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Demand-Based Option Pricing

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Abstract

We model the demand-pressure effect on prices when options cannot be perfectly hedged. The model shows that demand pressure in one option contract increases its price by an amount proportional to the variance of the unhedgeable part of the option. Similarly, the demand pressure increases the price of any other option by an amount proportional to the covariance of their unhedgeable parts. Empirically, we identify aggregate positions of dealers and end users using a unique dataset, and show that demand-pressure effects help explain well-known option-pricing puzzles. First, end users are net long index options, especially out-of-money puts, which helps explain their apparent expensiveness and the smirk. Second, demand patterns help explain the prices of single-stock options.

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1 Introduction

> "[Option implied volatility] skew is heavily influenced by supply and demand factors."
> — *Gray, director of global equity derivatives, Dresdner Kleinwort Benson*

> "The number of players in the skew market is limited. ... there's a huge imbalance between what clients want and what professionals can provide."
> — *Belhadj-Soulami, head of equity derivatives trading for Europe, Paribas*

> "To blithely attribute divergences between objective and risk-neutral probability measures to the free 'risk premium' parameters within an affine model is to abdicate one's responsibilities as a financial economist. ... a renewed focus on the explicit financial intermediation of the underlying risks by option market makers is needed."
> — *Bates (2003)*

We take on this challenge by providing a model of option intermediation, and by showing empirically using a unique dataset that demand pressure can help explain the main option-pricing puzzles.

The starting point of our analysis is that options are traded because they are useful and, therefore, options cannot be redundant for all investors (Hakansson (1979)). We denote the agents who have a fundamental need for option exposure as "end users."

Intermediaries such as market makers and proprietary traders provide liquidity to end users by taking the other side of the end-user net demand. If intermediaries can hedge perfectly — as in a Black and Scholes (1973) and Merton (1973) economy — then option prices are determined by no-arbitrage and demand pressure has no effect. In reality, however, even intermediaries cannot hedge options perfectly because of the impossibility of trading continuously, stochastic volatility, jumps in the underlying, and transaction costs (Figlewski (1989)).[1]

To capture this effect, we depart from the standard no-arbitrage literature that follows Black-Scholes-Merton by considering explicitly how options are priced by competitive risk-averse dealers who cannot hedge perfectly. In our model, dealers trade an arbitrary number of option contracts on the same underlying at discrete times. Since the dealers trade many option contracts, certain risks net out, while others do not. The dealers can hedge part of the remaining risk of their derivative positions by trading the underlying security and risk-free bonds. We consider a general class of distributions for the underlying, which can accommodate stochastic volatility and jumps. Dealers trade with end users. The model is agnostic about the end users' reasons for trade.

We compute equilibrium prices as functions of demand pressure, that is, the prices that make dealers optimally choose to supply the options that the end users demand.

[1] Also, traders have capital constraints as emphasized, e.g., by Shleifer and Vishny (1997).

We show explicitly how demand pressure enters into the pricing kernel. Intuitively, a positive demand pressure in an option increases the pricing kernel in the states of nature in which an optimally hedged position has a positive payoff. This pricing-kernel effect increases the price of the option, which entices the dealers to sell it. Specifically, a marginal change in the demand pressure in an option contract increases its price by an amount proportional to the variance of the unhedgeable part of the option, where the variance is computed under a certain probability measure. Similarly, the demand pressure increases the price of any other option by an amount proportional to the covariance of their unhedgeable parts. Hence, while demand pressure in a particular option raises its price, it also raises the price of other options on the same underlying, especially similar contracts.

Empirically, we use a unique dataset to identify aggregate daily positions of dealers and end users. In particular, we define dealers as market makers and end users as proprietary traders and customers of brokers.[2] We find that end users have a net long position in S&P500 index options with large net positions in out-of-the-money puts. Hence, since options are in zero net supply, dealers are short index options. While it is conventional wisdom among option traders that Wall Street is short index volatility, this paper is the first to demonstrate this fact using data on option holdings. This can help explain the puzzle that index options appear to be expensive, and that low-moneyness options seem to be especially expensive (Longstaff (1995), Bates (2000), Coval and Shumway (2001), Amin, Coval, and Seyhun (2004), Bondarenko (2003)). In the time series, demand for index options is related to their expensiveness, measured by the difference between their implied volatility and the volatility measure of Bates (2005). Further, the steepness of the smirk, measured by the difference between the implied volatility of low-moneyness options and at-the-money options, is positively related to the skew of option demand, measured by the demand for low-moneyness options minus the demand for high-moneyness options.

Jackwerth (2000) finds that a representative investor's option-implied utility function is inconsistent with standard assumptions in economic theory.[3] Since options are in zero net supply, a representative investor holds no options. We reconcile this finding for dealers who have significant short index option positions. Intuitively, an investor will short index options, but only a finite number of options. Hence, while a standard-utility investor may not be marginal on options given a zero position, he is marginal given a certain negative position. We do not address why end users buy these options; their motives might be related to portfolio insurance and agency problems (e.g. between investors and fund managers) that are not well captured by standard utility theory.

Another option-pricing puzzle is that index option prices are so different from the prices of single-stock options despite the fact that the distributions of the underlying

[2]The empirical results are robust to classifying proprietary traders as either dealers or end users.
[3]See also Driessen and Maenhout (2003).

appear relatively similar (e.g., Bakshi, Kapadia, and Madan (2003) and Bollen and Whaley (2004)). In particular, single-stock options appear cheaper and their smile is flatter. Consistently, we find that the demand pattern for single-stock options is very different from that of index options. For instance, end users are net short single-stock options – not long, as in the case of index options.

Demand patterns further help explain the time-series and cross-sectional pricing of single-stock options. Indeed, individual stock options are cheaper at times when end users sell more options, and, in the cross section, stocks with more negative demand for options, aggregated across contracts, tend to have relatively cheaper options.

The paper is related to several strands of literature. First, the literature on option pricing in the context of trading frictions and incomplete markets derives bounds on option prices. Arbitrage bounds are trivial with any transaction costs; for instance, the price of a call option can be as high as the price of the underlying stock (Soner, Shreve, and Cvitanic (1995)). This serious limitation of no-arbitrage pricing has led Cochrane and Saa-Requejo (2000) and Bernardo and Ledoit (2000) to derive tighter option-pricing bounds by restricting the Sharpe ratio or gain/loss ratio to be below an arbitrary level, and stochastic dominance bounds for small option positions are derived by Constantinides and Perrakis (2002) and extended and implemented empirically by Constantinides, Jackwerth, and Perrakis (2005). Rather than deriving bounds, we compute explicit prices based on the demand pressure by end users. We further complement this literature by taking portfolio considerations into account, that is, the effect of demand for one option on the prices of other options.

Second, the literature on utility-based option pricing ("indifference pricing") derives the option price that would make an agent (e.g., the representative agent) indifferent between buying the option and not buying it. See Rubinstein (1976), Brennan (1979), Stapleton and Subrahmanyam (1984), Hugonnier, Kramkov, and Schachermayer (2005) and references therein. While this literature computes the price of the first "marginal" option demanded, we show how option prices change when demand is non-trivial.

Third, Stein (1989) and Poteshman (2001) provide evidence that option investors misproject changes in the instantaneous volatility of underlying assets by examining the price changes of shorter and longer maturity options. Our paper shows how the cognitive biases of option end users can translate (via their option demands) into option prices even if market makers are not subject to any behavioral biases. By contrast, under standard models like Black-Scholes-Merton, market makers who can hedge their positions perfectly will correct the mistakes of other option market participants before they affect option prices. As a result, this paper provides a foundation for the application of behavioral finance to the option market.

Fourth, the general idea of demand-pressure effects goes back, at least, to Keynes (1923) and Hicks (1939) who considered futures markets. Our model is the first to apply this idea to option pricing and to incorporate the important features of option markets, namely dynamic trading of many assets, hedging using the underlying and bonds, stochastic volatility, and jumps. The generality of our model also makes it applicable to other markets. Consistent with our model's predictions, Wurgler and Zhuravskaya (2002) find that stocks that are hard to hedge experience larger price jumps when included into the S&P 500 index. Greenwood (2005) considers a major redefinition of the Nikkei 225 index in Japan and finds that stocks that are not affected by demand shocks, but that are correlated with securities facing demand shocks, experience price changes. Similarly in the fixed income market, Newman and Rierson (2004) find that non-informative issues of telecom bonds depress the price of the issued bond as well as correlated telecom bonds, and Gabaix, Krishnamurthy, and Vigneron (2004) find related evidence for mortgage-backed securities. Further, de Roon, Nijman, and Veld (2000) find futures-market evidence consistent with the model's predictions.

The most closely related paper is Bollen and Whaley (2004), which demonstrates that changes in implied volatility are correlated with signed option volume. These empirical results set the stage for our analysis by showing that *changes* in option demand lead to *changes* in option prices while leaving open the question of whether the *level* of option demand impacts the overall *level* (i.e., expensiveness) of option prices or the overall shape of implied-volatility curves.[4] We complement Bollen and Whaley (2004) by providing a theoretical model and by investigating empirically the relationship between the level of end user demand for options and the level and overall shape of implied volatility curves. In particular, we document that end users tend to have a net long SPX option position and a short equity-option position, thus helping to explain the relative expensiveness of index options. We also show that there is a strong downward skew in the net demand of index but not equity options which helps to explain the difference in the shapes of their overall implied volatility curves.

The rest of the paper is organized as follows. Section 2 describes the model, and Section 3 derives its pricing implications. Section 4 provides descriptive statistics on demand patterns for options, Section 5 tests the effect of demand pressure on option prices, and Section 6 concludes. The appendix contains proofs.

2 A Model of Demand Pressure

We consider a discrete-time infinite-horizon economy. There exists a risk-free asset paying interest at the rate of $R_f - 1$ per period, and a risky security that we refer to as the "underlying" security. At time t, the underlying has an exogenous strictly positive price[5] of S_t, dividend D_t, and an excess return of $R^e_t = (S_t + D_t)/S_{t-1} - R_f$ and the distribution of future prices and returns is characterized by a stationary Markov state

[4]Indeed, Bollen and Whaley (2004) find that a nontrivial part of the option price impact from day t signed option volume dissipates by day $t + 1$.

[5]All random variables are defined on a probability space $(\Omega, \mathcal{F}, Pr)$ with an associated filtration $\{\mathcal{F}_t : t \geq 0\}$ of sub-σ-algebras representing the resolution over time of information commonly available to agents.

variable $X_t \in \mathbb{X} \subset \mathbb{R}^n$, with $\mathbb{X}$ compact.[6] (The state variable could include the current level of volatility, the current jump intensity, etc.) The only condition we impose on the transition function $\pi : \mathbb{X} \times \mathbb{X} \to \mathbb{R}_+$ of X is that it have the Feller property.

The economy further has a number of "derivative" securities, whose prices are to be determined endogenously. A derivative security is characterized by its index $i \in I$, where i collects the information that identifies the derivative and its payoffs. For a European option, for instance, the strike price, maturity date, and whether the option is a "call" or "put" suffice. The set of derivatives traded at time t is denoted by I_t, and the vector of prices of traded securities is $p_t = (p_t^i)_{i \in I_t}$.

We assume that the payoffs of the derivatives depend on S_t and X_t. We note that the theory is completely general and does not require that the "derivatives" have payoffs that depend on the underlying. In principle, the derivatives could be any securities whose prices are affected be demand pressure.

The economy is populated by two kinds of agents: "dealers" and "end users." Dealers are competitive and there exists a representative dealer who has constant absolute risk aversion, that is, his utility for remaining life-time consumption is:

$$U(C_t, C_{t+1}, \ldots) = \mathrm{E}_t \left[\sum_{v=t}^{\infty} \rho^{v-t} u(C_v) \right],$$

where $u(c) = -\frac{1}{\gamma} e^{-\gamma c}$ and $\rho < 1$ is a discount factor. At any time t, the dealer must choose the consumption C_t, the dollar investment in the underlying θ_t, and the number of derivatives held $q_t = (q_t^i)_{i \in I_t}$, while satisfying the transversality condition $\lim_{t\to\infty} \mathrm{E}\left[\rho^{-t} e^{-kW_t}\right] = 0$, where the dealer's wealth evolves as

$$W_{t+1} \;=\; y_{t+1} + (W_t - C_t)R_f + q_t(p_{t+1} - R_f p_t) + \theta_t R_{t+1}^e,$$

$k = \gamma(R_f - 1)/R_f$, and y_t is the dealer's time-t endowment. We assume that the distribution of future endowments is characterized by X_t.[7]

In the real world, end users trade options for a variety of reasons such as portfolio insurance, agency reasons, behavioral reasons, institutional reasons etc. Rather than trying to capture these various trading motives endogenously, we assume that end users have an exogenous aggregate demand for derivatives of $d_t = (d_t^i)_{i \in I_t}$ at time t. We assume that R_t^e, D_t/S_t, and y_t are continuous functions of X_t. The distribution of future demand is characterized by X_t. Furthermore, for technical reasons, we assume that, after some time $\overline{T}$, demand pressure is zero, that is, $d_t = 0$ for $t > \overline{T}$.

Derivative prices are set through the interaction between dealers and end users in a competitive equilibrium.

[6]This condition can be relaxed at the expense of further technical complexity.

[7]More precisely, the distribution of $(y_{t+1}, y_{t+2}, \ldots)$ conditional on $\mathcal{F}_t$ is the same as the distribution conditional on X_t, i.e., $\mathcal{L}(y_{t+1}, y_{t+2}, \ldots \mid \mathcal{F}_t) = \mathcal{L}(y_{t+1}, y_{t+2}, \ldots \mid X_t)$.

Definition 1 *A price process $p_t = p_t(d_t, X_t)$ is a (competitive Markov) equilibrium if, given p, the representative dealer optimally chooses a derivative holding q such that derivative markets clear, i.e., $q + d = 0$.*

Our asset-pricing approach relies on the insight that, by observing the aggregate quantities held by dealers, we can determine the derivative prices consistent with the dealers' utility maximization. Our goal is to determine how derivative prices depend on the demand pressure d coming from end users. We note that it is not crucial that end users have inelastic demand. All that matters is that end users have demand curves that result in dealers holding a position of $q = -d$.

To determine the representative dealer's optimal behavior, we consider his value function $J(W; t, X)$, which depends on his wealth W, the state of nature X, and time t. Then, the dealer solves the following maximization problem:

$$\max_{C_t, q_t, \theta_t} \quad -\frac{1}{\gamma}e^{-\gamma C_t} + \rho \mathrm{E}_t[J(W_{t+1}; t+1, X_{t+1})] \tag{1}$$

$$\text{s.t.} \quad W_{t+1} = y_{t+1} + (W_t - C_t)R_f + q_t(p_{t+1} - R_f p_t) + \theta_t R^e_{t+1}. \tag{2}$$

The value function is characterized in the following proposition.

Lemma 1 *If $p_t = p_t(d_t, X_t)$ is the equilibrium price process and $k = \frac{\gamma(R_f - 1)}{R_f}$, then the dealer's value function and optimal consumption are given by*

$$J(W_t; t, X_t) = -\frac{1}{k}e^{-k(W_t + G_t(d_t, X_t))} \tag{3}$$

$$C_t = \frac{k}{\gamma}\left(W_t + G_t(d_t, X_t)\right) \tag{4}$$

and the stock and derivative holdings are characterized by the first-order conditions

$$0 = E_t\left[e^{-k\left(y_{t+1} + \theta_t R^e_{t+1} + q_t(p_{t+1} - R_f p_t) + G_{t+1}(d_{t+1}, X_{t+1})\right)} R^e_{t+1}\right] \tag{5}$$

$$0 = E_t\left[e^{-k\left(y_{t+1} + \theta_t R^e_{t+1} + q_t(p_{t+1} - R_f p_t) + G_{t+1}(d_{t+1}, X_{t+1})\right)} (p_{t+1} - R_f p_t)\right], \tag{6}$$

where, for $t \le T$, the function $G_t(d_t, X_t)$ is derived recursively using (5), (6), and

$$e^{-krG_t(d_t, X_t)} = R_f \rho E_t\left[e^{-k\left(y_{t+1} + q_t(p_{t+1} - R_f p_t) + \theta_t R^e_{t+1} + G_{t+1}(d_{t+1}, X_{t+1})\right)}\right] \tag{7}$$

and for $t > T$, the function $G_t(d_t, X_t) = \bar{G}(X_t)$ where $(\bar{G}(X_t), \bar{\theta}(X_t))$ solves

$$e^{-kr\bar{G}(X_t)} = R_f \rho E_t\left[e^{-k\left(y_{t+1} + \bar{\theta}_t R^e_{t+1} + \bar{G}(X_{t+1})\right)}\right] \tag{8}$$

$$0 = E_t\left[e^{-k\left(y_{t+1} + \bar{\theta}_t R^e_{t+1} + \bar{G}(X_{t+1})\right)} R^e_{t+1}\right]. \tag{9}$$

The optimal consumption is unique and the optimal security holdings are unique provided their payoffs are linearly independent.

While dealers compute optimal positions given prices, we are interested in inverting this mapping and compute the prices that make a given position optimal. The following proposition ensures that this inversion is possible.

Proposition 1 *Given any demand pressure process d for end users, there exists a unique equilibrium p.*

Before considering explicitly the effect of demand pressure, we make a couple of simple "parity" observations that show how to treat derivatives that are linearly dependent such as puts and calls with the same strike and maturity. For simplicity, we do this only in the case of a non-dividend paying underlying, but the results can easily be extended. We consider two derivatives, i and j such that a non-trivial linear combination of their payoffs lies in the span of exogenously-priced securities, i.e., the underlying and the bond. In other words, suppose that at the common maturity date T,

$$p_T^i = \lambda p_T^j + \alpha + \beta S_T$$

for some constants α, β, and λ. Then it is easily seen that, if positions $\left(q_t^i, q_t^j, b_t, \theta_t\right)$ in the two derivatives, the bond,[8] and the underlying, respectively, are optimal given the prices, then so are positions $\left(q_t^j + a, q_t^j - \lambda a, b_t - a\alpha R_f^{-(T-t)}, \theta_t - a\beta S_t^{-1}\right)$. This has the following implications for equilibrium prices:

Proposition 2 *Suppose that $D_t = 0$ and $p_T^i = \lambda p_T^j + \alpha + \beta S_T$. Then:*
(i) For any demand pressure, d, the equilibrium prices of the two derivatives are related by

$$p_t^i = \lambda p_t^j + \alpha R_f^{-(T-t)} + \beta S_t.$$

(ii) Changing the end user demand from $\left(d_t^i, d_t^j\right)$ to $\left(d_t^i + a, d_t^j - \lambda a\right)$, for any $a \in \mathbb{R}$, has no effect on equilibrium prices.

The first part of the proposition is a general version of the well-known put-call parity. It shows that if payoffs are linearly dependent then so are prices.

The second part of the proposition shows that linearly dependent derivatives have the same demand-pressure effects on prices. Hence, in our empirical exercise, we can aggregate the demand of calls and puts with the same strike and maturity. That is, a demand pressure of d^i calls and d^j puts is the same as a demand pressure of $d^i + d^j$ calls and 0 puts (or vice versa).

[8] This is a dollar amount; equivalently, we may assume that the price of the bond is always 1.

3 Price Effects of Demand Pressure

We now consider the main implication of the theory, namely the impact of demand pressure on prices. Our goal is to compute security prices p_t^i as functions of the current demand pressure d_t^j and the state variable X_t (which incorporates beliefs about future demand pressure).

We think of the price p, the hedge position θ_t in the underlying, and the consumption function G as functions of d_t^j and X_t. Alternatively, we can think of the dependent variables as functions of the dealer holding q_t^j and X_t, keeping in mind the equilibrium relation that $q = -d$. For now we use this latter notation.

At maturity date T, an option has a known price p_T. At any prior date t, the price p_t can be found recursively by "inverting" (6) to get

$$p_t \quad = \quad \frac{\mathrm{E}_t\left[e^{-k\left(y_{t+1}+\theta_t R^e_{t+1}+q_t p_{t+1}+G_{t+1}\right)}p_{t+1}\right]}{R_f \mathrm{E}_t\left[e^{-k\left(y_{t+1}+\theta_t R^e_{t+1}+q_t p_{t+1}+G_{t+1}\right)}\right]} \tag{10}$$

where the hedge position in the underlying, θ_t, solves

$$0 \quad = \quad \mathrm{E}_t\left[e^{-k\left(y_{t+1}+\theta_t R^e_{t+1}+q_t p_{t+1}+G_{t+1}\right)}R^e_{t+1}\right] \tag{11}$$

and where G is computed recursively as described in Lemma 1. Equations (10) and (11) can be written in terms of a demand-based pricing kernel:

Theorem 1 *Prices p and the hedge position θ satisfy*

$$p_t \quad = \quad E_t(m^d_{t+1}p_{t+1}) = \frac{1}{R_f}E^d_t(p_{t+1}) \tag{12}$$

$$0 \quad = \quad E_t(m^d_{t+1}R^e_{t+1}) = \frac{1}{R_f}E^d_t(R^e_{t+1}) \tag{13}$$

where the pricing kernel m^d is a function of demand pressure d:

$$m^d_{t+1} \quad = \quad \frac{e^{-k\left(y_{t+1}+\theta_t R^e_{t+1}+q_t p_{t+1}+G_{t+1}\right)}}{R_f E_t\left[e^{-k\left(y_{t+1}+\theta_t R^e_{t+1}+q_t p_{t+1}+G_{t+1}\right)}\right]} \tag{14}$$

$$\quad = \quad \frac{e^{-k\left(y_{t+1}+\theta_t R^e_{t+1}-d_t p_{t+1}+G_{t+1}\right)}}{R_f E_t\left[e^{-k\left(y_{t+1}+\theta_t R^e_{t+1}-d_t p_{t+1}+G_{t+1}\right)}\right]}, \tag{15}$$

and E^d_t is expected value with respect to the corresponding risk-neutral measure, i.e. the measure with a Radon-Nykodim derivative of $R_f m^d_{t+1}$.

To understand this pricing kernel, suppose for instance that end users want to sell derivative i such that $d_t^i < 0$, and that this is the only demand pressure. In equilibrium, dealers take the other side of the trade, buying $q_t^i = -d_t^i > 0$ units of this derivative, while hedging their position using a position of θ_t in the underlying. The pricing kernel is small whenever the "unhedgeable" part $q_t p_{t+1} + \theta_t R_{t+1}^e$ is large. Hence, the pricing kernel assigns a low value to states of nature in which a hedged position in the derivative pays off profitably, and it assigns a high value to states in which a hedged position in the derivative has a negative payoff. This pricing kernel-effect decreases the price of this derivative, which is what entices the dealers to buy it.

It is interesting to consider the first-order effect of demand pressure on prices. Hence, we calculate explicitly the sensitivity of the prices of a derivative p_t^i with respect to the demand pressure of another derivative d_t^j. We can initially differentiate with respect to q rather than d since $q^i = -d_t^i$.

For this, we first differentiate the pricing kernel[9]

$$\frac{\partial m_{t+1}^d}{\partial q_t^j} = -k m_{t+1}^d \left(p_{t+1}^j - R_f p_t^j + \frac{\partial \theta_t}{\partial q_t^j} R_{t+1}^e \right) \tag{16}$$

using the facts that $\frac{\partial G(t+1, X_{t+1}; q)}{\partial q_t^j} = 0$ and $\frac{\partial p_{t+1}}{\partial q_t^j} = 0$. With this result, it is straightforward to differentiate (13) to get

$$0 = \mathrm{E}_t \left(m_{t+1}^d \left(p_{t+1}^j - R_f p_t^j + \frac{\partial \theta_t}{\partial q_t^j} R_{t+1}^e \right) R_{t+1}^e \right) \tag{17}$$

which implies that the marginal hedge position is

$$\frac{\partial \theta_t}{\partial q_t^j} = -\frac{\mathrm{E}_t \left(m_{t+1}^d \left(p_{t+1}^j - R_f p_t^j \right) R_{t+1}^e \right)}{\mathrm{E}_t \left(m_{t+1}^d (R_{t+1}^e)^2 \right)} = -\frac{\mathrm{Cov}_t^d (p_{t+1}^j, R_{t+1}^e)}{\mathrm{Var}_t^d (R_{t+1}^e)} \tag{18}$$

Similarly, we derive the price sensitivity by differentiating (12)

$$\begin{aligned}
\frac{\partial p_t^i}{\partial q_t^j} &= -k \mathrm{E}_t \left[m_{t+1}^d \left(p_{t+1}^j - R_f p_t^j + \frac{\partial \theta_t}{\partial q_t^j} R_{t+1}^e \right) p_{t+1}^i \right] && (19) \\
&= -\frac{k}{R_f} \mathrm{E}_t^d \left[\left(p_{t+1}^j - R_f p_t^j - \frac{\mathrm{Cov}_t^d (p_{t+1}^j, R_{t+1}^e)}{\mathrm{Var}_t^d (R_{t+1}^e)} R_{t+1}^e \right) p_{t+1}^i \right] && (20) \\
&= -\gamma (R_f - 1) \mathrm{E}_t^d \left[\bar{p}_{t+1}^j \bar{p}_{t+1}^i \right] && (21) \\
&= -\gamma (R_f - 1) \mathrm{Cov}_t^d \left[\bar{p}_{t+1}^j, \bar{p}_{t+1}^i \right] && (22)
\end{aligned}$$

where $\bar{p}_{t+1}^i$ and $\bar{p}_{t+1}^j$ are the *unhedgeable* parts of the price changes as defined in:

[9] We suppress the arguments of functions. We note that p_t, θ_t, and G_t are functions of (d_t, X_t, t), and m_{t+1}^d is a function of $(d_t, X_t, d_{t+1}, X_{t+1}, y_{t+1}, R_{t+1}^e, t)$.

Definition 2 *The unhedgeable price change of any security k is*

$$\bar{p}^k_{t+1} \;=\; R_f^{-1}\left(p^k_{t+1} - R_f p^k_t - \frac{Cov^d_t(p^k_{t+1}, R^e_{t+1})}{Var^d_t(R^e_{t+1})}R^e_{t+1}\right). \tag{23}$$

Equation (22) can also be written in terms of the demand pressure, d, by using the equilibrium relation $d = -q$:

Theorem 2 *The price sensitivity to demand pressure is*

$$\frac{\partial p^i_t}{\partial d^j_t} \;=\; \gamma(R_f - 1)E^d_t\left(\bar{p}^i_{t+1}\bar{p}^j_{t+1}\right) = \gamma(R_f-1)\,Cov^d_t\left(\bar{p}^i_{t+1}, \bar{p}^j_{t+1}\right)$$

This result is intuitive: it says that the demand pressure in an option j increases the option's own price by an amount proportional to the variance of the unhedgeable part of the option and the aggregate risk aversion of dealers. We note that since a variance is always positive, the demand-pressure effect on the security itself is naturally always positive. Further, this demand pressure affects another option i by an amount proportional to the covariation of their unhedgeable parts. For European options, we can show, under the condition stated below, that a demand pressure in one option also increases the price of other options on the same underlying:

Proposition 3 *Demand pressure in any security j:*
(i) increases its own price, that is, $\frac{\partial p^j_t}{\partial d^j_t} \geq 0$.
(ii) increases the price of another security i, that is, $\frac{\partial p^i_t}{\partial d^j_t} \geq 0$, provided that $E^d_t\left[p^i_{t+1}\middle|\, S_{t+1}\right]$ and $E^d_t\left[p^j_{t+1}\middle|\, S_{t+1}\right]$ are convex functions of S_{t+1} and $Cov^d_t\left(p^i_{t+1}, p^j_{t+1}\middle|\, S_{t+1}\right) \geq 0$.

The conditions imposed in part *(ii)* are natural. First, we require that prices inherit the convexity property of the option payoffs in the underlying price. Second, we require that $\mathrm{Cov}^d_t\left(p^i_{t+1}, p^j_{t+1}\middle|\, S_{t+1}\right) \geq 0$, that is, changes in the other variables have a similar impact on both option prices — for instance, both prices are increasing in the volatility or demand level. Note that both conditions hold if both options mature after one period. The second condition also holds if option prices are homogenous (of degree 1) in (S, K), where K is the strike, and S_t is independent of X_t.

It is interesting to consider the total price that end users pay for their demand d_t at time t. Vectorizing the derivatives from Theorem 2, we can first-order approximate the price around a zero demand as follows

$$p_t \;\approx\; p_t(d_t = 0) + \gamma(R_f - 1)\mathrm{E}^d_t\left(\bar{p}_{t+1}\bar{p}'_{t+1}\right)d_t \tag{24}$$

Hence, the total price paid for the d_t derivatives is

$$d'_t p_t \;=\; d'_t p_t(d_t = 0) + \gamma(R_f - 1)d'_t\mathrm{E}^d_t\left(\bar{p}_{t+1}\bar{p}'_{t+1}\right)d_t \tag{25}$$

$$\;=\; d'_t p_t(0) + \gamma(R_f - 1)\mathrm{Var}^d_t\left(d'_t\bar{p}_{t+1}\right) \tag{26}$$

The first term $d_t' p_t(d_t = 0)$ is the price that end users would pay if their demand pressure did not affect prices. The second term is total variance of the unhedgeable part of all of the end users' positions.

While Proposition 3 shows that demand for an option increases the prices of all options, the size of the price effect is, of course, not the same for all options. Nor is the effect on implied volatilities the same. Under certain conditions, demand pressure in low-strike options has a larger impact on the implied volatility of low-strike options, and conversely for high strike options. The following proposition makes this intuitively appealing result precise. For simplicity, the proposition relies on unnecessarily restrictive assumptions. We let $p(p, K, d)$, respectively $p(c, K, d)$, denote the price of a put, respectively a call, with strike price K and 1 period to maturity, where d is the demand pressure. It is natural to compare low-strike and high-strike options that are "equally far out of the money." We do this be considering an out-of-the-money put with the same price as an out-of-the-money call.

Proposition 4 *Assume that the one-period risk-neutral distribution of the underlying return is symmetric and consider demand pressure $d > 0$ in an option with strike $K < R_f S_t$ that matures after one trading period. Then there exists a value $\bar{K}$ such that, for all $K' \leq \bar{K}$ and K'' such that $p(p, K', 0) = p(c, K'', 0)$, it holds that $p(p, K', d) > p(c, K'', d)$. That is, the price of the out-of-the-money put $p(p, K', \cdot)$ is more affected by the demand pressure than the price of out-of-the-money call $p(c, K'', \cdot)$. The reverse conclusion applies if there is demand for a high-strike option.*

Future demand pressure in a derivative j also affects the current price of derivative i. As above, we consider the first-order price effect. This is slightly more complicated, however, since we cannot differentiate with respect to the unknown future demand pressure. Instead, we "scale down" the future demand pressure, that is, we consider future demand pressures $\tilde{d}_s^j = \epsilon d_s^j$ for fixed d (equivalently, $\tilde{q}_s^j = \epsilon q_s^j$) for some $\epsilon \in \mathbb{R}$, $\forall s > t$, and $\forall j$.

Theorem 3 *Let $p_t(0)$ denote the equilibrium derivative prices with 0 demand pressure. Fixing a process d with $d_t = 0$ for all $t > T$ and a given T, the equilibrium prices p with a demand pressure of ϵd is*

$$p_t \;=\; p_t(0) + \gamma(R_f - 1)\left[E_t^0\left(\bar{p}_{t+1}\bar{p}_{t+1}'\right)d_t + \sum_{s>t} R_f^{-(s-t)} E_t^0\left(\bar{p}_{s+1}\bar{p}_{s+1}' d_s\right)\right]\epsilon + O(\epsilon^2)$$

This theorem shows that the impact of current demand pressure d_t on the price of a derivative i is given by the amount of hedging risk that a marginal position in security i would add to the dealer's portfolio, that is, it is the sum of the covariances of its unhedgeable part with the unhedgeable part of all the other securities, multiplied by their respective demand pressures. Further, the impact of future demand pressures d_s

is given by the expected future hedging risks. Of course, the impact increases with the dealers' risk aversion.

Next, we discuss how demand is priced in connection with three specific sources of unhedgeable risk for the dealers: discrete-time hedging, jumps in the underlying stock, and stochastic volatility risk. We focus on small hedging periods Δ_t and derive the results informally while relegating a more rigorous treatment to the appendix. The continuously compounded riskfree interest rate is denoted r, i.e. the riskfree return over one Δ_t time period is $R_f = e^{r\Delta_t}$. We assume throughout that S is an bounded semi-martingale with smooth transition density.

3.1 Price Effect of Risk due to Discrete-Time Hedging

To focus on the specific risk due discrete-time trading (rather than continuous trading), we consider a stock price that is a diffusion process driven by a Brownian motion with no other state variables. In this case, markets would be complete with continuous trading, and, hence, the dealer's hedging risk arises solely from his trading only at discrete times, spaced Δ_t time units apart.

We are interested in the price of option i as a function of the stock price S_t and demand pressure d_t, $p_t^i = p_t^i(S_t, d_t)$. We denote the price without demand pressure by f, that is, $f^i(t, S_t) := p_t^i(S_t, d = 0)$ and assume throughout that f is smooth for $t < T$. The change in the option price evolves approximately according to

$$p_{t+1}^i \cong f^i + f_S^i \Delta S + \frac{1}{2} f_{SS}^i (\Delta S)^2 + f_t^i \Delta_t \tag{27}$$

where $f^i = f^i(t, S_t)$, $f_t^i = \frac{\partial}{\partial t} f^i(t, S_t)$, $f_S^i = \frac{\partial}{\partial S} f^i(t, S_t)$, $f_{SS}^i = \frac{\partial^2}{\partial S^2} f^i(t, S_t)$, and $\Delta S = S_{t+1} - S_t$. The unhedgeable option price change is

$$\begin{aligned} e^{r\Delta_t} \bar{p}_{t+1}^i &= p_{t+1}^i - e^{r\Delta_t} p_t^i - f_S^i (S_{t+1} - e^{r\Delta_t} S_t) & (28) \\ &\cong -r\Delta_t f^i + f_t^i \Delta_t + r\Delta_t f_S^i S_t + \frac{1}{2} f_{SS}^i (\Delta S)^2 & (29) \end{aligned}$$

where we expand p_{t+1} and use $e^{r\Delta_t} \cong 1 + r\Delta_t$. To consider the impact of demand d_t^j in option j on the price of option i, we need the covariance of their unhedgeable parts:

$$\mathrm{Cov}_t(e^{r\Delta_t} \bar{p}_{t+1}^i \, , \, e^{r\Delta_t} \bar{p}_{t+1}^j) \cong \frac{1}{4} f_{SS}^i f_{SS}^j Var_t((\Delta S)^2)$$

Hence, by Theorem 2, we get the following result. (Details of the proof are in the appendix.)

Proposition 5 *If the underlying asset price follows a Markov diffusion and the period length is Δ_t, the effect on the price of demand at $d = 0$ is*

$$\frac{\partial p_t^i}{\partial d_t^j} = \frac{\gamma r Var_t((\Delta S)^2)}{4} f_{SS}^i f_{SS}^j + o(\Delta_t^2) \tag{30}$$

and the effect on the Black-Scholes implied volatility $\hat{\sigma}^i_t$ *is:*

$$\frac{\partial \hat{\sigma}^i_t}{\partial d^j_t} = \frac{\gamma r Var_t((\Delta S)^2)}{4} \frac{f^i_{SS}}{\nu^i} f^j_{SS} + o(\Delta_t^2), \tag{31}$$

where ν^i *is the Black-Scholes vega.*

Interestingly, the Black-Scholes gamma over vega, f^i_{SS}/ν^i, does not depend on moneyness so the first-order effect of demand with discrete trading risk is to change the level, but not the slope, of the implied-volatility curves.

Intuitively, the impact of the demand for options of type j depends on the gamma of these options, f^j_{SS}, since the dealers cannot hedge the non-linearity of the payoff.

The effect of discrete-time trading is small if hedging is frequent. More precisely, the effect is of the order of $Var_t((\Delta S)^2)$, namely Δ_t^2. Hence, if we add up T/Δ_t terms of this maginitude — corresponding to demand in each period between time 0 and maturity T — then the total effect is order Δ_t, which approaches zero as the Δ_t approaches zero. This is consistent with the Black-Scholes-Merton result of perfect hedging in continuous time. As we show next, the risks of jumps and stochastic volatility do not vanish for small Δ_t (specifically, they are of order Δ_t).

3.2 Jumps in the Underlying

To study the effect of jumps in the underlying, we suppose next that S is a discretely traded jump diffusion with iid. bounded jump size, independent of the state variables, and jump intensity π (i.e. jump probability over a period of $\pi\Delta_t$).

The unhedgeable price change is

$$e^{r\Delta_t}\bar{p}^i_{t+1} \cong -r\Delta_t f^i + f^i_t \Delta_t + r\Delta_t f^i_S S_t + (f^i_S S_t - \theta^i)\Delta S 1_{(\text{no jump})} + \kappa^i 1_{(\text{jump})}$$

where

$$\kappa^i \quad = \quad f^i(S_t + \Delta S) - f^i - \theta^i \Delta S. \tag{32}$$

is the unhedgeable risk in case of a jump of size ΔS.

Proposition 6 *If the underlying asset price can jump, the effect on the price of demand at* $d = 0$ *is*

$$\frac{\partial p^i_t}{\partial d^j_t} = \gamma r \left[(f^i_S S_t - \theta^i)(f^j_S S_t - \theta^j)\, Var_t(\Delta S) + \pi \Delta_t E_t \left(\kappa^i \kappa^j \right) \right] + o(\Delta_t) \tag{33}$$

and the effect on the Black-Scholes implied volatility $\hat{\sigma}^i_t$ *is:*

$$\frac{\partial \hat{\sigma}^i_t}{\partial d^j_t} = \frac{\gamma r \left[(f^i_S S_t - \theta^i)(f^j_S S_t - \theta^j)\, Var_t(\Delta S) + \pi \Delta_t E_t \left(\kappa^i \kappa^j \right) \right]}{\nu^i} + o(\Delta_t), \tag{34}$$

where ν^i *is the Black-Scholes vega.*

The terms of the form $f^i_S S_t - \theta^i$ arise because the optimal hedge θ differs from the optimal hedge without jumps, $f^i_S S_t$, which means that some of the local noise is being hedged imperfectly. If the jump probability is small, however, then this effect is small (i.e., it is second order in π). In this case, the main effect comes from the jump risk κ. We note that while conventional wisdom holds that Black-Scholes gamma is a measure of "jump risk," this is true only for the small local jumps considered in Section 3.1. Large jumps have qualitatively different implications captured by κ. For instance, a far-out-of-the-money put may have little gamma risk, but, if a large jump can bring the option in the money, the option may have κ risk. It can be shown that this jump-risk effect (34) means that demand can affect the slope of the implied-volatility curve to the first order and generate a smile.[10]

3.3 Stochastic-Volatility Risk

To consider stochastic volatility, we let the the state variable be $X_t = (S_t, \sigma_t)$, where the stock price S is a diffusion with volatility σ_t, which is diffusion driven by an independent Brownian motion. The option price $p^i_t = f^i(t, S_t, \sigma_t)$ has unhedgeable risk given by

$$\begin{aligned} e^{r\Delta_t}\bar{p}^i_{t+1} &= p^i_{t+1} - e^{r\Delta_t}p^i_t - \theta^i R^e_{t+1} \\ &\cong -r\Delta_t f^i + f^i_t\Delta_t + f^i_S S_t r\Delta_t + f^i_\sigma \Delta\sigma_{t+1} \end{aligned}$$

Proposition 7 *With stochastic volatility, the effect on the price of demand at $d = 0$ is*

$$\frac{\partial p^i_t}{\partial d^j_t} = \gamma r\, Var(\Delta\sigma) f^i_\sigma f^j_\sigma + o(\Delta_t) \tag{35}$$

and the effect on the Black-Scholes implied volatility $\hat{\sigma}^i_t$ is:

$$\frac{\partial \hat{\sigma}^i_t}{\partial d^j_t} = \gamma r\, Var(\Delta\sigma) \frac{f^i_\sigma}{\nu^i} f^j_\sigma + o(\Delta_t), \tag{36}$$

where ν^i is the Black-Scholes vega.

Intuitively, volatility risk is captured to the first order by f_σ. This derivative is not exactly the same as Black-Scholes vega, since vega is the price sensitivity to a permanent volatility change whereas f_σ measures the price sensitivity to a volatility change that mean reverts at the rate of ϕ. For an option with maturity at time $t + T$, we have

$$f^i_\sigma \;\cong\; \nu^i \frac{\partial}{\partial \sigma_t} E\left(\frac{\int_t^{t+T} \sigma_s ds}{T} \;\Big|\; \sigma_0 \right) \cong \nu^i \frac{1 - e^{-\phi T}}{\phi T}. \tag{37}$$

[10] Of course, the jump risk also generates smiles without demand-pressure effects; the result is that demand can exacerbate these.

Hence, if we combine (37) with (36), we see that stochastic volatility risk affects the level, but not the slope, of the implied volatility curves to the first order.

4 Descriptive Statistics

The main focus of this paper is the impact of net end-user option demand on option prices. We explore this impact both for S&P 500 index options and for equity (i.e., individual stock) options. Consequently, we employ data on SPX and equity option demand and prices.[11] Our data period extends from the beginning of 1996 through the end of 2001.[12] For the equity options, we limit the underlying stocks to those with strictly positive option volume on at least 80% of the trade days over the 1996 to 2001 period. This restriction yields 303 underlying stocks.

We acquire the data from two different sources. Data for computing net option demand were obtained directly from the Chicago Board Options Exchange (CBOE). These data consist of a daily record of closing short and long open interest on all SPX and equity options for public customers and firm proprietary traders.[13] The SPX options trade only at the CBOE while the equity options sometimes are cross-listed at other option markets. Our open interest data, however, include activity from all markets at which CBOE listed options trade. The entire option market is comprised of public customers, firm proprietary traders, and market makers. Hence, our data cover all non-market-maker option open interest.

Firm proprietary traders sometimes are end users of options and sometimes are liquidity suppliers. Consequently, we compute net end-user demand for an option in two different ways. First, we assume that firm proprietary traders are end users and compute the net demand for an option as the sum of the public customer and firm proprietary trader long open interest minus the sum of the public customer and firm proprietary trader short open interest. We refer to net demand computed in this way as *non-market-maker net demand*. Second, we assume that the firm proprietary traders are liquidity suppliers and compute the net demand for an option as the public customer long open interest minus the public customer short open interest. We refer to net demand computed in this second way as *public customer net demand*. The results are similar for non-market-maker net demand and public customer net demand. Therefore,

[11]Options on the S&P 500 index have many different option symbols. In this paper, *SPX options* always refers to all options that have SPX as their underlying asset, not only to those with option symbol SPX.

[12]During our data period, SPX options are cash-settled based on the SPX opening price on Friday of expiration week. Consequently, for purposes of measuring their time to maturity, we assume that they expire at the close of trading on the Thursday of expiration week.

[13]The total long open interest for any option always equals the total short open interest. For a given investor type (e.g., public customers), however, the long open interest is not equal to the short open interest in general.

for brevity, most results are reported only for non-market-maker net demand.

Even though the SPX and individual equity option market have been the subject of extensive empirical research, there is no systematic information on end-user demand in these markets. Consequently, we provide a somewhat detailed description of net demand for SPX and equity options. Over the 1996-2001 period the average daily non-market-maker net demand for SPX options is 103,254 contracts, and the average daily public customer net demand is 136,239 contracts. In other words, the typical end-user demand for SPX options during our data period is on the order of 125,000 SPX option contracts. For puts (calls), the average daily net demand from non-market makers is 124,345 ($-21,091$) contracts, while from public customers it is 182,205 ($-45,966$) contracts. These numbers indicate that most net option demand comes from puts. Indeed, end users tend to be net suppliers of on the order of 30,000 call contracts.

For the equity options, the average daily non-market-maker net demand per underlying stock is -2717 contracts, and the average daily public customer net demand is -4873 contracts. Hence, in the equity option market, unlike the index-option market, end users are net suppliers of options. This fact suggests that if demand for options has a first order impact on option prices, index options should on average be more expensive than individual equity options. Another interesting contrast with the index option market is that in the equity option market the net end-user demand for puts and calls is similar. For puts (calls), the average daily non-market-maker net demand is -1103 (-1614) contracts, while from public customers it is -2331 (-2543) contracts.

Panel A of Table 1 reports the average daily non-market-maker net demand for SPX options broken down by option maturity and moneyness (defined as the strike price divided by the underlying index level.) Panel A indicates that 39 percent of the net demand comes from contracts with fewer than 30 calendar days to expiration. Consistent with conventional wisdom, the good majority of this net demand is concentrated at moneyness where puts are out-of-the-money (OTM) (i.e., $moneyness < 1$.) Panel B of Table 1 reports the average option net demand per underlying stock for individual equity options from non-market makers. With the exception of some long maturity option categories (i.e, those with more than one year to expiration and in one case with more than six months to expiration), the non-market-maker net demand for all of the moneyness/maturity categories is negative. That is, non-market makers are net suppliers of options in all of these categories. This stands in stark contrast to the index option market in Panel A where non-market makers are net demanders of options in almost every moneyness/maturity category.

The other main source of data for this paper is the Ivy DB data set from OptionMetrics LLC. The OptionMetrics data include end-of-day volatilities implied from option prices, and we use the volatilities implied from SPX and CBOE listed equity options from the beginning of 1996 through the end of 2001. SPX options have European style exercise, and OptionMetrics computes implied volatilities by inverting the Black-Scholes formula. When performing this inversion, the option price is set to the

Maturity Range (Calendar Days)	0–0.85	0.85–0.90	0.90–0.95	0.95–1.00	1.00–1.05	1.05–1.10	1.10–1.15	1.15–2.00	All
	Moneyness Range (K/S)								
Panel A: SPX Option Non-Market Maker Net Demand									
1–9	6,014	1,780	1,841	2,357	2,255	1,638	524	367	16,776
10–29	7,953	1,300	1,115	6,427	2,883	2,055	946	676	23,356
30–59	5,792	745	2,679	7,296	1,619	-136	1,038	1,092	20,127
60–89	2,536	1,108	2,287	2,420	1,569	-56	118	464	10,447
90–179	7,011	2,813	2,689	2,083	201	1,015	4	2,406	18,223
180–364	2,630	3,096	2,335	-1,393	386	1,125	-117	437	8,501
365–999	583	942	1,673	1,340	1,074	816	560	-1,158	5,831
All	32,519	11,785	14,621	20,530	9,987	6,457	3,074	4,286	103,260
Panel B: Equity Option Non-Market Maker Net Demand									
1-9	-51	-25	-40	-45	-47	-31	-23	-34	-295
10-29	-64	-35	-57	-79	-102	-80	-55	-103	-576
30-59	-55	-31	-39	-55	-88	-90	-72	-144	-574
60-89	-47	-29	-37	-47	-60	-60	-55	-133	-469
90-179	-85	-60	-73	-84	-105	-111	-101	-321	-941
180-364	53	-19	-23	-24	-36	-35	-33	-109	-225
365-999	319	33	25	14	12	7	9	-56	363
All	70	-168	-244	-320	-426	-400	-331	-899	-2717

Table 1: Average non-market-maker net demand for put and call option contracts for SPX and individual equity options by moneyness and maturity, 1996-2001. Equity-option demand is per underlying stock.

midpoint of the best closing bid and offer prices, the interest rate is interpolated from available LIBOR rates so that its maturity is equal to the expiration of the option, and the index dividend yield is determined from put-call parity. The equity options have American style exercise, and OptionMetrics computes their implied volatilities using binomial trees that account for the early exercise feature and the timing and amount of the dividends expected to be paid by the underlying stock over the life of the options.

One of the central questions we are investigating is whether net demand pressure pushes option implied volatilities away from the volatilities that are expected to be realized over the remainder of the options' lives. We refer to the difference between implied volatility and a reference volatility estimated from the underlying security as *excess implied volatility*.

The reference volatility that we use for SPX options is the filtered volatility from the state-of-the-art model by Bates (2005), which accounts for jumps, stochastic volatility, and the risk premium implied by the equity market, but does not add extra risk premia to (over-)fit option prices.[14]

The reference volatility that we use for equity options is the predicted volatility over their lives from a GARCH(1,1) model estimated from five years of daily underlying stock returns leading up to the day of option observations. (Alternative measures using historical or realized volatility lead to similar results.) The daily returns on the

[14] We are grateful to David Bates for providing this measure.

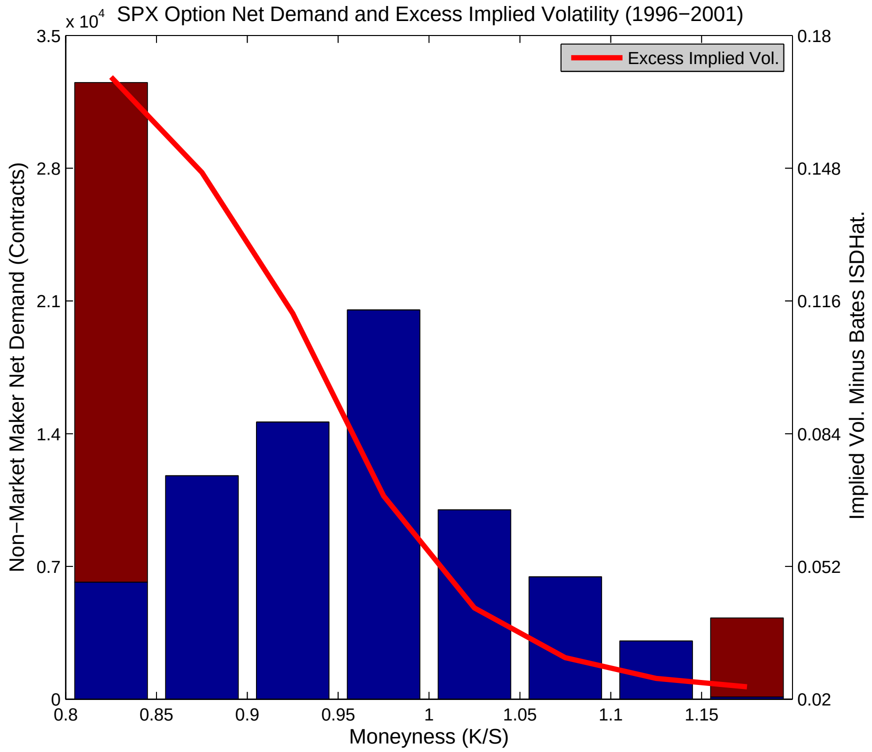

Figure 1: The bars show the average daily net demand for puts and calls from non-market makers for SPX options in the different moneyness categories (left axis). The top part of the leftmost (rightmost) bar shows the net demand for all options with moneyness less than 0.8 (greater than 1.2). The line is the average SPX excess implied volatility, that is, implied volatility minus the volatility from the underlying security, for each moneyness category (right axis). The data covers 1996-2001.

underlying index or stocks are obtained from the Center for Research in Security Prices (CRSP).

The daily average excess implied volatility for SPX options is 8.7%. To compute this number, on each trade day we average the implied volatilities on all SPX options that have at least 25 contracts of trading volume and then subtract the proxy for expected volatility. Consistent with previous research, on average the SPX options in our sample are expensive. For the equity options, the daily average excess implied volatility per underlying stock is -0.3%, which suggests that on average individual equity options are just slightly inexpensive. We required that an option trade at least 5 contracts and have a closing bid price of at least 37.5 cents in order to includes its implied volatility in the calculation.

Figure 1 compares SPX option expensiveness to net demands across moneyness categories. The line in the figure plots the average SPX excess implied volatility for eight moneyness intervals over the 1996-2001 period. In particular, on each trade date the average excess implied volatility is computed for all puts and calls in a moneyness interval. The line depicts the means of these daily averages. The excess implied volatility inherits the familiar downward sloping smirk in SPX option implied volatilities. The bars in Figure 1 represent the average daily net demand from non-market maker for SPX options in the moneyness categories, where the top part of the leftmost (rightmost) bar shows the net demand for all options with moneyness less than 0.8 (greater than 1.2).

The first main feature of Figure 1 is that index options are expensive (i.e. have a large risk premium), consistent with what is found in the literature, and that end users are net buyers of index options. This is consistent with our main hypothesis: end users buy index options and market makers require a premium to deliver them.

The second main feature of Figure 1 is that the net demand for low-strike options is greater than the demand for high-strike options. This can potentially help explain the fact that low-strike options are more expensive than high-strike options (Proposition 4). The shape of the demand across moneyness is clearly different from the shape of the expensiveness curve. We note, however, that our theory implies that demand pressure in one moneyness category impacts the implied volatility of options in other categories, thus “smoothing” the implied volatility curve and changing its shape. In fact, these average demands can give rise to a pattern of expensiveness similar to the one observed empirically using a version of the model with jump risk.

We also constructed a figure like Figure 1 except that both the excess implied volatilities and the net demands were computed only from put data. Unsurprisingly, the plot looked much like Figure 1, because (as was shown above) SPX option net demands are dominated by put net demands and put-call parity ensures that (up to market frictions) put and call options with the same strike price and maturity have the same implied volatilities. For brevity, we omit this figure from the paper. Figure 2 is constructed like Figure 1 except that only calls are used to compute the excess implied volatilities and the net demands. For calls, there appears to be a negative relationship between excess implied volatilities and net demand. This relationship suggests that call net demand cannot explain the call excess implied volatilities. Proposition 2(ii) predicts, however, that it is the total demand pressure of calls and puts that matters as depicted in Figure 1. Intuitively, the large demand for puts increases the prices of puts, and, by put-call parity, this also increases the prices of calls. The relatively small negative demand for calls cannot overturn this effect.

Figure 3 compares equity option expensiveness to net demands across moneyness categories. The line in the figure plots the average equity option excess implied volatility (with respect to the GARCH(1,1) volatility forecast) per underlying stock for eight moneyness intervals over the 1996-2001 period. In particular, on each trade date for

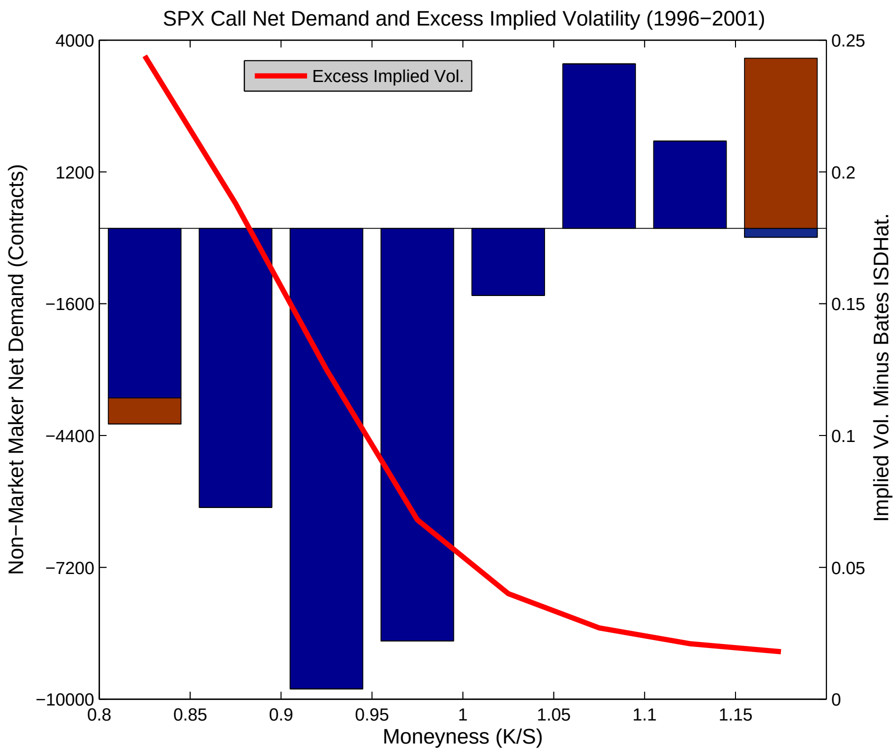

Figure 2: The bars show the average daily net demand for calls from non-market makers for SPX options in the different moneyness categories (left axis). The top part of the leftmost (rightmost) bar shows the net demand for all options with moneyness less than 0.8 (greater than 1.2). The line is the average SPX excess call implied volatility, that is, implied volatility minus the volatility from the underlying security, for each moneyness category (right axis). The data covers 1996-2001.

each underlying stock the average excess implied volatility is computed for all puts and calls in a moneyness interval. These excess implied volatilities are averaged across underlying stocks on each trade day for each moneyness interval. The line depicts the means of these daily averages. The excess implied volatility line is downward sloping but only varies by about 5% across the moneyness categories. By contrast, for the SPX options the excess implied volatility line varies by 15% across the corresponding moneyness categories. The bars in the figure represent the average daily net demand per underlying stock from non-market makers for equity options in the moneyness categories. The figure shows that non-market makers are net sellers of equity options on average, consistent with these options being cheap. Further, the figure shows that non-market makers sell mostly high-strike options, consistent with these options being

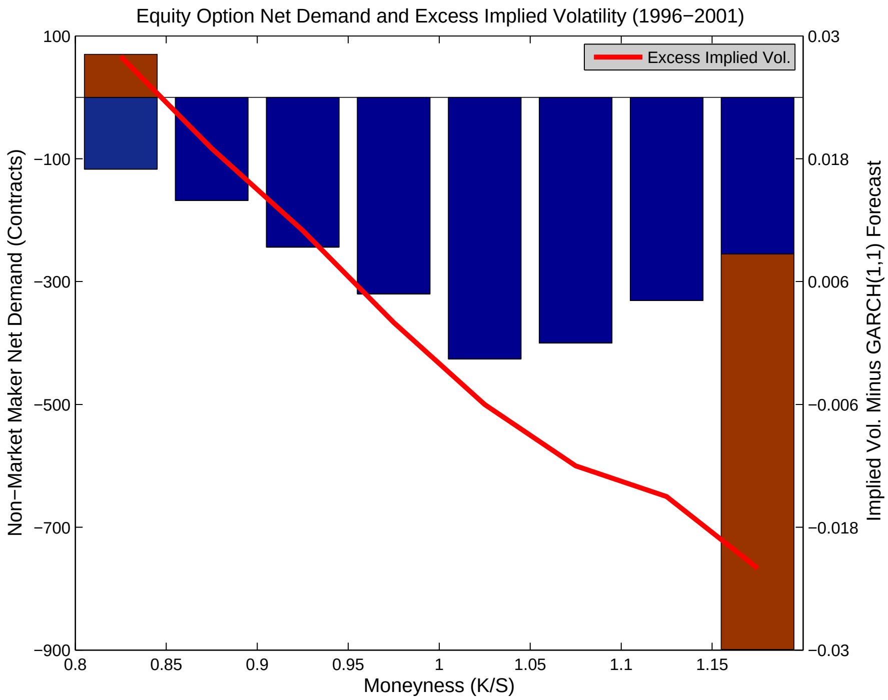

Figure 3: The bars show the average daily net demand per underlying stock from non-market makers for equity options in the different moneyness categories (left axis). The top part of the leftmost (rightmost) bar shows the net demand for all options with moneyness less than 0.8 (greater than 1.2). The line is the average equity option excess implied volatility, that is, implied volatility minus the GARCH(1,1) expected volatility, for each moneyness category (right axis). The data covers 1996-2001.

especially cheap. If the figure is constructed from only calls or only puts, it looks roughly the same (although the magnitudes of the bars are about half as large.)

Figure 4 plots the daily net positions (i.e., net demands) for SPX options aggregated across moneyness and maturity for public customers, firm proprietary traders, and market makers. The daily public customer net positions range from -1065 contracts to $+385,750$ contracts, and it tends to be larger over the first year or so of the sample. Although the public customer net position shows a good deal of variability, it is nearly always positive and never far from zero when negative. To a large extent, the market maker net option position is close to the public customer net position reflected across the horizontal axis. This is not surprising, because on each trade date the net positions of the three groups must sum to zero and the public customers constitute a much larger share of the market than the firm proprietary traders. The firm proprietary and market

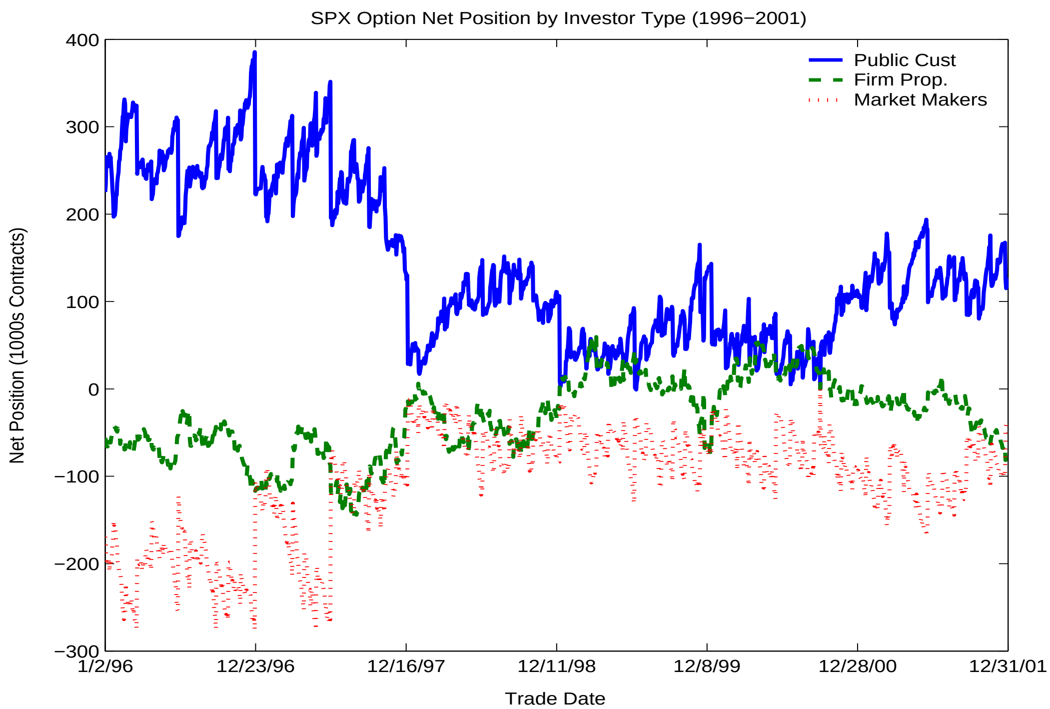

Figure 4: Time series of the daily net positions for SPX options aggregated across moneyness and maturity for public customers, firm proprietary traders, and market makers.

maker net positions roughly move with one another. In fact, the correlation between the two time-series is 0.44. This positive co-movement suggests that a non-trivial part of the firm proprietary option trading may be associated with supplying liquidity to the SPX option market. The correlations between the public customer time-series and those for firm proprietary traders and market makers on the other hand are, respectively, −0.78 and −0.90.

To illustrate the magnitude of the net demands, we compute approximate daily profits and losses (P&Ls) for the market makers' hedged positions assuming daily delta-hedging. The daily and cumulative P&Ls are illustrated in Figure 5, which shows that the group of market makers faces substantial risk that cannot be delta-hedged, with daily P&L varying between ca. $100M and $-100M. Further, the market makers make cumulative profits of ca. $800M over the 6-year period on their position taking.[15] With just over a hundred SPX market makers on the CBOE, this corresponds to a profit of approxiamtely $1M per year per market maker. Hence, consistent with the premise of

[15] This number does not take into account the costs of market making or the profits from the bid-ask spread on round-trip trades. A substantial part of market makers' profit may come from the latter.

our model, market makers face substantial risk and are compensated on average for the risk that they take.

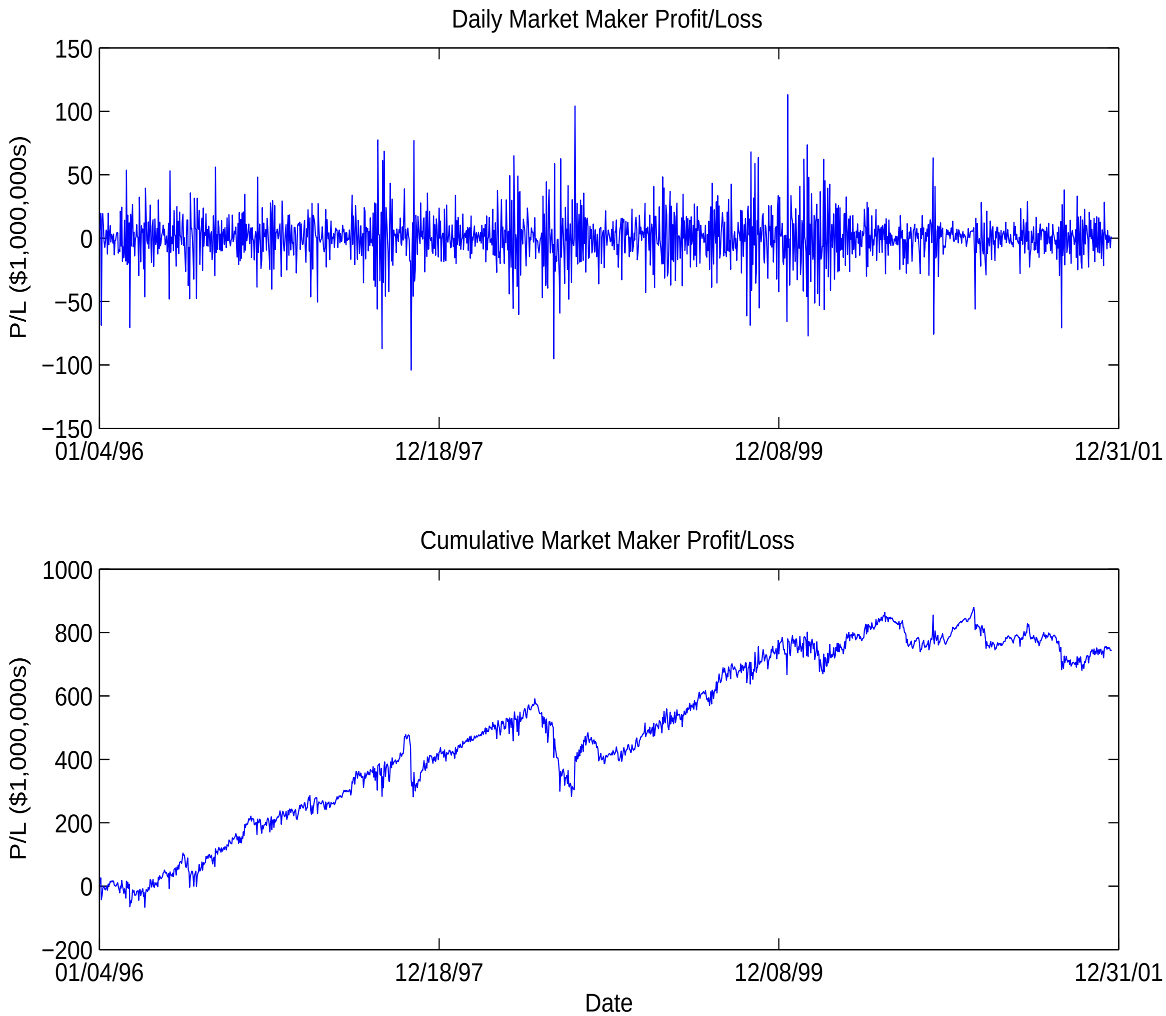

Figure 5: The top panel shows the market makers' daily profits and losses (P&L) assuming they delta-hedge their option positions once per day. The bottom panel shows the corresponding cumulative P&Ls.

5 Empirical Results

Proposition 3 states that positive (negative) demand pressure on one option increases (decreases) the price of all options on the same underlying asset, while Proposition 4 states that demand pressure on low (high) strike options has a greater price impact on low (high) strike options. The empirical work in this section of the paper examines these two predictions of the model by investigating whether overall excess implied volatility is higher on trade dates where net demand for options is higher and whether the excess implied volatility skew is steeper on trade dates where the skew in the net demand for options is steeper.

5.1 Excess Implied Volatility and Net Demand

We investigate first the time-series evidence for Proposition 3 by regressing a measure of excess implied volatility on one of various demand-based explanatory variables:

$$ExcessImplVol_t \quad = \quad a + b\, DemandVar_t + \epsilon_t \tag{38}$$

SPX:

We consider first this time-series relationship for SPX options, for which we define $ExcessImplVol$ as the average implied volatility of 1-month at-the-money SPX options minus the corresponding volatility of Bates (2005). When computing this variable, the SPX options included are those that have at least 25 contracts of trading volume, between 15 and 45 calendar days to expiration, and moneyness between 0.99 and 1.01. (We compute the excess implied volatility variable only from reasonably liquid options in order to make it less noisy in light of the fact that it is computed using only one trade date.)[16] By subtracting the volatility from the Bates (2005) model, we account for the direct effects of jumps, stochastic volatility, and the risk premium implied by the equity market.

The independent variable, $DemandVar$, is based on the aggregate net non-market-maker demand for SPX options that have 10–180 calendar days to expiration and moneyness between 0.8 and 1.20. (Similar, in fact stronger, results obtain when public-customer demand is used instead.) We employ, separately, four different independent variables. The first is simply the sum of all net demands. The other three independent variables correspond to "weighting" the net demands using the models based on the market maker risks associated with, respectively, discrete trading, jumps in the underlying, and stochastic volatility (Sections 3.1–3.3). Specifically, the net demands are weighted by the Black-Scholes gamma in the discrete-hedging model, by kappa computed using equally likely up and down moves of relative sizes 0.05 and 0.2 in the jump model, and by maturity-adjusted Black-Scholes vega in the stochastic volatility model.

[16] By contrast, in the previous section of the paper, when implied volatility statistics were computed from less liquid options or options with more extreme moneyness or maturity, they were averaged over the entire sample period.

Table 2: The relationship between the SPX Excess Implied Volatility (i.e. observed implied volatility minus volatility from the Bates (2005) model) and the SPX non-market-maker demand pressure weighted using either: *(i)* equal weights, *(ii)* weights based on discrete-time trading risk, *(iii)* weights based on jump risk, or *(iv)* weights based on stochastic-volatility risk. T-statistics computed using Newey-West are in parentheses.

	Before Structural Changes 1996/01–1996/10				After Structural Changes 1997/10–2001/12			
Constant	0.0001	0.0065	0.005	0.020	0.04	0.033	0.032	0.038
	(0.004)	(0.30)	(0.17)	(0.93)	(7.28)	(4.67)	(7.7)	(7.4)
#Contracts	2.1E-7				3.8E-7			
	(0.87)				(1.55)			
Disc. Trade		6.9E-10				2.8E-9		
		(0.91)				(3.85)		
Jump Risk			6.4E-8				3.2E-5	
			(0.79)				(3.68)	
Stoch. Vol.				8.7E-7				1.1E-5
				(0.27)				(2.74)
Adj. R^2	0.10	0.06	0.08	0.01	0.07	0.19	0.26	0.16
N	10	10	10	10	50	50	50	50

We run the regression on a monthly basis by averaging demand and expensiveness over each month. We do this because there are certain day-of-the-month effects for SPX options. (Our results are similar in a daily regression, not reported.)

The results are shown in Table 2. We report the results over two subsamples because, as seen in Figure 4, there appears to be a structural change in 1997. A structural change also happens around this time in the time series of open interest (not shown). These changes may be related to several events that changed the market for index options in the period from late 1996 to October 1997, such as the introduction of S&P500 e-mini futures and futures options on the competing Chicago Mercantile Exchange (CME), the introduction of Dow Jones options on the CBOE, and changes in margin requirements. Some of our results hold over the full sample, but their robustness and the explanatory power are smaller. Of course, we must entertain the possibility that the model's limited ability to jointly explain the full sample is due to problems with the theory.

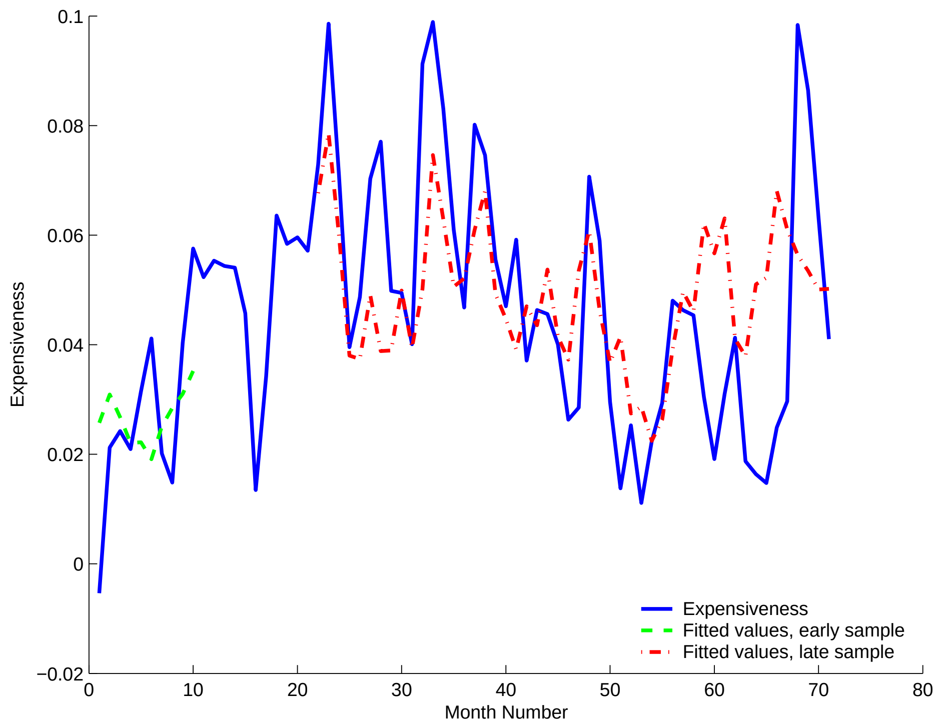

Figure 6: The solid line shows the expensiveness of SPX options, that is, implied volatility of 1-month at-the-money options minus the volatility measure of Bates (2005) which takes into account jumps, stochastic volatility, and the risk premium from the equity market. The dashed lines are, respectively, the fitted values of demand-based expensiveness using a model with underlying jumps, before and after certain structural changes (1996/01–1996/10 and 1997/10–2001/12).

We see that the estimate of the demand effect b is positive but insignificant over the first subsample, and positive and statistically significant over the second longer subsample for all three model-based explanatory variables.[17]

The expensiveness and the fitted values from the jump model are plotted in Figure 6, which clearly shows their comovement over the later sample. The fact that the b coefficient is positive indicates that, on average, when SPX net demand is higher (lower), excess SPX implied volatilities are also higher (lower). For the most successful model, the one based on jumps, changing the dependent variable from its lowest to its highest values over the late sub-sample would change the excess implied volatility by about 5.6 percentage points. A one-standard-deviation change in the jump variable

[17]The model-based explanatory variables work better than just adding all contracts (#*Contracts*), because they give higher weight to near-the-money options. If we just count contracts using a more narrow band of moneyness, then the #*Contracts* variable also becomes significant.

results in a one-half standard deviation change of excess implied volatility (the corresponding R^2 is 26%). The model is also successful in explaining a significant proportion of the level of the excess implied volatility. Over the late subsample, the average level is 4.9%, of which approximately one third – specifically, 1.7% – corresponds to the average level of demand, given the regression coefficient.

Further support for the hypothesis that the supply for options is upwardly sloping comes from the comparison between the estimated supply-curve slopes following market maker losses, respectively gains. If market maker risk aversion plays an important role in pricing options, then one would expect prices to be less sensitive to demand when market makers are well funded – in particular, following profitable periods – than not. This is exactly what we find. Breaking the daily sample[18] in two subsamples depending on whether the hedged market maker profits over the previous 20 trading days is positive or negative,[19] we estimate the regression (38) for each subsample and find that, following losses, the b coefficient is approximately twice as large as the coefficient obtained in the other subsample. For instance, in the jump model, the regression coefficient following losses is 2.6E-05 with a t-statistic 3.7 (330 observations), compared to a value of 1.1E-5 with t-statistic 6.1 in the complementary subsample (646 observations).

Equity Options:

We consider next the time-series relationship between demand and expensiveness for equity options. In particular, we run the time-series regression (38) for each stock, and average the coefficients across stocks. The results are shown in Table 3. We consider separately the subsample before and after the summer of 1999. This is because most options were listed only on one exchange before the summer of 1999, but many were listed on multiple exchanges after this summer. Hence, there was potentially a larger total capacity for risk taking by market makers after the cross listing. See for instance De Fontnouvelle, Fishe, and Harris (2003) for a detailed discussion of this well-known structural break.

We run the time-series regression separately for the 303 underlying stocks with strictly positive option volume on at least 80% of the trade days from the beginning of 1996 through the end of 2001. We compute the excess implied volatility as the average implied volatility of selected options,[20] minus the GARCH(1,1) volatility, and net demand as the total net non-market-maker demand for options with moneyness between 0.8 and 1.2 and maturity between 10 and 180 calendar days. We run the regression using monthly data on underlying stocks that have at least 12 months of

[18]Because of the structural changes discussed above, we restrict our attention to the period starting on October 1, 1997.

[19]Similar results obtain if the breaking point is the mean or median daily profit.

[20]That is, options with moneyness between 0.95 and 1.05, maturity between 15 and 45 calendar days, at least 5 contracts of trading volume, and implied volatilities available on OptionMetrics.

Table 3: The relationship between option expensiveness — i.e. implied volatility minus GARCH volatility — and non-market-maker net demand for equity options on 303 different underlying stock (Equation 38). We run time-series regressions for each underlying and report the average coefficients. The number p-val is the p-value of the binomial test that the coefficients are equally likely to be positive and negative.

	Before Cross-Listing of Options 04/1996–06/1999	After Cross-Listing of Options 10/1999–12/2001
Constant	-0.01	0.02
$NetDemand$	12E-6	8.5E-6
Average Adj. R^2	0.06	0.05
# positive	227	213
# negative	76	86
p-val	0.00	0.00

data available. (Daily regressions give stronger results.)

The average coefficient b measuring the effect of demand on expensiveness is positive and significant in both subsamples. This means that when the demand for equity options is larger their implied volatility is higher. The results are illustrated in Figure 7, which shows the expensiveness and fitted values of the demand effect on a monthly basis. The positive correlation is apparent. We note that the relation between average demand and average expensiveness is more striking if we do a single, pooled regression for these variables. It is comforting, however, that the relation also holds when we consider each stock separately.

Finally, we investigate the cross-sectional relationship between excess implied volatility and net demand in the equity option market. We multiply the net demand variable by the price volatility of the underlying stock (defined as the sample return volatility just described multiplied by the day's closing price of the stock.) We scale the net demand in this way, because market makers are likely to be more concerned about holding net demand in their inventory when the underlying stock's price volatility is greater.

We run the cross-sectional regression on each day and then employ the Fama-MacBeth method to compute point estimates and standard errors. We also use the Newey-West procedure to control for serial-correlation in the slope estimates. The slope coefficient is 5.9E-8 with a t-statistic of 6.44.

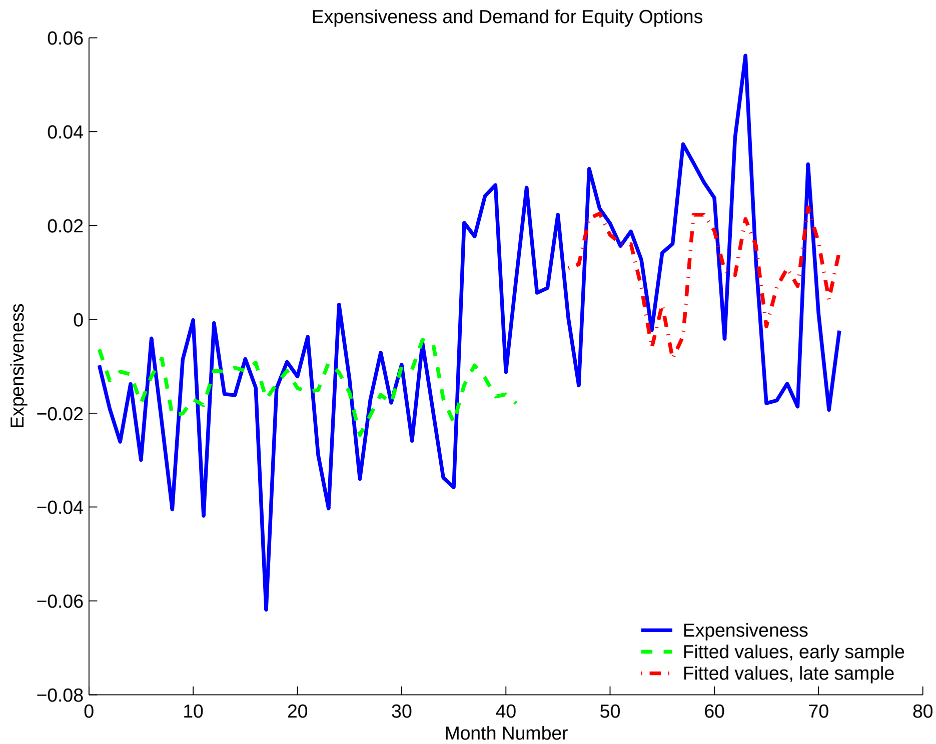

Figure 7: The solid line is the expensiveness of equity options, averaged across stocks. The dashed lines are, respectively, the fitted values of demand-based expensiveness before and after the cross-listing of options (1996/04–1999/05 and 1999/10–2001/12) using the average regression coefficients from stock-specific regressions and the average demand.

5.2 Implied Volatility Skew and Net Demand Skew

In order to investigate Proposition 4, we regress a measure of the steepness of the excess implied volatility skew on one of two demand-based explanatory variables:

$$ExcessImplVolSkew_t \quad = \quad a + b\, DemandVarSkew_t + \epsilon_t. \tag{39}$$

SPX:

For the SPX analysis, $ExcessImplVolSkew_t$ is the date-t implied volatility skew over and above the skew predicted by the jumps and stochastic volatility of the underlying index. The implied volatility skew is defined as the average implied volatility of options with moneyness between 0.93 and 0.95 that trade at least 25 contracts on trade date t and have more than 15 and fewer than 45 calendar days to expiration, minus the average implied volatility of options with moneyness between 0.99 and 1.01 that meet the same volume and maturity criteria. In order to eliminate the skew that is due

Table 4: The relationships between SPX implied-volatility skew and two potential explanatory variables: *(i)* the SPX non-market-maker demand skew and *(ii)* the jump-risk-model demand-based implied-volatility skew. T-statistics computed using Newey-West are in parentheses.

	Before Structural Changes 1996/01–1996/10		After Structural Changes 1997/10–2001/12	
Constant	0.032	0.04	0.04	0.032
	(57)	(6.4)	(20)	(13.2)
#Contracts	-1.2E-7		5.4E-7	
	(-0.9)		(3.4)	
Jump Risk		-3.0E-6		1.8E-5
		(-1.0)		(3.0)
Adj. R^2	0.08	0.14	0.22	0.28
N	10	10	50	50

to jumps and stochastic volatility of the underlying, we consider the implied volatility skew net of the similarly defined volatility skew implied by the objective distribution of Broadie, Chernov, and Johannes (2005) where the underlying volatility is that filtered from the Bates (2005) model.[21]

As explanatory variable, we use two measures. The first is the skew in net option demand, defined as the net non-market-maker demand for options with moneyness between 0.93 and 0.95 minus the net non-market-maker demand for options with moneyness between 0.99 and 1.01, using options with maturity between 10 and 180 calendar days. The second measure is the excess implied-volatility skew from the model with underlying jumps described in Section 3.2. (We do not consider the models with discrete trading and stochastic volatility since they do not have first-order skew implications as described in Sections 3.1 and 3.3.)

Table 4 reports the monthly OLS estimates of the skewness regression and Figure 8 illustrates the effects. As discussed in Section 5.1, we divide the sample into two subsamples because of structural changes. The slope coefficient is significantly positive in the late subsample using both the simple demand variable and the variable based on jump-risk.[22] Using the jump-risk model, variation between the minimum and maximum levels of the independent variable results in a skew change of about 3 percentage

[21]The model-implied skew is evaluated for one-month options with moneyness of, respectively, 0.94 and 1. We thank Mikhail Chernov for providing this time series.

[22]The slope coefficient is also significant over the full sample; the demand skewness is less non-stationary than the level of demand over the full sample.

points. Further, a one standard deviation move in the independent variable results in a change in the dependent variable of 0.53 standard deviations.

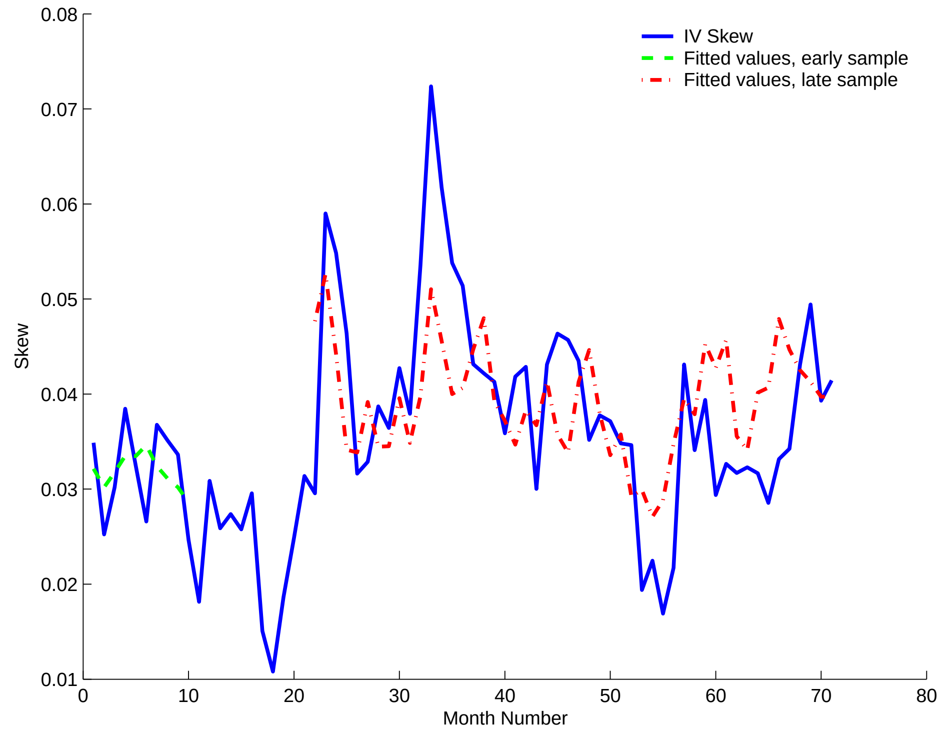

Figure 8: The solid line shows the implied volatility skew for SPX options. The dashed lines are, respectively, the fitted values from the skew in demand before and after certain structural changes (1996/01–1996/10 and 1997/10–2001/12).

Equity Options:

We consider next the time-series relationship for equity options between skew in demand and skew in implied volatility. In particular, we run the time-series regression (39) for each stock, and average the coefficients across stocks.[23] Once again, we consider separately the subsample before and after the summer of 1999, because of the widespread move toward cross-listing in the summer of 1999.

The results are shown in Table 5. The time-series regression is run separately for the same 303 underlying stocks as above. We compute the implied volatility skew as the average implied volatility of selected low moneyness minus that of near-the-money

[23]Now the dependent variable is simply the time t implied volatility from low moneyness minus near-the-money options.

Table 5: The relationship between implied volatility skew (i.e., implied volatility of low moneyness minus near-the-money options) and non-market-maker net demand skew (i.e., non-market-maker net demand for low moneyness minus high moneyness options) on 303 different underlying stocks (Equation 39). We run time-series regressions for each underlying and report the average coefficients. The number p-val is the p-value from the binomial test that the coefficients are equally likely to be positive and negative.

	Before Cross-Listing of Options 04/1996–06/1999	After Cross-Listing of Options 10/1999–12/2001
Constant	0.03	0.03
NetDemand	4.0E-6	4.2E-6
Average Adj. R^2	0.03	0.08
# positive	200	163
# negative	70	61
p-val	0.00	0.00

options.[24] The skew in demand is the total net non-market-maker demand for options with moneyness between 0.80 and 1.00 minus that for options with moneyness between 1.00 and 1.20.[25] We run the regression using monthly data on underlying stocks that have at least 12 months of data available.

The average coefficient b measuring the effect of skew in demand on skew in expensiveness is positive and significant in both subsamples. This results implies that when the demand for equity options is more skewed their implied volatility is more skewed.

6 Conclusion

Relative to the Black-Scholes-Merton benchmark, index and equity options display a number of robust pricing anomalies. A great deal of research has attempted to address these anomalies, in large part by generalizing the Black-Scholes-Merton assumptions about the dynamics of the underlying asset. While these efforts have met with some success, non-trivial pricing puzzles remain. Further, it is not clear that this approach can yield a satisfactory description of option prices. For example, index and equity op-

[24]That is, the average implied volatility from options with moneyness between 0.85 and 0.95 minus the average implied volatility from option with moneyness between 0.97 and 1.03. These options must have maturity between 15 and 45 calendar days, at least 1 contract of trading volume, and implied volatilities available on OptionMetrics.

[25]Options with maturity of 10 to 180 calendar day are included. The results are robust to variation in the definition in these demand categories.

tion prices display very different properties, although the dynamics of their underlying assets are quite similar.

This paper takes a different approach to option pricing. We recognize that in contrast to the Black-Scholes-Merton framework, in the real world options cannot be perfectly hedged. Consequently, if intermediaries such as market makers and proprietary traders who take the other side of end-user option demand are risk-averse, end-user demand for options will impact option prices.

The theoretical part of the paper develops a model of competitive risk-averse intermediaries who cannot perfectly hedge their option positions. We compute equilibrium prices as a function of net end-user demand and show that demand for an option increases its price by an amount proportional to the variance of the unhedgeable part of the option and that it changes the prices of other options on the same underlying asset by an amount proportional to the covariance of their unhedgeable parts.

The empirical part of the paper measures the expensiveness of an option as its Black-Scholes implied volatility minus a proxy for the expected volatility over the life of the option. We show that on average index options are quite expensive by this measure, and that they have high positive end-user demand. Equity options, on the other hand, are on average slightly inexpensive and have a small negative end-user demand. In accordance with the predictions of our theory, we find that options are overall more expensive when there is more end-user demand for options and that the expensiveness skew across moneyness is positively related to skew in end-user demand across moneyness. Finally, demand effects are stronger following recent market maker losses compared to times of recent market maker gains.

A Proofs

Proof of Lemma 1:

Note first that the boundedness of all the random variables considered (with the exception of S) ensures that all expectations are finite.

The Bellman equation is

$$\begin{aligned} J(W_t; t, X_t) &= -\frac{1}{k} e^{-k(W_t + G_t(d_t, X_t))} \\ &= \max_{C_t, q_t, \theta_t} \left\{ -\frac{1}{\gamma} e^{-\gamma C_t} + \rho \mathrm{E}_t \left[J(W_{t+1}; t+1, X_{t+1}) \right] \right\} \end{aligned} \tag{40}$$

Given the strict concavity of the utility function, the maximum is characterized by the first-order conditions (FOC's). Using the proposed functional form for the value function, the FOC for C_t is

$$0 = e^{-\gamma C_t} + k R_f \rho \mathrm{E}_t \left[J(W_{t+1}; t+1, X_{t+1}) \right] \tag{41}$$

which together with (40) yields

$$0 = e^{-\gamma C_t} + k R_f \left[J(W_t; t, X_t) + \frac{1}{\gamma} e^{-\gamma C_t} \right] \tag{42}$$

that is,

$$e^{-\gamma C_t} = e^{-k(W_t + G_t(d_t, X_t))} \tag{43}$$

implying (4). The FOC's for θ_t and q_t are (5) and (6). We derive G recursively as follows. First, we let $G(t+1, \cdot)$ be given. Then, θ_t and q_t are given as the unique solutions to Equations (5) and (6). Clearly, θ_t and q_t do not depend on the wealth W_t. Further, (42) implies that

$$0 = e^{-\gamma C_t} - R_f \rho \mathrm{E}_t \left[e^{-k\left(y_{t+1} + (W_t - C_t) R_f + q_t (p_{t+1} - R_f p_t) + \theta_t R^e_{t+1} + G_{t+1}(d_{t+1}, X_{t+1})\right)} \right], \tag{44}$$

that is,

$$e^{-\gamma C_t - krC_t + krW_t} = R_f \rho \mathrm{E}_t \left[e^{-k\left(y_{t+1} + q_t (p_{t+1} - R_f p_t) + \theta_t R^e_{t+1} + G_{t+1}(d_{t+1}, X_{t+1})\right)} \right], \tag{45}$$

which, using (4), yields the equation that defines $G_t(d_t, X_t)$ (since X_t is Markov):

$$e^{-krG_t(d_t, X_t)} = R_f \rho \mathrm{E}_t \left[e^{-k\left(y_{t+1} + q_t (p_{t+1} - R_f p_t) + \theta_t R^e_{t+1} + G_{t+1}(d_{t+1}, X_{t+1})\right)} \right] \tag{46}$$

At $t = \overline{T}$, we want to show the existence of a stationary solution. First note that the operator A defined by

$$AF(w;x) \;=\; \max_{C,\theta}\left\{-\frac{1}{\gamma}e^{-\gamma C} + \rho \mathrm{E}_t\left[F(W_{t+1}, X_{t+1})\right| W_t = w, X_t = x]\right\}$$

subject to

$$W_{t+1} \;=\; y_{t+1} + (W_t - C)R_f + \theta R^e_{t+1},$$

satisfies the conditions of Blackwell's Theorem,[26] and is therefore a contraction.

Furthermore, A maps any function of the type

$$F(w;x) \;=\; -\frac{1}{k}e^{-kw}g(x)$$

into a function of the same type, implying that the restriction of A to g, denoted also by A, is a contraction as well.

We now show that there exists $m > 0$ such that A maps the set

$$G^m = \{g : \mathbb{X} \to \mathbb{R} \,:\, g \text{ is continuous, } g \geq m\}$$

into itself.

Continuity holds by assumption (the Feller property). Let us look for $m > 0$. Since

$$\begin{aligned} Ag &\geq \inf_x \inf_\theta R_f \rho \mathrm{E}\left[e^{-k\left(y_{t+1}+\theta R^e_{t+1}\right)} g(X_{t+1})^{\frac{1}{R_f}} \,|\, X_t = x\right] \\ &\geq \inf_x \inf_\theta R_f \rho \mathrm{E}\left[e^{-k\left(y_{t+1}+\theta R^e_{t+1}\right)} \,|\, X_t = x\right] \left(\min_z g(z)\right)^{\frac{1}{R_f}} \\ &\geq B\left(\min_z g(z)\right)^{\frac{1}{R_f}} \end{aligned}$$

for a constant $B > 0$ (the inner infimum is a strictly positive, continuous function of $x \in \mathbb{X}$ compact), showing that the assertion for any m not bigger than $B^{\frac{R_f}{R_f - 1}}$.

Since G^m is complete, we conclude that A has a (unique) fixed point in G^m (which, therefore, is not 0).

It remains to prove that, given our candidate consumption and investment policy,

$$\lim_{t\to\infty} \mathrm{E}\left[\rho^{-t} e^{-kW_t}\right] = 0.$$

Start by noting that, for $t > T$,

$$W_{t+1} = W_t - (R_f - 1)G(X_t) + y_{t+1} + \theta_t R^e_{t+1},$$

[26]See, for instance, Stokey, Lucas, and Prescott (1989).

implying, by a repeated application of the iterated expectations, that

$$\mathrm{E}_T\left[e^{-k(W_t+G(X_t))}\right] \;=\; e^{-k(W_T+G(X_T))},$$

which is bounded. Since $G(X_t)$ is bounded, it follows that $\lim_{t\to\infty}\mathrm{E}\left[\rho^{-t}e^{-kW_t}\right]=0$.

The verification argument is standard, and particularly easy in this case given the boundedness of g.

□

Proof of Proposition 1:

Given a position process from date t onwards and a price process from date $t+1$ onward, the price at time t is determined by (6). It is immediate that p_t is measurable with respect to time-t information.

□

Proof of Proposition 2:

Part *(i)* is immediate, since prices are linear. Part *(ii)* follows because, for any $a\in\mathbb{R}$, the pricing kernel is kept exactly the same by the offsetting change in (q,θ).

□

Proof of Proposition 3:

Part a) is immediately since a variance is always positive. The proof of b) is based on the following result.

Lemma 2 *Given h_1 and h_2 convex functions on $\mathbb{R}$, $\forall\beta<0,\alpha,\gamma\in\mathbb{R}$, $\exists\,\alpha',\gamma'\in\mathbb{R}$ such that*

$$|h_1(x)-\alpha' x-\gamma'|\le|h_1(x)-\alpha x-\beta h_2(x)-\gamma|$$

$\forall x\in\mathbb{R}$. Consequently, under any distribution, regressing h_1 on h_2 and the identity function results in a positive coefficient on h_2.

Letting $\tilde{p}_{t+1}=p_{t+1}-\mathrm{E}^d_t\left[p_{t+1}\right]$ and suppressing subscripts, consider the expression

$$\Psi \;=\; \mathrm{E}^d\left[\tilde{p}^i\tilde{p}^j\right]\mathrm{Var}(R^e)-\mathrm{E}^d\left[\tilde{p}^iR^e\right]\mathrm{E}^d\left[\tilde{p}^jR^e\right],$$

which we want to show to be positive. Letting $\hat{p}^i=\mathrm{E}^d[\tilde{p}^i|\,S]$ and $\hat{p}^j=\mathrm{E}^d[\tilde{p}^j|\,S]$, we write

$$\begin{aligned}\Psi &= \mathrm{E}^d\left[\mathrm{Cov}\left(\tilde{p}^i,\tilde{p}^j|\,S\right)\mathrm{Var}(R^e)+\hat{p}^i\hat{p}^j\mathrm{Var}(R^e)-\mathrm{E}^d\left[\hat{p}^iR^e\right]\mathrm{E}^d\left[\hat{p}^jR^e\right]\right]\\ &= \mathrm{E}^d\left[\mathrm{Cov}\left(\tilde{p}^i,\tilde{p}^j|\,S\right)\mathrm{Var}(R^e)\right]+\mathrm{E}^d\left[\hat{p}^i\hat{p}^j\mathrm{Var}(R^e)-\mathrm{E}^d\left[\hat{p}^iR^e\right]\mathrm{E}^d\left[\hat{p}^jR^e\right]\right].\end{aligned}$$

The first term is positive by assumption, while the second is positive because $\hat{p}^i$ and $\hat{p}^j$ are convex and then using Lemma 2 .

□

Proof of Lemma 2: We consider three cases: *(i)* the intersection of the graphs of h_1 and $g \equiv \gamma + \alpha Id + \beta h_2$ is empty; *(ii)* their intersection is a singleton; *(iii)* their intersection contains more than one point.

(i) Since the graphs do not intersect, there exists a hyperplane that separates the convex sets $\{(x, y) : h_1(x) \leq y\}$ and $\{(x, y) : g(x) \geq y\}$.

(ii) The same is true if the two graphs have a tangency point. If they intersect in one point, $\hat{x}$, and are not tangent, then assume that, for $x < \hat{x}$, $h_1(x) > g(x)$ (a similar argument settles the complementary case). The convex set generated by $\{(\hat{x}, h_1(\hat{x})\} \vee \{(x, (h_1(x)) : x < \hat{x}\} \vee \{(x, g(x)) : x > \hat{x})\}$ and the one generated by $\{(\hat{x}, h_1(\hat{x})\} \vee \{(x, (h_1(x)) : x > \hat{x}\} \vee \{(x, g(x)) : x < \hat{x})\}$ have only one point in common, and therefore can be separated with a hyperplane.

(iii) Consider the line generated by the intersection. If there existed a point x at which the ordinate of the line was higher than $h_1(x)$ and than $g(x)$, then it would follow that at least one of the intersection points is actually interior to $\{(x, y) : h_1(x) \leq y\}$, which would be a contradiction. Similarly if the line was too low.

□

Proof of Theorem 3:

We compute the sensitivity of current prices to a deviation in future positions from 0 in the direction of demand $\tilde{d}_s = \epsilon_s d_s$ at time s by differentiating with respect to $\epsilon_s = \epsilon$ (evaluated at $\epsilon = 0$). We then aggregate the demands at all times to compute the total effect:

$$\frac{\partial p_t}{\partial \epsilon} = \sum_{s \geq 0} \frac{\partial p_t}{\partial \epsilon_s} \frac{\partial \epsilon_s}{\partial \epsilon} = \sum_{s \geq 0} \frac{\partial p_t}{\partial \epsilon_s}$$

To compute the price effect of expected demand at any time s, we note that it follows from the dealer's problem that

$$p_t = \mathrm{E}_t \left[\rho^{s-t} e^{-\gamma(C_s - C_t)} p_s \right]$$

which implies

$$\begin{aligned} \frac{\partial p_t}{\partial \epsilon_s} &= \mathrm{E}_t \left[\rho^{s-t} e^{-\gamma(C_s - C_t)} \frac{\partial p_s}{\partial \epsilon_s} \right] \\ &= R_f^{-(s-t)} \mathrm{E}_t^* \left[\frac{\partial p_s}{\partial \epsilon_s} \right] \\ &= R_f^{-(s-t)} \mathrm{E}_t^* \left[\frac{\partial p_s}{\partial \tilde{q}_s^j} \, q_s^j \right]. \end{aligned}$$

where we use that $\frac{\partial C_t}{\partial \epsilon_s} = \frac{\partial C_s}{\partial \epsilon_s} = 0$ at $q = 0$. The equality $\frac{\partial C_s}{\partial \epsilon_s} = 0$ follows from

$$\frac{\partial C_s}{\partial q_s^j} = \frac{k}{\gamma} \frac{\partial G(s, X_s; q)}{\partial q_s^j} = -\frac{k^2 R_f \rho}{\gamma} \mathrm{E}_s^* \left[p_{s+1}^j - R_f p_s^j + \frac{\partial \theta_s}{\partial q_s^j} R_{s+1}^e \right] = 0 \qquad (47)$$

and the other equality follows from differentiating the condition that marginal rates of substitution are equal

$$e^{-\gamma C_t} = e^{-\rho(s-t)} \mathrm{E}_t \left[e^{-\gamma C_s} \right],$$

which gives that

$$e^{-\gamma C_t} \frac{\partial C_t}{\partial \epsilon_s} = e^{-\rho(s-t)} \mathrm{E}_u \left[e^{-\gamma C_s} \frac{\partial C_s}{\partial \epsilon_s} \right] = 0$$

It remains to show that the price is a smooth (C^∞) function of ϵ. Consider a process d_t characterized by $d_t = 0$ for all $t > T$, and let the demand process be given by $\tilde{d}_t = \epsilon_t d_t = -q_t$. At time t, the following optimality conditions must hold:

$$e^{-\gamma C_t} = (R_f \rho)^{T+1-t} \, \mathrm{E}_t \left[e^{-k(W_{T+1} + G_{T+1})} \right] \qquad (48)$$

$$0 = \mathrm{E}_t \left[e^{-k(W_{T+1} + G_{T+1})} R_{t+1}^e \right] \qquad (49)$$

$$0 = \mathrm{E}_t \left[e^{-k(W_{T+1} + G_{T+1})} \left(p_{T+1} - R_f^{T+1-t} p_t \right) \right], \qquad (50)$$

with

$$W_{T+1} = (W_t - C_t) R_f^{T+1-t} + \sum_{s=t}^{T} \left(y_{s+1} - C_{s+1} + \theta_s R_{s+1}^e - \epsilon_s d_s \left(p_{s+1} - R_f p_s \right) \right) R_f^{T+1-s} + y_{T+1}.$$

We use the notation p_{T+1} for the time $T+1$-money value of the payoff of options expired by $T + 1$.

We show by induction that, given X_s, (p_s, θ_s, C_s) is a smooth function of $(\epsilon_s, \ldots, \epsilon_{s+1})$. Note that the statement holds trivially for $s > T$.

Assume therefore the statement for all $s > t$. There are $n_t + 2$ equations in (48)–(50), with n_{t+1} being the number of derivatives priced at time t. Note that the equations do not depend on ϵ_s for $s < t$ and that they are smooth in ϵ_s for all s , as well as in θ_t, C_t, and p_t. In order to prove the claim, we have to show that the derivative of the functions giving (48)–(50) with respect to (p_t, θ_t, C_t) is invertible (at $\epsilon = 0$ suffices), i.e., it has non-zero determinant. The implicit function theorem, then, proves the induction statement for $s = t$.

The non-zero determinant is shown as follows. If we let F_C, F_θ, respectively F_q denote the functions implicit in equations (48)–(50), it follows easily that

$$\begin{aligned} D_C F_C &\neq 0 \\ D_C F_\theta &= 0 \\ D_C F_q &= 0 \\ D_\theta F_\theta &\neq 0 \\ D_p F_\theta &= 0 \\ \det(D_p F_q) &\neq 0, \end{aligned}$$

implying that $D_{(C_t,\theta_t,p_t)}F$ has non-zero determinant. □

Proof of Proposition 4:

Consider an optimally hedged short put position with strike price $K < R_f S_t$. With $x = S_{t+1} - R_f S_t$, the payoff from this position is

$$\Pi(x) \;=\; -d(K - R_f S_t - x)^+ + \theta x.$$

The optimality of the hedge means that, under the risk-neutral measure,

$$\mathrm{E}^d\left[e^{-k\Pi(x)}x\right] \;=\; 0.$$

Note that, since $K < R_f S_t$, $\Pi(x) < 0$ for $x > 0$ and $\Pi(x) > 0$ for $K - R_f S_t < x < 0$. Consequently, given the symmetry of x around 0 and the zero-expectation condition above, with ξ denoting the density of x,

$$\int_K^\infty \left(e^{-k\Pi(x)}x - e^{-k\Pi(-x)}x\right)\xi(x)\,dx \;=\; -\int_0^K \left(e^{-k\Pi(x)}x - e^{-k\Pi(-x)}x\right)\xi(x)\,dx \;<\; 0.$$

It immediately follows that it cannot be true that $\Pi(-x) \geq \Pi(x)$ for all $x > |K - R_f S_t|$. In other words, for some value $x > |K - R_f S_t|$, $\Pi(-x) < \Pi(x)$, which then gives $d + \theta > -\theta$, or $|\theta| < \frac{1}{2}|d|$: the payoff is more sensitive to large downward movements in the underlying than to large upward movements. Thus, there exists $\bar{K}$ such that, for all $S_{t+1} < \bar{K}$,

$$\Pi(S_{t+1} - R_f S_t) < \Pi(-(S_{t+1} - R_f S_t)),$$

implying that, whenever $K' < \bar{K}$ and $K'' = 2R_f S_t - K'$,

$$\begin{aligned} p(p, K', d) &> p(c, K'', d) \\ p(p, K', 0) &= p(c, K'', 0), \end{aligned}$$

the second relation being the result of symmetry.

$\square$

Arbitrary Number of Time Units Δ_t between Hedging Dates:

Most of our analysis relies on the assumption that there is 1 time unit per period (which simplifies notation). For the results on frequent hedging (i.e. $\Delta_t \to 0$) in Propositions 5-7, it is useful to see how to adapt the results for an arbitrary Δ_t.

For simplicity, let the stock price be $S = X^1$, with no dividends, and let X follow a jump-diffusion with time-independent coefficient functions:

$$dX_t = (\mu(X_t)\,dt + v(X_t)\,dB_t + \eta(X_t)dN_t)\,, \tag{51}$$

where N_t is a counting process with arrival intensity $\lambda(X_t)$ and where the jumps size is drawn from a uniformly bounded distribution also depending on X_t.

Let Δ_t be arbitrary, provided that all derivatives mature at times that are integral multiples of Δ_t. The dealer's problem is to maximize

$$U(C_t, C_{t+\Delta_t}, \ldots) = \mathrm{E}_t\left[\sum_{l=0}^{\infty} \rho^{l\Delta_t} u\left(C_{t+l\Delta_t}\right)\Delta_t\right],$$

with $u(c) = -\frac{1}{\gamma}e^{-\gamma c}$. At any time t, the dealer must choose the consumption level C_t, the dollar investment θ_t in the underlying, and the number of derivatives held $q_t = (q_t^i)_{i\in I_t}$, while satisfying the transversality condition $\lim_{t\to\infty}\mathrm{E}\left[\rho^{-t}e^{-kW_t}\right] = 0$, where the dealer's wealth evolves as

$$W_{t+\Delta_t} = (Y_{t+\Delta_t} - Y_t) + (W_t - C_t\Delta_t)R_f + q_t(p_{t+\Delta_t} - rp_t) + \theta_t R^e_{t+\Delta_t}. \tag{52}$$

Here, $Y_t = \int_0^t y_s ds$ represents the cumulative endowment, and C is the annualized consumption.

The results of Lemma 1 become

$$J(W_t; t, X_t) = -\frac{1}{k}e^{-k(W_t + G_t(d_t, X_t))} \tag{53}$$

$$C_t = \frac{k}{\gamma}\left(W_t + G_t(d_t, X_t)\right), \tag{54}$$

with $k = \frac{R_f - 1}{R_f\Delta_t}\gamma = \frac{e^{r\Delta_t}-1}{e^{r\Delta_t}\Delta_t}\gamma$. The stock and derivative holdings are characterized by the first-order conditions

$$0 = \mathrm{E}_t\left[e^{-k\left(Y_{t+\Delta_t} + \theta_t R^e_{t+\Delta_t} + q_t(p_{t+\Delta_t} - rp_t) + G_{t+\Delta_t}(d_{t+\Delta_t}, X_{t+\Delta_t})\right)} R^e_{t+\Delta_t}\right] \tag{55}$$

$$0 = \mathrm{E}_t\left[e^{-k\left(y_{t+\Delta_t} + \theta_t R^e_{t+\Delta_t} + q_t(p_{t+\Delta_t} - rp_t) + G_{t+\Delta_t}(d_{t+\Delta_t}, X_{t+\Delta_t})\right)}\left(p_{t+\Delta_t} - rp_t\right)\right], \tag{56}$$

where, for $t \leq T$, the function $G_t(d_t, X_t)$ is derived recursively using (55), (56), and

$$e^{-kR_f G_t(d_t, X_t)} = R_f\rho\mathrm{E}_t\left[e^{-k\left(y_{t+\Delta_t} + q_t(p_{t+\Delta_t} - rp_t) + \theta_t R^e_{t+\Delta_t} + G_{t+\Delta_t}(d_{t+\Delta_t}, X_{t+\Delta_t})\right)}\right], \tag{57}$$

and, for $t > T$, the function $G_t(d_t, X_t) = \bar{G}(X_t)$ where $(\bar{G}(X_t), \bar{\theta}(X_t))$ solves

$$e^{-kR_f\bar{G}(X_t)} = R_f\rho \mathrm{E}_t\left[e^{-k\left(y_{t+\Delta_t}+\bar{\theta}_t R^e_{t+\Delta_t}+\bar{G}(X_{t+\Delta_t})\right)}\right] \quad (58)$$

$$0 = \mathrm{E}_t\left[e^{-k\left(y_{t+\Delta_t}+\bar{\theta}_t R^e_{t+\Delta_t}+\bar{G}(X_{t+\Delta_t})\right)} R^e_{t+\Delta_t}\right]. \quad (59)$$

Replicating calculations in the body of the paper, with the obvious modification, the statement of Theorem 3 is generalized to

$$p_t = p_t(0) + \gamma\frac{R_f-1}{\Delta_t}\left[E^0_t\left(\bar{p}_{t+\Delta_t}\bar{p}'_{t+\Delta_t}\right)d_t + \sum_{l>0} e^{-rl\Delta_t}E^0_t\left(\bar{p}_{t+(l+1)\Delta_t}\bar{p}'_{t+(l+1)\Delta_t}d_{t+l\Delta_t}\right)\right]\epsilon + O(\epsilon^2).$$

Proofs of Propositions 5-7:

In order to prove Propositions 5-7, we proceed with a few technical preliminaries. We assume that under the risk-neutral pricing measure given by $d = 0$ the process X_t is a jump-diffusion of the same functional form as given by Equation (51). For zero demand, an option price is defined by

$$p(S_t, t; X^{(1)}_t) = e^{-r(T-t)}\mathrm{E}^0\left[(S_T - K)^+ \mid X_t\right],$$

regardless of the frequency of trading, where $X^{(1)} = (X^2, \ldots, X^n)$.

For the computation of the covariances necessary, we shall be relying on applications of the following result. For sufficiently smooth functions $g, h : \mathbb{R}^n \to \mathbb{R}$,

$$d(g(X_t,t)h(X_t,t)) = m^{gh}_t\,dt + dM^{gh}_t,$$

where M^{gh} is a martingale and

$$m^{gh}_t = \left(g_X\mu + g_t + \frac{1}{2}tr\left(g_{XX}vv^\top\right)\right)h + \left(h_X\mu + h_t + \frac{1}{2}tr\left(h_{XX}vv^\top\right)\right)g + g_X vv^\top h_X^\top - \lambda\left(\mathrm{E}^0_t[g(X_t+\eta)h(X_t+\eta)] - g(X_t)h(X_t)\right).$$

It follows that

$$\mathrm{E}^0_t\left[g(X_{t+\Delta_t}, t+\Delta_t)h(X_{t+\Delta_t}, t+\Delta_t)\right] = g(X_t,t)h(X_t,t) + m^{gh}_t\Delta_t + O(\Delta_t^2). \quad (60)$$

We now apply this approximation result to compute the unhedgeable price covariances. We start with the hedge ratio θ, which requires estimating

$$\begin{aligned}\mathrm{E}^0_t\left[p_{t+\Delta_t}R^e_{t+\Delta_t}\right] &= \frac{1}{S_t}\left(v^{1\cdot}v^\top p_X^\top + \lambda p\mathrm{E}^0_t[\eta] - \lambda\left(\mathrm{E}^0_t[p(X_t+\eta)\eta]\right)\right)\Delta_t + O(\Delta_t^2)\\ &= \frac{1}{S_t}\left(v^{1\cdot}v^\top p_X^\top - \lambda\left(\mathrm{E}^0_t\left[(p(X_t+\eta) - p(X_t))\,\eta\right]\right)\right)\Delta_t + O(\Delta_t^2)\end{aligned}$$

and

$$\mathrm{E}_t^0\left[\left(R^e_{t+\Delta_t}\right)^2\right] \;=\; \frac{1}{S_t^2}\left(v^{1\cdot}(v^{1\cdot})^\top - \lambda \mathrm{E}^0\left[\eta^2\right]\right)\Delta_t + O(\Delta_t^2),$$

where $v^{1\cdot}$ denotes the first row of the matrix v. This gives θ up to order $O(\Delta_t)$.

Finally, compute the un-hedged payoff covariance using θ and applying (60) with appropriate choices for g and h:

$$\mathrm{Cov}_t^0\left[\bar{p}^i_{t+\Delta_t}, \bar{p}^j_{t+\Delta_t}\right] \;=\; \left(m_t^{p^i p^j} - \theta_t^i m_t^{p^j R^e} - \theta_t^j m_t^{p^i R^e} + \theta_t^i \theta_t^j m_t^{(R^e)^2}\right)\Delta_t + O(\Delta_t^2),$$

where the only quantity not computed above is

$$m_t^{p^i p^j} \;=\; 2rp^i p^j + p^j_X v (p^i_X v)^\top - \lambda \mathrm{E}_t^0\left[\left(p^i(X_t+\eta) - p^i(X_t)\right)\left(p^j(X_t+\eta) - p^j(X_t)\right)\right]$$

Here we used the fact that, since $e^{-rs}p(X_s, s)$ is a martingale, the fundamental pricing PDE holds, i.e., the drift of $p(X_s, s)$ equals $rp(X_s, s)$.

We now specialize the model to three choices, corresponding to each of Sections 3.1–3.3:

Proof of Proposition 5:

In the first model $S = X^1$ is a Markov diffusion, i.e., the jump component is 0. We obtain immediately that $\theta_t = S_t p_S(S_t, t) + O(\Delta_t) = S_t f_S(S_t, t) + O(\Delta_t)$, and therefore $\mathrm{Cov}_t^0\left[\bar{p}^i_{t+\Delta_t}\bar{p}^j_{t+\Delta_t}\right] = 0$ up to terms in $O(\Delta_t^2)$.

To obtain a more nuanced answer we work directly with an exact third-order Taylor expansion of the function $p(S_{t+\Delta_t}, t+\Delta_t)$ around the point (S_t, t). It follows that

$$\begin{aligned}
&\mathrm{Cov}_t^0\left[\bar{p}^i_{t+\Delta_t}, \bar{p}^j_{t+\Delta_t}\right] \\
&= \mathrm{Cov}_t^0\left[\frac{1}{2} f^i_{SS}\Delta S^2 + \Delta S\, O(\Delta_t) + O(\Delta_t^2), \frac{1}{2} f^j_{SS}\Delta S^2 + \Delta S\, O(\Delta_t) + O(\Delta_t^2)\right] \\
&= \frac{1}{4} f^i_{SS} f^j_{SS} \mathrm{Var}_t^0\left[\Delta S^2\right] + O\left(\Delta_t^{\frac{5}{2}}\right) \\
&= \frac{1}{2} f^i_{SS} f^j_{SS} v^4 \Delta_t^2 + O\left(\Delta_t^{\frac{5}{2}}\right).
\end{aligned}$$

□

Proof of Proposition 6:

In this model, S is a Markov jump-diffusion. Let $\bar{\eta} = S^{-1}\eta$ be the relative jump size and $\sigma = S^{-1}v$ be the relative volatility, giving

$$\theta_t \;=\; \frac{\sigma^2 f_S S_t - \lambda \mathrm{E}_t^0[(p(S_t(1+\bar{\eta})) - p(S_t))\,\bar{\eta}]}{\sigma^2 - \lambda \mathrm{E}^0[\bar{\eta}^2]} + O(\Delta_t)$$

and

$$\mathrm{Cov}_t^0\left[\bar{p}^i_{t+\Delta_t},\bar{p}^j_{t+\Delta_t}\right] = \left((f^i_S S_t-\theta^i)(f^j_S S_t-\theta^j)\sigma^2+\lambda \mathrm{E}_t^0\left(\kappa^i\kappa^j\right)\right)\Delta_t+O(\Delta_t^2),$$

with $\kappa^i = f^i(S_t(1+\bar{\eta}))-f^i-\theta^i\bar{\eta}$.

□

Proof of Proposition 7:

Here we let $X=(S,\sigma)^\top$ with

$$\begin{aligned} dS_t &= rS_t\,dt+\sigma_t S_t\,dB_{1,t} \\ d\sigma_t &= \phi(\bar{\sigma}-\sigma_t)\,dt+\sigma_\sigma(S_t,\sigma_t)\,dB_{2,t}. \end{aligned}$$

Since the Brownian motion driving the volatility, $B_{2,t}$ is independent from the one driving the underlying return, $B_{1,t}$, the hedge ratio is the same as in the 1-dimensional Markov-diffusion case,

$$\theta_t = f_S S_t+O(\Delta_t),$$

and

$$\mathrm{Cov}_t^0\left[\bar{p}^i_{t+\Delta_t},\bar{p}^j_{t+\Delta_t}\right] = f^i_\sigma f^j_\sigma\sigma^2_\sigma\Delta_t+O(\Delta_t^2),$$

where f_σ is the derivative of f with respect to the second argument. □

Empirical Implementation:

For the empirical implementation of the model, we do as follows. We weight the demand of the included options according to the model-implied covariances of unhedgeable parts, using the models of Sections 3.1–3.3. Specifically, for the diffusive risk we compute an option's Black-Scholes-Merton gamma (i.e. the second derivative with respect to the price of the underlying) evaluated at the option's moneyness and maturity, 20% volatility, $S=1$, an interest rate of $r=5\%$, dividend rate $q=1.5\%$, $\Delta_t=0.01$. The results are robust the choices of these parameters. We keep S, r, q, and σ constant throughout the sample to avoid biases arising from changes of these variables. Hence, changes in the model-implied demand effect is solely due to changes in demand patterns.

Similarly, we compute the demand effect with stochastic volatility risk by computing the Black-Scholes-Merton vega (i.e. the derivative with respect to the volatility) evaluated at the same parameters as above and adjusting for maturity. We use the maturity adjustment described in Equation 37, where the volatility mean-reversion parameter ϕ is set to 6 based on the estimate of Pan (2002).

We compute the covariance jump risk $E(\kappa^i\kappa^j)$ by computing for each option its unhedged profit/loss in the case of equally likely jumps of 5% and -20%, jump arrival rate of 5% per year, and the parameters above. For this, we compute a jump-adjusted option price function f without demand effects and without jump risk premia using a straightforward adaptation of Merton (1976) We use the delta hedging given by $\theta = f_S$. Finally, we compute the variables κ and their covariance using the definitions (32) and (34).

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