_t = q\overline{A}^N M_{\tau_N} \tag{19}$$

where $q > 0$ is a constant, $\overline{A}^N$ is the vintage specific productivity of trees in the current epoch and M_{τ_N} is the value of the historical maximum of θ_t at the start of the technological epoch. Note that these costs will always grow between epochs[16]. However they will stay constant within an epoch. This is attractive, because it will make it easier to isolate the channels that lead to variations in asset prices. Moreover, the costs will share the same trend growth as consumption[17], and hence

[16] Since N will grow when an epoch changes, and $M_{\tau_{N+1}}$ will be higher than M_{τ_N}

[17] To see this note that the trend growth in consumption is $\overline{A}^N\theta_t$ while the costs are given by $q_t = q\overline{A}^N M_{\tau_N} = q\overline{A}^N\theta_t\left(\frac{M_{\tau_N}}{\theta_t}\right)$. Since $\left(\frac{M_{\tau_N}}{\theta_t}\right)$ is stationary, it follows that (log) consumption and the (log of) q_t share the same trend and hence are cointegrated.

the compensation to gardeners will be cointegrated with aggregate consumption, while the amount of services provided will be stationary.

A final assumption that is made purely for technical convenience is that

$$\zeta(i) = \zeta_0 (1-i)^{\nu}, \quad i \in [0,1] \tag{20}$$

where $\zeta_0, \nu > 0$ are constants.

2.6 Equilibrium

The equilibrium definition is standard. It requires that all markets clear and that all actions be optimal given prices.

Definition 1 *A competitive equilibrium is a set of stochastic processes* $\langle C_t, K_{n,t}, H_t, dl_t, q_t \rangle$ *s.t.*

a) C_t, dl_t *solve the optimization problem (14) subject to (15)*

b) Firms determine the optimal time to plant a tree by solving the optimization problem (4)

c) The goods market clears[18]*:*

$$C_t = Y_t \text{ for all } t \geq 0 \tag{21}$$

where Y_t *is given by (9) and* $K_{n,t}$ *is given by*

$$K_{n,t} = \int_0^1 \widetilde{\chi}_{n,j,t} dj \tag{22}$$

where $\widetilde{\chi}_{n,j,t}$ *is an indicator that takes the value* 1 *if firm* j *has planted a tree in epoch* n *by time* t *and* 0 *otherwise.*

d) The market for gardening services clears, i.e. for all n, t :

$$dl_t = dK_{n,t} \tag{23}$$

e) The markets for all assets clear.

[18] This condition might seem surprising at first. One would expect that investment in new trees should introduce a wedge between output and consumption in this economy. The resolution of the puzzle is that new trees in this economy are created by the extra effort of gardeners without crowding out current consumption goods. See also footnote 12

If one could determine the optimal processes $K_{n,t}$, assuming that the costs of gardening are given by (13), then the optimal consumption process could be readily determined by (21), and (9). This would in turn imply that the equilibrium stochastic discount factor is given by:

$$H_t = e^{-\rho t} U_C \tag{24}$$

This observation suggests that the most natural way to proceed in order to determine an equilibrium is to make a conjecture about the stochastic discount factor H_t, solve for the optimal stopping times in equation (4), aggregate in order to obtain the processes $K_{n,t}$ for $n = N, ...\infty$, and verify that the resulting consumption process satisfies (24). This is done in section 3.

3 Equilibrium Allocations

3.1 Investment decisions by firms

We first start by making a guess about the stochastic discount factor in general equilibrium. In particular we assume that the equilibrium stochastic discount factor is:

$$H_t = e^{-\rho t} \left(\frac{\theta_t}{M_t} \right)^{-\gamma} \tag{25}$$

with M_t defined as in (18). In Proposition 1 in the appendix we present the closed form solution to the firm's optimal stopping problem under this stochastic discount factor. We also show that the consumption and investment process that results at the aggregate will satisfy (17) and hence constitute a competitive equilibrium.

The solution to the optimal stopping problem of the firm has an intuitive "threshold" form: Firm j in round N should plant a tree when the ratio of aggregate productivity θ_t to its running maximum at the beginning of the current epoch (M_{τ_N}) crosses the threshold $\overline{\theta}^{(j)}$ given by $\overline{\theta}^{(j)} = \Xi / \zeta(i_{N,j})$, where $\Xi > 0$ is an appropriate constant given explicitly in the appendix. Formally, the optimal time for firm j to plant a tree in epoch N is when:

$$\tau^*_{j,N} = \inf_{\tau_N \leq t < \tau_{N+1}} \left\{ t : \frac{\theta_t}{M_{\tau_N}} \geq \frac{\Xi}{\zeta(i_{N,j})} \right\} \tag{26}$$

The optimal policies of the firms possess three desirable and intuitive properties: First, no firm will find it optimal to plant a tree immediately when the new epoch arrives, as long as[19]:

$$\frac{\Xi}{\zeta(0)} > 1 \tag{27}$$

which we shall assume throughout.

Second, a key implication of (26) is that the firms that have the option to plant a more "productive" tree will always go first, since the investment threshold $\overline{\theta}^{(j)}$ will be lower for them. This is intuitive: A firm that can profit more from planting a tree has a higher opportunity cost of waiting and should always plant a tree first.

Third, and most importantly, there are going to be strong correlations between the optimal investment decisions of the firms. Conditional on $\frac{\theta_t}{M_{\tau_N}}$ reaching the relevant investment threshold $\frac{\Xi}{\zeta(0)}$ for the first firm, a number of other firms will also find it optimal to invest in close proximity.[20]

Figure 1 gives a visual impression of these facts by plotting the impulse response function of an increase in N (i.e. the arrival of a new epoch) on consumption.

As can be seen, in the short run consumption is unaffected, as all firms are waiting to invest. Once however the threshold for the first firm is reached, then the growth rate of consumption peaks and starts to decline thereafter. The intuition for this decline is the following: the most profitable firms start investing first, and hence the most productive investment opportunities are depleted. This leaves less attractive investment opportunities unexploited and hence a moderation in the anticipated growth rate of the economy going forward.

This delayed reaction of the economy to a major technological shock is consistent with recent findings in the macroeconomic literature (See e.g. Vigfusson [2004] and references therein).

3.2 Aggregate consumption and endogenous cycles

Another interesting implication of the behavior of aggregate consumption can be seen upon examining equation (9). Taking logs, this equation becomes:

$$\log(Y_t) = \log(C_t) = \log(\theta_t) + N\log(\overline{A}) + x_t \tag{28}$$

[19] To see why this condition is sufficient to induce waiting, examine (26) and note that at the beginning of an epoch $\frac{\theta_{\tau_N}}{M_{\tau_N}} \leq 1$. Hence all firms (even the most productive one) will be "below" their investment thresholds.

[20] This is simply because $\zeta(i)$ is a continuous function of i and θ_t is a continuous function of time.

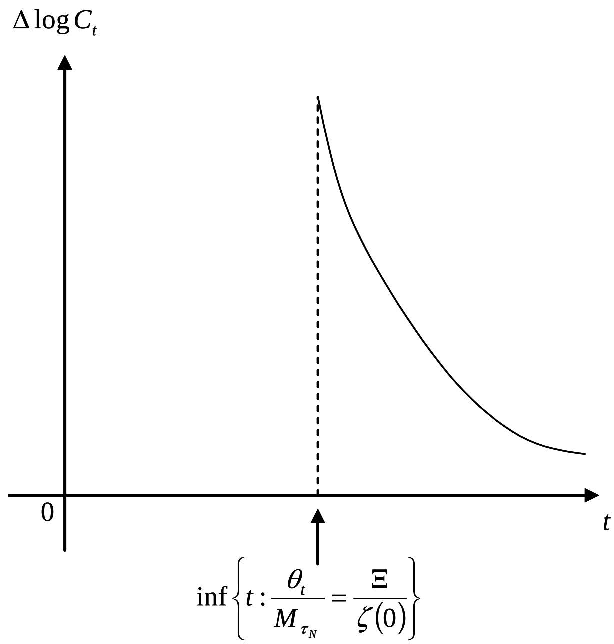

FIGURE 1: Impulse response function of a shock to the Poisson process N_t. The shock impacts the economy at time 0, which is given by the intersection of the x and y axes.

where x_t is equal to:

$$x_t = \log\left(\frac{X_t}{\overline{A}^N}\right) = \log\left[\sum_{n=-\infty..N-1} \overline{A}^{(n-N)} F\left(K_{n,\overline{\tau_n}}\right) + F(K_{N,t})\right] \tag{29}$$

The expression inside the square brackets of (29) is a geometrically declining average (at the rate $\frac{1}{\overline{A}}$) of the random terms $F(K_{n,\overline{\tau_n}})$. This means that e^{x_t} will behave approximately as an AR(1) process (across epochs)[21].

Hence, the model is able to produce *endogenous cycles*, on top of the pure random walk stochastic trend $\log(\theta_t) + N\log(\overline{A})$ that we assumed exogenously. The fact that consumption will exhibit a strong random walk component, is desirable from an empirical point of view, since variations in consumption are commonly believed to exhibit a stochastic (as opposed to a deterministic) trend. This presents an improvement over existing production based general equilibrium models, where consumption exhibits trend-stationary behavior. Most importantly, it will allow us to examine the reaction of asset returns to both trend and cyclical shocks to consumption, as is done in recent papers on consumption risk over different horizons (see e.g. Bansal, Dittmar, and Kiku [2005]).

Finally, since x_t is the difference between the (log) level of consumption and the (log) level of the trend $(\log(\theta_t) + N\log(\overline{A}))$ it follows that x_t has predictive power over future consumption growth. Using the same methods as in Cochrane [1994], one can show that:

$$x_t - E(x) = -\int_t^{\infty} \left[E_t\left(d\log(C_{t+s})\right) - E\left(d\log(C_{t+s})\right)\right] \tag{30}$$

This equation shows that $x_t - E(x)$ can be thought of as a measure of the distance between actual output and stochastic trend. Whenever this difference is negative, the economy has not absorbed the full benefit of existing technology that is captured in the stochastic trend. Therefore future growth rates will be large. By contrast whenever $x_t - E(x)$ is positive, this means that the economy is above its trend line, and the future growth rates will be moderate. Figure 2 illustrates these notions graphically.

4 Qualitative and Quantitative Implications

[21] The statement would be exact if the terms $F(K_{n,\overline{\tau_n}})$ were perfectly i.i.d. across epochs. However there is small but positive persistence in the stationary components $F(K_{n,\tau_n})$, that further amplifies the persistence in x_t.

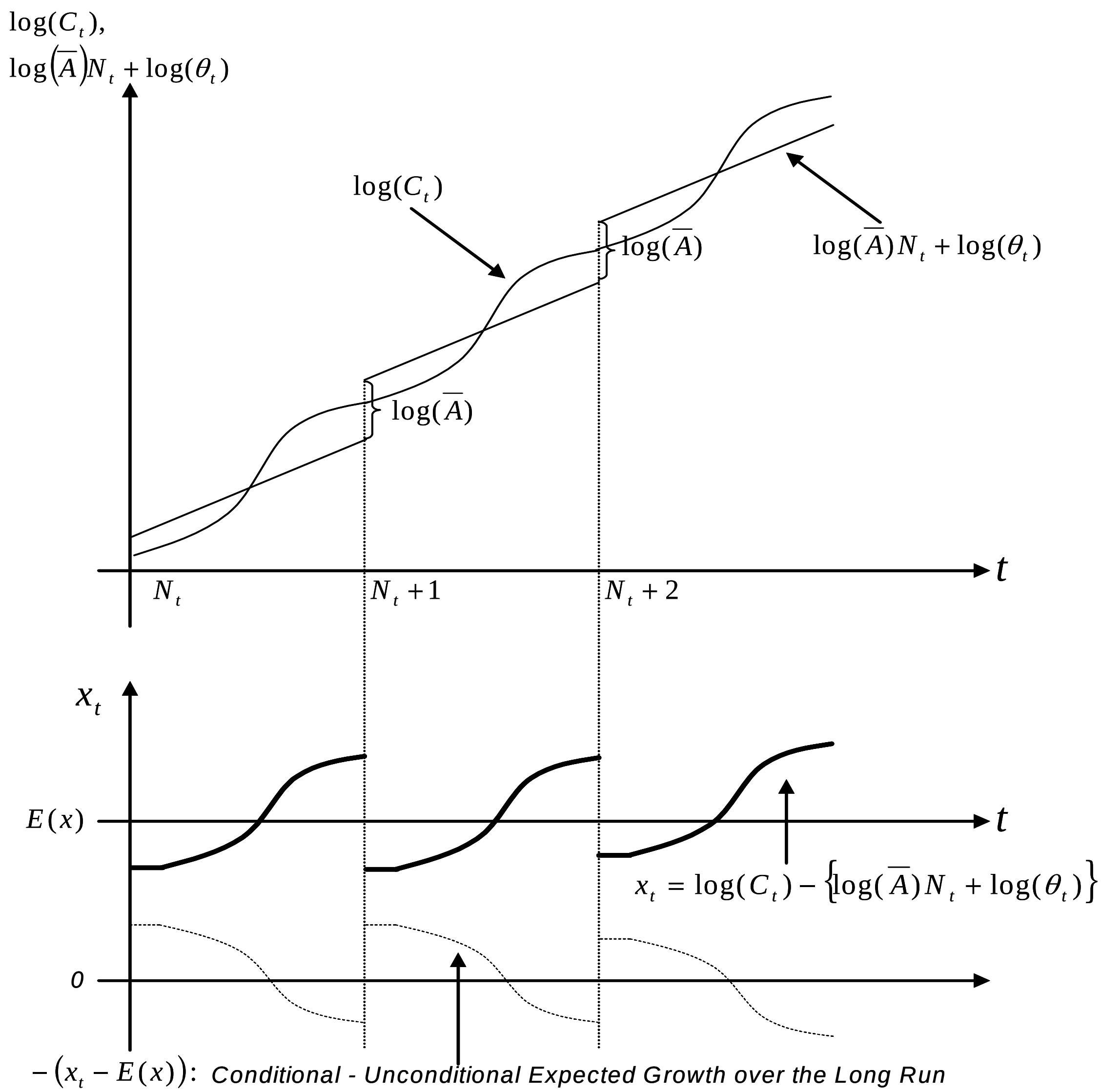

FIGURE 2: This figure depicts the trend $\log(\overline{A})N_t + \log(\theta_t)$ and the actual level of (log) consumption $\log(C_t)$, as well as the difference between the two. To illustrate the behavior of a "typical" path, we have set the Brownian increments (dB_t) to be equal to 0 so that $\log(\theta_t) = \left(\mu - \frac{\sigma^2}{2}\right) t$.

μ	0.009	γ	8	$\zeta(0)$	1
σ	0.030	ρ	0.05	v	2
λ	0.050	$\bar{A}$	1.50	q	32.5

TABLE 1: Parameters used for the calibration

The appendix presents closed form solutions for asset prices and related quantities in Proposition 2. In the body of the text we present a qualitative discussion of results along with a quantitative assessment of the stationary quantities implied by the model.

4.1 Calibration

Table 1 presents our choice of the 9 parameters for the baseline calibration exercise. These parameters were chosen so as to match as closely as possible 22 unconditional moments. These unconditional moments include first and second moments of consumption growth, the one year real interest rate, the yearly equity premium, the log (P/D) ratio and the aggregate book to market ratio. These 10 time series moments were complemented by another 12 cross sectional moments, which correspond to the cross sectional distribution of size quantiles in the model. Time series moments are given in table 2, whereas cross sectional moments on size are given in the bottom two rows of table 3 along with their empirical counterparts.

As can be seen from the Tables 2 and 3 the model fit is satisfactory. Most time series moments are within $20-50\%$ of their empirical counterparts. The cross sectional distribution of log size implied by the model is less disperse than in the data, especially so for the outlier portfolios.

The overall performance of the model in terms of unconditional time series moments is comparable to models of external habit formation such as Abel [1990], and Chan and Kogan [2002]. As in these models, the analytic tractability of keeping risk aversion constant comes at the cost of making the real rate relatively volatile. The benefit of the utility specification (16), however, is that it will facilitate closed form solutions and tractability, which are important given the complexity of the aggregation.

	Data	Model
Mean of consumption growth	0.021^a	0.029
Volatility of consumption growth	0.035^a	0.048
Mean of 1-year zero coupon yield	0.029^a	0.026
Volatility of 1-year zero coupon yield	0.052^a	0.068
Mean of Equity Premium	0.053^a	0.038
Volatility of Equity Premium	0.18^a	0.195
Mean (log) Price to Dividend Ratio	3.14^a	3.448
Volatility of (log) Price to Dividend Ratio	0.37^a	0.326
Mean of Book to Market	0.668^b	0.766
Volatility of Book to Market	0.230^b	0.312

TABLE 2: Unconditional Moments of the model and the data. (Annualized rates) All data labeled with ^{a} are from the website of Robert Shiller. The entire (1871-2005) sample was used in computing moments of the data. To compute the volatility of the (ex-ante) real interest rate, we used data from the Livingston Survey (available post 1946 from the website of the Philadelphia FED) to compute the standard deviation of the difference between expected and realized inflation in the postwar sample. That number is about 2.1%. The volatility of the ex-post real rate during that same period is 3.65%. Since $Var\left(r_t^f\right) = Var\left(E_{t-1}\left(r_t^f\right)\right) + Var\left(\pi_t - E_{t-1}\left(\pi_t\right)\right)$, (where π_t denotes inflation at time t) the volatility of the ex-ante real rate is $\sqrt{(3.65)^2 - (2.1)^2} \simeq 3\%$. Inflation surveys are not available pre-1946. Therefore, to compute the volatility of the (ex-ante) real interest rate for that period we made the assumption that the standard deviation of inflation errors is proportional to the realized standard deviation of inflation for pre-world war II data, an assumption that is supported in the data of the post world-war II sample. Using this assumption we imputed a standard deviation of inflation expectation errors of 4.28% for the pre-1946 sample. Given a volatility of the ex-post real rate of 8.25%, this resulted in a volatility of the ex-ante real rate of about $\sqrt{(8.25)^2 - (4.28)^2} \simeq 7\%$. In the table we report the weighted average of the two volatilities. The obtained 5.2% standard deviation is similar to the numbers given in Jermann [1998] (5.67%) and Campbell, Lo, and MacKinlay [1997] (Table 8.1) (5.44%). The Data labeled with ^{b} are from Pontiff and Schall [1998]. The unconditional moments for the model are computed from a Monte Carlo Simulation involving 20000 years of data, dropping the initial 8000 to ensure that initial quantitites are drawn from their stationary distribution. Simulated consumption moments are based on annualized quarterly data.

Portfolios formed on Size (Stationary Distribution)												
Deciles	1A	1B	2	3	4	5	6	7	8	9	10A	10B
Returns -Data	1.64	1.16	1.29	1.24	1.25	1.29	1.17	1.07	1.10	0.95	0.88	0.90
Returns -Simulated	0.71	0.70	0.69	0.68	0.66	0.65	0.63	0.62	0.61	0.60	0.60	0.60
Log Size - Data	1.98	3.18	3.63	4.10	4.50	4.89	5.30	5.73	6.24	6.82	7.39	8.44
Log Size - Simulated	1.68	2.24	2.58	2.86	3.16	3.47	3.79	4.11	4.44	4.79	5.09	5.41

TABLE 3: Portfolios sorted by size - model and data. The data are from Fama and French [1992], who report nominal montly returns, which are affected by the high inflation rates between 1963 and 1990. We report real montlhy returns for the simulations. For details on the number of simulations used, see the caption to table 2. To compare, note that the average monthly inflation between 1963 and 1990 was about 0.8, and hence this number should be subtracted from the Fama-French returns in order to make them comparable to the simulated numbers.

4.2 Time Series Properties of Aggregate Consumption

Several of the results that follow depend on the correlation between consumption and returns. Therefore, before discussing any implications of the model for returns, we first need to make sure that the model is able to match the features of the consumption data. Since the model produces some predictability in consumption growth, we need to make sure that this predictability is weak, as is the case in the data.

A useful visual depiction of the time series properties of (differences in log) consumption is facilitated by the log-periodogram (see Hamilton [1994] for details). A flat log-periodogram is an indication of white noise, while a downward sloping log periodogram is an indication of time series dependence.

The top subplot of Figure 3 depicts the smoothed log periodogram for consumption growth in the data along with the results obtained from multiple simulations of the model. The 2.5%, 97.5%, and 50% bands depict the respective quantiles of model simulations. The strong random walk component contained in the simulated consumption process allows us to match the very weak positive time series dependence of real-world consumption data.

Finally, the model matches the strong negative correlation between shocks to trend and shocks to the cyclical component of consumption, that has been observed by Morley, Nelson, and Zivot

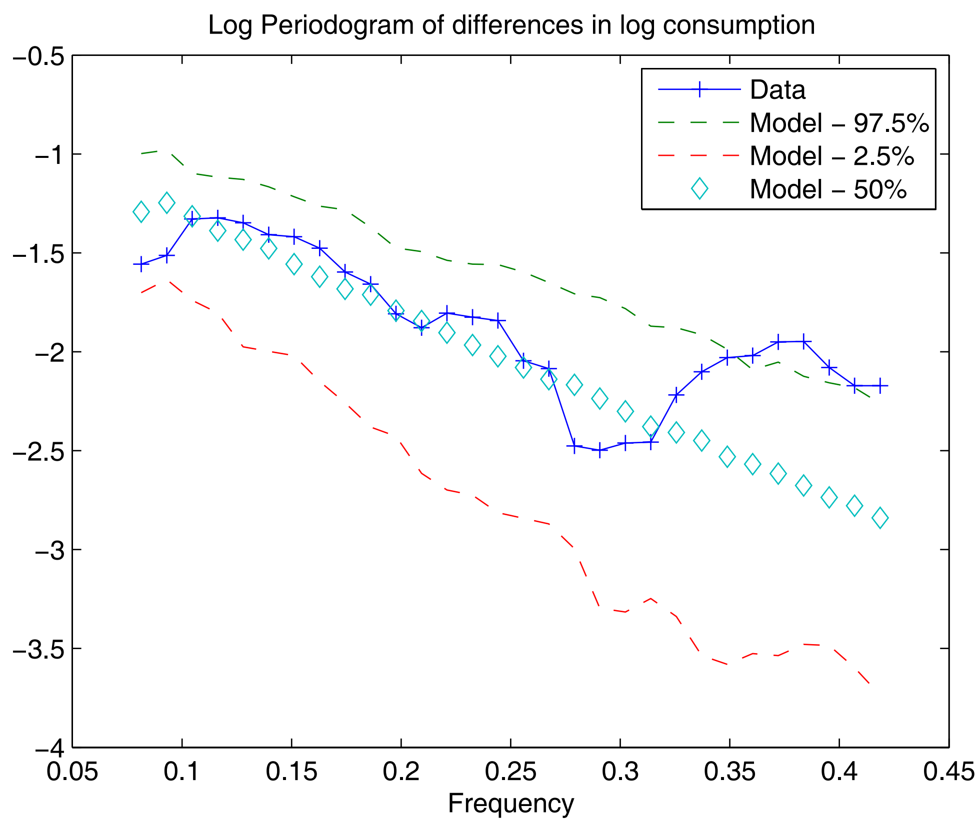

FIGURE 3: Log Periodogram of the consumption process for the data and the model. The line labeled "data" depicts the log periodogram for post- world war I yearly differences in log consumption. The other three lines give the distribution of log-periodograms for simulated data based on 100 repetitions of 87 year long consumption paths. An equally weighted "nearest neighbor" kernel was used to perform the smoothing, equally weighting the 7 nearest frequencies.

[2002]. Morley, Nelson, and Zivot [2002] report a negative correlation of -0.9 while our simulated data exhibit a correlation of -0.78. They interpret this negative correlation as an indication that the economy absorbs permanent innovations with a lag. Indeed our model supports this conclusion: When a new epoch arrives, the trend line in the economy jumps up instantaneously. However, the level of consumption remains unchanged. Since the cycle is the difference between level and trend (by definition), this means that the cyclical component exhibits an offsetting negative jump. Of course, as time passes, positive shocks to the trend θ_t make firms invest, and hence translate into positive cyclical shocks, offsetting the perfect negative correlation.

4.3 Countercyclical Variation in Expected Returns

Having obtained the quantitative and qualitative properties of the model for aggregate consumption, we now turn to a discussion of the model implications for asset returns, which are the main focus of this paper.

The price of a firm in general equilibrium is given by (6). Equation (6) decomposes the price of a firm into three components: 1) the value of assets in place, 2) The value of growth options in the current technological epoch and 3) The value of growth options in all subsequent technological epochs. In the appendix (Proposition 2) we give closed form expressions for both the value of a single firm and the value of the aggregate stock market.

In analogy to an individual firm, one can add up the value of all firms to arrive at the value of the aggregate stock market. Subsequently, one can decompose its value into assets in place, current epoch and future epoch options. Such a decomposition shows that the relative weight of growth options is countercyclical at the aggregate. When the current level of consumption is below its stochastic trend, this implies that there is a large number of unexploited investment opportunities for firms. Accordingly, the relative weight of growth options will be substantial. By contrast, when consumption is above its trend level, the most profitable investment opportunities have been exploited, and the relative importance of growth options is small.

The importance of growth options in the aggregate stock market is central for asset pricing, since the expected return on the aggregate stock market is equal to the expected returns of assets in place and growth options weighted by their relative importance. In particular, letting w_t^{o+f} be the fraction of the value of the aggregate stock market that is attributable to current and future epoch options, w_t^o be the fraction that is due to current epoch options and w_t^f be the fraction due to future epoch options, the expected instantaneous excess return is given by:

$$\mu - r = (1 - w_t^{o+f}) \left(\mu^A - r\right) + w_t^{o+f} \left[\frac{w_t^o}{w_t^{o+f}} (\mu^o - r) + \frac{w_t^f}{w_t^{o+f}} \left(\mu^f - r\right) \right] \tag{31}$$

where $\mu - r$ is the excess return on the aggregate stock market, $\mu^A - r$ is the excess return on assets in place and $\mu^f - r$ is the excess return on future growth options. The following Lemma compares these excess returns:

Lemma 1 *The expected (excess) return of current epoch growth options* $(\mu^o - r)$ *is strictly larger than the expected (excess) return of future growth options* $\left(\mu^f - r\right)$*, which in turn is larger than the expected (excess) return of assets in place* $\left(\mu^A - r\right)$*.*

There is a simple intuition to understand why growth options are riskier than assets in place. The dividends of assets in place are linear in θ_t. However, growth options are non-linear claims: they deliver payoffs if and only if θ_t increases sufficiently, else they are worthless. Since they deliver their payoffs only in "good" times and not in bad times, they are riskier claims.

At a practical level, Lemma 1 shows how countercyclical variation in the relative importance of growth options translates into countercyclical variation in expected returns: When the economy is below its stochastic trend, there are numerous growth options, which are risky in light of Lemma 1. This pushes aggregate expected excess returns upwards. However, as growth opportunities get exploited, the relative importance of growth options and hence the expected excess returns in the stock market decline.

Finally, the countercyclical variation in expected returns helps explain why valuation ratios (such as the Price to Dividend ratio) do not strongly predict (per share) dividend growth[22] in this model, but rather returns: In simulations of the model, the R-squared of regressions of future (per share) dividend growth on the log P/D ratio is about 12.9% at the 5 year horizon, which compares well with the 12% number[23] in the data. As Lettau and Ludvigson [2005], Larrain and Yogo [2005] and Menzly, Santos, and Veronesi [2004] explain in detail, countercyclical variation in expected

[22] Note that consumption and dividends per share are not the same thing in this model, because of the constant equity issuance that is taking place.

[23] See Cochrane [2005], Table 20.1

Correlations	Data	Model
quarterly cons. growth and quarterly returns	0.17	0.34
3-year cons. growth and 3-year returns	0.30	0.64
Bandpass filtered returns and consumption (high frequency)	0.15	0.31
Bandpass filtered returns and consumption (low frequency)	0.44	0.57

TABLE 4: Correlations between consumption growth and returns. Consumption data include the full post WWII sample on non-durables and services as provided by the St. Louis FED, and returns are value weighted CRSP returns. The first two rows report correlations of consumption and return data over intervals of a quarter, while the second row reports the respective correlation over 3-year (overlapping) intervals. The last two rows report correlation of band-pass filtered consumption and returns. We used the Baxter and King [1999] filter to isolate "high frequencies" (swings<1.5 years) and low frequencies (swings between 1.5 and 8 years).

returns imply that the ability of valuation ratios to predict dividend growth will be attenuated. The model reproduces this finding for exactly the same reason.

4.4 The correlation between consumption and returns

The new feature introduced by the present model is the delayed reaction of the economy to a major technological shock. Sofar we have shown how this intuitive feature can help explain certain observations in macroeconomics (such as the negative correlation between trend and cycle shocks) and finance (the countercyclical behavior of expected returns). Interestingly, besides addressing these univariate properties of consumption and returns, the model has some strong implication for the joint time series properties of consumption and returns. In particular, this section shows how the model can help address an issue that has proved hard to reconcile with rational one-factor models of asset pricing, namely a correlation between consumption and returns that is low at high frequencies, and higher at lower frequencies.

Table 4 illustrates the low correlation puzzle that has been documented by Cochrane and Hansen [1991]. The correlation between quarterly consumption growth and quarterly (excess) returns in the data is small (about 0.17). As Daniel and Marshall [1999] document, the correlation between consumption and excess returns increases at lower frequencies. An intuitive way to see this is to compute correlations between 3-year consumption growth and returns and note that the correlation increases to 0.30. A more thorough way of documenting this fact is to filter out high frequencies by applying a Baxter and King [1999] filter. The patterns that we document in the table confirm the findings in Daniel and Marshall [1999]. The correlation between consumption and returns is higher at lower frequencies.

The model can reproduce these patterns. To understand why, recall that in this model there are two types of technological shocks. Shocks to θ_t increase both consumption and returns on impact. However, the arrival of technological epochs produces different reactions in consumption and returns in the short run and in the long run. In the short run, the arrival of a new epoch will raise expected returns, as the new growth options raise the riskiness of the stock market. However, average consumption growth will decline in the short run, since the old growth options become obsolete and it is not profitable to plant the new vintages yet. It is only after the passage of some time that the new technology will boost output and consumption growth.

The interplay of these two shocks helps explain why consumption is weakly correlated with returns in the short run, whereas the correlation becomes stronger in the long run. Table 4 illustrates these effects, by comparing correlations in the data by the equivalent correlations in simulated data. The last two rows show that the model produces correlations that are only about 15% higher than their counterparts in the data at the respective frequencies. Importantly, the model is able to reproduce the increase in correlation as one moves to lower frequencies.

4.5 P/D Predictability

We conclude the discussion of the time series properties of returns by performing the usual predictability regressions of aggregate excess returns on the aggregate log P/D ratio. Table 5 tabulates the results of these regressions, and compares them to the data. We simulate 100 years of data and obtain several independent samples of such 100-year spans of artificial data. We run predictability regressions for each of these samples and report the average coefficient along with a 95% distribution band. We then compare these simulations to the equivalent point estimates in the data.

The coefficients in the simulations have the right sign, but are about 1/3 of their empirical counterparts. Moreover, the empirical point estimates are within the 95% distribution band according to the model.

P/D Predictive Ability

	Data		Model	
Horizon(years)	Coefficient	R-square	Coefficent	R-square
1	-0.120	0.040	-0.054	0.012
			(-0.239, 0.057)	(0.000, 0.058)
2	-0.300	0.100	-0.102	0.023
			(-0.420, 0.112)	(0.000, 0.116)
3	-0.350	0.110	-0.149	0.034
			(-0.559, 0.176)	(0.000, 0.154)
5	-0.640	0.230	-0.224	0.052
			(-0.925, 0.273)	(0.000, 0.270)
7	-0.730	0.250	-0.273	0.063
			(-1.165, 0.355)	(0.000, 0.353)

TABLE 5: Results of predictive Regressions. Excess returns in the aggregate stock market between t and $t+T$ for $T = 1, 2, 3, 5, 7$ are regressed on the P/D ratio at time t. A constant is included but not reported. The data column is from Chan and Kogan [2002]. The simulations were performed by drawing 100 time series of a length equal to the data and performing the same predictive regressions. For each draw out of the 100, we simulate 5000 years of data and only keep the last 100 years of data to run the regressions. We report the means of these simulations next to the data. The numbers in parentheses are the 95% confidence interval of the estimates obtained in the simulations.

The results of table 5 may seem surprising at first. One would anticipate that the P/D ratio will be high when growth options have not been exercised, and low if they have. Hence that should imply a positive relationship between expected excess returns and the P/D ratio, rather than the negative relation that we obtain in the simulations.

The resolution of the puzzle lies in the difference between long horizon and instantaneous expected returns. The easiest way to see this is to consider *an individual firm* and study the evolution of its P/D ratio over a technological epoch: The top plot of figure 4 depicts the P/D ratio and the *instantaneous* expected return of the firm. Clearly, the two are positively correlated: As long as a firm has not planted a tree, the fraction of growth options in its price is large and so is its P/D ratio and its expected return in light of Lemma 1. Once the firm plants a tree, its P/D ratio experiences a discontinuous drop, and so does its instantaneous expected return. This reflects the transformation of growth options into assets in place.

To compare, the bottom plot depicts the P/D ratio against the *average instantaneous* expected return between t and $t+T$, for any T that is larger or equal to the average time it takes to plant a tree[24], starting at the beginning of an epoch. Now there is a negative relation between the P/D and the average expected return, at least before a firm decides to invest. The reason is that a high P/D ratio anticipates the decline in expected returns that will occur over the long run, when the firm plants the new tree.

By aggregating over firms we can extend these results to the aggregate stock market, since the investment decisions of firms are strongly correlated. The main difference between the picture at the aggregate level and the individual firm level is that the decline in the P/D ratio does not occur in a discontinuous fashion, but is more gradual.

In conclusion, as long as we predict returns over long horizons, we should expect a negative relationship between the P/D return and expected returns, as the one found in the data[25].

[24] This qualitative pattern for the average expected return would hold as long as we averaged over any $T_1 > T$ periods. For intervals shorter than T we would obtain a hump shaped pattern for the average expected return and hence no clear positive or negative relationship.

[25] There is a caveat here: The predictability is not exclusively due to the cyclical forces described above. As Abel [2005] shows, models of external habit formation can produce predictability in discrete data, and our model is no exception to this rule, since we report the results of regressions performed with discrete data. However, as Abel [2005] finds, the predictability due to external habit formation is likely to be small.

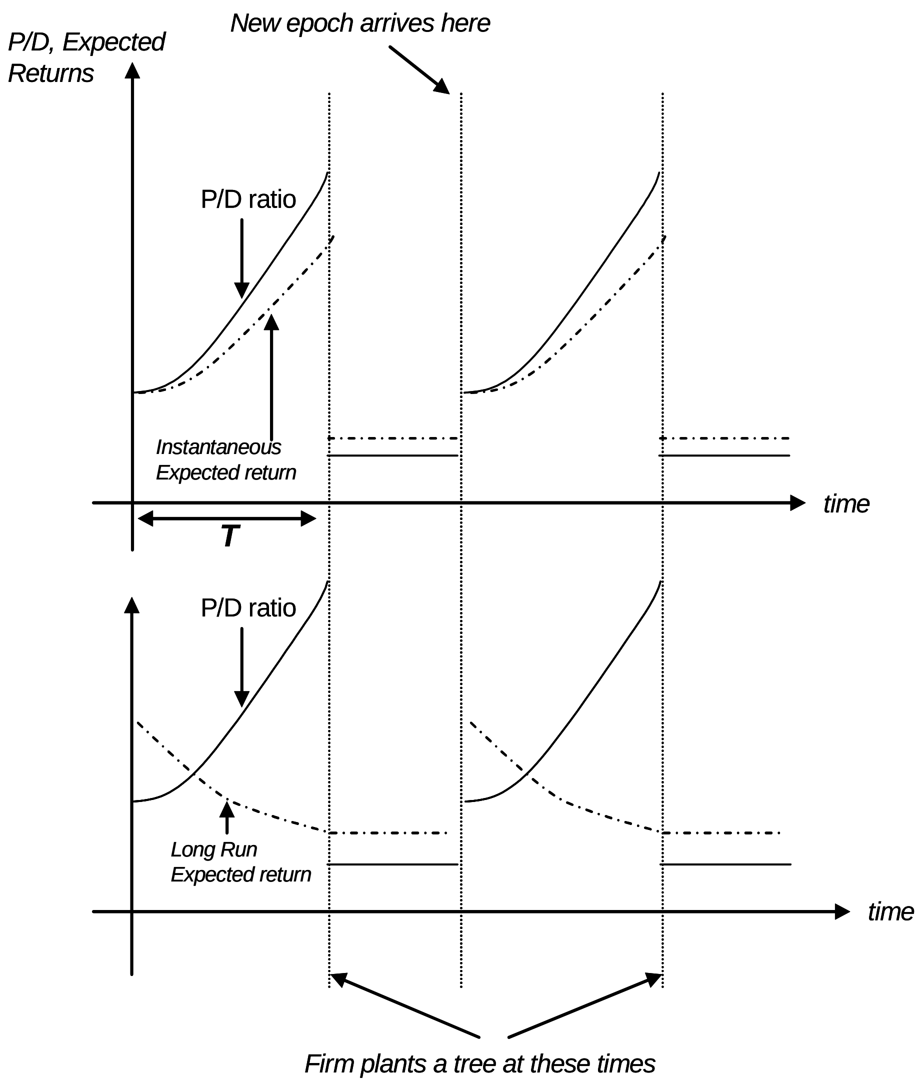

FIGURE 4: The top plot depicts the P/D ratio and the instantaneous expected return. The bottom plot depicts the P/D ratio against the average expected return over T periods, where T is the average time it takes to plant a tree. To pick a "typical" path we set the Brownian increments (dB_t) equal to 0.

4.6 Cross Sectional Predictability

Sofar we have developed the implications of the model for the time series. However, one of the major motivations for using production based models is that they can endogenously produce implications for the properties of returns in the cross section, since the correlation between the returns of individual firms and the sources of risks are endogenous.

Our focus in this subsection will be to show why the model is able to produce a size and a value premium. In the next subsection we discuss why these cross sectional phenomena can be explained by a consumption CAPM including "long run" consumption growth instead of quarterly consumption growth.

To show why the model is able to produce a size premium, it will be easiest to consider a firm j that has a higher market value of equity (size) than firm j':

$$P_{N,j,t} > P_{N,j',t}$$

To simplify the analysis, assume further that both of these firms have exercised their growth option in the current epoch, so that

$$P^o_{N,j,t} = P^o_{N,j',t} = 0.$$

Since the future growth options are the same for both firms, the relative importance of growth options for firm j must be smaller. Using Lemma 1, and applying formula (31), firm j must therefore have a lower expected return. Hence, assuming that one could safely ignore current epoch growth options, a sorting of companies based on size will produce a size premium: Companies with higher market value will have lower expected returns.

The presence of current epoch growth options will distort the perfect ranking of expected returns implied by size[26]. For the calibrations that we consider, we find however that current epoch growth options are not quantitatively important enough to affect the size effect.

The model is also consistent with the value premium. This may seem counterintuitive at first, since one would expect that firms with a high market to book ratio should have a substantial fraction of their value tied up in growth options, and hence should be riskier. The resolution of the

[26] Intuitively, high market values may be associated with a valuable current period growth option instead of numerous assets in place. Therefore knowing that a firm has a high market value might mean that it has a valuable current period growth option, in which case its return should be high.

puzzle is that trees are heterogenous in this economy, and accordingly the market to book ratio of a given firm will primarily reflect the average productivity of its existing trees, and not just the share of growth options.

The easiest way to see this, is to consider two firms j and j', that have planted a tree in every single epoch, including the current one. Clearly, the two firms will have identical book values and identical growth options. However, suppose that firm j has always been "luckier" than firm j' in terms of the productivity of the trees it has had the opportunity to plant. Then the market value of firm j will be higher than the market value of firm j', because the value of its assets in place will be higher:

$$P^A_{N,j,t} > P^A_{N,j',t} \tag{32}$$

The growth options of the two firms are identical, and hence it must be the case that the total value of firm j is larger than the total value of firm j' :

$$P_{N,j,t} > P_{N,j',t} \tag{33}$$

But (32) and (33) imply that:

$$w_t^{(j),o+f} = \frac{P^f_{N,t}}{P^A_{N,j,t} + P^f_{N,t}} = \frac{P^f_{N,t}}{P_{N,j,t}} < \frac{P^f_{N,t}}{P_{N,j',t}} = w_t^{(j'),o+f}$$

and hence firm j has a *smaller* fraction of its value tied up in growth options. Accordingly firm j has a lower expected return than firm j'. Also, since the book values of the two firms are identical, equation (33) implies that firm j has a lower book to market ratio than firm j'. This is consistent with the well known fact that firms with a low book to market ratio have a low expected return (the value premium).

We note in passing that the model is also able to reproduce additional properties of the cross sectional data: Since high size (and/or high growth) firms are firms that will typically have trees with higher productivity on average, the model is consistent with the empirical evidence reported in Fama and French [1995], who show that sorting on size and value will produce predictability for a firm's profitability (earnings to book ratio). The model is also consistent with the evidence that small firms will tend to grow faster than large firms. The reason is mean reversion: In expectation all firms have the same book value of trees (after detrending by $\overline{A}^N \theta_t$) in the long run. Hence firms who are below that stationary value at a given time can be expected to grow faster and vice versa. Finally, the model also predicts that firms with a low book to market ratio (high Tobin's q) will tend to exhibit stronger investment activity (as measured by the growth in the book value of assets). The intuition for this is simple: A high Tobin's q (low book to market) will reflect a) the productivity of existing trees, but also b) the magnitude of future growth options compared to the current capital stock of the firm. The first component will drive expected returns down as we showed above, but will be irrelevant (pure noise) for predicting the growth rate in the capital stock. However, the second component will predict the growth in the capital stock. The interplay of these two forces can help explain why regressions of the growth rate of trees on Tobin's q will produce a positive but low coefficient, and a low R-squared. In simulated data these regressions produced a coefficient of 0.06 and an R-squared of 0.01, which is very close to what Abel and Eberly [2002a] find in the data[27].

However, the model cannot produce a size and a value effect as independent effects: Sorting on size will leave little or no room for a value effect and vice versa. For this reason, we focus on the size effect henceforth, and note that sorting on value produces similar results. We note in passing that modifications of the model that introduce stochastic depreciation of the existing trees could be used to explain the value and the size premium jointly. However they are beyond the scope of the present paper.

Quantitatively, the cross sectional distribution of expected returns is smaller than in the data. Table 3 presents the average returns on size sorted portfolios and compares them to the returns reported in Fama and French [1992]. Ignoring portfolio 1A of Fama and French [1992], the difference in monthly returns between the highest and the lowest size portfolio in our model is about a third of the equivalent value in the data. Hence, the model has a similar performance to Gomes, Kogan, and Zhang [2003] in terms of obtaining a quantitatively plausible size premium. This is partly driven by the fact that the model produces a smaller spread in log size than what is observed in the data.

[27] Abel and Eberly [2002a] report coefficients between 0.03 and 0.11 and an R-squared of $0.02 - 0.08$.

4.7 Consumption Risk in the short and in the long run

The conditional consumption CAPM, which holds in this model, asserts that the following relationship determines the expected return of any firm j :

$$\mu_t^{(j)} - r = -cov_t\left(\frac{dP_t^{(j)}}{P_t^{(j)}}; \frac{dH_t}{H_t}\right) = \gamma cov_t\left(\frac{dP_t^{(j)}}{P_t^{(j)}}; \frac{d\theta_t}{\theta_t}\right) = \gamma\sigma\left(\sigma\theta_t \frac{\frac{\partial P_t^{(j)}}{\partial\theta_t}}{P_t^{(j)}}\right) \tag{34}$$

The first equality in (34) is the usual CAPM relationship in continuous time (Karatzas and Shreve [1998], Chapter 4). The second equality follows from (25) and the fact that the running maximum of θ_t is an increasing process, and hence has bounded variation. Accordingly, it has no quadratic variation and no covariation[28] with the increments in $P_t^{(j)}$. The final equality exploits the homoskedasticity in the increments of θ_t.

An important implication of (34) is that *only the covariation between increments to θ_t and returns matter for pricing purposes.* Moreover, the conditional CAPM implied by the present model, conditions "down" to an unconditional CAPM:

$$E\mu_t^{(j)} - r = \gamma cov\left(\frac{dP_t^{(j)}}{P_t^{(j)}}; \frac{d\theta_t}{\theta_t}\right) \tag{35}$$

because the price of risk is constant in this model. (See Cochrane [2005], Page 138).

Equation (35) asserts that only the covariance between shocks to the trend θ_t and returns matter for asset pricing in the present model. This is in line with the findings in Bansal, Dittmar, and Kiku [2005] who document the dominant role of trend shocks for the determination of expected returns.

A practical implication of (35) is that the regular consumption CAPM with discretely observed data for consumption and returns need not hold. To see this, let Δ be a difference operator over a short interval of time, say a quarter. By equation (28):

$$\Delta\log(C_t) = \Delta\log(\theta_t) + \Delta\left[x_t + N\log(\overline{A})\right]$$

and hence quarterly (log) consumption differences $\Delta\log(C_t)$ measure quarterly changes in $\log(\theta_t)$ with error. Intuitively, consumption growth captures not only increments to the trend θ_t, but also to the stationary component $\Delta\left[x_t + N\log(\overline{A})\right]$.

[28] For details on these notions, see Karatzas and Shreve [1991].

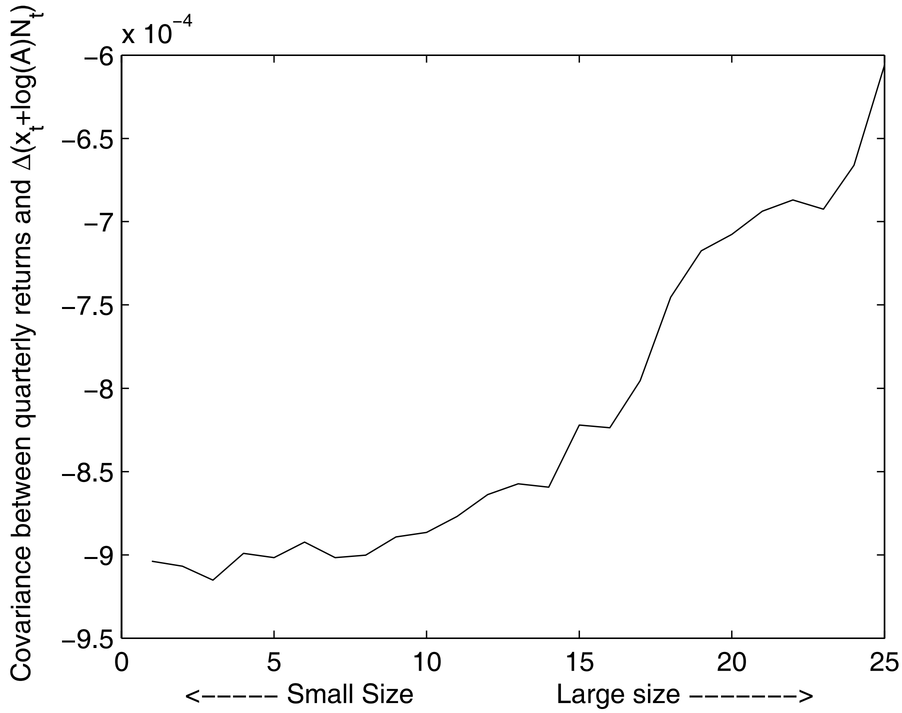

FIGURE 5: The covariance between excess returns and $\Delta\left[x_t + N\log(\overline{A})\right]$. Portfolios are arranged along the $x-$axis ranging from small size to large size.

The covariance between asset returns and $\Delta\left[x_t + N\log(\overline{A})\right]$ turns out to be negative in this model, as long as Δ is not very large (say a quarter). The reason is identical to the one in section 4.4: At the beginning of an epoch expected returns are high, while $\Delta\left[x_t + N\log(\overline{A})\right]$ is practically 0 as very few firms are planting trees. Once firms start planting trees, the term $\Delta\left[x_t + N\log(\overline{A})\right]$ becomes large while expected returns become low. Hence, there is a negative correlation between expected returns and $\Delta\left[x_t + N\log(\overline{A})\right]$ that makes the correlation between returns and $\Delta\log(C_t)$ a downward biased estimate of the covariance between returns and $\Delta\log(\theta_t)$.

Figure 5 gives a visual impression of this effect. It plots the covariance between $\Delta\left[x_t + N\log(\overline{A})\right]$ and the excess returns on various portfolios sorted on size. As can be seen, all covariances are negative. This implies that the presence of the term $\Delta\left[x_t + N\log(\overline{A})\right]$ makes the covariance between returns and consumption a downward biased estimate between returns and increments to the shock $\log(\theta_t)$:

$$cov\left(R_t^{(j)}, \Delta\log(C_t)\right) < cov\left(R_t^{(j)}, \Delta\log(\theta_t)\right)$$

The practical implication of this observation is that a consumption CAPM using quarterly consumption growth $\Delta \log(C_t)$ to proxy for increments to $\Delta \log(\theta_t)$ fails to capture the cross section of returns in this model. The top part of figure 6 illustrates this effect by plotting the average returns on size sorted portfolios against the returns that would be predicted by a regular consumption CAPM using quarterly consumption growth.

Noteworthy, the model also predicts that these biases will be mitigated if one uses the covariance between returns and consumption growth over longer intervals as Parker and Julliard [2005] do. The bottom subplot of figure 6 illustrates that an econometrician who would estimate the consumption CAPM using the covariance between quarterly returns and consumption growth over the subsequent 5 years, would uncover a tight link between these covariances and average returns on size sorted portfolios.

This finding is similar to the key result in section 4.4: At the beginning of an epoch returns will immediately increase, whereas consumption will follow with a lag. Hence, by lengthening the observation interval for consumption one can exploit the fact that the increase in expected returns will predict consumption growth over the long run. Hence the covariance between long run consumption and quarterly returns will be higher, which helps eliminate the bias contained in short run correlations.

We also note that we obtained practically identical results when we used covariances between 5-year returns and 5-year consumption growth, instead of covariances between quarterly returns and 5-year consumption growth. The reason is intuitive: Covariances between "long run" returns and "long run" consumption will isolate comovements in trends between the two quantities and will eliminate the effects of cyclical variations in consumption growth as shown in Bansal, Dittmar, and Kiku [2005].

Finally, we point out that the results presented here rely on three features of the model: 1) Consumption adjusts in a sluggish manner in the short run, while expected returns immediately reflect anticipations of consumption growth over the long run through growth options 2) Only the trend-increments $d \log \theta_t$ are priced, and 3) the fact that discrete time consumption data over different horizons will provide proxies for the unobserved covariances between returns and the innovations to trend $d \log \theta_t$ over various horizons. Importantly, all three properties would continue

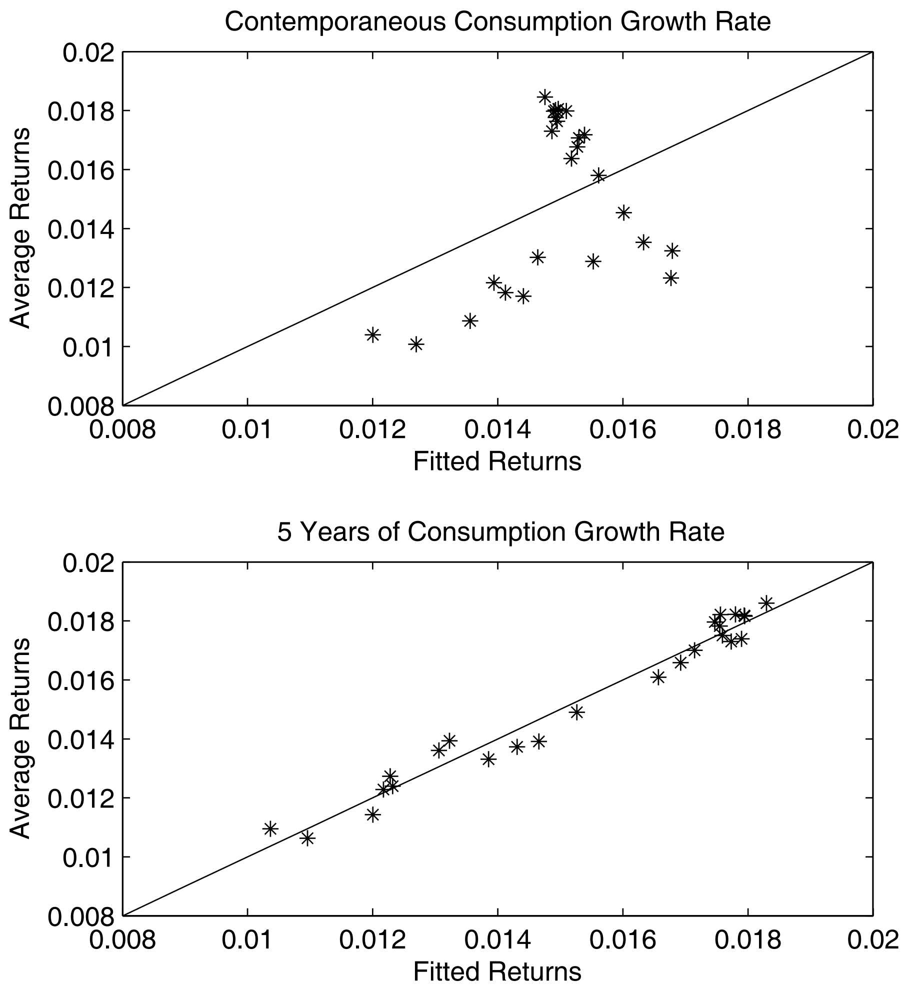

FIGURE 6: In both subplots average returns are plotted against the returns that would be predicted by the consumption CAPM using short and long consumption differences. The top subplot depicts results for the consumption CAPM using 1-quarter consumption growth and the bottom subplot using 5-year consumption growth rates to evaluate the covariation between consumption growth and quarterly excess returns.

to hold if one were to set up the model in discrete time[29]: it is not just discrete time that produces the results, but the interaction of all of the above factors, in particular properties 1) and 2).

5 Conclusion

Why does consumption based asset pricing seem to work better over longer horizons, than over short horizons? In this paper we proposed a general equilibrium framework that served as a laboratory in order to investigate this question. The key ingredient of our model is the joint presence of "small" frequent disembodied productivity shocks and "large" infrequent embodied technology shocks. The first type of shocks affect the economy on impact and behave exactly as a random walk. The latter arrive also in a random walk fashion. However, there are delays between their impact and their effects on the economy.

This setup allowed us to examine several of the stylized facts about consumption and asset prices in a *unified* framework.

First, we showed how the delayed reaction to a major technology shock can propagate the otherwise i.i.d. shocks of the model, so as to produce endogenous cycles. A key feature of the model is the different reaction of consumption growth to a major technology shock in the short run and in the long run. In the short run, consumption growth is moderate, as the planting of old trees stops, while the new trees are still not profitable. However, in the long run the new technology starts being adopted widely and this leads to an acceleration of growth.

Second, we argued that the arrival and eventual depletion of growth options over the cycle will lead to countercyclical expected returns. The extent to which the economy has absorbed a major technological shock, will determine the relative weight of growth options and hence the riskiness and the expected returns of various firms.

Third, we combined the above two observations in order to study the correlation between consumption and returns in the short and in the long run. In response to the arrival of an epoch,

[29] The main convenience allowed by continuous time is that the only source of quadratic variation in the stochastic discount factor comes from θ_t. In discrete time, one would also have to account for the covariance between returns and the running maximum of θ_t, since the log of the stochastic discount factor is given by $\gamma\left[\log(\theta_t) - \log(M_t)\right]$. Accounting for this force would however strengthen the results further, since the correlation between the cyclical component and increments to the running maximum of θ_t is negative.

expected consumption growth and expected returns will move in opposite directions in the short run. This will attenuate the correlation between consumption and returns at high frequencies. Hence, by introducing two shocks, the model can account for the so-called "low correlation puzzle" at high frequencies, which has been difficult to explain in one factor rational asset pricing models. It can also account for the success of consumption based asset pricing at lower frequencies, as has been observed in recent literature.

Needless to say, several aspects of the model can be improved in future research. Perhaps the most interesting extension would be to extend the model to an Epstein Zin utility specification. Working out such an extension is a technically challenging task. However, it can be reasonably conjectured that the major intuitions outlined in the paper would not only continue to hold, but would likely be substantially strengthened under such an extension.

A Appendix

A.1 Propositions and Proofs

A.1.1 Main Proposition

Proposition 1 *Define the constants* Z^*, γ_1, γ_1^*, γ_2^* *and* Ξ *by*

$$Z^* = \frac{1}{\rho - \mu(1-\gamma) - \frac{\sigma^2}{2}\gamma(\gamma-1)} \tag{36}$$

$$\gamma_1 = \frac{\sqrt{\left(\mu - \frac{\sigma^2}{2}\right)^2 + 2\sigma^2(\rho+\lambda)} - \left(\mu - \frac{\sigma^2}{2}\right)}{\sigma^2} \tag{37}$$

$$\gamma_2 = \frac{-\sqrt{\left(\mu - \frac{\sigma^2}{2}\right)^2 + 2\sigma^2(\rho+\lambda)} - \left(\mu - \frac{\sigma^2}{2}\right)}{\sigma^2} \tag{38}$$

$$\gamma_1^* = \frac{\sqrt{\left(\mu - \frac{\sigma^2}{2}\right)^2 + 2\sigma^2\rho} - \left(\mu - \frac{\sigma^2}{2}\right)}{\sigma^2} \tag{39}$$

$$\gamma_2^* = \frac{-\sqrt{\left(\mu - \frac{\sigma^2}{2}\right)^2 + 2\sigma^2\rho} - \left(\mu - \frac{\sigma^2}{2}\right)}{\sigma^2} \tag{40}$$

$$\Xi = \frac{q}{Z^*}\frac{\gamma_1}{\gamma_1 - 1}\frac{\gamma_1^* - 1}{\gamma_1^* + \gamma - 1} \tag{41}$$

and assume that:

$$\gamma_1^* > 1$$

$$\gamma_2^* < 1 - \gamma$$

$$\frac{\Xi}{\zeta(0)} > 1$$

Assume moreover that H_t *is given by (25), and* q_t *is given by (19). Then, firm* j *faced with the optimal stopping problem (4) will plant a tree the first time that* θ_t *crosses the threshold* $\overline{\theta}$

$$\overline{\theta} = M_{\tau_N}\frac{\Xi}{\zeta(i_{N,j})} \tag{42}$$

where M_{τ_N} *is the value of the running maximum of* θ_t *at time* τ_N *(i.e. when round* N *begins). Finally, if firms follow the above threshold policies, then*

$$\frac{C_t}{M_t^C} = \frac{\theta_t}{M_t} \tag{43}$$

with M_t, M_t^C *defined in (18) and (11). Therefore*

$$H_t = e^{-\rho t}U_C = e^{-\rho t}\left(\frac{C_t}{M_t^C}\right)^{-\gamma} = e^{-\rho t}\left(\frac{\theta_t}{M_t}\right)^{-\gamma}$$

as conjectured in (25).

Remark 1 *The assumption $\gamma_1^* > 1$ implies $\gamma_1 > 1$. Additionally, the assumption that $\gamma_2^* < 1-\gamma$ implies that $\gamma_2 < 1-\gamma$. Finally $\gamma_1^* > 1$ and $\gamma_2^* < 1-\gamma$ imply that $Z^* > 0$. These observations are used repeatedly in the proofs.*

This is the key Proposition of the paper. We start by assuming that the state price density is indeed given by:

$$H_t = e^{-\rho t}\left(\frac{\theta_t}{M_t}\right)^{-\gamma} \tag{44}$$

and prove that (42) provides the solution to the firm's optimal stopping problem. A useful intermediate first result is the following:

Lemma 2 *The conditional expectation:*

$$Z(\theta_t, M_t) \equiv E_t\left[\int_t^\infty e^{-\rho(s-t)}\left(\frac{\theta_s}{M_s}\right)^{-\gamma}\theta_s ds\right] \tag{45}$$

can be computed explicitly as

$$Z\left(\theta_t, M_t\right) = Z^*\left(\frac{\theta_t}{M_t}\right)^{-\gamma}\theta_t\left[1+\frac{\gamma}{\gamma_1^*-1}\left(\frac{\theta_t}{M_t}\right)^{\gamma+\gamma_1^*-1}\right] \tag{46}$$

with Z^* and γ_1^* as defined in (36) and (39) respectively.

Proof of Lemma 2. One can verify directly that the function Z in equation (46) satisfies:

$$\left(\frac{\theta_t}{M_t}\right)^{-\gamma}\theta_t + \mu\theta Z_\theta + \frac{\sigma^2}{2}\theta^2 Z_{\theta\theta} - \rho Z = 0 \tag{47}$$

whenever $\theta_t < M_t$ and it also satisfies a reflection condition:

$$Z_M(\theta_t, M_t) = 0 \tag{48}$$

at $\theta_t = M_t$. By Ito's Lemma,

$$\begin{aligned} e^{-\rho T}Z\left(\theta_T, M_T\right) - e^{-\rho t}Z\left(\theta_t, M_t\right) &= \int_t^T e^{-\rho s}\left[\mu\theta Z_\theta + \frac{\sigma^2}{2}\theta^2 Z_{\theta\theta} - \rho Z\right]ds + \\ &+ \int_t^T e^{-\rho s}\sigma\theta_s Z_\theta dB_s + \int_t^T e^{-\rho s}Z_M dM_s \end{aligned} \tag{49}$$

Using (47), (48), noting that $dM_s \neq 0$ if and only if $\theta_s = M_s$, taking expectations and letting $T \to \infty$ shows that (46) leads to (45). ■

Corollary 1 *The value of assets in place for firm j is given by*

$$P_{j,t}^A = Z^* X_{j,t}\theta_t\left[1+\frac{\gamma}{\gamma_1^*-1}\left(\frac{\theta_t}{M_t}\right)^{\gamma+\gamma_1^*-1}\right]$$

Proof of Corollary 1. Combine (46) and (5). ■

Given this Lemma we are now in a position to discuss the solution to the firm's optimization problem.

The option to plant a tree in epoch N does not affect the option to plant a tree in any subsequent epoch. Therefore, the firm chooses its optimal strategy to plant a tree "epoch by epoch".

The individual firm takes the state price density (44) and the costs of planting a tree (19) as given. With these functional specifications, the optimization problem (4) becomes:

$$P^o_{N,j,t} = \max_{\tau} E_t \left\{ 1_{\{\tau \le \tau_{N+1}\}} e^{-\rho(\tau-t)} \frac{\left(\frac{\theta_\tau}{M_\tau}\right)^{-\gamma}}{\left(\frac{\theta_t}{M_t}\right)^{-\gamma}} \left[\int_\tau^\infty e^{-\rho(s-\tau)} \frac{\left(\frac{\theta_s}{M_s}\right)^{-\gamma}}{\left(\frac{\theta_\tau}{M_\tau}\right)^{-\gamma}} \zeta\left(i_{j,N}\right) \bar{A}^N \theta_s ds - q\bar{A}^N M_{\tau_N} \right] \right\}$$

Using Lemma 2, the law of iterated expectations and simplifying, this optimization problem can be rewritten as

$$\begin{aligned} P^o_{N,j,t} &= \bar{A}^N \left(\frac{\theta_t}{M_t}\right)^{\gamma} M_{\tau_N} \times \qquad (50) \\ & \times \max_{\tau \ge t} E_t \left[e^{-(\rho+\lambda)(\tau-t)} \left(\zeta\left(i_{j,N}\right) Z^* \left[1 + \frac{\gamma}{\gamma_1^* - 1} \left(\frac{\theta_\tau}{M_\tau}\right)^{\gamma+\gamma_1^*-1} \right] \frac{\theta_\tau}{M_{\tau_N}} \left(\frac{\theta_\tau}{M_\tau}\right)^{-\gamma} - q \left(\frac{\theta_\tau}{M_\tau}\right)^{-\gamma} \right) \right] \end{aligned}$$

To solve the optimization problem inside the square brackets we proceed as follows: We start by restricting our attention to trigger strategies, i.e. strategies where the firm invests the first time that the ratio $\frac{\theta_t}{M_{\tau_N}}$ crosses a threshold $\bar{\theta}$. Formally, consider strategies of the form:

$$\tau_{\bar{\theta}} = \inf\{s \ge t : \frac{\theta_s}{M_{\tau_N}} \ge \bar{\theta}\} \qquad (51)$$

The proof of the following result is standard and is omitted[30]:

$$E_t \left(e^{-(\rho+\lambda)(\tau_{\bar{\theta}}-t)} \right) = \left(\frac{\theta_t}{\bar{\theta} M_{\tau_N}} \right)^{\gamma_1} \qquad (52)$$

where γ_1 is defined in (37). Defining

$$\phi(\theta_t, M_t, M_{\tau_N}; \bar{\theta}) = E_t \left[e^{-(\rho+\lambda)(\tau_{\bar{\theta}}-t)} \left(\zeta\left(i_{j,N}\right) Z^* \left[1 + \frac{\gamma}{\gamma_1^* - 1} \left(\frac{\theta_{\tau_{\bar{\theta}}}}{M_{\tau_{\bar{\theta}}}}\right)^{\gamma+\gamma_1^*-1} \right] \frac{\theta_{\tau_{\bar{\theta}}}}{M_{\tau_N}} \left(\frac{\theta_{\tau_{\bar{\theta}}}}{M_{\tau_{\bar{\theta}}}}\right)^{-\gamma} - q \left(\frac{\theta_{\tau_{\bar{\theta}}}}{M_{\tau_{\bar{\theta}}}}\right)^{-\gamma} \right) \right]$$

and using (52) we obtain (assuming that $\theta_t / M_{\tau_N} \le \bar{\theta}$):

$$\begin{aligned} \phi(\theta_t, M_t, M_{\tau_N}; \bar{\theta}) &= \zeta\left(i_{j,N}\right) Z^* \left\{ 1 + \frac{\gamma}{\gamma_1^* - 1} \left[\min\left(\frac{\bar{\theta}}{M_t / M_{\tau_N}}, 1 \right) \right]^{\gamma_1^*+\gamma-1} \right\} \left[\min\left(\frac{\bar{\theta}}{M_t / M_{\tau_N}}, 1 \right) \right]^{-\gamma} \left(\frac{\theta_t}{M_{\tau_N}} \right)^{\gamma_1} \bar{\theta}^{1-\gamma_1} \\ & -q \left[\min\left(\frac{\bar{\theta}}{M_t / M_{\tau_N}}, 1 \right) \right]^{-\gamma} \left(\frac{\theta_t}{M_{\tau_N}} \right)^{\gamma_1} \bar{\theta}^{-\gamma_1} \end{aligned}$$

[30] See e.g. Karatzas and Shreve [1991] page 197 for details.

The term $\min\left(\frac{\bar{\theta}}{M_t/M_{\tau_N}},1\right)$ arises because either $M_{\tau_{\bar{\theta}}} = \theta_{\tau_{\bar{\theta}}}$ or $M_{\tau_{\bar{\theta}}} = M_t$.

Given the simple structure of the policies that we consider, finding the optimal stopping strategy in the constrained set (51) amounts to a simple one-dimensional optimization maximization over $\bar{\theta}$.

The next Lemma determines the optimal strategies in the constrained set (51). We omit the proof both in order to save space, and because the steps of the proof are fairly straightforward differentiations. Most importantly, the next Lemma is only useful in that it will allow us to form a conjecture on the value function and the optimal stopping policies for arbitrary stopping policies. We then show via a formal verification theorem that the conjectured value function is indeed the appropriate value function.

It will be useful to refer to figure 7 that gives a graphical depiction of the regions described in the proof.

Lemma 3 *Let θ_1, $\underline{\theta}_2$ and $\bar{\theta}_2$ defined as*

$$\begin{aligned} \underline{\theta}_2 &= \frac{q}{\zeta\left(i_{j,N}\right)Z^*}\frac{\gamma+\gamma_1}{\left(\gamma_1+\gamma-1\right)+\frac{\gamma\left(\gamma_1-\gamma_1^*\right)}{\gamma_1^*-1}} \\ \bar{\theta}_2 &\equiv \frac{q}{\zeta\left(i_{j,N}\right)Z^*}\frac{\gamma+\gamma_1}{\gamma_1+\gamma-1} \\ \theta_1 &= q\frac{1}{Z^*}\frac{\gamma_1}{\gamma_1-1}\frac{\gamma_1^*-1}{\gamma_1^*+\gamma-1}\frac{1}{\zeta\left(i_{j,N}\right)} \end{aligned}$$

Then $\theta_1 < \underline{\theta}_2 < \bar{\theta}_2$ and there exists a unique positive solution to the nonlinear equation

$$h\left(\theta_2; M_t/M_{\tau_N}\right) \equiv \left(1-\gamma_1-\gamma\right)\theta_2 - \frac{\gamma\left(\gamma_1-\gamma_1^*\right)}{\gamma_1^*-1}\left(\frac{\theta_2}{M_t/M_{\tau_N}}\right)^{\gamma_1^*+\gamma-1}\theta_2 + \left(\gamma+\gamma_1\right)q\frac{1}{\zeta\left(i_{j,N}\right)Z^*} = 0 \qquad (53)$$

as long as $M_t/M_{\tau_N} > \underline{\theta}_2$. Let $\theta_2(M_t/M_{\tau_N})$ denote this unique positive solution. Then

$$\theta_2(M_t/M_{\tau_N}) < \bar{\theta}_2 \qquad (54)$$

and the solution to the problem

$$v(\theta_t, M_t, M_{\tau_N}) = \max_{\bar{\theta}} \phi(\theta_t, M_t, M_{\tau_N}; \overline{\theta})$$

is given as follows:

In region I, i.e. when $\theta_t/M_{\tau_N} > \theta_2(M_t/M_{\tau_N})$, then:

$$v(\theta_t, M_t, M_{\tau_N}) = \zeta\left(i_{j,N}\right)Z^*\frac{\theta_t}{M_{\tau_N}}\left(\frac{\theta_t}{M_t}\right)^{-\gamma} + \frac{\gamma}{\gamma_1^*-1}\zeta\left(i_{j,N}\right)Z^*\frac{\theta_t}{M_{\tau_N}}\left(\frac{\theta_t}{M_t}\right)^{\gamma_1^*-1} - q\left(\frac{\theta_t}{M_t}\right)^{-\gamma}$$

In region II, i.e. when $M_t/M_{\tau_N} \geq \underline{\theta}_2$ and $\theta_2(M_t/M_{\tau_N}) \geq \theta_t/M_{\tau_N}$, then:

$$\begin{aligned} v(\theta_t, M_t, M_{\tau_N}) = & \left[\zeta\left(i_{j,N}\right)Z^*\left(\frac{\theta_2}{M_t/M_{\tau_N}}\right)^{-\gamma}\theta_2^{1-\gamma_1} + \zeta\left(i_{j,N}\right)Z^*\frac{\gamma}{\gamma_1^*-1}\left(\frac{\theta_2}{M_t/M_{\tau_N}}\right)^{\gamma_1^*-1}\theta_2^{1-\gamma_1}\right]\left(\frac{\theta_t}{M_{\tau_N}}\right)^{\gamma_1} \\ & -q\left(\frac{\theta_2}{M_t/M_{\tau_N}}\right)^{-\gamma}\theta_2^{-\gamma_1}\left(\frac{\theta_t}{M_{\tau_N}}\right)^{\gamma_1} \end{aligned}$$

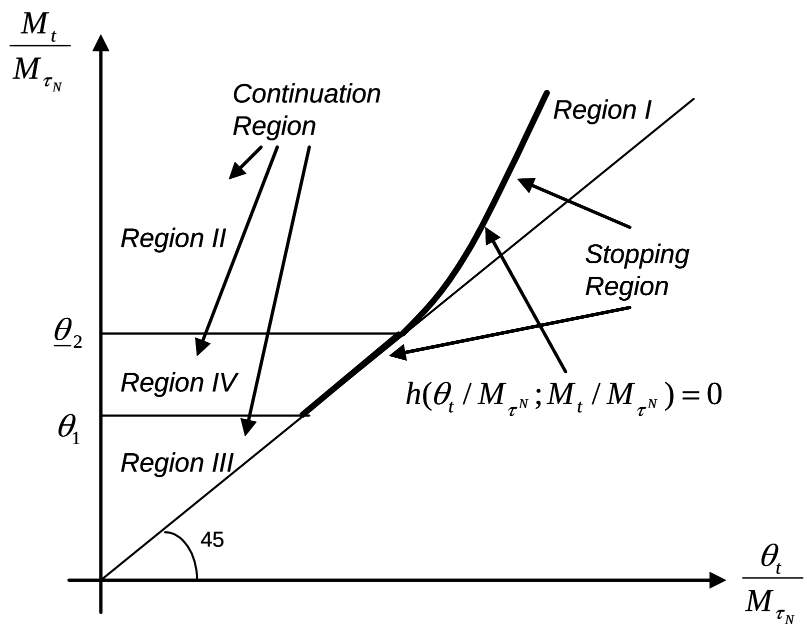

Figure 7: Depiction of the various regions. The continuation region is separated from the stopping region by the bold line.

In region III, i.e. when $M_t/M_{\tau_N} \leq \theta_1$, then

$$v(\theta_t, M_t, M_{\tau_N}) = \frac{1}{\gamma_1 - 1} q \left(\frac{\theta_t}{M_{\tau_N}} \right)^{\gamma_1} \theta_1^{-\gamma_1} \tag{55}$$

In region IV, i.e. when $\theta_1 \leq M_t/M_{\tau_N} \leq \underline{\theta}_2$, then

$$v(\theta_t, M_t, M_{\tau_N}) = \zeta \left(i_{j,N} \right) Z^* \frac{\gamma + \gamma_1^* - 1}{\gamma_1^* - 1} \frac{M_t}{M_{\tau_N}} \left(\frac{M_t}{\theta_t} \right)^{-\gamma_1} - q \left(\frac{M_t}{\theta_t} \right)^{-\gamma_1}$$

The stopping region is $S = \{(\theta_t, M_t) : \theta_1 \leq \theta/M_{\tau_N} = M/M_{\tau_N} \leq \theta_2 \text{ or } \theta/M_{\tau_N} \geq \theta_2\}$. The optimal stopping time is the first time t such that (θ_t, M_t) enters the stopping region.

Figure 7 gives a graphical depiction of all the possible regions. Clearly $\theta_t \leq M_t$ and we need only concern ourselves with the region above the 45 degree line. The bold line depicts the boundary between the stopping and the continuation region. Given that this is a two dimensional optimal stopping problem the decision rule will be a mapping between (θ_t, M_t) and the discrete decision {stop - continue}. The depicted stopping region is the set of points where the firm would choose to invest. The complement of this region is the set of points where it would wait.

We are now ready to state and prove the key result:

Lemma 4 *The function v given in Lemma 3 is the value function of problem (50) and the optimal stopping policy given in Lemma 3 is optimal among all stopping times.*

Proof of Lemma 4. Let the "value of immediate exercise" $f(\theta_t, M_t; M_{\tau_N})$ be defined as

$$\begin{aligned} f(\theta_t, M_t; M_{\tau_N}) \quad &\equiv \quad \zeta\left(i_{j,N}\right) Z^* \frac{\theta_t}{M_{\tau_N}} \left(\frac{\theta_t}{M_t}\right)^{-\gamma} \\ &+\zeta\left(i_{j,N}\right) Z^* \frac{\gamma}{\gamma_1^*-1} \left(\frac{\theta_t}{M_t}\right)^{\gamma_1^*-1} \frac{\theta_t}{M_{\tau_N}} - q\left(\frac{\theta_t}{M_t}\right)^{-\gamma} \end{aligned}$$

As a first step we show that the function v as given in Lemma 3 satisfies the following properties:

$$v(\theta_t, M_t, M_{\tau_N}) \geq f(\theta_t, M_t, M_{\tau_N}) \tag{56}$$

$$v_M(M_t, M_t, M_{\tau_N}) \leq 0 \quad \text{for } \theta_t = M_t \tag{57}$$

and

$$\mathcal{A}v(\theta_t, M_t, M_{\tau_N}) \leq 0 \tag{58}$$

where $\mathcal{A}v = \frac{\sigma^2}{2}\theta^2 v_{\theta\theta} + \mu\theta v_\theta - (\rho+\lambda)\, v$. Moreover, we shall demonstrate that v is continuous and differentiable throughout its domain. Finally we will employ a verification Theorem for optimal stopping similar to (Øksendal [2003]) to conclude.

Equation (56) holds by construction of the value function $v(\theta_t, M_t, M_{\tau_N})$. To see this note that according to Lemma 3:

$$\begin{aligned} v(\theta_t, M_t, M_{\tau_N}) \quad &= \quad \max_{\overline{\theta}} \phi(\theta_t, M_t, M_{\tau_N}; \overline{\theta}) \geq \\ &\geq \quad \phi(\theta_t, M_t, M_{\tau_N}; \frac{\theta_t}{M_{\tau_N}}) = \\ &= \quad f(\theta_t, M_t, M_{\tau_N}) \end{aligned}$$

Next we shall check equation (57) and equation (58) in all four regions.

In region I, $\theta_t / M_{\tau_N} > \theta_2\left(M_t / M_{\tau_N}\right)$, and hence

$$v_M(M_t, M_t, M_{\tau_N}) = -\gamma q \frac{1}{M_t} < 0$$

To show (58) we note that:

$$\begin{aligned} \mathcal{A}v(\theta_t, M_t, M_{\tau_N}) \quad &= \quad -\zeta\left(i_{j,N}\right) \frac{Z^*}{Z} \frac{\theta_t}{M_{\tau_N}} \left(\frac{\theta_t}{M_t}\right)^{-\gamma} - \frac{\gamma\lambda}{\gamma_1^*-1} \zeta\left(i_{j,N}\right) Z^* \frac{\theta_t}{M_{\tau_N}} \left(\frac{\theta_t}{M_t}\right)^{\gamma_1^*-1} - \frac{q}{Y}\left(\frac{\theta_t}{M_t}\right)^{-\gamma} \\ &= \quad \left[-\zeta\left(i_{j,N}\right) \frac{Z^*}{Z} \frac{\theta_t}{M_{\tau_N}} - \frac{\gamma\lambda}{\gamma_1^*-1} \zeta\left(i_{j,N}\right) Z^* \frac{\theta_t}{M_{\tau_N}} \left(\frac{\theta_t}{M_t}\right)^{\gamma_1^*+\gamma-1} - \frac{q}{Y}\right] \left(\frac{\theta_t}{M_t}\right)^{-\gamma} \end{aligned} \tag{59}$$

where

$$
\begin{aligned}
Z &= \frac{-1}{\frac{\sigma^2}{2}(\gamma-1)\gamma+\mu(1-\gamma)-\rho-\lambda} \\
Y &= \frac{1}{\frac{\sigma^2}{2}\gamma^2-\gamma\left(\mu-\frac{\sigma^2}{2}\right)-\rho-\lambda}
\end{aligned}
$$

Before proceeding, note that $\gamma_1\gamma_2 = -\frac{2(\rho+\lambda)}{\sigma^2}$and $\gamma_1+\gamma_2 = 1-\frac{2\mu}{\sigma^2}$. Using these facts along with remark 1 gives

$$
\begin{aligned}
Z &= \frac{1}{\frac{\sigma^2}{2}\left(\gamma_1+\gamma-1\right)\left(1-\gamma_2-\gamma\right)} > 0 \qquad (60) \\
Y &= \frac{1}{\frac{\sigma^2}{2}\left(\gamma_1+\gamma\right)\left(\gamma_2+\gamma\right)}
\end{aligned}
$$

Hence,

$$
\frac{1}{Y} = \frac{\gamma+\gamma_2}{1-\gamma-\gamma_2}\frac{\gamma_1+\gamma}{\gamma_1+\gamma-1}\frac{1}{Z} \qquad (61)
$$

From equation (53), we obtain:

$$
\frac{\gamma\zeta\left(i_{j,N}\right)Z^*}{\gamma_1^*-1}\left(\frac{\theta_2}{M_t/M_{\tau_N}}\right)^{\gamma_1^*+\gamma-1}\theta_2 = \left(\gamma_1+\gamma-1\right)\theta_2\frac{\zeta\left(i_{j,N}\right)Z^*}{\gamma_1^*-\gamma_1} - q\frac{\gamma+\gamma_1}{\gamma_1^*-\gamma_1} \qquad (62)
$$

Hence, by $\theta_t/M_{\tau_N} > \theta_2\left(M_t/M_{\tau_N}\right)$, equation (61), equation (62), the assumption $\gamma_1^* > 1$ and $Z^* > 0$ we can

express the terms inside the square bracket of (59) as

$$
\begin{aligned}
& -\zeta\left(i_{j,N}\right)\frac{Z^*}{Z}\frac{\theta_t}{M_{\tau_N}} - \frac{\gamma\lambda}{\gamma_1^*-1}\zeta\left(i_{j,N}\right)Z^*\frac{\theta_t}{M_{\tau_N}}\left(\frac{\theta_t}{M_t}\right)^{\gamma_1^*+\gamma-1} - \frac{1}{Y}q \\
\leq \; & -\zeta\left(i_{j,N}\right)\theta_2\frac{Z^*}{Z} - \lambda\left(\left(\gamma_1+\gamma-1\right)\theta_2\frac{\zeta\left(i_{j,N}\right)Z^*}{\gamma_1^*-\gamma_1} - q\frac{\gamma+\gamma_1}{\gamma_1^*-\gamma_1}\right) - \frac{1}{Y}q \\
= \; & \left[\frac{\lambda\left(\gamma_1+\gamma-1\right)}{\gamma_1-\gamma_1^*} - \frac{1}{Z}\right]Z^*\zeta\left(i_{j,N}\right)\theta_2 + q\left[-\frac{\left(\gamma+\gamma_1\right)\lambda}{\gamma_1-\gamma_1^*} - \frac{\gamma+\gamma_2}{1-\gamma-\gamma_2}\frac{\gamma_1+\gamma}{\gamma_1+\gamma-1}\frac{1}{Z}\right]
\end{aligned}
$$

To arrive from equation (59) to equation (58), we only need to show that

$$
\left[\frac{\lambda\left(\gamma_1+\gamma-1\right)}{\gamma_1-\gamma_1^*} - \frac{1}{Z}\right]Z^*\zeta\left(i_{j,N}\right)\theta_2 + q\left[-\frac{\left(\gamma+\gamma_1\right)\lambda}{\gamma_1-\gamma_1^*} - \frac{\gamma+\gamma_2}{1-\gamma-\gamma_2}\frac{\gamma_1+\gamma}{\gamma_1+\gamma-1}\frac{1}{Z}\right] \leq 0 \qquad (63)
$$

We first want to show that

$$
\frac{\lambda\left(\gamma_1+\gamma-1\right)}{\gamma_1-\gamma_1^*} > \frac{1}{Z} \qquad (64)
$$

By equation (60), this is equivalent to showing that

$$
\frac{\lambda\left(\gamma_1+\gamma-1\right)}{\gamma_1-\gamma_1^*} > \left(\gamma_1+\gamma-1\right)\left(1-\gamma_2-\gamma\right)\frac{\sigma^2}{2} \qquad (65)
$$

which amounts to showing that

$$
\left(1-\gamma_2-\gamma\right)\left(\gamma_1-\gamma_1^*\right) \quad < \frac{2\lambda}{\sigma^2} = \left(-\gamma_2+\gamma_1^*\right)\left(\gamma_1-\gamma_1^*\right) \qquad (66)
$$

which in turn follows because $\gamma_1^* > 1-\gamma$. Hence, we have proven equation (64). By equation (54), $\theta_2 \leq \bar{\theta}_2 = q\frac{1}{Z^*}\frac{1}{\zeta(i_{j,N})}\frac{\gamma+\gamma_1}{(\gamma_1+\gamma-1)}$. Moreover, $Z>0$, and remark 1 imply that

$$\begin{aligned} & \left[\frac{\lambda(\gamma_1+\gamma-1)}{\gamma_1-\gamma_1^*}-\frac{1}{Z}\right] Z^*\zeta(i_{j,N})\,\theta_2 + q\left[-\frac{(\gamma+\gamma_1)\lambda}{\gamma_1-\gamma_1^*}-\frac{\gamma+\gamma_2}{1-\gamma-\gamma_2}\frac{\gamma_1+\gamma}{\gamma_1+\gamma-1}\frac{1}{Z}\right] \\ \leq & \left[\frac{\lambda(\gamma_1+\gamma-1)}{\gamma_1-\gamma_1^*}-\frac{1}{Z}\right] q\frac{\gamma_1+\gamma}{(\gamma_1+\gamma-1)} + q\left[-\frac{(\gamma+\gamma_1)\lambda}{\gamma_1-\gamma_1^*}-\frac{\gamma+\gamma_2}{1-\gamma-\gamma_2}\frac{\gamma_1+\gamma}{\gamma_1+\gamma-1}\frac{1}{Z}\right] \\ = & \left[\frac{\gamma+\gamma_2}{\gamma+\gamma_2-1}-1\right] q\frac{1}{Z}\frac{\gamma_1+\gamma}{\gamma_1+\gamma-1} < 0 \end{aligned}$$

Hence, we showed equation (63), and hence, $\mathcal{A}v(\theta_t, M_t, M_{\tau_N}) \leq 0$.

In region II, $M_t/M_{\tau_N} \geq \underline{\theta}_2$ and $\theta_2(M_t/M_{\tau_N}) \geq \theta_t/M_{\tau_N}$. A direct calculation yields

$$\mathcal{A}v(\theta_t, M_t, M_{\tau_N}) = 0$$

since $\frac{\sigma^2}{2}\gamma_1(\gamma_1-1)+\mu\gamma_1-(\rho+\lambda)=0$. It is equally straightforward to check that in region II, equation (57) is satisfied automatically, since $\theta_t < M_t$.

In region III, $M_t/M_{\tau_N} \leq \theta_1$ and

$$v(\theta_t, M_t, M_{\tau_N}) = \frac{1}{\gamma_1-1}q\left(\frac{\theta_t}{M_{\tau_N}}\right)^{\gamma_1}\theta_1^{-\gamma_1}$$

Since $\frac{\sigma^2}{2}\gamma_1(\gamma_1-1)+\mu\gamma_1-(\rho+\lambda)=0$, we obtain $\mathcal{A}v(\theta_t, M_t; M_{\tau_N}) = 0$. Moreover, since v is independent of M_t in this region, $v_M(M_t, M_t; M_{\tau_N}) = 0$.

In region IV, $\theta_1 \leq M_t/M_{\tau_N} \leq \underline{\theta}_2$. Again, from $\frac{\sigma^2}{2}\gamma_1(\gamma_1-1)+\mu\gamma_1-(\rho+\lambda)=0$, we have $\mathcal{A}v(\theta_t, M_t, M_{\tau_N}) = 0$. To verify equation (57),

$$\begin{aligned} v_M(M_t, M_t, M_{\tau_N}) &= \zeta(i_{j,N})\,Z^*\frac{\gamma+\gamma_1^*-1}{\gamma_1^*-1}(1-\gamma_1)\frac{1}{M_{\tau_N}}+\gamma_1 q\frac{1}{M_t} \\ &= \left[\zeta(i_{j,N})\,Z^*\frac{\gamma+\gamma_1^*-1}{\gamma_1^*-1}(1-\gamma_1)\frac{M_t}{M_{\tau_N}}+\gamma_1 q\right]\frac{1}{M_t} \\ &\leq \left[\zeta(i_{j,N})\,Z^*\frac{\gamma+\gamma_1^*-1}{\gamma_1^*-1}(1-\gamma_1)\,\theta_1+\gamma_1 q\right]\frac{1}{M_t} \\ &= \left[(1-\gamma_1)\,q\frac{\gamma_1}{\gamma_1-1}+\gamma_1 q\right]\frac{1}{M_t} \\ &= 0 \end{aligned}$$

Having shown (56), (57), (58) in all regions of the state space, we are left with showing continuity and continuous differentiability of v with respect to θ_t. Since both of these conditions are satisfied in the interior of the four regions, it remains to check continuity and differentiability at the boundaries of regions I and II, II and IV and IV and III. This is a matter of straightforward computation of left and right limits and we leave the details of the computations out[31] in order to save space.

[31] Available upon request.

Given the properties of v it is now straightforward to take any stopping time τ and use Ito's formula to obtain:

$$
\begin{aligned}
e^{-(\rho+\lambda)\tau}v(\theta_\tau, M_\tau; M_{\tau_N}) &= e^{-(\rho+\lambda)t}v(\theta_t, M_t; M_{\tau_N}) + \int_t^\tau \mathcal{A}v(\theta_s, M_s; M_{\tau_N})e^{-(\rho+\lambda)s}ds \qquad (67) \\
&+ \int_t^\tau \sigma\theta_s v_\theta(\theta_s, M_s; M_{\tau_N})e^{-(\rho+\lambda)s}dB_s + \int_t^\tau v_M(\theta_s, M_s; M_{\tau_N})e^{-(\rho+\lambda)s}dM_s
\end{aligned}
$$

From equation (56), (57) and (58), we have

$$
\begin{aligned}
e^{-(\rho+\lambda)t}v(\theta_t, M_t; M_{\tau_N}) &\geq -E_t\left[\int_t^\tau \mathcal{A}v(\theta_s, M_s; M_{\tau_N})e^{-(\rho+\lambda)s}ds\right] + E_t\left[e^{-(\rho+\lambda)\tau}v(\theta_\tau, M_\tau; M_{\tau_N})\right] \\
&\geq E_t\left[e^{-(\rho+\lambda)\tau}v(\theta_\tau, M_\tau; M_{\tau_N})\right] \geq E_t\left[e^{-(\rho+\lambda)\tau}f(\theta_\tau, M_\tau; M_{\tau_N})\right]
\end{aligned}
$$

That is,

$$
v(\theta_t, M_t; M_{\tau_N}) \geq E_t\left[e^{-(\rho+\lambda)(\tau-t)}f(\theta_\tau, M_\tau; M_{\tau_N})\right]
$$

Hence, the conjectured value function $\nu(\theta_t, M_t; M_{\tau_N})$ is an upper bound for all value functions, and it is attainable by the proposed stopping rule. Hence it must be the value function of the optimal stopping problem (50). ■

This establishes that the stopping rule (42) is optimal. The next step is to establish that if firms follow the threshold policies just described, then equation (43) is satisfied.

To see this let $t^* \leq t$ be the smallest time prior to t such that $\theta_{t^*} = M_t$. Since in the proposed equilibrium firms plant new trees only at[32] $\theta_t = M_t$, then there are no new trees planted between time t^* and t. Hence,

$$
\int_0^1 X_{j,t^*}dj = \int_0^1 X_{j,t}dj
$$

Since

$$
\theta_{t^*}\int_0^1 X_{j,t^*}dj \leq M_t^C \leq M_t\int_0^1 X_{j,t}dt = \theta_{t^*}\int_0^1 X_{j,t^*}dj
$$

it follows that

$$
M_t^C = M_t\int_0^1 X_{j,t}dt
$$

Hence,

$$
\frac{C_t}{M_t^C} = \frac{\theta_t\int_0^1 X_{j,t}dt}{M_t\int_0^1 X_{j,t}dt} = \frac{\theta_t}{M_t}
$$

This concludes the proof of the main proposition.

Having constructed the equilibrium state price density, the rest of the verification that $\langle C_t, K_{n,t}, H_t\rangle$ constitute an equilibrium in the sense of definition 1 is standard. The reader is referred to Basak [1999] and

[32] Note that according to the proposed equilibrium all firms are always in Region III.

the monograph of Karatzas and Shreve [1998] Chapter 4 for details. This concludes the proof of Proposition 1.

A.1.2 Propositions and Proofs for section 4

The next proposition determines the price of a firm.

Proposition 2 *Let $\widehat{\gamma}_1, \widehat{Z}$ be given by*

$$\hat{\gamma}_1 = \frac{\left(\frac{\sigma^2}{2} - \mu\right) + \sqrt{\left(\frac{\sigma^2}{2} - \mu\right)^2 + 2\sigma^2(\rho + \lambda\left(1 - \overline{A}\right))}}{\sigma^2}$$

$$\widehat{Z} = -\frac{1}{\frac{\sigma^2}{2}\gamma_1(\gamma_1 - 1) + \mu\gamma_1 - \left[\rho + \lambda(1 - \bar{A})\right]}$$

and assume that:

$$\hat{\gamma}_1 > 1, \ \widehat{Z} < 0$$

Then, the price of firm j in technological epoch N is given by (6) where

$$P^A_{j,t} = Z^* X_{j,t}\theta_t \left[1 + \frac{\gamma}{\gamma_1^* - 1}\left(\frac{\theta_t}{M_t}\right)^{\gamma + \gamma_1^* - 1}\right] \tag{68}$$

$$P^o_{N,j,t} = Z^* \bar{A}^N \theta_t \left[\frac{1}{\gamma_1 - 1}\left(\frac{\theta_t}{M_t}\right)^{\gamma + \gamma_1 - 1}\left(\frac{M_t}{M_{\tau_N}}\right)^{\gamma_1 - 1}\left(\frac{q}{Z^*}\right)\psi(i_{j,N})^{-\gamma_1}\right]\left(1 - 1_{\{\widetilde{\chi}_{N,j} = 1\}}\right) \tag{69}$$

$$P^f_{N,t} = Z^* \bar{A}^N \theta_t \left\{\frac{\lambda \bar{A}\hat{Z}}{\gamma_1 - 1}\left[\left(\frac{\theta_t}{M_t}\right)^{\gamma + \gamma_1 - 1} - \frac{\gamma_1 - 1}{\hat{\gamma}_1 - 1}\left(\frac{\theta_t}{M_t}\right)^{\gamma + \widehat{\gamma}_1 - 1}\right]\left(\frac{q}{Z^*}\right)\left(\int_0^1 \psi(i)^{-\gamma_1} di\right)\right\} \tag{70}$$

The constants $Z^*, \gamma_1^*, \gamma_1$ are given in Proposition 1. $X_{j,t}$ is given by (3), $1_{\{\widetilde{\chi}_{N,j}=1\}}$ is the indicator function used in equation (3) that takes the value 1 *if firm j has planted a tree in the current epoch and* 0 *otherwise and $\psi(i_{j,N})$ is given by*

$$\psi(i_{j,N}) \equiv \frac{\Xi}{\zeta(i_{j,N})} = \frac{q}{Z^*}\frac{\gamma_1}{\gamma_1 - 1}\frac{\gamma_1^* - 1}{\gamma_1^* + \gamma - 1}\frac{1}{\zeta(i_{j,N})}$$

Proof of Proposition 2. Equation (68) follows from Corollary 1. To compute (69), one can use equation (50) along with the definition of v in equation(55). (Note that in equilibrium, the firm is always in region III).

To compute (70), let $\tau^*_{N,j}$ be optimal time for firm j to plant a new tree in the current epoch and τ_N be the time when epoch N arrived. Since $M_{\tau^*_{N,j}} = \theta_{\tau^*_{N,j}} = \psi(i_{j,N})M_{\tau_N}$, the value of all the future growth options is,

$$P^f_{N,j,t} = \frac{E\left(E_t\left[\sum_{n=N+1}^{\infty} e^{-\rho\left(\tau^*_{n,j} - t\right)} I_{\tau^*_{n,j} \leq \tau_{n+1}} \bar{A}^n \left(\zeta(i_{j,n})\int_{\tau^*_{n,j}}^{\infty} e^{-\rho\left(s - \tau^*_{n,j}\right)} M_s^{\gamma}\theta_s^{1-\gamma} ds - qM_{\tau_n}M^{\gamma}_{\tau^*_{n,j}}\theta^{-\gamma}_{\tau^*_{n,j}}\right)\right]\right)}{M_t^{\gamma}\theta_t^{-\gamma}}$$

$$= \frac{\sum_{n=N+1}^{\infty} \bar{A}^n E\left(E_t\left[e^{-\rho\left(\tau^*_{n,j} - t\right)} I_{\tau^*_{n,j} \leq \tau_{n+1}} \left(\zeta(i_{j,n})\int_{\tau^*_{n,j}}^{\infty} e^{-\rho\left(s - \tau^*_{n,j}\right)} M_s^{\gamma}\theta_s^{1-\gamma} ds - qM_{\tau_n}\right)\right]\right)}{M_t^{\gamma}\theta_t^{-\gamma}}$$

where E denote the expectation with respect to the unknown draws $i_{j,n}$ in future periods. To calculate $P^f_{N,j,t}$, we first find the following conditional expectation for any $n > N$ and any given $\zeta(i_{j,n})$:

$$\begin{aligned}
& E_t\left[e^{-\rho\left(\tau^*_{n,j}-t\right)} I_{\tau^*_{n,j}\le\tau_{n+1}}\left(\zeta(i_{j,n})\int_{\tau^*_{n,j}}^{\infty} e^{-\rho\left(s-\tau^*_{n,j}\right)} M_s^{\gamma}\theta_s^{1-\gamma}ds - qM_{\tau_n}\right)\right] \\
= & \; E_t\left[e^{-\rho\left(\tau^*_{n,j}-\tau_n+\tau_n-t\right)} I_{\tau^*_{n,j}\le\tau_{n+1}}\left(\zeta(i_{j,n})\int_{\tau^*_{n,j}}^{\infty} e^{-\rho\left(s-\tau^*_{n,j}\right)} M_s^{\gamma}\theta_s^{1-\gamma}ds - qM_{\tau_n}\right)\right] \\
= & \; E_t\left[E_{\tau^*_{n,j}}\left[e^{-\rho\left(\tau^*_{n,j}-\tau_n+\tau_n-t\right)} I_{\tau^*_{n,j}\le\tau_{n+1}}\left(\zeta(i_{j,n})\int_{\tau^*_{n,j}}^{\infty} e^{-\rho\left(s-\tau^*_{n,j}\right)} M_s^{\gamma}\theta_s^{1-\gamma}ds - qM_{\tau_n}\right)\right]\right] \\
= & \; E_t\left(e^{-(\rho+\lambda)\left(\tau^*_{n,j}-\tau_n\right)-\rho(\tau_n-t)} M_{\tau_n}\right)\left[\zeta(i_{j,n})\, Z^* \frac{\gamma+\gamma_1^*-1}{\gamma_1^*-1}\psi(i_{j,n}) - q\right] \\
= & \; q\frac{1}{\gamma_1-1} E_t\left(e^{-(\rho+\lambda)\left(\tau^*_{n,j}-\tau_n\right)-\rho(\tau_n-t)} M_{\tau_n}\right) \\
= & \; q\frac{1}{\gamma_1-1} E_t\left(\left(\frac{\psi(i_{j,n})}{\theta_{\tau_n}/M_{\tau_n}}\right)^{-\gamma_1} e^{-\rho(\tau_n-t)} M_{\tau_n}\right) \\
= & \; q\frac{1}{\gamma_1-1}\psi(i_{j,n})^{-\gamma_1} E_t\left[e^{-\rho(\tau_n-t)}\theta_{\tau_n}^{\gamma_1} M_{\tau_n}^{1-\gamma_1}\right]
\end{aligned}$$

where the second to last equality follows from

$$E_{\tau_n}\left(e^{-(\rho+\lambda)\left(\tau^*_{n,j}-\tau_n\right)}\right) = \left(\frac{\psi(i_{j,n})}{\theta_{\tau_n}/M_{\tau_n}}\right)^{-\gamma_1}$$

$\tau_n \ge t$, and the property of iterated conditional expectation. Hence,

$$\begin{aligned}
P^f_{N,j,t} &= \frac{\sum_{n=N+1}^{\infty} \bar{A}^n E\left(q\frac{1}{\gamma_1-1}\psi(i_{j,n})^{-\gamma_1} E_t\left[e^{-\rho(\tau_n-t)}\theta_{\tau_n}^{\gamma_1} M_{\tau_n}^{1-\gamma_1}\right]\right)}{M_t^{\gamma}\theta_t^{-\gamma}} \\
&= \frac{q}{\gamma_1-1}\left(\frac{\theta_t}{M_t}\right)^{\gamma} \bar{A}^N \; E\left(\psi(i_{j,n})^{-\gamma_1}\right) \sum_{n=N+1}^{\infty} \bar{A}^{n-N} E_t\left[e^{-\rho(\tau_n-t)}\theta_{\tau_n}^{\gamma_1} M_{\tau_n}^{1-\gamma_1}\right] \\
&= \frac{q}{\gamma_1-1}\left(\frac{\theta_t}{M_t}\right)^{\gamma} \bar{A}^N \; \left(\int_0^1 \psi(i)^{-\gamma_1} di\right) \sum_{n=N+1}^{\infty} \bar{A}^{n-N} E_t\left[e^{-\rho(\tau_n-t)}\theta_{\tau_n}^{\gamma_1} M_{\tau_n}^{1-\gamma_1}\right] \\
&\equiv \frac{q}{\gamma_1-1}\left(\frac{\theta_t}{M_t}\right)^{\gamma} \bar{A}^N \; \left(\int_0^1 \psi(i)^{-\gamma_1} di\right) \cdot \hat{P}^f_{N,j,t}(\theta_t, M_t)
\end{aligned}$$

where

$$\hat{P}^f_{N,j,t}(\theta_t, M_t) \equiv \sum_{n=N+1}^{\infty} \bar{A}^{n-N} E_t\left[e^{-\rho(\tau_n-t)}\theta_{\tau_n}^{\gamma_1} M_{\tau_n}^{1-\gamma_1}\right] = \tag{71}$$

$$= E_t\left(e^{-\rho(\tau_{N+1}-t)} \sum_{n=N+1}^{\infty} \bar{A}^{n-N}\left[e^{-\rho(\tau_n-\tau_{N+1})}\theta_{\tau_n}^{\gamma_1} M_{\tau_n}^{1-\gamma_1}\right]\right) \tag{72}$$

Now, we only need to find $\hat{P}^f_{N,j,t}$. It will be easiest to re-express $\hat{P}^f_{N,j,t}$ in recursive form:

$$\hat{P}^f_{N,j,t}(\theta_t, M_t) = E_t\left(e^{-\rho(\tau_{N+1}-t)}\left[\bar{A}\theta_{\tau_{N+1}}^{\gamma_1} M_{\tau_{N+1}}^{1-\gamma_1} + \bar{A}\hat{P}^f_{N,j,t}\left(\theta_{\tau_{N+1}}, M_{\tau_{N+1}}\right)\right]\right) \tag{73}$$

The solution to $\hat{P}^f_{N,j,t}$ is given by

$$\hat{P}^f_{N,j,t}(\theta_t, M_t) = \lambda \bar{A} \hat{Z} \theta_t^{\gamma_1} M_t^{1-\gamma_1} \left[1 - \frac{\gamma_1 - 1}{\hat{\gamma}_1 - 1} \theta_t^{\hat{\gamma}_1 - \gamma_1} M_t^{\gamma_1 - \hat{\gamma}_1}\right] \tag{74}$$

where

$$\hat{Z} = \frac{-1}{\frac{\sigma^2}{2}\gamma_1(\gamma_1 - 1) + \mu\gamma_1 - \left[\rho + \lambda - \lambda\bar{A}\right]}$$

One can verify that this is true by checking that

$$0 = \frac{\sigma^2}{2}\theta_t^2 \hat{P}^f_{\theta\theta} + \mu\theta_t \hat{P}^f_\theta - (\rho + \lambda)\hat{P}^f + \lambda\left[\bar{A}\theta_t^{\gamma_1} M_t^{1-\gamma_1} + \bar{A}\hat{P}^f\right]$$

and

$$\hat{P}^f_M(\theta_t, M_t) = 0 \text{ at } \theta_t = M_t$$

and then employing a similar verification step to the one given in Lemma 2. Hence, the value of all the future growth options is

$$\begin{aligned}
P^f_{N,j,t} &= \frac{q}{\gamma_1 - 1}\left(\frac{\theta_t}{M_t}\right)^{\gamma} \bar{A}^N \left(\int_0^1 \psi(i)^{-\gamma_1} di\right) \cdot \hat{P}^f_{N,j,t}(\theta_t, M_t) = \\
&= \frac{q}{\gamma_1 - 1}\left(\frac{\theta_t}{M_t}\right)^{\gamma} \bar{A}^N \left(\int_0^1 \psi(i)^{-\gamma_1} di\right) \lambda\bar{A}\hat{Z}\theta_t^{\gamma_1} M_t^{1-\gamma_1}\left[1 - \frac{\gamma_1 - 1}{\hat{\gamma}_1 - 1}\theta_t^{\hat{\gamma}_1 - \gamma_1} M_t^{\gamma_1 - \hat{\gamma}_1}\right] &(75)\\
&= Z^* \bar{A}^N \theta_t \left\{\frac{\lambda\bar{A}\hat{Z}}{\gamma_1 - 1}\left[\left(\frac{\theta_t}{M_t}\right)^{\gamma+\gamma_1-1} - \frac{\gamma_1 - 1}{\hat{\gamma}_1 - 1}\left(\frac{\theta_t}{M_t}\right)^{\gamma+\hat{\gamma}_1-1}\right]\left(\frac{q}{Z^*}\right)\left(\int_0^1 \psi(i)^{-\gamma_1} di\right)\right\} &(76)
\end{aligned}$$

This completes the proof of the proposition. ■

Proof of Lemma 1. A feature of the model is that risk aversion is constant[33]. Hence the Sharpe ratio is a constant equal to $\gamma\sigma$, and the instantaneous excess return on any asset is given by:

$$\mu^{(j)} - r = \gamma\sigma\sigma_{j,t}$$

where:

$$\sigma_{j,t} = \left(\frac{\sigma\theta_t}{P_{N,j,t}}\right)\frac{\partial P_{N,j,t}}{\partial\theta_t}$$

Hence in order to establish the result is suffices to compare the instantaneous volatility of assets in place, current epoch growth options and future growth options. Using Proposition 2 and starting with the volatility of asset in place, we obtain:

$$\begin{aligned}
\sigma^A_{j,t} &= \left(\frac{\sigma\theta_t}{P^A_{N,j,t}}\right)\frac{\partial P^A_{N,j,t}}{\partial\theta_t} \\
&= \sigma\left[w_1 + (1 - w_1)(\gamma + \gamma_1^*)\right] \\
&= \sigma\left[(\gamma + \gamma_1^*) + w_1(1 - \gamma - \gamma_1^*)\right] &(77)
\end{aligned}$$

[33] The running maximum of a diffusion has no quadratic variation, and hence all of the quadratic variation of the pricing kernel is driven by variations in θ_t and not M_t.

where

$$
\begin{aligned}
w_1 &= \frac{\theta_t}{\theta_t + \frac{\gamma}{\gamma_1^* - 1} M_t^{1-\gamma-\gamma_1^*} \theta_t^{\gamma+\gamma_1^*}} \\
&= \frac{\gamma_1^* - 1}{\gamma_1^* - 1 + \gamma \left(\frac{M_t}{\theta_t}\right)^{1-\gamma-\gamma_1^*}}
\end{aligned}
$$

It is clear that $1 \geq w_1 \geq 0$. For the volatility of the current epoch growth option, we obtain similarly

$$
\begin{aligned}
\sigma_{j,t}^{o} &= \left(\frac{\sigma\theta_t}{P_{N,j,t}^{o}}\right) \frac{\partial P_{N,j,t}^{o}}{\partial \theta_t} \\
&= \sigma(\gamma + \gamma_1) \qquad (78)
\end{aligned}
$$

For the volatility of future options,

$$
\begin{aligned}
\sigma_{j,t}^{f} &= \left(\frac{\sigma\theta_t}{P_{N,j,t}^{f}}\right) \frac{\partial P_{N,j,t}^{f}}{\partial \theta_t} \\
&= \sigma\left[w_3(\gamma + \gamma_1) + (1 - w_3)(\gamma + \hat{\gamma}_1)\right] \\
&= \sigma\left[(\gamma + \hat{\gamma}_1) + w_3(\gamma_1 - \hat{\gamma}_1)\right] \qquad (79)
\end{aligned}
$$

where

$$
\begin{aligned}
w_3 &= \frac{\theta_t^{\gamma+\gamma_1} M_t^{1-\gamma-\gamma_1}}{\theta_t^{\gamma+\gamma_1} M_t^{1-\gamma-\gamma_1} - \frac{\gamma_1 - 1}{\hat{\gamma}_1 - 1} \theta_t^{\gamma+\hat{\gamma}_1} M_t^{1-\gamma-\hat{\gamma}_1}} \\
&= \frac{1}{1 - \frac{\gamma_1 - 1}{\hat{\gamma}_1 - 1} \left(\frac{M_t}{\theta_t}\right)^{\gamma_1 - \hat{\gamma}_1}}
\end{aligned}
$$

Since $\gamma_1 \geq \hat{\gamma}_1$ and $w_3 \leq 0$, equation (78) and (79) implies

$$
\sigma_{j,t}^{f} \leq \sigma(\gamma + \hat{\gamma}_1) < \sigma(\gamma + \gamma_1) = \sigma_{j,t}^{o}
$$

To develop the relation between the volatility of assets in place and future growth options, we define the function $d(x)$ as follows,

$$
\begin{aligned}
d\left(\frac{M_t}{\theta_t}\right) &\equiv \frac{\sigma_{j,t}^{A} - \sigma_{j,t}^{f}}{\sigma} \\
&= \left[(\gamma + \gamma_1^*) + w_1(1 - \gamma - \gamma_1^*)\right] - \left[(\gamma + \hat{\gamma}_1) + w_3(\gamma_1 - \hat{\gamma}_1)\right] \\
&= (\gamma_1^* - \hat{\gamma}_1) + \frac{\gamma_1^* - 1}{\gamma_1^* - 1 + \gamma\left(\frac{M_t}{\theta_t}\right)^{1-\gamma-\gamma_1^*}}(1 - \gamma - \gamma_1^*) - \frac{\hat{\gamma}_1 - 1}{\hat{\gamma}_1 - 1 - (\gamma_1 - 1)\left(\frac{M_t}{\theta_t}\right)^{\gamma_1 - \hat{\gamma}_1}}(\gamma_1 - \hat{\gamma}_1)
\end{aligned}
$$

It is clear that

$$
\begin{aligned}
d(1) &= (\gamma_1^* - \hat{\gamma}_1) + \frac{\gamma_1^* - 1}{\gamma_1^* - 1 + \gamma}(1 - \gamma - \gamma_1^*) - \frac{\hat{\gamma}_1 - 1}{\hat{\gamma}_1 - 1 - (\gamma_1 - 1)}(\gamma_1 - \hat{\gamma}_1) \\
&= 0
\end{aligned}
$$

and

$$\begin{aligned} d'(x) &= -\frac{(\gamma_1^* - 1)\gamma x^{-\gamma-\gamma_1^*}}{(\gamma_1^* - 1 + \gamma x^{1-\gamma-\gamma_1^*})^2}(1-\gamma-\gamma_1^*)^2 - \frac{(\hat{\gamma}_1 - 1)(\gamma_1 - 1)x^{\gamma_1-\hat{\gamma}_1-1}}{[\hat{\gamma}_1 - 1 - (\gamma_1 - 1)x^{\gamma_1-\hat{\gamma}_1}]^2}(\gamma_1 - \hat{\gamma}_1)^2 \\ &< 0 \end{aligned}$$

It follows that $d(x) < 0$ for all $x > 1$. That is, $\sigma_{j,t}^A \leq \sigma_{j,t}^f$ and the equality holds only if $\theta_t = M_t$. Therefore, we have

$$\sigma_{j,t}^A \leq \sigma_{j,t}^f < \sigma_{j,t}^o$$

and the equality holds only if $\theta_t = M_t$. ■

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