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Abstract

We study the optimal portfolio choice of hedge fund managers who are compensated by high-water mark contracts – contracts that pay only when the value of the fund exceeds its previously recorded maximum. Surprisingly, we find that even risk-neutral managers will not place unboundedly large weights on the risky assets, despite the option-type features of the contract. We show that the resulting optimal portfolio will place a constant fraction in a mean-variance efficient portfolio and the rest in a riskless asset, as in the portfolio of an investor with constant relative risk aversion. This result is a consequence of the in(de)finite horizon of the contract. We argue more generally that the risk-seeking incentives of option-type compensation contracts rely on the interaction of convex compensation and finite horizons, rather than on the convexity of the compensation scheme.

Keywords: Performance evaluation, Hedge funds, Option-type compensation, High-water marks, Continuous time

JEL Codes: G11, G2

1 Introduction

A popular intuition in financial economics[1] is that contracts with convex payoffs can potentially increase the incentive to take risks – risk-shifting. According to this commonly held view, if a manager is rewarded for gains but not punished for losses, that manager will only gain from increasing underlying volatility.

In this paper, we ask the question: How are the incentives of the manager affected if the horizon of the compensation contract is infinite or indefinite?

Our interest in this question is both practical and theoretical. At a practical level, we want to understand the incentives provided by certain contracts called high-water mark contracts, which have proliferated with the growth of the hedge fund industry. As we show, such contracts can be thought of as a perpetually renewed call option on the value of the fund, and so they present a particularly interesting type of option contract with an in(de)finite horizon. At the theoretical level, we want to understand whether the in(de)finite horizon of the contract will mitigate risk-shifting incentives.

Under the typical high-water mark contract, hedge fund managers receive a fraction of the increase in fund value in excess of the last recorded maximum – the high-water mark – provided that such an increase took place. Such performance fees are typical in the hedge fund industry, and they can lead to spectacular compensation for the hedge fund managers – when the fund is successful[2]. They also seem to avoid the problem of biasing portfolios toward replicating an index, as is the case of relative performance evaluation.[3]

[1] This intuition has been challenged by Carpenter (2000) and Ross (2004). We discuss their work and its relation to ours in the literature review section of the introduction.

[2] These fees can range from 15% to 50% of the recent increase and can earn billions of dollars. There have been several academic articles on the characteristics of hedge fund fees, such as Fung and Hsieh (1999), Ang, Rhodes-Kropf, Zhao (2005) as well as several articles in the Wall Street Journal, such as Zuckerman (2004). An example (reported in Goetzmann, Ingersoll and Ross [2003]) is the Quantum Fund, which receives a high-water mark fee of 20% and a regular annual fee of 1%. For a (pre-fee) performance of 49% in 1995 this resulted in an estimated compensation of $393 million. In contrast, when the fund lost 1.5% in 1996 and did not reach its high-water mark, the compensation was just 1% on the net asset value, or $54 million dollars.

[3] For an analysis of the incentives to bias the portfolio in the direction of a particular index and take on

The high-water mark contract resembles an option in that the manager is paid when fund value increases above a given level. To keep with option terminology, we call this level the "strike price". When the fund gains value, the manager is paid and the strike price increases, but when the fund loses money, the strike price remains unchanged and the manager retains her implied option at the old strike price. The high-water mark contract should not be thought of as one option, but as a sequence of options with a changing strike price. When the fund declines in value, the value of all the implied future options declines as well. Importantly, most hedge fund management contracts do not have a pre-specified termination date.

At a first pass, high-water mark contracts seem to be subject to standard risk-shifting problems. Because the manager can determine the fund's volatility – and increasing underlying asset volatility increases the value of an option – one might think that a manager would take an unboundedly large position in risky assets. We show that this need not be the case and provide examples to the contrary. To illustrate our point, we compute the optimal portfolio of a risk-neutral hedge fund manager who maximizes the expected discounted value of compensation fees. Surprisingly, we demonstrate that her portfolio coincides with the portfolio that would have been chosen by a CRRA (constant relative risk aversion) investor.

The intuition for our results is straightforward. A bolder portfolio today could help overcome the current high-water mark more quickly, but it will also increase the likelihood that next period the manager will start far underwater[4]. A hedge fund manager sees a trade-off between the two effects, and she is led to the choice of an interior portfolio. Interestingly, this portfolio places a constant fraction of wealth in the risky asset and the rest in a mean-variance efficient portfolio of stocks. Hence, the risk-neutral hedge fund manager behaves exactly as a Merton-type investor with a specific constant relative risk aversion coefficient, that we determine explicitly.

Our analysis supports two broad conclusions:

additional risk, see also Chevalier and Ellison (1997), Carpenter (2000), Cuoco and Kaniel (2001), Basak, Shapiro and Tepla (2002), and Basak, Pavlova and Shapiro (2003).

[4] "Underwater" means that the value of the fund would be below the high-water mark.

First, the commonly given risk-shifting intuition does not rely only on the implicit option contract *per se*, but on *the finiteness of the horizon* as well. In our model the manager is risk-neutral, and yet she ends up choosing a bounded portfolio, despite the option-type character to her compensation. The only disciplining force in our model is the presence of a continuation value function on an infinite horizon.

Second, absolute performance evaluation[5] seems to induce portfolio managers to invest in mean variance efficient portfolios as opposed to portfolios that contain a bias towards a specific index. This result has been confirmed empirically by studies such as Fung and Hsieh (1997). Somewhat surprisingly, the fraction invested in the risky and the riskless asset are constant over time in much the same way that they would be for an investor with constant relative risk aversion.

We make several assumptions so as to sharpen our results. While they are not completely realistic, they either serve to make our results more surprising or to preserve comparability with the existing literature. First, we assume the manager is risk neutral and we place no restrictions on the manager's portfolio choice. This increases her incentives to risk shift while giving her the ability to take extremely leveraged positions. We use these assumptions to illustrate circumstances where one would think that incentives should have the most "bite", leading the manager to take unboundedly large risks.

Second, the fund has an effectively infinite horizon. It is liquidated either at a random time or when its value drops to 0, whichever comes first. This assumption is a key departure from most of the existing literature on portfolio choice by professional fund managers. A common thread behind all of our results is the disciplining effect of the infinite horizon.[6]

[5] We use the term absolute performance evaluation to mean that outcomes are not compared to an index. This contrasts with relative performance evaluation in which compensation is determined by evaluating performance relative to an index.

[6] Liquidating the fund when wealth drops to zero (and not at a higher threshold) is not realistic. However, endogenous termination – perhaps after several periods of bad performance – would provide its own disciplining effect. This assumption also provides the most useful comparison to Goetzmann, Ingersoll and Ross (2003), who demonstrate that in this case a manager who is trying to maximize the implicit market value of fees would try to maximize volatility.

Third, and in line with the preexisting literature, we completely restrict the manger's ability to trade on her own account[7]. At a practical level this means that the manager's consumption stream and her compensation fees coincide and the manager is maximizing the (subjective) expected discounted sum of future incentive fees rather than their implied market value.

This paper is related to a number of strands in the literature. The paper closest to this is Goetzmann, Ingersoll, and Ross (2003) (henceforth "GIR"). They estimate the implied market value of hedge fund management fees in a setting where the fund's value follows a simple log-normal process and the fund managers have no discretion over the choice of portfolio. However, they don't explicitly model the effects of hedge fund compensation schemes on portfolio choice. In their framework, maximizing the market value of the fund manager's compensation would lead to unbounded volatility, absent other constraints. Our analysis supports and extends their result: We can show that a manager would indeed place an unbounded weight on risky assets if she tried to maximize the implied market value of fees. However, we show that maximization of expected discounted compensation can lead to different outcomes: The key difference is that in our paper the continuation value is

[7] The usual choice in the literature is to assume that the compensation is paid out at some time T and the manager derives utility $U(W_T)$, where W_T is her compensation and U is some concave function. However, it is implicitly assumed that the manager cannot trade on a private account using these fees as collateral. For papers in this line of literature see Carpenter (2000), Cuoco and Kaniel (2001), Basak, Shapiro and Tepla (2002), and Basak, Pavlova and Shapiro (2003).

The inability of the manager to trade privately is necessary in order to make the problem interesting. If we allowed the manager to trade freely, then separation results in complete markets would imply that a manager should maximize the (implicit) market value of her fees. This in turn could be done by setting volatility to the maximum possible level. This follows directly from the results in Goetzmann, Ingersoll, and Ross (2003), who show that the market value of the management fees is strictly increasing in the volatility of the underlying asset. Hence, a manager who would be given the option to choose a portfolio in this framework would clearly choose an unbounded portfolio. Therefore, in line with all preexisting literature on portfolio choice, we assume that the manager cannot borrow or enter any contingent contracts with respect to her fees.

sufficiently higher, as to discipline the manager away from risk-shifting.

The basic intuition that convex compensation schemes will increase risk seeking has been challenged by both Carpenter (2000) and Ross (2004) who have addressed this problem in finite horizon settings. In both cases, a risk averse manager may not be made more risk seeking by an option-like contract. However, it appears that in both papers a risk neutral manager would choose unbounded volatility. Our setup is different, since we have an infinite horizon setting. In particular, we show that even a risk neutral manager would not seek unbounded risks. This helps us demonstrate the new intuition produced by this paper: what mitigates risk seeking is *exclusively* the disciplining force of the continuation value on an infinite horizon.

The paper is also related to the literature on incentive fees for fund managers.[8] In this literature, it is typically assumed that managers receive management fees at the end of some finite period. In most of these papers there is also some form of benchmarking. By contrast, this paper examines high-water mark incentive schemes on an infinite horizon. Our results are different because in our model there is no explicit terminal time. Moreover, the "benchmarking" is not exogenous, but depends on the past performance of the fund.

On a methodological level, this paper shares many similarities to Browne (1997) on the maximization of the expected discounted reward from attaining a goal. In fact, the problem that we consider in this paper and the problem considered by Browne (1997) are identical in the region where the high-water mark is not increasing. We also use insights from Heinricher and Stockbridge (1991) to provide a new verification theorem that is appropriate for problems of the kind considered in this paper.

The structure of the paper is as follows: Section 2 presents the model. Section 3 presents a heuristic way of arriving at the solution and a discussion of the results. Section 4 contains an extension to the risk-averse case, and Section 5 contains an extension to multiple assets and a drift in the high-water mark. Section 6 concludes. All proofs are contained in the

[8] Recent paper in this literature include Carpenter (2000), Cuoco and Kaniel (2001), Basak, Shapiro and Tepla (2002), and Basak, Pavlova and Shapiro (2003), Hodder and Jackwerth (2004), among many others.

appendix.

2 The Model

We begin with a model in which the manager is risk-neutral, there is one risky asset, no intermediate withdrawal of funds by investors, and the high-water mark is adjusted only when the value of the fund exceeds the previously recorded high-water mark. A number of possible extensions are considered in later sections.

2.1 Investment Opportunities

The hedge fund manager has two investment opportunities. The first opportunity is the money market, where she receives a fixed interest rate of $r > 0$. Formally, the price of an asset invested in the money market evolves as

$$\frac{dP_0}{P_0} = rdt$$

We place no limits on the positions that can be taken in the money market. In addition, the fund manager can invest in a risky security with a price per share that evolves as a geometric Brownian motion

$$\frac{dP_1}{P_1} = \mu dt + \sigma dB_t$$

where $\mu > r$ and $\sigma > 0$ are constants and B is a one-dimensional Brownian motion.[9]

We make no statement about the source of the hedge fund manager's investment opportunities. In particular they could reflect the prevailing risk premia, or simply reflect the manager's subjective assessment of expected returns. We study only how the manager exploits investment opportunities, not how and why these investment opportunities exist.

[9] B is a one dimensional Brownian motion on a complete probability space (Ω, F, P). We denote by $F = \{F_t\}$ the P-augmentation of the filtration generated by B.

2.2 Portfolio and Wealth Processes

The portfolio process π_t is the fraction of fund wealth (W_t) invested in the risky asset at time t. The remainder of fund wealth, $W_t(1-\pi_t)$, is invested in the riskless asset. We allow short selling and both borrowing and lending at the riskless rate.[10]

An important modification of the standard dynamic budget constraint is introduced by the presence of a high-water mark. This component is related to the running maximum process

$$H_t = \max\{W_s; s \in [0,t]\}$$

Whenever $dH_t \neq 0$ – meaning that the running maximum increases – fund wealth declines by kdH_t to reflect the fraction of the increase that is given to the fund manager as compensation. Under these assumptions, the evolution of the wealth process is

$$dW_t = W_t \pi_t \left(\mu dt + \sigma dB_t\right) + W_t \left(1 - \pi_t\right) r dt - k dH_t \tag{1}$$

2.3 Optimization Problem

The hedge fund manager is assumed to be maximizing the expected net present value of her fees. Mathematically, these are represented by an objective function of the form

$$\max_{\pi} E\left[\int_0^{\tau^0 \wedge \infty} e^{-(\beta+\lambda)t} k dH_t\right] \tag{2}$$

The agent's discount factor, $\beta > 0$, is adjusted by the probability of fund termination. In order to perform some useful analysis later, we assume that the hedge fund is subject to exogenous random termination. This termination process is assumed to be Poisson with constant intensity $\lambda > 0$. At termination, the investor is assumed to withdraw all of her

[10] To have a well defined problem, we also require that for $\theta_t = \pi_t W_t$, it is the case that

$$\int_0^T \theta_t^2 dt < \infty \text{ for all } T > 0$$

This is a common assumption throughout the portfolio choice literature and is used to avoid doubling strategies (see Duffie[1996]).

invested funds, the hedge fund management contract is terminated, and the manager's continuation value is 0. We also assume that the fund will be terminated the first time wealth drops to 0 – i.e. at τ^0 such that

$$\tau^0 = \inf\{t : W_t \leq 0\}$$

although under the optimal portfolio policy this will never occur in finite time.

3 Solution

3.1 A Heuristic Derivation

While the optimization problem (2) is somewhat nonstandard, one can heuristically derive some conditions that candidate value functions should satisfy. In the body of the text we make no attempt to formally justify any of the steps that we take. Instead, after we derive these conditions, we use a rigorous verification theorem (proved in the appendix) to show that they are sufficient to identify the value function.

We start in the usual fashion by rewriting the value function for the optimization problem (2) in time separable form as

$$e^{-(\beta+\lambda)t}V(W_t, H_t)$$

where

$$V(W_t, H_t) = \max_{\{\pi_s\}_{s=t}^{\infty}} E_t\left[\int_t^{\tau^0\wedge\infty} e^{-(\beta+\lambda)(s-t)}kdH_s\right] \tag{3}$$

The first step is to study the value function in the region $\{W_t = H_t\}$. We assume – holding H_t fixed – that W_t increases instantaneously above H_t by an amount ε. Then the value function will satisfy

$$V(W_t + \varepsilon, H_t) = k\varepsilon + V(W_t + (1 - k)\varepsilon, H_t + \varepsilon)$$

for small ε. The right-hand side captures the fact that once the high-water mark is reached, an amount of $k\varepsilon$ is paid out to the manager as part of her incentive compensation. Simultaneously, wealth is reduced to reflect the outflow towards the manager, and the high-water mark is reset to $H_t + \varepsilon$. If we assume differentiability, we can expand in a Taylor fashion around $(W_t, H_t)|_{W_t=H_t}$ to obtain

$$V + \varepsilon V_W = k\varepsilon + V + V_W(1-k)\varepsilon + \varepsilon V_H$$

Subtracting V from both sides and dividing by ε, we find that the value function on the set $\{W_t = H_t\}$ should satisfy

$$kV_W = k + V_H \tag{4}$$

The next step is to study the value function on the set $\{W_t < H_t\}$. In this region $dH_t = 0$ and $V(W_t, H_t)$ behaves as a discounted martingale that satisfies the standard Hamilton-Jacobi-Bellman relation

$$0 = -(\beta+\lambda)V + \max_{\pi_t}\{V_W W_t(r + \pi_t(\mu - r)) + \frac{1}{2}V_{WW}W_t^2\sigma^2\pi_t^2\} \tag{5}$$

Observe that terms associated with dH_t do not appear in this relation because the running maximum does not increase in this region.

To proceed, we assume that the value function is concave with respect to W_t, and we carry out the maximization in (5) to obtain

$$\pi_t^* = -\frac{V_W}{V_{WW}W_t}\frac{\mu - r}{\sigma^2} \tag{6}$$

which is the standard Merton-type solution. After substituting this optimal π_t^* into (5) and simplifying, we obtain

$$0 = -(\beta+\lambda)V + rV_W W_t - \frac{1}{2}\frac{(V_W)^2}{V_{WW}}\frac{(\mu - r)^2}{\sigma^2}$$

A solution is

$$V(W_t, H_t) = K(H_t)W_t^\eta$$

where $K(H_t)$ is a function that depends on H_t only, and the constant η solves

$$\eta^2 r - \eta(r+\beta+\lambda+\omega) + (\beta+\lambda) = 0 \tag{7}$$

where ω is defined to be

$$\omega = \frac{1}{2}\frac{(\mu - r)^2}{\sigma^2} \tag{8}$$

Remember that $\mu > r$ and $\omega > 0$ by assumption.

Equation (7) admits two solutions:

$$\eta^{\pm} = \frac{r+\beta+\lambda+\omega \pm \sqrt{(r+\beta+\lambda+\omega)^2 - 4r(\beta+\lambda)}}{2r} \tag{9}$$

We can rewrite the term inside the square root as

$$(\beta+\lambda+\omega-r)^2 + 4r\omega > 0 \tag{10}$$

which shows that both roots of (9) are real. The roots are also both positive since $\eta^+ > 0$ and $\eta^+\eta^- = \frac{\beta+\lambda}{r} > 0$. Additionally, $\eta^- < 1$ and $\eta^+ > 1$ which can be seen by evaluating (7) and observing that the expression is positive for $\eta = 0$ and negative for $\eta = 1$.

In the Appendix we prove a "verification" theorem. This theorem states that if a function satisfies a set of conditions, then it is the value function. One of these conditions is concavity with respect to W_t. In order to ensure such concavity, we restrict attention to the root $\eta^- < 1$, and our conjecture for the value function becomes

$$V(W_t, H_t) = K(H_t) W_t^{\eta^-} \tag{11}$$

To simplify notation, from now on we shall write η instead of η^- and we will implicitly refer always to the root of the polynomial in (9) associated with the negative sign.

Finally, we must find $K(H_t)$, so that the boundary condition (4) is satisfied. Taking derivatives and evaluating at $W_t = H_t$, we find that

$$k\eta K(H_t) H_t^{\eta-1} = k + K'(H_t) H_t^{\eta}$$

This leads to the ordinary differential equation

$$K'(H_t) = k\left(\eta\frac{K(H_t)}{H_t} - \frac{1}{H_t^\eta}\right) \tag{12}$$

with the (general) solution

$$K(H_t) = \left[\frac{k}{\eta(1+k)-1}H^{1-(1+k)\eta} + C\right]H^{k\eta} \tag{13}$$

for an arbitrary constant C. To determine the constant C, we assume, hypothetically, that the fund will be terminated whenever $H_t = \overline{H}$ for a sufficiently high $\overline{H}$. This implies the boundary condition:

$$V(W_t, \overline{H}) = 0$$

which leads us to set the constant in (13) equal to

$$C = -\frac{k\overline{H}^{(1-\eta(1+k))}}{\eta(1+k)-1} \tag{14}$$

To obtain the solution to the infinite horizon problem, let $\overline{H} \to \infty$ and assume that

$$\eta(1+k) > 1 \tag{15}$$

Then the constant in equation (13) converges to 0 and the (conjectured) value function[11] is

$$V(W_t, H_t) = \frac{kH_t}{\eta(1+k)-1}\left(\frac{W_t}{H_t}\right)^\eta \tag{16}$$

The following proposition, proved in the appendix, verifies our conjecture:

Proposition 1 *Assume that condition (15) holds and that $\omega > 0$. Then the value function is given by (16) and the optimal portfolio by (6).*

[11] The conjecture for the value function (16) also satisfies the properties

$$\lim_{H\to\infty} V_H(H_t, W_t) = 0$$
$$\lim_{W_t\to 0} V(H_t, W_t) = 0$$

In deriving this value function, the key assumptions that we made on the parameters were equation (15) and $\omega > 0$. Both conditions are easy to interpret economically. Condition (15) is an inherently technical condition and says that discounting of future payoffs has to be sufficiently strong to ensure that the value function of the hedge fund manager remains finite. $\omega > 0$ implies that the manager perceives that there exist some investment opportunities in the market that will yield more than r. We will discuss this latter condition in section 3.3.

Combining (6) and (16), we obtain the optimal portfolio

$$\pi_t^* = \frac{1}{1-\eta}\frac{\mu - r}{\sigma^2} \tag{17}$$

As already mentioned, this portfolio places a constant fraction in the risky assets independent of any state variable of the problem. In this sense, a risk-neutral hedge fund manager behaves as if she were a Merton-type investor with constant relative risk aversion[12] of $1-\eta$. This effective risk aversion and its magnitude are endogenous rather than assumed.

In the next subsections we develop the intuition behind equation (17) by identifying the key assumptions of the analysis and performing comparative statics exercises.

3.2 The Horizon Effect

As we have argued in the introduction to the paper, the infinite horizon is key for our results. The goal of this subsection is to show that managers with finite horizons will opt

[12] Our portfolio is similar to Browne (1997). Browne (1997) considers the optimization problem of maximizing expected discounted gain from attaining a target level of wealth and finds the same portfolio as in (17). Interestingly, since the value function takes (endogenously) the form of a power function in fund wealth, the portfolio of the manager is identical to the portfolio that would be chosen by a Merton investor with risk aversion $1-\eta$. Our similarity to Browne (1997) is no coincidence: Our hedge fund manager understands that her compensation increases whenever she attains the previously recorded high-water mark and tries to maximize the expected discounted gain from attaining that threshold. As a result, the two problems coincide whenever $W_t < H_t$. This correspondence explains the somewhat surprising fact that the optimal portfolio does not depend on k. In contrast to Browne (1997), however, the reward from attaining the target in our problem is endogenous because it depends on the last recorded high-water mark. Moreover, it is reset each time the watermark is reached. Browne (1997) by contrast assumes an exogenous reward.

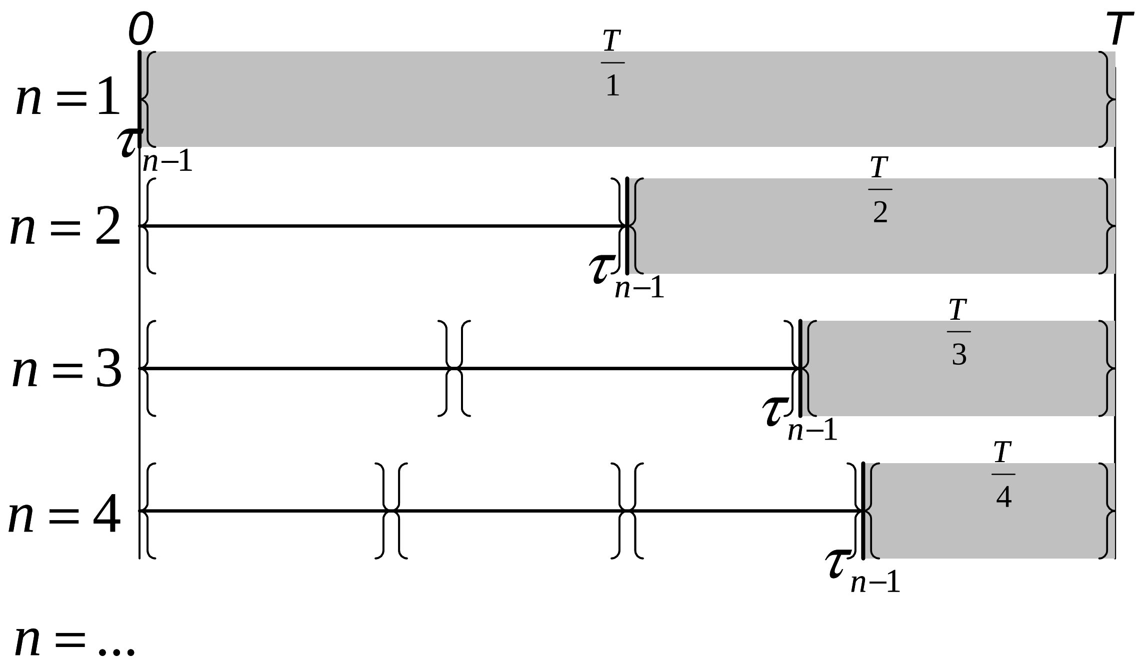

Figure 1: This figure illustrates the construction of the limit in section 3.2. The curly brackets indicate the expected arrival time of Poisson increments, given by $\frac{T}{n} = \frac{1}{\lambda_{(n)}}$, for different values of n. By construction the expected time of termination $E(\tau_n)$ is equal to T for all $n = 1, 2...\infty$. The shaded areas indicate the last "period" before termination. In the paper we study the portfolio choice of the manager in this last period.

for unbounded volatility as they approach the termination time T.

To preserve tractability, we examine a sequence of stochastic termination times. We choose this sequence so that the stochastic termination time (τ_n) will be equal to T with probability approaching 1 as $n \to \infty$. The method that we employ has been used very effectively in the pricing of American put options on finite horizons by Carr (1998) and we refer the reader to that paper for details.

To start, let N_t be a sequence of standard Poisson processes with interarrival intensity $\lambda_n = \frac{n}{T}$. Now define the termination time τ_n as

$$\tau_n = \inf\{t : N_t = n\}$$

This means that N_t measures the number of times that a Poisson "counter" with arrival intensity λ_n has increased, and the fund is terminated when $N_t = n$. Note that as n increases both the interrarival intensity λ_n and the number of Poisson jumps needed to trigger termination increase proportional to n. Hence the expected termination time is equal to T for all n. However, as n increases the variance of the termination time converges to 0. As Carr (1998) shows, the termination time τ_n has an Erlang distribution

$$\Pr(\tau_n \in dt) = \frac{(\lambda)^n}{(n-1)!} t^{n-1} e^{-\lambda t} dt$$

where $\lambda = \lambda_n = \frac{n}{T}$.Moreover, $\Pr(\tau_n \in dt)$ converges to a Dirac delta function centered at T.[13] Let us now fix a number n and consider the portfolio that the manager chooses when $N_t = n-1$, i.e. at the next to last "period", τ_{n-1}. Figure 1 provides a graphical illustration of this construction for different values of n. At that time, the arrival of termination is exponential with arrival intensity $\lambda_n = \frac{n}{T}$. Because this is the same setup we used in the previous section, we can re-use the formulas of that section to show that the portfolio of the manager will be given by (17) with η determined by (9)

$$\eta = \frac{r + \beta + \lambda_n + \omega - \sqrt{(r + \beta + \lambda_n + \omega)^2 - 4r(\beta + \lambda_n)}}{2r}$$

As n approaches ∞, both τ_n and τ_{n-1} will be arbitrarily close to T. So, if we want to study portfolio choice as the manager approaches the termination time T, it suffices to examine the portfolio that she will choose at the (random) time τ_{n-1}:

Lemma 1 *Let $\pi_{(n-1)}$ be the portfolio that a manager chooses at time τ_{n-1}. Then*

$$\lim_{n\to\infty} \pi_{(n-1)} = \infty$$

The intuition for this result comes from the fact that as the termination time approaches, the manager becomes effectively more impatient. In the infinite horizon setting, the manager

[13] Intuitively, by splitting the random termination into n i.i.d. subperiods, we can control the randomness associated with the time of arrival of each subperiod, while keeping the mean arrival rate fixed. With a simple exponential distribution this is impossible because the mean and the variance are controlled by the same parameter. However, an Erlang distribution allows one to "control" its mean and its variance separately.

behaves as if she were "effectively" risk averse, since her future prospects for fees are hurt whenever the level of wealth under management declines. By contrast, when the horizon is finite and near, there is no difference to the manager if she ends up moderately or extremely below the high-water mark.

3.3 An Alternative Derivation

In this section, we provide an alternative derivation of the results of section 3.1. This alternative derivation shows our connection to the option valuation techniques in Goetzmann, Ingersoll and Ross (2003) (GIR) and allows us to explain the significance of our assumption that $\mu > r$.

To begin, we concentrate on policies that invest an exogenous constant fraction of wealth in the risky asset: $\pi_t = \pi$ for all t. Since the optimal policy belongs to this class, we will be able to recover our earlier results.

As in section 3.1, in the region $\{W_t < H_t\}$ the value function for any fixed portfolio π satisfies

$$0 = -(\beta + \lambda)V + V_W W_t \left[r + \pi(\mu - r)\right] + \frac{1}{2} V_{WW} \sigma^2 \pi^2 W_t^2$$

This ODE has a general solution of the form

$$V(W_t, H_t) = K_1(H_t) W_t^{\alpha^+} + K_2(H_t) W_t^{\alpha^-}$$

where $K_1(H_t)$ and $K_2(H_t)$ represent arbitrary functions that can depend at most on H_t. The constants α^+ and α^- solve the quadratic equation

$$-(\beta + \lambda) + \alpha \left(r + \pi(\mu - r) - \frac{1}{2}\sigma^2\pi^2\right) + \frac{1}{2}\sigma^2\pi^2\alpha^2 = 0 \tag{18}$$

Define the mean growth rate of log-wealth as

$$l(\pi) = \left(r + \pi(\mu - r) - \frac{1}{2}\sigma^2\pi^2\right) \tag{19}$$

so that the solution to (18) can be written as

$$\alpha^{\pm}(\pi) = \frac{-l(\pi) \pm \sqrt{l(\pi)^2 + 2(\beta + \lambda)\sigma^2\pi^2}}{\sigma^2\pi^2}$$

We will study the roots $\alpha^{\pm}$ as functions of π. Because α^{-} will be negative and we wish to satisfy the boundary condition

$$\lim_{W_t \to 0} V(W_t, H_t) = 0$$

we will drop the negative root by setting $K_2(H_t) = 0$. Since we are focusing on the positive root α^{+}, we shall use the letter α instead of α^{+}. From this point on, the value function can be obtained by repeating the same sequence of steps as in equations (11)-(16) in section 3.1. We arrive at an expression for V that takes the portfolio policy π as given:

$$V(W_t, H_t; \pi) = \frac{kH_t}{a(\pi)(1+k) - 1} \left(\frac{W_t}{H_t} \right)^{\alpha(\pi)} \tag{20}$$

This replicates the setup and the results in GIR with the two important exceptions: μ can be different than r, and the log growth rate of the fund and its volatility depends on the choice of π. To continue, we can maximize the value function (20) over π:

Proposition 2 *Assume that Condition (15) holds and that $\mu > r$. Then $V(W_t, H_t; \pi)$ is maximized over π independently of (W_t, H_t) when*

$$\pi = \pi^* = \frac{1}{1-\eta} \frac{\mu - r}{\sigma^2} \tag{21}$$

A key assumption of the above proposition is that $\mu > r$. Clearly, this is identical to the assumption of a non-zero Sharpe ratio ($\omega > 0$) that we made in the previous section. This assumption is substantive. If we let $\mu \to r$, then the portfolio of the manager becomes unbounded:

Lemma 2 *Assume that $\beta + \lambda > r$ and $\mu > r$. Then*

$$\lim_{\mu \to r} \eta = 1$$

$$\lim_{\mu \to r} \pi^* = \infty$$

This is similar to the high-water mark valuation result shown in GIR. They value the managers fees under the risk neutral probability measure where, by construction, $\beta = r = \mu$ and hence $\beta + \lambda > r$. Therefore, the desired portfolio has infinite volatility. (Strictly speaking, the maximization problem is not well defined in this case).

In the context of portfolio choice, however, these results may seem very surprising. How is it that moving the Sharpe ratio towards zero leads to extremely bold portfolios? The resolution of the puzzle can be obtained by remembering that the manager's investment opportunities affect her continuation value and thus they affect the relative importance of short and long run objectives.

To make this basic intuition more explicit, we start by considering expression (20). Note that π affects this expression only through $\alpha(\pi)$. One can show that V is decreasing[14] in α, so the manager will choose π so as to *minimize* $\alpha(\pi)$.

If $\mu = r$, then V represents the net present value of the manager's fees. This implies that there is an upper bound to $V(W_t, H_t; \pi)$, namely W_t. Indeed, the whole analysis in GIR revolves around the share of fund wealth that the manager appropriates due to the high water mark contract. Clearly, this share cannot be above 1.

Next, observe that the bound $V(W_t, H_t; \pi) \leq W_t$ implies that $\alpha(\pi) \geq 1$. To see this, note that $\alpha(\pi) = 1$ implies $V(W_t, H_t; \pi) = W_t$ and remember that V is *decreasing* in $\alpha(\pi)$. As a result, it cannot be that there exists a π for which $\alpha(\pi)$ is below 1 because if there were, we would have $V(W_t, H_t; \pi) > W_t$. As a result, only values of $\alpha(\pi) \geq 1$ make sense when $\mu = r$ and the manager's value function will be (weakly) convex. Hence, the manager will want to take unbounded portfolio positions.

The key difference between the $\mu = r$ and the $\mu > r$ case is that when $\mu > r$, the bound $V \leq W_t$ is no longer valid. Intuitively, this results from the fact that V is no longer the "fair value" of a payoff for which the value cannot exceed W. Instead V is the expectation of the discounted fees *under the natural measure*. Under the natural measure, there is no reason V

[14] To see this differentiate $\frac{kH_t}{\alpha(1+k)-1}$ with respect to α which gives $-\frac{k(1+k)H_t}{(\alpha(1+k)-1)^2} < 0$ and also differentiate $\left(\frac{W_t}{H_t}\right)^{\alpha}$ with respect to α to obtain: $\left(\frac{W_t}{H_t}\right)^{\alpha}(\log(W) - \log(H)) \leq 0$.

must be less than W_t, and, in fact, (20) shows that V can exceed W_t. Thus, $\alpha(\pi)$ can attain values lower than 1, and this has the effect of introducing concavity into the problem and making the optimal portfolio finite.

This reasoning explains what is so special about the $\mu = r$ case: When $\mu = r$, the continuation value function of the problem isn't sufficient to make the manager fear the long run costs associated with increasing volatility in the short run. To see this, examine (20) when $W_t = H_t$:

$$V(H_t, H_t; \pi) = \frac{kH_t}{\alpha(\pi)(1+k) - 1}$$

The reciprocal of $\alpha(\pi)(1+k) - 1$ expresses the value function in multiples of the current payout kH_t, much like the P/E ratio of a stock expresses the price of the stock in multiples of current earnings. Hence higher values of V are associated with lower values of $\alpha(\pi)$. This is equivalent to the statement that the P/E ratio of a stock will increase (ceteris paribus) when either the discount rate increases or the growth of future payoffs increases. In both of these cases the future looks more important than the present and hence the ratio of price to current earnings increases.

A similar effect is at operation here. A higher Sharpe ratio makes the future look more important than the present. We shall refer to this effect schematically as "patience" or "long-termism". "Patience" is reflected in lower values of $\alpha(\pi^*)$, which is identical to η in light of Proposition 2. However as we have argued before, lower values of η can be equivalently thought of as increases in the "effective" risk aversion of the manager. The conclusion is that lower values of η can be interpreted in a *dual* way: a) as making the long run payoffs look more important than the short run or b) as decreases in the "effective" risk aversion of the manager.

We develop this basic intuition further in the next section where we perform comparative statics.

3.4 Comparative Statics

Given the very simple form of the optimal portfolio and the value function, one can obtain a number of comparative statics results. We isolate a few that seem to be central in understanding both the properties of the solution and the main intuition.

Lemma 3 *If (15) holds, $\omega > 0$, and $\beta + \lambda > r$, then*

$$\frac{\partial \pi_t}{\partial \beta} = \frac{\partial \pi_t}{\partial \lambda} > 0$$

If, in addition, $\beta + \lambda > r + \omega$, then

$$\frac{\partial \pi_t}{\partial \mu} < 0$$

These comparative statics are an extension of lemmas 1 and 2 and they reinforce our findings in the previous two sections. The parameters β and λ control how short-termist a manager is: The more a manager discounts the future, the bolder is her optimal portfolio. The reasoning in lemma 1 was that there was no continuation value to the problem as the exogenous termination time became near. Here we show, more generally, that managers who discount the future at higher rates will exhibit stronger risk-shifting behavior.

The result $\frac{\partial \pi_t}{\partial \mu} < 0$ is an extension of lemma 2, and it also seems surprising at first sight. In most Merton-type setups one would expect the opposite, namely that a higher μ will lead to bolder portfolios. The resolution of the puzzle is that in our setup the "effective" risk aversion of the manager is not exogenous, but instead is endogenous and depends on the relative importance of the present payoffs compared to future payoffs.

Recall that the optimal portfolio is given by

$$\pi = \frac{1}{1-\eta} \frac{\mu - r}{\sigma^2}$$

When $\mu - r$ increases, the second fraction clearly increases. Hence, if η were to remain constant, the portfolio would increase, as in any standard Merton setup. We shall refer to this as the "direct" effect of an increase in μ. However, one also expects that when μ increases, the value function will also increase; in our model, μ enters the value function through the parameter η, which is equal to one minus the coefficient of "effective" risk aversion. So, an increase in the value of μ has the dual effect of increasing both the value function through a decrease in η, and, accordingly increasing the coefficient of "effective" risk aversion $1-\eta$. Our assumption $\beta+\lambda>r+\omega$ guarantees that the indirect effect of increased effective risk aversion is stronger than the direct effect of μ on portfolio choice[15].

We conclude this subsection with two features of the value function that appears odd at first: $V_H>0$ and $V_k<0$. We start with a discussion of the first result: One would expect the opposite, namely that increases in the water mark would reduce the value function of the manager. It turns out that this unusual property is the result of 1) the risk-neutrality assumption and 2) the reduction in the fund growth rate that results from payments to the manager.

To understand why an increase in the high-water mark actually ends up increasing the expected discounted fees, evaluate the value function (16) at $W_t=H_t$. One obtains

$$V(H_t,H_t)=\frac{kH_t}{\eta(1+k)-1}$$

which is linear in H_t. This equation means that the expected gain to a manager that has just reached her high-water mark is proportional to the value of funds under management. We call this the "scale effect": at the point at which the manager begins to collect her fees, she would prefer the fund to be as large as possible. However, the scale effect needs to be balanced against the "waiting effect" that is captured by the term $\left(\frac{W_t}{H_t}\right)^{\eta}$. An increase in H_t for a fixed W_t means that it will take more time to hit the high-water mark. The fact that $\eta<1$ implies that the scale effect will dominate the waiting effect – the manager is "patient".

Figure 1 depicts an alternative interpretation of the $V_H>0$ result. We fix a wealth of W_0 and examine how fast fund wealth W_t can reach H_2. We compare two cases: In the first,

[15] As a matter of fact this result is to be anticipated in light of Lemma 2. There we showed that the optimal portfolio will diverge to infinity when ω is close to 0. Hence we should expect the portfolio to actually decline (for ω close to 0) as μ (and hence ω) increases.

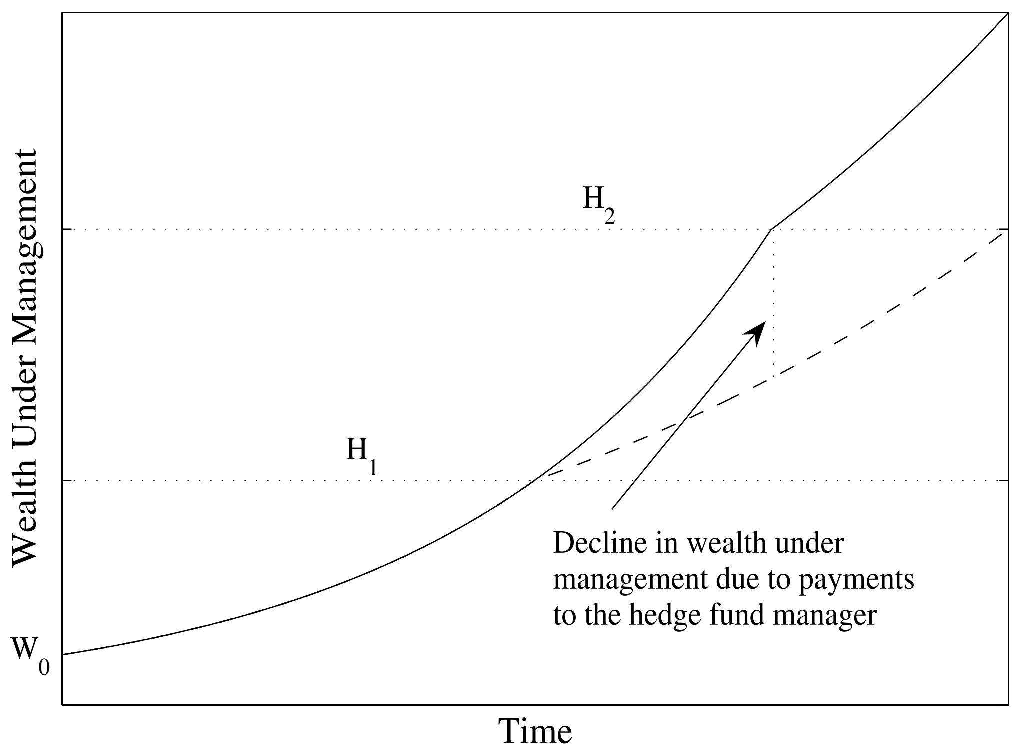

Figure 2: Reduction in the growth of fund wealth W_t due to payments to the hedge fund manager.

the current high-water mark is H_1, and in the second, it is at H_2. For any given portfolio policy, the growth of the fund value between W_0 and H_1 will be unaffected by whether the water mark is at H_1 or H_2. However, once H_1 is reached, there is an important difference due to fund wealth being partially diverted to management fees. If the high-water mark is at H_2, then the elapsed time between H_1 and H_2 will be less (solid line) than in the case when the high-water mark is H_1 (dashed line). This is because if the high-water mark is H_1, then fund growth will be lower between H_1 and H_2 by the amount of the payments provided to the manager. As a result, a manager can attain a higher fund value in shorter time when she faces an increased high-water mark. The trade-off for the manager is between payments now and increased wealth under management and correspondingly higher payments later.

The fact that $V_k < 0$ is due to a similar reasoning. Assuming that k is large enough to satisfy condition (15), then a direct differentiation yields

$$V_k = \frac{(\eta - 1)H_t}{(\eta(1+k) - 1)^2} \left(\frac{W_t}{H_t}\right)^{\eta} < 0$$

A higher k will increase the payouts to the manager, but it will also reduce the growth rate of W_t, as money gets diverted to the manager. If a manager puts heavy emphasis on the future, then the latter effect will dominate.

The purpose of the next section is to demonstrate how both of these counterintuitive results are linked with the extreme assumption of risk neutrality (which keeps the marginal utility of a dollar constant). With the more empirically relevant assumption of risk aversion neither of the two results will hold.

4 Risk Aversion

In this section we extend the model to allow for risk aversion. To motivate our choice of objective function, let us begin with the risk-neutral objective that we have been analyzing throughout. Integrating it by parts, we obtain

$$\int_0^\infty e^{-\beta s} k dH_s = \int_0^\infty e^{-\beta s} \beta k \left(H_s - H_0\right) ds \tag{22}$$

This follows because the manager is indifferent to obtaining a lump sum kdH_s now versus obtaining a payment in perpetuity of βkdH_s. Adding up those perpetuities yields $\beta k(H_s - H_0)$. It is also the case that if $\beta = r$, then the manager could literally exchange the two consumption paths by buying such a perpetuity.

Motivated by this rewriting of the objective of the manager in the risk neutral case, we can now postulate the objective of the manager in the risk averse case to be

$$V = \max_{\pi_s} E\left[\int_0^{\infty} e^{-\beta s} U(rk\,(H_s - H_0))ds\right] \tag{23}$$

for a continuously differentiable and concave function U. For technical convenience we shall also assume that[16] $U(0) = 0$.

The economic intuition behind (23) is straightforward. Suppose that the manager receives her compensation in the form of a perpetuity: if the watermark increases by kdH_t, the manager buys a perpetuity of the same value which from that point on delivers a consumption flow $rkdH_t$ forever. In line with the "market exclusion" assumption we have made so far, the manager has no other means of accessing markets beyond the purchase of these perpetuities, so that her consumption at time t is the sum of the flows paid out by the perpetuities that she has purchased by t, namely

$$rk\int_0^t dH_s = rk(H_t - H_0)$$

The reason why we have to go one step beyond the complete market exclusion assumption in the risk averse case is that $U()$ in equation (23) is not linear. The manager would not be able to benefit from a "lumpy" consumption stream – like the one implied by a high-water mark compensation arrangement – unless she was able to spread the profits into the future. This is precisely what we allow for with the purchase of a perpetuity. However, in order to keep the model simple, we shall not allow borrowing, trading, or selling of perpetuities or any other securities on the manager's personal account.

[16] This assumption is purely for convenience and can easily be relaxed, by introducing an exogenous flow of consumption in addition to the management fees.

To solve the problem in (23), we will again apply integration by parts and obtain

$$V = \max_{\pi} E\left[\int_0^{\infty} e^{-\beta s} U'(rk\,(H_s - H_0))\frac{r}{\beta} k dH_s\right]$$

The objective has now the same familiar form as in section 3.1. Accordingly, we can follow the same steps and obtain the HJB equation

$$0 = -\beta V + \max_{\pi_t}\{V_W W_t(r + \pi_t(\mu - r)) + \frac{1}{2} V_{WW} W_t^2 \sigma^2 \pi_t^2\} \tag{24}$$

which holds whenever $W_t < H_t$. We shall conjecture a solution of the form

$$V(W_t, H_t) = F(H_t) W_t^{\eta} \tag{25}$$

for some appropriate function F. Plugging into (24), we obtain

$$\pi_t^* = \frac{1}{1-\eta}\frac{\mu - r}{\sigma^2}$$

Notice that the portfolio allocation has not changed from our original model. The reason is that the hedge fund manager experiences increases in her utility only if $W_t = H_t$. Hence, her goal is to maximize the expected discounted reward from the next time that the fund value exceeds the current high-water mark. Her objective when $W_t < H_t$ remains the same as for the risk-neutral case, and it is the $\{W_t < H_t\}$ region that drives portfolio policy.

What does change, however, is the behavior of the value function at the boundary $\{W_t = H_t\}$. To study the value function at that boundary, we can apply the same logic as in section 3.1 to arrive at

$$V_W k = U'(rk\,(H_t - H_0))\frac{r}{\beta} k + V_H$$

Using our conjecture (25), we obtain

$$\eta k F(H_t) H_t^{\eta - 1} = F'(H_t) H_t^{\eta} + U'(rk\,(H_t - H_0))\frac{rk}{\beta} \tag{26}$$

This ODE can be solved numerically for any U, but to keep the analysis simple we will assume that

$$U(x) = \frac{x^{1-\gamma}}{1-\gamma}$$

We will also assume that $0 < \gamma < 1$ in order to maintain the property that $U(0) = 0$. Then, the solution to (26) is given by

$$F(H_t) = \frac{(rk)^{1-\gamma}}{\beta} \left(\int_{H_t}^{\infty} x^{-\eta(1+k)} (x - H_0)^{-\gamma} dx \right) H_t^{\eta k}$$

With the solution for $F(H_t)$, we can verify that $V = F(H_t) W_t^{\eta}$ is indeed the value function under the additional assumption that

$$\gamma + \eta(1 + k) - 1 > 0 \tag{27}$$

by following the exact same steps as in section 3.1. This last condition plays the same role as assumption (15) in our risk neutral derivation: it is a technical assumption that keeps the value function well-defined and finite.

One new property of the value function is that if γ is large enough, then $V_H > 0$, in line with common intuition. We attributed the earlier $V_H < 0$ result (in the risk neutral case) to a "scaling" effect that dominates the "waiting" effect. With risk aversion, however, the marginal value to additional consumption declines asymptotically, and so we would expect the "scaling" effect to be less important. Indeed,

Lemma 4 *If $\gamma > 1 - \eta$, then $V_H < 0$.*

Another counterintuitive property of the value function under risk neutrality is that $V_k < 0$. With risk aversion, this property will no longer be necessarily true. In particular we have the following result:

Lemma 5 *Assume the technical condition*[17]

$$\sqrt{\eta k} < \frac{\eta}{2}(1 + k) < 1 \tag{28}$$

Then, there exists an open set of γ in $(0, 1)$ such that

$$F_k > 0$$

and hence $V_k > 0$.

[17] This condition is used in order to simplify the proofs.

5 Extensions

The extension of the model to the multiple asset case is straightforward. Assume that there are n risky assets all of which follow a log-normal process

$$\frac{dP_i}{P_i} = \mu_i dt + \sum_{j=1}^{n} \sigma_{ij} dB_t^{(j)}, i = 1...n$$

where $dB_t^{(j)}$ denotes an n-dimensional Brownian Motion. We continue to assume that the riskless asset evolves as

$$\frac{dP_0}{P_0} = rdt$$

The wealth evolution equation (for a given vector π) is given as

$$dW_t^{\pi} = W_t^{\pi} \left(\left[r + \sum_{i=1}^{n} \pi_i(\mu_i - r) \right] dt + \sum_{i=1}^{n} \sum_{j=1}^{n} \pi_i \sigma_{ij} dB_t^{(j)} \right) - kdH_t$$

Denoting as σ the matrix σ_{ij}[18] and as μ, π the respective vectors of means and portfolios, one can define the infinitesimal generator[19] of any function $f(W, H)$ as:

$$\mathcal{A}f(W, H) = (\pi'(\mu - r1_n) + r)W f_W + \frac{1}{2}\pi'\sigma\sigma'\pi f_{WW} W^2$$

Now it is straightforward to verify that all of the results of section 3 hold with the constant ω redefined to be

$$\omega = \frac{1}{2} (\mu - r1_n)' (\sigma\sigma')^{-1} (\mu - r1_n)$$

The optimal portfolio is still the mean-variance efficient portfolio

$$\pi^* = \frac{1}{1 - \eta} (\sigma\sigma')^{-1} (\mu - r1_n)$$

This confirms our previous conclusion that the portfolio manager will choose a (potentially leveraged) but efficient mean-variance portfolio that keeps a constant exposure to the risky assets.

[18] We assume througout the invertibility of $\sigma\sigma'$.

[19] See Oksendal (1998)

It is equally straightforward to extend the model to allow for a more general adjustment to the high water mark. In particular, a common stipulation is to allow the high water mark to increase at the rate g. It turns out that we can allow for this extension within our framework by appropriately adjusting both the interest rate and the discount rate of the manager.

To see this result, notice that allowing the water-mark to increase at the rate g will change the Bellman equation (5) to

$$0 = V_H H g - (\beta + \lambda) V + \max_{\pi_t} \{ V_W W_t (r + \pi_t (\mu - r)) + \frac{1}{2} V_{WW} W_t^2 \sigma^2 \pi_t^2 \} \tag{29}$$

The boundary condition (4) remains the same. We next conjecture a value function of the form

$$V(H_t, W_t) = C H_t^{1-\eta} W_t^{\eta} \tag{30}$$

for appropriate constants C and η. Plugging (30) into (29) leads to

$$\eta^2 (r - g) - (r + \beta + \lambda - 2g + \omega)\eta + (\beta + \lambda - g) = 0$$

Notice that this equation is identical to (7) with the exception that $\beta + \lambda - g$ replaces $\beta + \lambda$ and $r - g$ replaces r. Accordingly, if the high water mark increases at the rate g, this is observationally equivalent to reducing both the discount of the manager and the interest rate by g and then repeating the analysis in section 3.1.

6 Conclusion

The purpose of this paper is to study the portfolio choice incentives that are implied by high-water mark fees for hedge fund managers. We showed that under certain regularity conditions even a risk-neutral hedge fund manager has an incentive to choose an interior mean-variance efficient portfolio. The surprising result is that the optimization problem of the hedge fund manager allows an interior solution despite risk-neutrality and the fact that termination is enforced either when wealth reaches 0 or at some stochastic time that is independent of asset market events.

Our analysis yields two conclusions:

First, this paper draws a distinction between the effects of *option-type compensation* and those of *finite horizons*. In our model, compensation has option-type features, yet the portfolio is constant and bounded. By contrast, were the hedge fund to terminate at some pre-determined time, the manager's preferred portfolio would become unbounded as she approached termination. We believe that this intuition extends beyond the specifics of the option-type compensation contract considered here. It appears that one can mitigate risk-shifting by making sure that there is a trade-off between increasing variance in the "current" period and some sort of "punishment" in terms of the options that will follow after the current period.

Second, high-water mark contracts seem to provide the right incentives for portfolio choice, in the sense that the portfolio chosen exhibits two-fund separation: A constant fraction is invested in the riskless asset and the rest in a mean-variance efficient portfolio (given the manager's information set). By contrast, most of the literature on portfolio choice in the presence of benchmarking demonstrates the presence of biases in the portfolio towards a particular index.

Our conclusions are robust across several extensions of the basic framework to multiple assets, risk averse preferences, and changes in the way the high-water mark is calculated.

We would like to conclude with a caveat. The purpose of this paper was not to provide an exhaustive analysis of the incentive effects of high-water mark contracts. We made some extreme assumptions in order to demonstrate most clearly the disciplining effect of a continuation value on an in(de)finite horizon. In terms of future research, we believe that there are at least two interesting directions: First to relax the exclusion of private accounts, but instead impose the constraint that the manager cannot borrow against the future net present value of her income. And second, to allow for optimal early termination of the manager by the investor.

A Proof of Proposition 1

We present a verification theorem that can be invoked to verify that the conjecture that we made about the value function in (16) is actually correct. In doing this, we focus on the setup of section 3. Extending the proof to the setup of sections 4 and 5 is straightforward. The proof proceeds in two steps. First, we prove the verification theorem assuming that the fund is terminated when the high-water mark reaches a level $\overline{H}$. This is done in proposition 3. Then we let $\overline{H} \to \infty$ and use the monotone convergence theorem.

It will save some notation to give the proof in terms of $\theta = \pi W$, the total holdings of stock, instead of π.

Proposition 3 *Let* $\tau^0 = \inf\{t : W_t = 0\}$ *and* $\tau^H = \inf\{t : H_t = \overline{H}\}$. *Suppose that there exists a function that is twice continuously differentiable in* W_t *and once continuously differentiable in* H_t*, such that*

$$0 = -(\beta + \lambda)V + \max_{\theta}\{V_W(\theta\mu + (W_t - \theta)r) + \frac{1}{2}V_{WW}\sigma^2\theta^2\} \tag{31}$$

and that

$$kV_W = k + V_H \tag{32}$$

whenever $W_t = H_t$. *In addition, assume that the contract is terminated when* $H_t = \overline{H}$*, so that* $V(W_t, \overline{H}) = 0$ *for all* $W_t \geq 0$*,* $V(0, H_t) = 0$ *for all* H_t*, and that* $V(W_t, H_t) \geq 0$. *Suppose also that* $V_W > 0$ *and* $V_{WW} < 0$ *and let*

$$\theta^* = -\frac{V_W}{V_{WW}}\frac{(\mu - r)}{\sigma^2} \tag{33}$$

Assume finally that θ^* *and* $\theta^* V_W$ *are bounded in* $[0, \overline{H}]$. *For any admissible policy* θ_t*, denote by* H_t^θ *and* W_t^θ *the resulting processes for* W_t *and* H_t.

Then

$$V(W_0, H_0) \geq E\left[\int_0^{\tau^0 \wedge \tau^{\overline{H}}} e^{-(\beta+\lambda)s} k dH_s^\theta\right] \tag{34}$$

for all admissible controls θ*, and equality holds for the optimal control* (33).

Remark 1 *It is straightforward to verify that the function*

$$V(W_t, H_t) = W_t^{\eta} H_t^{k\eta} \left(\frac{kH^{1-(1+k)\eta}}{\eta(1+k)-1} - \frac{k\overline{H}^{1-(1+k)\eta}}{\eta(1+k)-1} \right) \tag{35}$$

with η given in (9) satisfies all the conditions of the above proposition.

Proof. The proof proceeds in essentially the same way as every Verification Theorem (see Fleming and Soner [1993] Chapter III, Oksendal [1998], Chapter 10 and 11, Browne [1997]). Applying Ito's Lemma to

$$M(t, W_t, H_t) = e^{-(\beta+\lambda)t} V(W_t, H_t) + \int_0^t e^{-(\beta+\lambda)s} k dH_s \tag{36}$$

for a time $t < \tau^0 \wedge T \wedge \tau^{\overline{H}}$, one obtains

$$\begin{aligned} M(t, W_t, H_t) &= M(0, W_0, H_0) + \int_0^t e^{-(\beta+\lambda)s} \left[\mathcal{A}V(W_s, H_s; \theta_s)\right] ds + \int_0^t e^{-(\beta+\lambda)s} \sigma\theta_s V_W dB_s \\ &\quad + \int_0^t e^{-(\beta+\lambda)s} \left(k - V_W k + V_H\right) dH_s \end{aligned}$$

where

$$\mathcal{A}V(W_s, H_s; \theta_s) = \frac{1}{2}\sigma^2\theta_s^2 V_{WW} + \theta_s \left[(\mu - r) V_W\right] + rWV_W - (\beta+\lambda)V$$

By condition (32), one can rewrite this as

$$M(t, W_t, H_t) = M(0, W_0, H_0) + \int_0^t e^{-(\beta+\lambda)s} \left[\mathcal{A}V(W_s, H_s; \theta_s)\right] ds + \int_0^t e^{-(\beta+\lambda)s} \sigma\theta_s V_W dB_s \tag{37}$$

By assumption $V_{WW} < 0$, and so $\mathcal{A}V(W_s, H_s; \theta_s)$ is a concave quadratic expression in θ_s which attains its maximum at

$$\theta^* = -\frac{V_W}{V_{WW}} \frac{(\mu - r)}{\sigma^2}$$

with corresponding maximal value

$$\mathcal{A}V(W_s, H_s; \theta_s^*) = rWV_W - \omega\frac{V_W^2}{V_{WW}} - (\beta+\lambda)V = 0$$

where the second equality follows from (31). This observation implies that the second term on the right-hand side of (37) is less than or equal to 0 for any portfolio policy. Accordingly

$$\begin{aligned}\int_0^{\tau^0\wedge T\wedge\tau^{\overline{H}}} e^{-(\beta+\lambda)s}\sigma\theta_s V_W dB_s &= M(\tau^0\wedge T\wedge\tau^{\overline{H}}, W_{\tau^0\wedge T\wedge\tau^{\overline{H}}}, H_{\tau^0\wedge T\wedge\tau^{\overline{H}}}) \\ &\quad -M(0,W_0,H_0) - \int_0^{\tau^0\wedge T\wedge\tau^{\overline{H}}} e^{-(\beta+\lambda)s}\left[\mathcal{A}V(W_s,H_s;\theta_s)\right] ds \\ &\geq M(\tau^0\wedge T\wedge\tau^{\overline{H}}, W_{\tau^0\wedge T\wedge\tau^{\overline{H}}}, H_{\tau^0\wedge T\wedge\tau^{\overline{H}}}) - M(0,W_0,H_0)\end{aligned} \tag{38}$$

Now, by virtue of the assumption $V \geq 0$ for all H_t and W_t, it follows that the left-hand side is a local martingale that is bounded below by $(-M(0,W_0,H_0))$ and is thus a supermartingale. Because of this we can take expectations in (37) and claim that

$$\begin{aligned}E\left[M(\tau^0\wedge T\wedge\tau^{\overline{H}}, W_{\tau^0\wedge T\wedge\tau^{\overline{H}}}, H_{\tau^0\wedge T\wedge\tau^{\overline{H}}})\right] &\leq E\int_0^{\tau^0\wedge T\wedge\tau^{\overline{H}}} e^{-(\beta+\lambda)s}\left[\mathcal{A}V(W_s,H_s;\theta_s)\right] ds \\ &\quad +M(0,W_0,H_0) \\ &\leq E\int_0^{\tau^0\wedge T\wedge\tau^{\overline{H}}} e^{-(\beta+\lambda)s}\sup_{\theta_s}\left[\mathcal{A}V(W_s,H_s;\theta_s)\right] ds \\ &\quad +M(0,W_0,H_0) \\ &= M(0,W_0,H_0)\end{aligned}$$

Because θ^* and $\theta^* V_W$ are bounded in $[0,\overline{H}]$, we can replace inequalities with equalities for the optimal policy as the left-hand side in (38) is a martingale. Accordingly, for any admissible policy

$$\begin{aligned}V(W_0,H_0) &= M(0,W_0,H_0) \\ &\geq E\left[e^{-(\beta+\lambda)\left(\tau^0\wedge T\wedge\tau^{\overline{H}}\right)}V(W_{\tau^0\wedge T\wedge\tau^{\overline{H}}}, H_{\tau^0\wedge T\wedge\tau^{\overline{H}}}) + \int_0^{\tau^0\wedge T\wedge\tau^{\overline{H}}} e^{-(\beta+\lambda)s} k dH_s\right]\end{aligned}$$

Taking limits as $T\to\infty$, and using the fact that for any policy

$$\lim_{T\to\infty} E\left[e^{-(\beta+\lambda)\left(\tau^0\wedge T\wedge\tau^{\overline{H}}\right)}V(W_{\tau^0\wedge T\wedge\tau^{\overline{H}}}, H_{\tau^0\wedge T\wedge\tau^{\overline{H}}})\right] = 0$$

the monotone convergence theorem implies

$$\begin{aligned}
& \lim_{T\to\infty} E\left[e^{-(\beta+\lambda)\left(\tau^0\wedge T\wedge\tau^{\overline{H}}\right)}V(W^{\theta^*}_{\tau^0\wedge T\wedge\tau^{\overline{H}}}, H^{\theta^*}_{\tau^0\wedge T\wedge\tau^{\overline{H}}}) + \int_0^{\tau^0\wedge T\wedge\tau^{\overline{H}}} e^{-(\beta+\lambda)s} k dH^{\theta^*}_s\right] \\
= \; & E\left[\int_0^{\tau^0\wedge\tau^{\overline{H}}} e^{-(\beta+\lambda)s} k dH^{\theta^*}_s\right] \\
\geq \; & \lim_{T\to\infty} E\left[e^{-(\beta+\lambda)\left(\tau^0\wedge T\wedge\tau^{\overline{H}}\right)}V(W^{\theta}_{\tau^0\wedge T\wedge\tau^{\overline{H}}}, H^{\theta}_{\tau^0\wedge T\wedge\tau^{\overline{H}}}) + \int_0^{\tau^0\wedge T\wedge\tau^{\overline{H}}} e^{-(\beta+\lambda)s} k dH^{\theta}_s\right] \\
= \; & E\left[\int_0^{\tau^0\wedge T} e^{-(\beta+\lambda)s} k dH^{\theta}_s\right]
\end{aligned}$$

This concludes the proof. ■

To obtain a verification theorem for the infinite horizon problem one needs to observe that the expression in (35) approaches our conjecture for the value function as $\overline{H} \to \infty$. The result that the solution to the infinite horizon problem is given by (16) in section 3 follows by the monotone convergence theorem.

B Other Proofs

Proof of Lemma 1. In light of (17), it suffices to show that $n \to \infty$ implies $\eta \to 1$. Remember that $0 \leq \eta \leq 1$ for any n, as shown in the discussion following formula (10) in section 3.1. From equation (7) and $\lambda_n = \frac{n}{T}$, we have that η solves

$$\eta^2 r - (r + \beta + \frac{n}{T} + \omega)\eta + (\beta + \frac{n}{T}) = 0$$

Dividing by n and rearranging gives

$$\eta = \frac{\frac{1}{T} + \frac{\eta^2 r}{n} + \frac{\beta}{n}}{\frac{1}{T} + \frac{r+\beta+\omega}{n}}$$

Because η is bounded between 0 and 1, so is η^2. Since T is constant, we obtain the limit

$$\lim_{n\to\infty} \eta = 1$$

The fact that $\pi \to \infty$ is then a direct implication of (17). ■

Proof of Lemma 2. Remember that the optimal portfolio is given as

$$\pi^* = \frac{1}{1-\eta}\frac{\mu - r}{\sigma^2}$$

and that η is described by

$$\eta^2 r - (r + \beta + \lambda + \omega)\eta + (\beta + \lambda) = 0$$

Since $\mu \to r$ implies $\omega \to 0$, we have that $\eta \to 1$ by inspection. (Remember that the second quadratic root is disregarded).

Now, $\mu \to r$ implies that both the numerator and denominator of the optimal portfolio policy go to zero, so we will apply L'Hospital's rule. This gives

$$\lim_{\mu \to r} \pi^* = \frac{1}{\sigma^2} \lim_{\mu \to r} \frac{1}{-\eta_\omega \omega_\mu} \tag{39}$$

where the subscripts represent derivatives. We apply the implicit function theorem to the equation describing η and we find that

$$\frac{d\eta}{d\omega} = -\frac{-\eta}{2\eta r - (r + \beta + \lambda + \omega)}$$

When $\mu = r$, we have $\omega = 0$ and $\eta = 1$, so that

$$\lim_{\mu \to r} \frac{d\eta}{d\omega} = \frac{1}{r - (\beta + \lambda)}$$

By assumption, we have $\beta + \lambda > r$, and so equation (39) becomes

$$\lim_{\mu \to r} \pi^* = \frac{1}{\sigma^2} \lim_{\mu \to r} \frac{1}{-\eta_\omega \omega_\mu} = \frac{\beta + \lambda - r}{\sigma^2} \lim_{\mu \to r} \frac{1}{\frac{\mu - r}{\sigma^2}} = \infty$$

■

Proof of Proposition 2. The value function is

$$\frac{kH_t}{a(\pi)(1+k) - 1} \times \left(\frac{W_t}{H_t}\right)^{\alpha(\pi)}$$

Because of condition (15) and $\frac{W_t}{H_t} \leq 1$, both terms are separately maximized when $\alpha(\pi)$ is minimized. Hence, the value function will be maximized when $\alpha(\pi)$ is minimized. We will show that the minimum value of $\alpha(\pi)$ is achieved when equation (21) is met.

One can directly differentiate $\alpha(\pi)$ w.r.t. π, but this appears to be cumbersome. However, there is an easier way to proceed. First, we start with the observation that the π that minimizes $\alpha(\pi)$ must be positive as long as $\mu > r$. To see this, assume otherwise: Assume that the π that minimizes $\alpha(\pi)$ is negative, and consider $\alpha(-\pi)$ which is given by

$$\alpha(-\pi) = \frac{\sqrt{l(-\pi)^2 + 2(\beta+\lambda)\sigma^2\pi^2} - l(-\pi)}{\sigma^2\pi^2}$$

and also

$$l(-\pi) = \left(r + (-\pi)(\mu - r) - \frac{1}{2}\sigma^2\pi^2\right) > l(\pi)$$

since $\mu > r$. Note now that any expression of the form

$$\sqrt{l^2 + c} - l$$

is decreasing in l for any l and $c > 0$, which means that

$$\alpha(-\pi) < \alpha(\pi)$$

which is a contradiction. Hence, as long as $\mu > r$ we can constrain attention to the non-negative axis to determine the value of π that minimizes $\alpha(\pi)$. The next step is to use the implicit function theorem on equation (18) to obtain

$$\frac{d\alpha}{d\pi} = -\frac{\alpha\left[(\mu - r) - \sigma^2\pi\right] + \sigma^2\pi\alpha^2}{\left(r + \pi(\mu - r) - \frac{1}{2}\sigma^2\pi^2\right) + \sigma^2\pi^2\alpha} \tag{40}$$

Setting the numerator equal to 0 yields

$$\pi^* = \frac{1}{1 - \alpha(\pi^*)}\frac{(\mu - r)}{\sigma^2}$$

We now show that $\alpha(\pi^*) = \eta$. Replace π^*in equation (18) and use the definition of ω from (8) to obtain

$$-(\beta+\lambda) + \alpha\left(r + \frac{1}{1-\alpha}2\omega - \frac{\omega}{(1-\alpha)^2}\right) + \frac{\omega\alpha^2}{(1-\alpha)^2} = 0$$

Re-arranging yields

$$-\alpha^2 r + \alpha\left(r + \beta + \lambda + \omega\right) - (\beta + \lambda) = 0$$

which is exactly equation (7), which defines η^- and η^+. Since we want $\pi^* > 0$ we can restrict attention to $\eta^- = \eta < 1$.

To check the second order conditions, the uniqueness of the local optimum implies we need only verify

$$\alpha(\pi^*) \quad < \quad \lim_{\pi \to 0} \alpha(\pi) \tag{41}$$

$$\alpha(\pi^*) \quad < \quad \lim_{\pi \to \infty} \alpha(\pi) \tag{42}$$

Since $\lim_{\pi \to \infty} \alpha(\pi) = 1$, equation (42) holds because $\alpha(\pi^*) < 1$. Since $\lim_{\pi \to 0} \alpha(\pi) = \infty$, equation (41) holds. ■

Proof of Lemma 3. We prove the first result only for β. The result for λ follows similarly. Remember that $0 \leq \eta \leq 1$. Using the implicit function theorem on

$$\eta^2 r - (r + \beta + \lambda + \omega)\eta + (\beta + \lambda) = 0 \tag{43}$$

yields

$$\eta_\beta = \frac{\eta(1-\eta)}{(\beta + \lambda) - \eta^2 r} > 0$$

where the inequality follows from $\eta \in (0, 1)$ and the assumption $r < \beta + \lambda$. This means that

$$\frac{\partial \left(\frac{1}{1-\eta}\right)}{\partial \beta} > 0$$

and $\frac{\partial}{\partial \beta}\pi^* > 0$ follows from (17). Identical steps lead to $\frac{\partial}{\partial \lambda}\pi^* > 0$.

We now examine $\frac{\partial}{\partial \mu}\pi^* > 0$. Differentiating (17) yields

$$\frac{\partial \pi}{\partial \mu} = \frac{1}{\sigma^2}\left(\frac{1 - \eta + \eta_\omega \omega_\mu (\mu - r)}{(1-\eta)^2}\right) \tag{44}$$

In the proof of lemma 2, we obtain

$$\frac{d\eta}{d\omega} = \frac{\eta}{2\eta r - (r + \beta + \lambda + \omega)}$$

and from the definition of ω in (8), we obtain

$$\omega_\mu (\mu - r) = 2\omega$$

Using these in (44), we have

$$sgn(\frac{\partial\pi}{\partial\mu}) = sgn\left[1-\eta+\frac{2\eta\omega}{2\eta r-(r+\beta+\lambda+\omega)}\right]$$

Using (7) and rearranging gives

$$1-\eta+\frac{2\eta\omega}{2\eta r-(r+\beta+\lambda+\omega)} = \frac{(\beta+\lambda-r-\omega)(1-\eta)}{2\eta r-(r+\beta+\lambda+\omega)}$$

which is negative when $\beta+\lambda > r+\omega$. ■

Proof of Lemma 4. The derivative $F'(H_t)$ is given by

$$\begin{aligned} F'(H_t) &= \frac{(rk)^{1-\gamma}}{\beta}\left[-H_t^{-(\gamma+\eta)}\left(1-\frac{H_0}{H_t}\right)^{-\gamma}+\eta k H_t^{\eta k-1}\int_{H_t}^{\infty} x^{-\eta(1+k)-\gamma}\left(1-\frac{H_0}{x}\right)^{-\gamma}dx\right] = \\ &= \frac{(rk)^{1-\gamma}}{\beta}H_t^{-(\gamma+\eta)}\left(1-\frac{H_0}{H_t}\right)^{-\gamma}\left[\eta k H_t^{\eta(1+k)+\gamma-1}\int_{H_t}^{\infty} x^{-\eta(1+k)-\gamma}\left(\frac{1-\frac{H_0}{x}}{1-\frac{H_0}{H_t}}\right)^{-\gamma}dx-1\right] \end{aligned}$$

Since $H_0 \le H_t \le x$ it follows that

$$\frac{H_0}{x} \le \frac{H_0}{H_t}$$

so that

$$\left(\frac{1-\frac{H_0}{x}}{1-\frac{H_0}{H_t}}\right)^{-\gamma} \le 1$$

As a result,

$$\begin{aligned} F'(H_t) &\le \frac{(rk)^{1-\gamma}}{\beta}H_t^{-(\gamma+\eta)}\left(1-\frac{H_0}{H_t}\right)^{-\gamma}\left[\eta k H_t^{\eta(1+k)+\gamma-1}\int_{H_t}^{\infty} x^{-\eta(1+k)-\gamma}dx-1\right] = \\ &= \frac{(rk)^{1-\gamma}}{\beta}H_t^{-(\gamma+\eta)}\left(1-\frac{H_0}{H_t}\right)^{-\gamma}\left[\frac{\eta k}{\eta(1+k)+\gamma-1}-1\right] = \\ &= \frac{(rk)^{1-\gamma}}{\beta}H_t^{-(\gamma+\eta)}\left(1-\frac{H_0}{H_t}\right)^{-\gamma}\left[\frac{1-\gamma-\eta}{\eta(1+k)+\gamma-1}\right] \\ &< 0 \end{aligned}$$

and therefore $V_H < 0$ ■

Proof of Lemma 5. The derivative F_k is given by:

$$\begin{aligned} F_k &= \frac{(r)^{1-\gamma}}{\beta}(1-\gamma)k^{-\gamma}\left(\int_{H_t}^{\infty} x^{-\eta(1+k)}(x-H_0)^{-\gamma}dx\right)H_t^{\eta k} \\ &\quad -\eta\frac{(rk)^{1-\gamma}}{\beta}\left(\int_{H_t}^{\infty}\left[\log(x)-\log(H_t)\right]x^{-\eta(1+k)}(x-H_0)^{-\gamma}dx\right)H_t^{\eta k} \end{aligned}$$

which can be re-expressed as:

$$F_k = \frac{r^{1-\gamma} H_t^{\eta k}}{\beta k^{\gamma}} \left(\int_{H_t}^{\infty} \left[(1-\gamma) - \eta k (\log(x) - \log(H_t) \right] \left(1 - \frac{H_0}{x} \right)^{-\gamma} x^{-\eta(1+k)-\gamma} dx \right)$$

Examining the terms inside the integral we observe that $1-\gamma > 0$ and that $-\eta k(\log(x) - \log(H_t)) < 0$. Moreover it is clear that $1-\gamma-\eta k(\log(x) - \log(H_t)) > 0$ if $x = H_t$ and that $1-\gamma-\eta k(\log(x)-\log(H_t)) < 0$ as $x \to \infty$. Hence the term $(1-\gamma)-\eta k(\log(x)-\log(H_t)$ will be positive close to the lower integration limit and will become negative for large enough x. Let $\overline{x}$ denote the (unique) value of x where $(1-\gamma) - \eta k(\log(x) - \log(H_t) = 0$.

Note also that $\left(1 - \frac{H_0}{x}\right)^{-\gamma}$ is a declining function of x. This implies then

$$\begin{aligned} F_k &= \frac{r^{1-\gamma} H_t^{\eta k}}{\beta k^{\gamma}} \left(1 - \frac{H_0}{\overline{x}} \right)^{-\gamma} \left(\int_{H_t}^{\infty} \left[(1-\gamma) - \eta k (\log(x) - \log(H_t) \right] \frac{\left(1 - \frac{H_0}{x}\right)^{-\gamma}}{\left(1 - \frac{H_0}{\overline{x}}\right)^{-\gamma}} x^{-\eta(1+k)-\gamma} dx \right) \\ &> \frac{r^{1-\gamma} H_t^{\eta k}}{\beta k^{\gamma}} \left(1 - \frac{H_0}{\overline{x}} \right)^{-\gamma} \left(\int_{H_t}^{\infty} \left[(1-\gamma) - \eta k (\log(x) - \log(H_t)) \right] x^{-\eta(1+k)-\gamma} dx \right) \end{aligned}$$

since the declining function $\frac{\left(1-\frac{H_0}{x}\right)^{-\gamma}}{\left(1-\frac{H_0}{\overline{x}}\right)^{-\gamma}}$ will "weight" by a factor more than 1 the positive parts of

$$\left[(1-\gamma) - \eta k (\log(x) - \log(H_t) \right]$$

and by a factor less than 1 the negative parts of it. Hence, we are left with showing that

$$\int_{H_t}^{\infty} \left[(1-\gamma) - \eta k (\log(x) - \log(H_t)) \right] x^{-\eta(1+k)-\gamma} dx > 0$$

This integral can be rewritten as:

$$\left[(1-\gamma) + \eta k \log(H_t) \right] \int_{H_t}^{\infty} x^{-\eta(1+k)-\gamma} dx - \eta k \int_{H_t}^{\infty} \log(x) x^{-\eta(1+k)-\gamma} dx$$

which -after using integration by parts on the second term - is equal to:

$$\begin{aligned} &-\frac{(1-\gamma) + \eta k \log(H_t)}{1-\eta(1+k)-\gamma} H_t^{1-\eta(1+k)-\gamma} + \\ &+\eta k \left[\log(H_t) \frac{H_t^{1-\eta(1+k)-\gamma}}{1-\eta(1+k)-\gamma} \right] + \frac{\eta k}{1-\eta(1+k)-\gamma} \int_{H_t}^{\infty} x^{-\eta(1+k)-\gamma} dx \end{aligned}$$

Computing the integral associated with the third term and simplifying we can reduce this expression to

$$\left(-\frac{1-\gamma}{1-\eta(1+k)-\gamma}-\frac{\eta k}{(1-\eta(1+k)-\gamma)^2}\right)H_t^{1-\eta(1+k)-\gamma}$$

In order to establish the claim it suffices to show that:

$$-\frac{1-\gamma}{1-\eta(1+k)-\gamma}-\frac{\eta k}{(1-\eta(1+k)-\gamma)^2}>0$$

or:

$$\frac{(1-\gamma)(\gamma+\eta(1+k)-1)-\eta k}{(1-\eta(1+k)-\gamma)^2}>0$$

The denominator is clearly positive. The sign of the numerator

$$(1-\gamma)(\gamma+\eta(1+k)-1)-\eta k \tag{45}$$

will depend on the parameters of the problem. However, we can show that there will always exist values of γ that will make the numerator positive as long as condition (28) holds. To see this, maximize (45) over γ to obtain the optimal γ^*:

$$\gamma^*=1-\frac{\eta}{2}(1+k)$$

Given assumption (28) γ^* is between 0 and 1.Substituting γ^* into (45) gives:

$$\frac{\eta^2(1+k)^2}{4}-\eta k$$

which is larger than 0 by assumption (28). ■

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