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The Neoclassical Theory of Investment in Speculative Markets

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Abstract

In this paper I investigate whether firms' physical investments should react to the speculative overpricing of their securities. I introduce investment subject to quadratic adjustment costs (along the lines of Abel and Eberly [1994]) in an infinite horizon continuous time model with short sale constraints and heterogeneous beliefs (along the lines of Scheinkman and Xiong [2003]). Under standard assumptions, I show that the neoclassical "q" theory of investment will continue to hold despite the presence of (endogenous) speculative mispricing in the stock market. Strikingly, the welfare implications of the theory will also continue to hold, despite the presence of a speculative bubble. I show how the model provides a new formalization of the notions of "short-termist" and "long-termist" investment policies and also how the behavior of investment can be used to disentangle rational and behavioral approaches to so-called asset pricing anomalies.

JEL Codes: E2, G1

Keywords: Investment, q-theory, speculation, short sales constraints, heterogenous beliefs, bubbles, mis-pricing

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1 Introduction

> ...certain classes of investment are governed by the average expectation of those who deal on the Stock Exchange as revealed in the price of shares, rather than by the genuine expectations of the professional entrepreneur (J.M.Keynes, The General Theory, 1936, p.151)
>
> Further, it is difficult to know how firms in making investment, and financing decisions, should react to changes in the market's valuation of risk which reflect speculative movements..... (J. Tobin and W. Brainard, Private Values and Public Policy, Essays in Honor of William Fellner, 1977, p. 248)

Standard neoclassical theory predicts that investment is inherently tied with the stock market through Tobin's "q". The essence of "q" theory is the following argument: If the repurchase cost of capital is less than the net present value of additional profits it will bring at the margin, the company should then invest and vice versa. The only reason preventing the ratio of the two values (known as q) from always being equal to 1 are adjustment costs[1]: It is expensive to install new capital and thus a deviation of q from 1 can exist, but it should diminish over time. The link between investment and the stock market follows: The value of a company is the net present value of its profits and thus whenever one sees the stock market rising, one should simultaneously observe an increase in investment in order to bring the numerator and the denominator of the "q" ratio in line.

There is however a concern with this line of reasoning. Namely, what happens if, at times, the stock market valuation does not reflect the net present value of profits but instead contains terms that are unrelated to "fundamentals"? Will the "q" theory continue to hold, or will firm's decision makers adopt an eclectic approach about which of the components of valuation they will pay attention to?

As the two quotations at the beginning of the text suggest, this question is probably one of the oldest questions in the neoclassical theory of investment. Economists and market participants are repeatedly reminded of the importance of this question every time that stock markets exhibit rapid increases in value. Typically, such movements raise concerns about the extent to which they are justified by fundamentals alone. Also apparent during most of these episodes is that business' physical investment is the main nexus connecting the real economy with the financial sector. During the recent boom and bust of the stock market, investment was the main macroeconomic aggregate that co-moved with the stock market. The behavior of investment in the US during the late twenties, as well as the behavior of the Japanese economy during the late eighties provide further sources of alarming empirical evidence in this direction[2].

[1] Under Hayashi's (1982) conditions.

[2] For empirical evidence during those episodes see Panageas (2005), Chirinko and Schaller (2001)

Whether investment will react only to fundamental sources, or whether it will also react to speculative overpricing in the stock market could have important policy implications. If investment "filters" out speculative components and only reacts to fundamental variations in the stock market, it is seemingly difficult to make an argument for activism by the monetary and/or fiscal authorities. Indeed, in most of the existing discussions, activism is justified on the basis of "misguided" physical investment.[3]

The aim of this paper is to answer the question of whether investment should react to both fundamentals and speculative components and study the implications of this issue. I start with an explicit reason for why assets can deviate from fundamentals, namely heterogenous beliefs along with a constraint on short sales. I then introduce investment subject to quadratic adjustment costs and study investors' holding horizons, optimal investment, and the resulting equilibrium prices in a unified dynamic framework.

More specifically, I use short sales constraints and heterogenous beliefs as two of the main building blocks of the model in order to derive positive deviations of prices from "fundamentals". It is intuitive that the presence of a short sale constraint can cause the price of an asset to deviate from its fundamental value if market participants do not have homogenous beliefs. Agents who believe that the current price is above the net present value of dividends, would have to go short in order to take advantage of what they perceive to be mis-pricing. However, they cannot do this because of the short sale constraint. Accordingly, for pricing purposes, these skeptical agents do not matter, and the price will only reflect the views of the most optimistic market participants.

Furthermore, if the most optimistic agent perceives that there might be an even more optimistic agent "down the road", then the equilibrium price will contain a speculative component. This component will discount the (purely speculative) gains from reselling shares to these more optimistic agents in the future.

This basic intuition was first expressed in a formal intertemporal model by Harrison and Kreps (1978). A number of papers extended the intuition into various directions. A partial listing includes Allen, Morris, and Postlewaite (1993), Morris (1996), Detemple and Murthy (1997), Hong and Stein (2002) and most recently Scheinkman and Xiong (2003). All these papers study an exchange setting without a role for investment.

On the other hand there are numerous papers that study investment, but do not allow for deviations from fundamentals. The formal neoclassical theory of investment in a dynamic setting with adjustment costs goes back at least to Brainard and Tobin (1968), Lucas and Prescott (1971), Hayashi (1982), and Abel (1983). A unified approach to this theory was provided in the seminal paper by Abel and Eberly (1994).

In this paper, I link these two strands of the literature into one unified model. In particular the framework presented in this paper nests Scheinkman and Xiong (2003) and Abel and Eberly (1994, 1997) as special cases. The most important advantage of providing this link is that all quantities of interest are endogenous to the model. In particular, the extent of overpricing, the holding horizons of the investors, and the optimal

[3] For example see Blanchard (2000), Dupor (2002), Dupor and Conley (2004).

investment strategies of the firm are all jointly determined. Hence the model can address a novel set of issues:

a) Will investment be affected by speculative overpricing, and under which assumptions? The answer proposed by the model is simple and intuitive. Assuming no frictions of any sort (i.e. transaction costs, shareholder-management conflicts etc.), *the neoclassical q theory will hold despite the presence of speculation*. Marginal q will contain (endogenously) a purely speculative component and as a result investment will be increased. However, the assumption of frictionless trading is sometimes implausible. For example, in many companies the controlling shareholders are likely to face larger costs of liquidating their positions compared to retail investors[4] and hence they have longer holding horizons. The model then predicts that the effect of speculative valuations on investment policy will be attenuated. A major advantage of the present framework is that horizons and resale premia are endogenous: The horizon of an investor will be determined as a function of the costs that she is facing in accessing markets, the extent of mispricing etc. As a result, a new insight of the analysis is that *short-termist and long-termist investment policies are not mutually exclusive alternatives*, but rather the *two extremes of a continuum of possibilities*, that depend on the ease of access to financial markets by controlling shareholders. This provides *a new formalization* of these notions as well as *a unified view* of neoclassical investment theory in speculative markets.

b) The second issue that I address is efficiency: Is the share price maximizing investment level efficient from a welfare perspective? This is a particularly important question for economic policy, since a negative answer would open the possibility for welfare enhancing intervention. Somewhat surprisingly, the answer of the present model is that *share price maximizing investment is efficient*, under standard assumptions. This is a striking result, because the price of the company and marginal q are above their "fundamental" values *under any agent's beliefs*.

c) The third issue that I address is: Which investments are likely to be most affected by speculation? The present model derives the speculative components in marginal q endogenously and makes the simple prediction that marginal q and investment will be highest *when the disagreement about the marginal product of capital of such investments is highest*. Hence the model provides a potential rationalization for the high sensitivity of younger companies' investment to the stock market, or the investment of companies that employ new and untested technologies ("high tech" companies).

d) A final issue concerns the importance of investment in disentangling rational vs. behavioral explanations of certain asset pricing "anomalies". There is a growing literature that argues theoretically and empirically that predictability in asset returns is compatible with rational variations in risk premia because investment co-moves with variations in expected returns. Implicitly, this co-movement is seen as evidence consistent with rational explanations of return predictability[5]. What I show in this paper is that behavioral

[4] Because of loss of private benefits, adverse price pressure on the stock, or capital gains taxes.

[5] See for example Lamont (2000) for an empirical investigation and Zhang (2004) for a theoretical exposition. Both papers

and rational theories *become observationally equivalent* in many respects if investment reacts perfectly to speculative overpricing. The present model can deliver both predictability and co-movement between investment and expected returns, along with numerous other asset pricing facts. On the negative side, this observational equivalence suggests caution in the interpretation of certain empirical findings. On the positive side however, it suggests one promising approach to disentangle rational and behavioral theories. Namely, to identify companies with investment policies determined by "long termist" investors, and *study whether investment and expected returns co-move for these companies.* As these companies' investment would "filter" out the speculative components, any evidence of co-movement between expected returns and investment in this subset would be consistent *only* with rational variation in expected returns.

As the model allows for tractable closed form solutions one can address also purely quantitative issues. I demonstrate in particular that investment significantly *amplifies* speculative overpricing, compared to a model without investment. The reason is that firms will react to speculative overpricing of their securities by increasing investment. This will make the so-called "growth options" or "rents to the adjustment technology" embedded in the price of the firm more valuable. Calibrated versions of the model, suggest that growth options[6] gain significant relative importance in the price. This could help explain the astoundingly large measures of growth options (like market-to-book or price-to-earnings ratios) that are commonly observed during speculative episodes.

Because of the closed form solutions it is also computationaly feasible to simulate artificially large datasets like CRSP or COMPUSTAT and determine the degree of belief heterogeneity that would be consistent with the predictability observed in empirical studies. Simply put, *I investigate whether a behavioral explanation of predictability, would or would not require strongly diverging beliefs by certain investors* . In order to account for the observed predictability in the data, I find that one needs to assume relatively strong belief divergence, but only for a small subset of all the companies present in the artificial (simulated) dataset.

To summarize, the present paper makes it possible to study two feedbacks jointly: from speculative valuations to investment and the reverse. Thus it makes it possible to analyze a richer set of economic questions than models that would contain only investment but no room for speculation (like Abel and Eberly [1994]) or speculative overpricing but no investment (like Scheinkman and Xiong [2003])

The paper is related to a number of strands in the literature. A number of papers have addressed similar issues, including: Fischer and Merton (1984), Morck, Shleifer, and Vishny (1990), Blanchard, Rhee, and Summers (1993), Stein (1996), Chirinko and Schaller (1996) and more recently Polk and Sapienza (2002)

point out that their results *are consistent with rational theories*, without making the claim that they can disentangle behavioral from rational theories. Other papers that use investment to account for predictability in a rational framework are Berk, Green and Naik (1999) and Gomes, Kogan and Zhang (2003) among others.

[6] Growth options are defined as the difference between the equilibrium price when investment is determined optimally and the equilibrium price when investment is set to 0 throughout.

and Gilchrist, Himmelberg, and Huberman (2002). A central theme of this literature is the importance of investor's horizons. However, the models in the existing literature do not allow for dynamic trading, dynamic market clearing and dynamic investment. By allowing for optimal dynamic trading and market clearing, the present paper makes the investor holding horizon *an endogenous quantity, unlike overlapping generations models (OLG) that assume finite horizons.* This is a crucial advantage if one is to speak *of the welfare properties of the model*, since all agents are present in the markets at all times (even if they may choose to not participate in certain instances). Furthermore, speculative premia *arise endogenously* as the result of dynamic trading, and do not need to be assumed exogenously. In brief, the present framework allows *a more natural formalization of the notion of short-termism versus long-termism* along with the possibility to quantify speculative components and analyze their efficiency properties. Additionally, since the model is intertemporal, it is possible to derive *an explicit relation* between speculation and neoclassical investment theory in the "marginal q" tradition of Brainard and Tobin (1965).

The paper is also related to literature in financial economics that uses insights from investment theory to address issues such as the predictability of returns, the role of book to market ratios, etc. A partial listing would include Jermann (1998), Cochrane (1991,1996), Naik (1994), Berk, Green, and Naik (1999), Lamont (2000). Berk, Green, and Naik (1999), Gomes, Kogan and Zhang (2003), and Zhang (2005) in particular show how a model with investment can account for some apparent irregularities in asset pricing as the power of the book-to-market ratio to predict returns[7]. Most of these models are perfectly rational. However, this paper presents a *quantifiable* behavioral model that can explain the same set of facts. Having a quantifiable behavioral model with the ability to explain certain features of asset pricing data *can help identify the observationally equivalent parts of the two theories* from the parts that *distinguish* them and thus guide empirical research. As already mentioned, a promising approach suggested by the theory developed in this paper is to study the co-movement between investment and expected returns for companies whose investment policies are influenced mostly by large "long-termist" shareholders.

The paper is also complementary to the strand in the macroeconomics literature that models bubbles in the framework of overlapping generations models. Tirole (1985) is the seminal paper in this literature, whereas Olivier (2000), Ventura (2003), Caballero, Farhi and Hammour (2005), are some recent contributions. This literature assumes short horizons, whereas in the present paper short horizons arise endogenously. Most importantly, the setup of this model allows for standard welfare definitions, since all agents are infinitely lived. This is in contrast with OLG models, which preclude conventional definitions of welfare since certain groups of agents never get a chance to trade. It is also noteworthy that in the standard OLG model bubbles can exist and be efficient, if and only if the economy is dynamically inefficient and thus bubbles *crowd out*

[7] This fact is documented in the cross section by Fama and French (1992,1995) and in the time series dimension by Kothari and Shanken (1997) among others.

investment. This is at odds with the data, since bubbles and investment booms seem to coincide. Recent models in the OLG tradition (Olivier [2000], Ventura [2003]) overcome this difficulty by introducing some other background inefficiency or externality that the bubble helps correct. Interestingly in these models it is sometimes efficient to create a bubble if none exists. Hence there is always room for some potential policy intervention. By contrast, the present framework suggests that bubbles will boost investment, without having to assume anything about dynamic inefficiency or other sources of externalities or constraints. Moreover, the level of investment chosen by a share price maximizing firm will be efficient *no matter whether speculative bubbles exist or not.* Therefore there appears to be no room for policy intervention. Finally, as a practical matter, in OLG type models it is somewhat hard to reconcile the fact that holding horizons conicide with biological lifespans of generations: In the real world bubbles are associated with typical holding horizons of a few days, not decades. However, it is true that the simpler setup of overlapping generations allows one to address a richer set of issues (related e.g. to savings and fiscal policy) that would be difficult in the present setup. In a sense, models with heterogenous beliefs and shorting constraints provide micro-foundations for the bubbles that arise in the OLG literature[8].

The outline of the paper is as follows: Section 2 contains the model setup. Section 3 contains a discussion of the basic equilibrium definitions and conditions of optimality along with the solution to the model. Section 4 contains a discussion of the model's implications for investment theory. Section 5 contains the implications of the model for asset prices and section 6 concludes.

2 The model

This section contains the building blocks of the model: a) A standard investment framework with quadratic adjustment costs along the lines of Abel (1983), Abel and Eberly (1994), (1997) and b) A model where valuations endogenously deviate from "long-run" fundamentals due to short selling constraints and heterogenous beliefs along the lines of Scheinkman and Xiong (2003). In this section, I introduce the basic concepts. The next section derives a closed form solution for equilibrium prices and investment.

2.1 Company Profits and Investment

There is a single company and the goal will be to determine its value as part of the (partial) equilibrium solution of the model. In line with Scheinkman and Xiong (2003), the company's cumulative earnings process (dD_t) is given by:

$$dD_t = K_t f_t dt + K_t \sigma_D dZ_t^D \qquad (1)$$

[8] Even though different in scope, the seminal paper by Woodford (1991) is similar in spirit in using borrowing constraints to arrive at results that are similar to OLG models, even though his agents are infinitely lived.

where K_t is the capital stock, f_t is the marginal product of capital, (which I will refer to as "productivity") and dZ_t^D is a standard one dimensional Brownian Motion. σ_D is a constant controlling the "noise" in earnings. The presence of such noise prevents market participants from precisely inferring f_t (since they can only observe K_t and dD_t). The variable f_t is *not* observable and evolves according to an Ornstein Uhlenbeck process as:

$$df_t = -\lambda(f_t - \overline{f})dt + \sigma\sqrt{\frac{f_t}{\overline{f}}}dZ_t^f \tag{2}$$

where $\lambda > 0$ is a mean reversion parameter, $\overline{f} > 0$ is a long-run productivity rate, σ is a constant controlling the volatility of the process and dZ_t^f is a second Brownian motion that is independent of dZ_t^D. For simplicity I will assume that the company is fully financed by equity and there is a finite number of company shares whose supply I normalize to 1.

The company can invest in physical capital at the rate i_t, while depreciation is given by δ. Accordingly, the evolution of the capital stock is given by:

$$dK_t = (-\delta K_t + i_t)\, dt$$

Investment is subject to quadratic adjustment costs, so that the cumulative company earnings net of investment costs are given by:

$$d\Pi_t = dD_t - \left(pi_t + \frac{\chi}{2}(i_t^2)\right) dt$$

where χ is a constant controlling the significance of adjustment costs and p is the cost of purchasing capital. It will be useful for later purposes to define

$$\widetilde{p} = p\left(\frac{r+\delta}{\overline{f}}\right)$$

The assumption of adjustment costs that are independent of K_t has the benefit of allowing tractable solutions, however it comes at the cost of breaking down the equivalence between average and marginal "q" [9]. Panageas (2005) demonstrates how to generalize to the case where marginal and average q coincide, without however being able to compute explicit closed form solutions. It is likely that one can add further realistic features (for example irreversibility, more general adjustment cost functions). However, one would have to give up on closed form solutions.

2.2 Agents and Signals

The information structure is very similar to Scheinkman and Xiong (2003). There are two continuums of risk neutral agents that I will call type A and type B agents. Risk neutrality is convenient both in terms of

[9] This assumption has been made by several authors in the literature. See Abel and Eberly (1994) and the references therein (especially footnote 19)

simplifying the calculations and abstracting from considerations related to spanning, etc. In addition to the earnings process (1) both agents observe two signals that I will denote as signal s^A and signal s^B. These signals evolve according to:

$$\begin{aligned} ds_t^A &= f_t dt + \sigma_s \phi dZ_t^f + \sigma_s \sqrt{1-\phi^2} dZ_t^A \\ ds_t^B &= f_t dt + \sigma_s dZ_t^B \end{aligned} \tag{3}$$

where $(dZ_t^A, dZ_t^B, dZ_t^f, dZ_t^D)$ are standard mutually orthogonal Brownian motions and $0 < \phi < 1$ is a parameter controlling the informativeness of signal s^A, while σ_s is a constant controlling the noise of the two signals. To see why ϕ controls the informativeness of signal s^A, note that ϕ is just the correlation between the observed ds_t^A and the unobserved df_t.[10]

Agents have heterogenous perceptions about the informativeness of the various signals. Agents in group A have the correct beliefs, while agents in group B assume that the innovations to the s_t^B process are more informative and the innovations to the s_t^A process less informative than in reality. In particular they believe that the signals evolve according to:

$$\begin{aligned} ds_t^A &= f_t dt + \sigma_s dZ_t^A \\ ds_t^B &= f_t dt + \sigma_s \phi dZ_t^f + \sigma_s \sqrt{1-\phi^2} dZ_t^B \end{aligned}$$

Finally, and in line with both Harrison and Kreps (1978) and Scheinkman and Xiong (2003) I assume that the total wealth of each group is infinite.[11]

There are two remarks about this setup. First, there is no asymmetric information of any sort in this model. The observable quantities (signals, prices, capital and earnings) are in every agent's information set. Disagreement among agents lies on the interpretation of the signals. Second, and in line with Scheinkman and Xiong (2003), I assume that agents of type B do not try to update their beliefs about ϕ, mostly for simplicity. Scheinkman and Xiong (2003) motivate this assumption by appealing to overconfidence.[12]

In the appendix I establish an approximate filter for this setup.[13] In particular, I show the following

[10] To see this, note that dZ_t^f appears in both (2) and (3)

[11] This assumption is made by both Harrison and Kreps (1978) and Scheinkman and Xiong (2003) and is used to drive prices to the reservation value of each group. The motivation behind it is that an individual company is "small" compared to the entire economy. It encapsulates the partial equilibrium nature of the exercise. In particular, irrational traders can survive. Harrison and Kreps (1978) contain an extensive discussion on relaxations of this assumption. The reader is referred to that paper for details.

[12] It is important to note that even if one allowed updating about ϕ, the results wouldn't change qualitatively, as long as agents start with different priors on ϕ.

[13] In contrast to Scheinkman and Xiong (2003) I assume a square root process for f_t in (2) instead of a standard OU process in order to guarantee f_t is positive, thus allowing a lower bound on f_t (which is required for some of the proofs in the appendix). The downside of this assumption is that filtering becomes much more involved and I have to settle for an approximate filter, the properties of which seem to be very good. On this issue see also the next footnote.

result:

Proposition 1 *Let the posterior mean of agent A about* f_t *be denoted* $\widehat{f}_t^A$. *Then agent* A'*s beliefs about* f, *evolve approximately according to:*

$$d\widehat{f}_t^A = -\lambda\left(\widehat{f}_t^A - \overline{f}\right)dt + \sqrt{\frac{\widehat{f}_t^A}{\overline{f}}}\sigma_f dB_t^A \tag{4}$$

where σ_f *is an appropriate constant and* dB_t^A *is an appropriate linear combination of the processes* $\left(ds_t^A - \widehat{f}_t^A dt\right)$, $\left(ds_t^B - \widehat{f}_t^A dt\right)$, $\left(\frac{dD_t}{K_t} - \widehat{f}_t^A dt\right)$ *with the property that the volatility of* dB_t^A *is 1. (Details are given in the appendix). Similarly for agent* B*:*

$$d\widehat{f}_t^B = -\lambda\left(\widehat{f}_t^B - \overline{f}\right)dt + \sqrt{\frac{\widehat{f}_t^B}{\overline{f}}}\sigma_f dB_t^B \tag{5}$$

where dB_t^A *is an appropriate linear combination of the processes* $\left(ds_t^A - \widehat{f}_t^B dt\right)$, $\left(ds_t^B - \widehat{f}_t^B dt\right)$, $\left(\frac{dD_t}{K_t} - \widehat{f}_t^B dt\right)$ *with the property that the volatility of* dB_t^B *is 1.*

This proposition is almost identical to the result established in Scheinkman and Xiong (2003) and the interested reader is referred to this paper for details. For the purposes of this work, it suffices to provide a basic intuition behind the above proposition. The proposition asserts that agents will update their beliefs about f_t by appropriately weighting the information in the signals and the information contained in the earnings process. Both Scheinkman and Xiong (2003) and the appendix to this paper derive these weights and demonstrate that agents of type A will place more weight on signal s_t^A compared to agents of type B. Similarly agents of type B will overweigh signal s_t^B compared to type A agents. This is intuitive. An agent of type A believes that signal ds_t^A is correlated with the unobserved df_t and hence a positive shock to ds_t^A will make her adjust her beliefs about the location of f_t. The exact opposite is true of agent B, whose beliefs about f_t will be more sensitive to changes in ds_t^B.

A quantity that will be central for what follows is the disagreement process defined as the difference between the posterior means of agents A and B about f_t :

$$g_t^A = \widehat{f}_t^B - \widehat{f}_t^A$$

The appendix demonstrates that g_t^A can be approximated by a simple OU process:

$$dg_t^A = -\rho g_t^A dt + \sigma_g dW_t^A \tag{6}$$

with ρ an appropriate constant (determined in the appendix). Moreover, the correlation between dg_t^A and $d\widehat{f}_t^A$ is[14]:

$$cov(dg_t^A, d\widehat{f}_t^A) = -\frac{\sigma_g^2}{2}dt$$

[14] I would like to thank Bernard Dumas for providing me with a set of notes on how to perform this computation.

The situation for agent B is symmetric. She perceives that the process:

$$g_t^B = -g_t^A$$

evolves approximately as an OU process (exactly the same OU process as for agent A) and $cov(dg_t^B, d\widehat{f}_t^B) = -\frac{\sigma_g^2}{2}dt$.

Obviously, knowing $\widehat{f}_t^A, g_t^A$ allows one to compute $\widehat{f}_t^B = \widehat{f}_t^A + g_t^A$. As agents are risk neutral and adjustment costs are quadratic, the posterior means of the two types of agents will be sufficient statistics for the entire belief structure. Hence the state variables of the model are $\left(K_t, \widehat{f}_t^A, g_t^A\right)$. It is very important to note that approximations are confined only to obtaining the dynamics of the *exogenous* state variables $\left(\widehat{f}_t^A, g_t^A\right)$. *Conditional on these approximate dynamics for the belief processes, the rest of the analysis is exact.* In particular, no approximation is necessary for the *endogenous quantities* of the model (prices, investment policies and trading strategies). Alternatively put, if one *outright assumed* that the posterior mean of agent A about f_t behaved as in equation (4), while the difference of her beliefs with agent B behaved as postulated in (6), one could proceed with the rest of the analysis without any sort of approximation.[15]

3 Equilibrium Investment, Trading and Pricing

To determine equilibrium prices it is instructive to first analyze equilibrium prices and investment assuming that all agents are of the same type (A without loss of generality). This will provide a "no-speculation" benchmark. I then turn to the question of determining equilibrium prices and investment in the presence of heterogenous beliefs.

3.1 Homogenous Beliefs

I start with the simplest possible case where every agent is of type A and accordingly everyone agrees on the interpretation of the signals. The results in this subsection are fairly standard and the reader is referred for details to Abel and Eberly (1994,1997).

The goal is to maximize shareholder value

$$P_t = \max_{i_s} E^A \int_t^\infty e^{-r(s-t)} d\Pi_s \tag{7}$$

[15] An alternative approach to the one pursued in this section that avoids all approximations is the following: In order to enforce a "lower bound" on $\widehat{f}_t^A$, one could assume the same dynamics and derive the same exact filters as in Scheinkman and Xiong (2003) with the addition of an appropriate termination value of the company once $\widehat{f}_t^A$ hits a lower barrier (e.g. 0). This would make the entire analysis exact at the cost of imposing an ad-hoc termination value. In either case the expression for marginal q is the same, while there are some insignificant differences for the "growth options".

where P_t is the price of the company and $r > 0$ is the riskless rate. (7) can be rewritten as[16]

$$P_t = \max_{i_s} E^A \int_t^{\infty} e^{-r(s-t)} \left(f_s K_s - p i_s - \frac{\chi}{2}(i_s^2) \right) ds$$

One can further rewrite the above objective as[17]

$$P_t = \max_{i_s} E^A \int_t^{\infty} e^{-r(s-t)} \left(\widehat{f}_s^A K_s - p i_s - \frac{\chi}{2}(i_s^2) \right) ds \tag{8}$$

This is a problem of exactly the same form as the ones considered in Abel and Eberly (1994),(1997). The next proposition gives the solution

Proposition 2 *The solution to (7) is:*

$$P_t\left(\widehat{f}_t^A, K_t\right) = \left(\frac{\overline{f}}{r+\delta} + \frac{\widehat{f}_t^A - \overline{f}}{r+\delta+\lambda} \right) K_t + \left[C_1 \left(\widehat{f}_t^A - \overline{f} \right)^2 + C_2 \left(\widehat{f}_t^A - \overline{f} \right) + C_3 \right] \tag{9}$$

for appropriate constants C_1, C_2, C_3 given in the appendix. Optimal investment is given by:

$$i_t = \frac{1}{\chi}(P_K - p) = \frac{1}{\chi} \left(\frac{\overline{f}(1-\widetilde{p})}{r+\delta} + \frac{\widehat{f}_t^A - \overline{f}}{r+\delta+\lambda} \right) \tag{10}$$

Exactly as in Abel and Eberly (1997), the equilibrium price is a linear function of the capital stock. The term multiplying the capital stock in (9) is marginal "q". There are two observsations about this term. First, it is equivalent to the net present value of the marginal product of capital discounted at the interest rate plus the depreciation rate. Indeed, the appendix demonstrates that:

$$E\left(\int_t^{\infty} e^{-(r+\delta)(s-t)} \widehat{f}_s^A ds \right) = \frac{\overline{f}}{r+\delta} + \frac{\widehat{f}_t^A - \overline{f}}{r+\delta+\lambda} \tag{11}$$

Second, marginal q is the derivative of the equilibrium price w.r.t. the capital stock, and hence it captures the "incentive to invest". Alternatively put, it summarizes the benefits of investing an inframarginal unit of capital and hence affects the optimal investment choice as equation (10) reveals. Assuming also that:

$$\widetilde{p} < \frac{\lambda}{\lambda + r + \delta} \equiv \overline{p} \tag{12}$$

it is easy to see that $i_t > 0$ for all $\widehat{f}_t^A$and thus $K_t > 0$ for all $t > 0$ (since $\widehat{f}_t^A > 0$). Assumption (12) is the *only parameter restriction* that is needed for some of the results and I will assume it throughout. The term

[16] Throughout I will restrict attention to investment policies that satisfy the requirement:

$$E\left[\int_t^{\infty} e^{-r(s-t)} K_s dZ_s^D \right] = 0$$

which amounts to a standard square integrability condition on the allowed capital stock processes. Indeed in the present setup the capital stock turns out to be stationary and as such this is an easily verifiable condition.

[17] This is true since the objective is linear in the state and quadratic only in the control i_t, hence certainty equivalence holds. For details on such problems see Bertsekas (1995).

inside the square brackets of (9) captures the "rents to the adjustment technology" or "growth options", i.e. the value of being able to adjust the capital stock in the future. The appendix establishes that

$$\left(C_1\left(\widehat{f}_t^A-\overline{f}\right)^2+C_2\left(\widehat{f}_t^A-\overline{f}\right)+C_3\right)=\frac{1}{2\chi}E\left(\int_t^{\infty}e^{-r(s-t)}\left(P_K(s)-p\right)^2ds\right) \tag{13}$$

The right hand side of (13) shows that small adjustment costs (i.e. low values of χ) will tend to increase the value of the adjustment technology and vice versa. This is intuitive: the less it costs to adjust the capital stock, the more a company is able to invest and take advantage of temporary increases in fundamentals $\left(\widehat{f}_t^A\right)$. Hence the more valuable is its adjustment technology.

Of interest is to also examine how "marginal q" and "growth options" depend on $\widehat{f}_t^A$. It is clear that marginal q is increasing in $\widehat{f}_t^A$. The same is true for the growth options:

Lemma 1 *Let:*

$$u^F\left(\widehat{f}_t^A\right)=\left(C_1\left(\widehat{f}_t^A-\overline{f}\right)^2+C_2\left(\widehat{f}_t^A-\overline{f}\right)+C_3\right)$$

Then

$$u^F_{\widehat{f}_t^A}>0$$

The above proposition establishes that the value of "growth options" increases as $\widehat{f}_t^A$ increases. This is because $\widehat{f}_t^A$ is persistent and hence an increasing $\widehat{f}_t^A$ makes it more likely that large investments will need to be undertaken in the near future and thus the technology to adjust the capital stock becomes more valuable.

3.2 Heterogenous Beliefs: Optimal Investment, Trading, and Equilibrium Prices

The following definition provides a starting point for the analysis:

Definition 1 *An equilibrium is defined as a collection of a (stochastic) price process P_t, an investment policy i_t, and optimal (selling) stopping times τ for investors $\{A,B\}$ such that:*

$$P_t=\max_{o\in\{A,B\}}\left\{\sup_{i_s,\tau}E_t^o\left[\int_t^{t+\tau}e^{-r(s-t)}\left(dD_s-\left(pi_s+\frac{\chi}{2}i_s^2\right)ds\right)+e^{-r\tau}P_{t+\tau}\right]\right\} \tag{14}$$

This recursion is almost identical to the one originally proposed by Harrison and Kreps (1978) with the difference being an additional optimization over investment strategies.

From a purely asset pricing perspective, (14) states that the equilibrium price will be the maximum of the two "private" valuations of a company share by agents A,B. These private valuations are inside the curly brackets and are in turn comprised of two components: a) the net present value of profits *up to the optimally chosen time at which a share is resold* $(t+\tau)$ and b) the equilibrium (discounted) price obtained at time $t+\tau$ by reselling the stock. Moreover the investment policy is determined so as to maximize the share price.

To understand the above equilibrium definition it is useful to consider two thought experiments. Suppose that an investor takes the equilibrium price process as given. Suppose moreover, that the right hand side in (14) is larger than the left hand side. There would then exist either a (physical) investment strategy or a "selling" policy for at least one of the investors that would make her want to bid more than the current price in the market. Similarly, if the right side was smaller than the left side, then there would be neither an investor, nor an investment policy, nor a selling strategy that would allow either of the investors to earn (in expectation) a rate of return equal to the interest rate. Hence left and right side must equal.

It is interesting to note that (14) collapses to the usual net present value expression when all investors have the same beliefs. To see this suppose that all investors are of type A. Then, (14) simplifies to:

$$\begin{aligned} P_t &= \sup_{i_s,\tau} E_t^A \left[\int_t^{t+\tau} e^{-r(s-t)} \left(dD_s - \left(p i_s + \frac{\chi}{2} i_s^2 \right) ds \right) + e^{-r\tau} P_{t+\tau} \right] = \\ &= \sup_{i_s} E_t^A \left[\int_t^{\infty} e^{-r(s-t)} \left(dD_s - \left(p i_s + \frac{\chi}{2} i_s^2 \right) ds \right) \right] \end{aligned} \tag{15}$$

The second line follows from Bellman's Optimality Principle which asserts that the continuation value from *any* stopping time onwards must be equal to the infinite horizon value of the dynamic optimization problem[18].

In general however, the key difference between (15) and (14) is that the identity of the investor holding the company stock will change dynamically through time, and hence (14) will not collapse to (15).

Solving for an equilibrium is not straightforward. In contrast to Scheinkman and Xiong (2003), the dividend process is endogenous because of investment and the presence of an additional state variable (capital). Hence it is impossible to collapse the problem to a single one-dimensional optimal stopping problem as in Scheinkman and Xiong (2003). Fortunately, an alternative argument can be used (see appendix) that preserves the tractability of the framework of Scheinkman and Xiong (2003).

The next proposition demonstrates how to construct an equilibrium to (14). For most of the results that will be discussed in the body of the paper, it suffices to present the general structure of the solution. In particular, as the next proposition shows, the equilibrium price is still an affine function of the capital stock. Simply put, the equilibrium price continues to have the same simple form as (9). For the body of the text, I concentrate on a discussion of marginal q and relegate the exact computation of growth options to the appendix.

Proposition 3 *Let*

$$q_t = \frac{\overline{f}}{r+\delta} + \frac{\widehat{f}_t^A - \overline{f}}{r+\delta+\lambda} + 1\{g_t^A > 0\} \frac{g_t^A}{r+\delta+\lambda} + \frac{y_1(-|g_t^A|)}{2(r+\delta+\lambda) y_1'(0)} \tag{16}$$

[18] See e.g. Fleming and Soner (1993)

where $1\{\cdot\}$ *is the indicator function, and* y_1 *is a positive, convex, increasing and differentiable function*[19]*, which solves the ordinary differential equation:*

$$\frac{\sigma_g^2}{2} y_1'' - \rho g_t^A y_1' - (r+\delta) y_1 = 0 \tag{17}$$

subject to the boundary conditions:

$$\lim_{x \to \infty} y_1(x) = \infty \tag{18}$$

$$\lim_{x \to -\infty} y_1(x) = 0 \tag{19}$$

Then an equilibrium is given by the function $P\left(\widehat{f}_t^A, g_t^A, K_t\right)$[20]*:*

$$P\left(\widehat{f}_t^A, g_t^A, K_t\right) = q_t\left(\widehat{f}_t^A, g_t^A\right) K_t + G\left(\widehat{f}_t^A, g_t^A\right)$$

for an appropriate function $G\left(\widehat{f}_t^A, g_t^A\right)$ *computed explicitly in the appendix. The optimal investment strategy is*

$$i_t = \frac{1}{\chi}\left(q_t - p\right) \tag{20}$$

and the optimal stopping time for investor A is to resell the asset immediately once $\widehat{f}_t^A < \widehat{f}_t^B$ *and to hold it otherwise. Similarly, the optimal strategy for investor B is to resell the asset once* $\widehat{f}_t^B < \widehat{f}_t^A$ *and to hold it otherwise.*

There are several observations associated with this proposition. First, as already noted, the equilibrium price is a simple affine function of the capital stock. Just as in the case of homogenous beliefs, one obtains a decomposition of the price into marginal q (times the capital stock) and growth options. And as in the case for homogenous beliefs, marginal q continues to be the key quantity driving investment as (20) demonstrates. There is also a tight linkage between marginal q and the results obtained in Scheinkman and Xiong (2003), which will be explored subsequently.

A final remark concerns the optimal selling strategies implied by the above proposition. Just as in Scheinkman and Xiong (2003), the identity of the investor who is holding the shares of the company will change an infinite number of times. Simply put, the company is always in the hands of the most "optimistic" investor, i.e. the investor who currently has the highest posterior mean about f_t. By equation (6) the difference in beliefs between the two agents $\left(g_t^A\right)$ will mean revert to 0. Moreover, since g_t^A is a diffusion, it will "cross" 0 infinitely often and therefore the identity of the person holding the stock will change infinitely often. This basic result will be responsible for the presence of speculative components in the equilibrium price, since investors will expect to resell the shares to each other with probability 1.

[19] The explicit solution to y is given by a Kummer function. Functions of this sort are discussed in Abramowitz and Stegun (1965), Chapter 13.

[20] Under some mild restrictions on the allowed parameters discussed in the appendix.

3.2.1 Marginal "q" and speculation

As already asserted, the key quantity driving investment is marginal q. Comparing the expression for marginal q in the presence of heterogenous beliefs to the equivalent expression in the presence of homogenous beliefs one observes an extra term, namely:

$$b(g_t^A) \equiv 1\{g_t^A > 0\}\frac{g_t^A}{r+\delta+\lambda} + \frac{y_1(-\left|g_t^A\right|)}{2(r+\delta+\lambda)y_1'(0)} \tag{21}$$

The function y_1 is positive, increasing, convex, continuously differentiable and asymptotes to 0 at $-\infty$ and $+\infty$ at $+\infty$. The key property that it satisfies is (17). To see why this property is important, assume first that $g_t^A < 0$ so that $\widehat{f}_t^A > \widehat{f}_t^B$ and thus agent A holds the company by proposition 3. Apply Ito's Lemma to $b(g_t^A)$, and use (17) to obtain:

$$\frac{dE(b(g_t^A))}{dt} = \frac{\sigma_g^2}{2}b'' - \rho g_t^A b' = (r+\delta)\,b$$

This calculation shows that $b(\cdot)$ is a positive term that grows (in expectation) at the interest rate plus the rate of depreciation. Hence, from the perspective of agent A it behaves as if it were a pure "bubble" that grows at a constant rate (in expectation). It is also interesting to use the identities $\widehat{f}_t^B = \widehat{f}_t^A + g_t^A$ and $g_t^A = -g_t^B$ to express q_t in (16) as:

$$q_t = \frac{\overline{f}}{r+\delta} + \frac{\widehat{f}_t^B - \overline{f}}{r+\delta+\lambda} + b(g_t^B)$$

where

$$b(g_t^B) = 1\{g_t^B > 0\}\frac{g_t^B}{r+\delta+\lambda} + \frac{y_1(-\left|g_t^B\right|)}{2(r+\delta+\lambda)y_1'(0)} \tag{22}$$

One can now apply the same reasoning to show that whenever agent B holds the company, (which means $\widehat{f}_t^B > \widehat{f}_t^A$ and thus $g_t^B < 0$) (22) gives

$$\frac{dE(b(g_t^B))}{dt} = \frac{\sigma_g^2}{2}b'' - \rho g_t^B b' = (r+\delta)\,b$$

Hence, when agent B holds the stock, $b(\cdot)$ is growing as a bubble according to the beliefs of agent B. The same result is true for growth options.

In sum, no matter who is owning the company there is always a component built into the price that behaves *like an endogenous bubble*. In contrast to the bubbles that arise in OLG models however, one need not assume that agents are finitely lived, or that the economy is dynamically inefficient. *Effectively, the fact that the identity of the stockholder changes dynamically introduces finite horizons and therefore "bubbles" into marginal q endogenously.*

An implication of the above analysis is that marginal q will be higher than *any* agents' beliefs about the appropriately discounted net present value of the marginal product of capital. The easiest way to see this is to use $\widehat{f}_t^B = \widehat{f}_t^A + g_t^A$ in order to express q_t as:

$$q_t = \max_{o\in\{A,B\}} \left(\frac{\overline{f}}{r+\delta} + \frac{\widehat{f}_t^o - \overline{f}}{r+\delta+\lambda} \right) + \frac{y_1(-\left|g_t^A\right|)}{2(r+\delta+\lambda)y_1'(0)} > \max_{o\in\{A,B\}} \left(\frac{\overline{f}}{r+\delta} + \frac{\widehat{f}_t^o - \overline{f}}{r+\delta+\lambda} \right) \tag{23}$$

It is very interesting to relate the results obtained in this section to Scheinkman and Xiong (2003) and Abel and Eberly (1994,1997). Even though the equilibrium price of the firm is significantly different than in Scheinkman and Xiong (2003) due to the presence of growth options[21], the marginal q obtained in this section is just the equilibrium price one would obtain in the model of Scheinkman and Xiong (2003) if one introduced an asset with payoffs equal to the marginal product of capital and used the interest rate $r + \delta$ instead of r. There is also a connection to Abel and Eberly (1994,1997) who establish that marginal q is just the net present value of the marginal product of capital discounted at $r + \delta$, when adjustment costs are independent of the capital stock. However, Abel and Eberly (1994,1997) cannot distinguish between market prices and net present values since agents have homogenous beliefs and thus the two notions coincide.

What this section demonstrates, is that *q theory essentially continues to hold, even though marginal q contains an endogenous "speculative bubble"*. The reason is intuitive. Investment in the present model serves two purposes. One is to increase the "long run fundamentals" of the company according to the beliefs of the current owners. This motive is captured by the term:

$$\max_{o\in\{A,B\}} \left(\frac{\overline{f}}{r+\delta} + \frac{\widehat{f}_t^o - \overline{f}}{r+\delta+\lambda} \right)$$

in equation (23). However, since there is disagreement about the true marginal product of capital, there is a second and purely speculative reason for the increased investment, which is captured by the term:

$$\frac{y_1(-\left|g_t^A\right|)}{2(r+\delta+\lambda)y_1'(0)}$$

in (23). This term is "speculative" since it captures the resale premium that current owners will obtain once they resell the firm. By investing more they can increase not only the "long run" fundamental value of the firm, but also the resale price that they can obtain once they decide to liquidate their holdings of company stock. This basic observation and its implications for investment theory are discussed in greater detail in the next section.

4 Implications for Investment Theory

This section discusses some of the key implications of the model for investment theory. The three questions that I will try to answer using the framework developed are: 1) Is the investment level that is chosen as a result of share price maximization efficient? 2) Which are the key conditions for neoclassical investment

[21] See the appendix for a closed form solution to the growth options.

theory to hold in the presence of speculation - what happens if one relaxes these assumptions? and 3) Which types of investments are likely to be more affected by speculation?

4.1 Efficiency

One of the cornerstone results of neoclassical investment theory is the efficiency of the investment level that results when firms maximize their share price. Does this result continue to hold in a speculative market?

To answer this question, it is important to give a more precise definition of efficiency. As is common, I will introduce a central planner. This central planner will be constrained by the structure of the asset market. In particular she will not be able to alleviate the shorting constraint. In particular, I will assume that:

1. Her *only* choice variable is to dictate the investment policy i_t of the firm. In particular she takes the interest rate r as exogenously given, like a central planner in a small open economy

2. Given the investment policy, market participants make trading and consumption decisions. The equilibrium price of the firm is given by:
$$P_t = \max_{o \in \{A,B\}} \left\{ \sup_{\tau} E_t^o \left[\int_t^{t+\tau} e^{-r(s-t)} \left(dD_s - \left(pi_s + \frac{\chi}{2} i_s^2 \right) ds \right) + e^{-r\tau} P_{t+\tau} \right] \right\} \tag{24}$$
(Notice that there is no longer an optimization step over i_t since market participants take investment as given)

3. The central planner maximizes a weighted average of the expected utility of the continuum of markets participants[22]:
$$V = \int \lambda(x) V(x) dx \tag{25}$$
where $\lambda(x) > 0$ is the weight of participant x and $V(x)$ is agent $x's$ expected utility:
$$V(x) = E^x \int_t^{\infty} e^{-r(s-t)} dC_s^{(x)}$$
and $C_t^{(x)}$ is the consumption process for agent x.

This definition of the central planner's actions and objectives attempts to isolate the investment decision, while keeping the structure of the asset market unaltered. The following proposition gives a key result, that is quite general. In particular, it holds for any group of agents, structure of beliefs, disagreement etc.

[22] I shall constrain attention to $\lambda(x)$ that keeps this objective finite for all i_t. There is no contradiction in assuming infinite (collective) wealth of the market participants within each group and a well defined value function for the central planner. One can always choose $\lambda(x)$ that gives a bounded value function for the central planner while preserving the assumption of infinite (collective) wealth within each group.

Proposition 4 *Suppose prices are determined by (24). For any investment policy i_t, there exist two constants (depending on x only) $\alpha_0(x)$ and $\alpha_1(x) \geq 0$ such that the value function of agent x can be written as:*

$$V(x) = \alpha_0 + \alpha_1 P_t \tag{26}$$

with $\alpha_1 > 0$ if agent x is endowed with stock at time t.

This result immediately implies the efficiency of share price maximization as an objective for the central planner, since (26) and (25) imply that:

$$V = \int \lambda(x) V(x) dx = \left(\int \lambda(x) a_0(x) dx \right) + \left(\int \lambda(x) a_1(x) dx \right) P_t = A_0 + A_1 P_t$$

for $A_1 = \left(\int \lambda(x) a_1(x) dx \right) \geq 0$[23]. Hence the same investment policy that maximizes the company price also maximizes the central planner's objective, since her objective is an affine function of the price of the company.

This result is somewhat surprising: Everyone in the economy agrees that prices are above their fundamental values under any market participants' subjective (infinite horizon) valuation. Yet, the resulting investment is efficient. It should also be emphasized that the statement of the proposition does not depend on the specifics of the setup. The only equation of the whole setup, that is used in the proof is (24).

The intuition is simple: For agents that hold stock, their *subjective* assessment of their utility depends on the price of the stock: A higher price of the stock (of which they have non-negative holdings because of the shorting constraint) increases their wealth and therefore their *expected* consumption in the future, irrespective of whether they form that expectation under correct or incorrect beliefs.

A somewhat constraining but simultaneously simplifying feature of the present framework is risk neutrality. It allows one to arrive at equation (26) which obviously wouldn't hold under risk aversion.[24] The fact remains however, that the model presents a simple *counterexample* to the claim that is commonly made in the press and the academic literature: Namely that one should take active measures to combat speculation because it will lead to misplaced investment and misallocation of resources. The above analysis suggests that the conventional approach to allocative efficiency does not necessarily imply such a conclusion[25].

It is also noteworthy that the present setup allows an assessment of efficiency in a context where stock prices are above everyone's fundamental valuations, agents are infinitely lived and always present in the market. Models with heterogenous beliefs, risk averse consumers and complete markets would also imply

[23] There are a few technical remarks: First $A_1 > 0$ as long as the central planner places a non-degenerate weight $\lambda(x)$ on agents that are endowed with some stock at time t. Second, even in the degenerate case where the central planner places 0 weight on time t stockholders, $A_1 > 0$ as long as the central planner can effect a once and for all transfer payment between the two groups at time t.

[24] However, it is not clear whether risk aversion would add new intuitions in this context.

[25] However, it appears that paternalistic approaches to welfare would immediately imply activism.

efficient investment allocations, but then prices would coincide with every agent's fundamental valuation (since markets are complete), so the efficiency of share price maximizing investment would be an obvious result. Similarly, OLG models would allow deviations from fundamental (infinite horizon) valuations, but addressing welfare in those models is hard, because no agent can participate in the market at all the times. This precludes standard welfare criteria and creates inherent inefficiencies irrespective of the investment policy chosen.

4.2 Long-termism, Short-termism, and Neoclassical Investment Theory

Which are the key assumptions in the model that make the neoclassical (marginal q) theory hold? Alternatively, which conditions would have to fail to invalidate the conclusion? In this model the most important channel that connects investment to speculative overpricing is the assumption of share price maximization.

In reality however, one might think that there are certain key investors that do not have the same ease of access to financial markets as the average retail investor. To make an example, the average retail investor of company X can liquidate her shares without having to worry that her decision will adversely affect the price, result in significant transactions costs or capital gains taxes. However, if a major shareholder tried to liquidate her position in shares of company X, that would in all likelihood cause a severe price reaction, be associated with large transactions costs and substantial capital gains taxes[26]. At the same time it is unlikely that a retail investor and a major stockholder will have the same power in affecting the investment decisions of company X.

To be precise, suppose that company X has one large investor (I) that would face a cost of T (per share) if she tried to liquidate her shares. However, there are also a number of retail investors who face no cost in reselling their shares and are of type $\{A, B\}$ as previously described. Finally, assume that the large investor controls the investment of the company as long as she doesn't resell her shares.

Then, it seems reasonable to modify the definition of equilibrium as follows:

Definition 2 *Suppose I holds enough shares that allow her to control the investment policy of the company. Then an equilibrium is defined as a collection of a (stochastic) price process (for retail stock) P_t, an investment policy i_t, and optimal (selling) stopping times τ for investors $\{I, A, B\}$ such that:*

1) (Optimality of the investment policy for the large investor) The investment policy i_t is determined so that:

$$\sup_{i_t,\tau} E_t^I \left[\int_t^{t+\tau} e^{-r(s-t)} \left(dD_s - \left(pi_s + \frac{\chi}{2} i_s^2 \right) ds \right) + e^{-r\tau} \left(P_{t+\tau} - T \right) \right] \tag{27}$$

[26]Especially if the "key" investor was holding shares in the company for a much longer time than retail investors and potentially before the onset of disagreement.

2) (Asset Pricing given i_t). The equilibrium price for stock in the "open market" is determined so that:

$$P_t = \max_{o \in \{A,B\}} \left\{ \sup_{\tau} E_t^o \left[\int_t^{t+\tau} e^{-r(s-t)} \left(dD_s - \left(pi_s + \frac{\chi}{2} i_s^2 \right) ds \right) + e^{-r\tau} P_{t+\tau} \right] \right\} \tag{28}$$

If I resells her shares, suppose that equilibrium reverts back to the definition given in (14). The major difference between the definitions of equilibrium given above and (14) is the presence of a cost T that makes the large investor different from anyone else. In a sense, a large investor is "entrenched" inside the company and cannot profit from speculation as much as retail investors can.

To make an extreme point suppose that the large investor I would be facing a sufficiently high T, that would make it optimal to never resell her shares. (To achieve that let $T \to \infty$) Then it is optimal to set $\tau = \infty$ in (27) so that it becomes:

$$\sup_{i_t} E_t^I \left[\int_t^{\infty} e^{-r(s-t)} \left(dD_s - \left(pi_s + \frac{\chi}{2} i_s^2 \right) ds \right) \right] \tag{29}$$

Notice that this infinite horizon objective is perfectly equivalent to (8). Alternatively, even though the stock prices at which retail investors buy and sell stock will continue to be above their "fundamental values" (because of equation [28]) the investment policy will not at all be affected by the overpricing .

This is a key observation that allows a formalization of discussions contained in e.g. Blanchard, Rhee and Summers (1993) and Stein (1996) about long-termism and short-termism. Here, I allow an explicit modelling of these notions in a framework where markets clear dynamically. Long-termism is just the extent of "entrenchment" of current "major" shareholders in the firm. If current major shareholders cannot resell their shares (because of capital gains taxes, or purely for control reasons or any other reason captured in the cost T) then mispricing will not affect the investment policy of the firm. If however, the firm is maximizing share price, then "overpricing" will affect investment as described in the previous section.

One of the most significant advantages of the present framework is that short horizons are not imposed exogenously (as in OLG models), but result endogenously due to the short-selling constraint. Moreover, for major shareholders the horizon is also an endogenous quantity that depends on their ability to liquidate their holdings (i.e the cost T).

An important implication is that both short-termism and long-termism are not two mutually exclusive possibilities. They present the two extremes of a continuum of possibilities that depend on the cost T. The usual short-termist hypothesis presents the limit as $T \to 0$, while the long-termist alternative presents the limit as $T \to \infty$. The theory is silent about which is the more empirically relevant region, but makes a precise prediction about the factors that determine the answer to this question (namely the cost T). In reality one would expect the truth to lie between the two extremes and this may be an explanation for the literature that finds some, yet limited, "pass through" of speculative components to investment (see e.g. Blanchard Rhee and Summers [1993], Morck, Shleifer and Vishny [1990]).

It is important to note that the same results would hold if "major shareholders" could only partially liquidate their holdings of stock in the company (presumably because they value control and they don't want to liquidate their shares beyond some level). Then their objective would be to maximize some weighted average between share price and long term value.

A very practical implication[27] of the above discussion concerns the necessary features of a tax system that would aim to limit the "pass through" of speculative components to investment. Sometimes it is argued that one should raise a tax on transactions (a so-called "Tobin's" tax) so as to force investors to hold stocks longer and limit their speculative trading motives. The above analysis suggests that it is sufficient to raise taxes on "key" shareholders. It is interesting that the current system of capital gains taxes has such effects. To see this, assume that certain key investors own a large fraction of the stock and have been owning it for a long time. Furthermore, assume that speculation raises the price of their shares. Since they have been owning shares before the onset of speculation, they will face substantial capital gains taxes if they try to liquidate their shares compared to the average "small" investor who has been holding company stock for a shorter duration of time. Since capital gains taxes *are due only upon resale of the shares* they end up taxing large investors who have been holding stock for a long time more than the average small investor, and hence achieve exactly what is needed in order to induce "long termism".

4.3 Heterogeneity in Beliefs and Investment

What types of investments are likely to be most affected by speculative overpricing? To answer this question, assume that the firm is maximizing share price (short-termism). Then the model predicts that investment will be most affected, when the heterogeneity of beliefs about the future marginal product of capital is most severe.

Formally, marginal q is a strictly increasing function of the variability in belief differences (σ_g) [28] all else equal. As a result, investment will be higher when σ_g is higher.

This is a key prediction of the model that can potentially help explain a number of empirical facts. The first empirical fact is that speculation seems to affect investment in "high tech" and relatively young companies as was recently the case for many companies in the NASDAQ. It seems plausible to assume that there was increased divergence in beliefs about the future marginal product of capital of many of these companies. Combined with shorting constraints[29] this created fertile ground for speculative overpricing.

The predictions of the model are also consistent with the increased sensitivity of young firms' investment to the stock market[30]. One would expect that young firms introduce innovations thus leaving more room for

[27] I would like to thank Andy Abel for helpful discussions on this issue.

[28] This can be shown by exactly the same steps as in Scheinkman and Xiong (2003).

[29] See Ofek and Richardson (2003) for empirical evidence on this.

[30] For empirical evidence see e.g. Farhi and Panageas (2005) and references therein.

disagreement about their future marginal product of capital. Hence, both their investment and their share price are likely to be influenced more by speculation.

A final possible application of this framework (that is not pursued here) is the following: In the present model, speculation has the effect of also increasing the volatility of marginal q. If one combines this with the "q" theory of mergers developed in Jovanovic and Rousseau (2002) one could also explain why speculation leads to such increases in merger activity as the ones observed in the data.

In summary, the model makes predictions not only about whether investment is going to be influenced by speculation, but also which types of investment are likely to be affected. Hetereogeneity in beliefs about the future marginal product of capital is a key condition, which appears to be particularly relevant for younger firms and firms in "new tech" sectors.

5 Implications for Asset Prices

Thusfar, the focus has been on the effects "inflated asset prices" have on investment. In this section I explore the reverse link, namely from investment to asset prices. The equilibrium price in this paper differs significantly from Scheinkman and Xiong (2003) because of growth options (which capture the value of the ability to adjust the capital stock). Moreover, the presence of a time varying capital stock and a time varying q (the market-to-book ratio in finance jargon) will make it possible to use simulated moments of the model to match certain asset pricing facts.

The first subsection presents a theoretical discussion of predictability of returns in the present model. The discussion there is intended to provide the intuitions for the results that follow in the second subsection.

The second subsection presents a quantitative exercise: Using the closed form solution to the model, I simulate an artificial dataset containing roughly the same number of observations as CRSP and calibrate the model so as to match certain well known asset pricing facts. The purpose of this exercise is to measure in a quantitative sense: a) by how much does investment and the presence of growth options affect the equilibrium prices and b) how strong heterogeneity in beliefs would have to be in order to match the predictability of returns by book-to-market ratios observed in the data.

5.1 Predictability and Endogenous Investment

Up to this stage, there was no need to assume that one of the two agents should have objectively rational beliefs about the data generating process. The only key assumption was heterogeneity in beliefs. To talk about the implications of the model for expected returns however, one must make an additional assumption about the objective data generating process. In particular, assume from now on that agents of type A have the "correct" beliefs, in the sense that their understanding of the data generating process coincides with the objective data generating process. This enables me to replace the objective probability measure with the belief system of agent A in computing the expected returns that follow.

Given that all agents are risk neutral and there is perfect competition, the expected return perceived by any agent who holds stock has to be equal to the interest rate. Similarly, an agent liquidates her position in stock, whenever she feels that the expected return on the stock from that point on will be less than the interest rate[31].

To see this formally, define:

$$\mathcal{Z}P = \max_i \left(\frac{1}{2}\frac{\sigma_f^2}{\overline{f}}\widehat{f}_t^A P_{ff} - \lambda(\widehat{f}_t^A - \overline{f})P_f + \frac{1}{2}\sigma_g^2 P_{gg} - \frac{1}{2}\sigma_g^2 P_{fg} - \rho g P_g + P_K(-\delta K_t + i_t) + \widehat{f}_t^A K_t - i\left(p + \frac{\chi}{2}i\right)\right)$$

Applying Ito's Lemma to P and using the optimal investment policy i^*, one obtains:

$$E^A(P_{t+dt} - P_t) + \left[\widehat{f}_t^A K_t - i^*\left(p + \frac{\chi}{2}i^*\right)\right]dt = \mathcal{Z}Pdt$$

The appendix establishes that[32]

$$\mathcal{Z}P \begin{cases} = rP \text{ if } g_t^A < 0 \\ < rP \text{ if } g_t^A \geq 0 \end{cases} \tag{30}$$

This last equation shows that agents of type A will hold stock if and only if they perceive its expected return to be equal to r, else they will resell to agents of type B. One can also show that the "gap" between the expected return and rP will be increasing in g_t^A. More importantly *there will be predictability* in this market: Define the Book to Market ratio as:

$$\frac{K_t}{P_t} = \frac{1}{q_t\left(\widehat{f}_t^A, g_t^A\right)} + \frac{K_t}{G\left(\widehat{f}_t^A, g_t^A\right)} \tag{31}$$

Assume for a moment that the second term in (31) can be ignored. By (16), q_t is increasing in g_t^A and hence $\frac{K_t}{P_t}$ is decreasing in g_t^A. Keeping $\widehat{f}_t^A$ fixed, this implies that a low $\frac{K_t}{P_t}$ will be associated with high values of g_t^A and low returns (from [30])[33]. Whether investment is high in these times or not, will depend on the assumption that one makes about short-termism and long-termism. If investors are long-termist, investment will not be affected. On the contrary, if investors are short-termist then investment will also be high.

These observations suggest that the model at least qualitatively is able to explain a number of facts about asset prices documented in Fama and French (1992). A high market to book ratio will predict low expected returns going forward and high investment. Moreover, it suggests caution with the argument made in the literature that one can use investment to tell apart rational and irrational theories of predictability[34]. If the firm pursues short-termist objectives, then investment and expected returns will co-move, even though the

[31] Proposition 7 establishes exactly that.

[32] This can be established by using equations (79) and (80) in the appendix along with: $\mathcal{Z}P - rP = \mathcal{L}P$.

[33] It turns out that a similar result holds for growth options.

[34] See also Lamont [2000] for a careful discussion on this.

variation in expected returns is driven by wrong beliefs. Thus, finding that investment predicts returns is suggestive of *neither irrationality or rationality*, since it is compatible with both types of theories.

This observational equivalence seems to be a robust result. Zhang (2005) contains a discussion of asset pricing anomalies in a perfectly rational q-theoretic model. As long as q-theory continues to hold in the presence of speculation, most of the results in Zhang (2005) continue to hold in the present framework[35], even though the driving force of predictability is behavioral.

On the positive side however, the model suggests a promising path to disentangle the two types of explanations. The discussion about long-termism and short-termism suggests that one should study separately the co-movement between investment and expected returns for companies with large controlling shareholders and those with diverse stockholder bodies. According to the model, there should be little co-movement between the investment behavior and the expected returns of companies with large "long-termist" controlling shareholders. This is a manifestation of the fact that long-termist investors will ignore variations in the difference in beliefs g_t^A in making investment decisions, which are the source of predictability. Obviously, this will not hold true for share price maximizing firms. Hence, if the variation in expected returns observed in the market is due to behavioral influences, there should be a different extent of co-movement between expected returns and investment for firms pursuing short-termist and long-termist objectives. If however, variations in expected returns are due to purely rational variations in risk or risk aversion, there appears to be no reason to make such a distinction[36].

In conclusion, the present model suggests that if investment is affected by speculative motives, then it becomes fairly hard to use investment in order to distinguish rational from irrational theories of asset pricing. The model will become observationaly equivalent in many respects to rational models of predictability, since most of these models operate through an investment channel (Berk Green and Naik [1999], Gomes Kogan and Zhang [2003], Zhang [2005] etc.). On a more positive note, this section also suggests a promising route to construct tests of rational vs. irrational explanations for asset pricing facts.

5.2 A Quantitative Exercise

The benefit of having a closed form solution for the value of the firm is that one can simulate a very large (artificial) dataset, such as CRSP and COMPUSTAT, and then calibrate some key parameters of the model so as to match certain well documented asset pricing facts.

In this section I try to address two questions in particular: 1) Does investment amplify the effect of

[35] Zhang (2005) uses a framework where marginal and average q are equal. The framework developed here will produce the same equivalence if adjustment costs are taken to be linear homogenous in the capital stock. (Panageas [2005])

[36] In the presence of risk aversion one would need the added assumption that "entrenched" long-termist investors should also have access to markets that will allow them to hedge the risks associated with their holdings of shares of the company that they cannot easily resell.

mispricing compared to a model without investment and by how much? And 2) how large does the difference in beliefs between the rational and the irrational agent have to be in order to account for the variation and predictability in returns observed in the data? Throughout this section, assume that investment is determined as part of the share price maximization problem of the firm (short-termism).

Before proceeding with the calibration, one should note an important fact about the model: "Overpricing" is controlled solely by one (and only one) parameter. This parameter is ϕ (defined in section 2.2). It captures the extent to which each investor believes that signal A (resp. signal B) is more informative than signal B (resp. signal A). When $\phi = 0$ for instance there is no difference in beliefs between the two agents and there are no speculative components as a result.

Hence, the base case $\phi = 0$ presents a good departure point. Under this restriction the model becomes a standard Abel and Eberly [1994] framework. In order to calibrate the rest of the parameters, I match simulated moments of the model to the data. Then I vary ϕ in order to study the effects of speculative mispricing.

As a first step, I set some parameters following standard choices in the literature. The parameter δ is the depreciation rate and is set to 0.07. The interest rate is set to $r = 0.06$. The reason for this choice is explained below. The parameter $\overline{f}$ captures the marginal product of capital "in the long run" and is set to $r + \delta$. As a result of these choices, the mean of the stationary distribution of marginal q is equal to 1, since:

$$E^A(q) = E^A\left(\frac{\overline{f}}{r+\delta} + \frac{f-\overline{f}}{r+\delta+\lambda}\right) = \frac{\overline{f}}{r+\delta} = 1$$

In the above expression I have deliberately dropped time subscripts in order to denote expectation under the stationary measure.

It is straightforward to verify that in this model the capital stock and the price will have a stationary distribution because investment itself is stationary. One can then determine p and the stationary standard deviation of $\widehat{f}_t^A$, so as to match the average book-to-market ratio in the data. Furthermore one can use the first two moments of the dividend-to-price ratio to match parameters related to mean reversion in f_t and the volatility of dividends. Finally χ is determined so that the model can match empirical regressions of the investment to capital ratio on average q in the data. Table 1 reports the parameters that were chosen to match the moments in the data (reported in Table 2). To compute the (stationary) moments implied by the model I used a Monte Carlo experiment to simulate 184.000 years of artificial data[37].

It is important to note that these are *time series moments* of the model. Hence one should match them to time series moments in the data. The data in the second column are based on several empirical studies: The study of Kothari and Shanken (1997) reports an average B/M of 0.69 with a standard deviation of

[37] dt was chosen to be $1/12$ in the simulations. Several initial observations were discarded to ensure that initial conditions are drawn from the stationary distribution.

Parameter	Value	Parameter	Value
λ	0.10	p	0.38
r	0.06	σ	$0.25\overline{f}$
δ	0.07	σ_s	σ
$\overline{f}$	0.13	σ_D	0.5σ
χ	2.00	ϕ (basecase)	0

Table 1: Parameters used in calibration

Statisitic:	Model	Data
Mean (B/M)	0.730	0.690
S.D. (B/M)	0.160	0.220
Mean (I/K)	0.070	0.069
S.D.(I/K)	0.024	0.020
Mean(D/P)	0.050	0.036
S.D.(D/P)	0.019	0.014
Mean (annualized Return)	0.065	0.061
S.D. (annualized Return)	0.165	0.239
Mean (marginal q)	1.000	-
S.D. (marginal q)	0.250	-
$\frac{cov(I/K,q)}{var(q)}$	0.060	(0.03; 0.11)

Table 2: Data and model implied moments. For data sources see text.

0.22 using a sample of companies from 1926 to 1990. The dividend yield is given as 0.036 with a standard deviation of 0.014.

The interest rate was chosen at 6% in order to match the return reported in Kothari and Shanken (1997) who report a value weighted return of 9.4 % (for the companies in their sample) with standard deviation of 23.9 %. However these returns are not real returns. Adjusting for an average annual inflation of 3.27% from 1926 to 1990 produces an average real return of 6.13%.

It should be noted that the present model cannot simultaneously match the real interest rate and the real return on the stock market, since risk neutrality precludes any risk premium. Thus one should interpret "the interest rate" in this paper as proxying for both a discount rate and a risk premium. Using this interpretation, I chose $r = 0.06$, in order to match the absolute level of (real) returns in the sample of Kothari and Shanken (1997).[38] It is reasonable to conjecture that in a richer model with risk aversion one would obtain similar

[38] The fact that the model does not allow a risk premium does not mean however that it cannot be used to study predictability

results to this model as long as there is little variation in relative risk aversion of agents and the correlations between stock payoffs and the stochastic discount factor.

To compare the average investment to capital ratio to the data, I used the data in Blanchard, Rhee and Summers (1993) to compute the average investment to capital ratio and its standard deviation. They report (annual) data on (aggregate) investment and capital in the US economy from 1900-1990. In Farhi and Panageas (2005) this series is extended to 2003. The average investment to capital ratio for this extended series is 6.85%, while its standard deviation is 1.95 percent. The final statistic is the coefficient obtained by regressing the investment to capital ratio on average q. These regressions are performed routinely in microeconomic studies. For comparison purposes, Abel and Eberly (2002) report an average coefficient of 0.03 (when they use simple OLS estimates) and 0.11 when they use analyst forecasts as instruments.

The baseline model has a descent performance in terms of matching some basic moments of the data. It should be emphasized that ϕ was taken to be 0 in calibrating the above parameters. Hence the above simulated moments correspond to a world without disagreement of any sort.

The first question one may ask is how large is the overpricing that would result if ϕ is varied. Instead of speaking of ϕ directly, it is probably more informative to study the implications of different levels of ϕ for the average deviation of irrational from rational beliefs. A simple way to measure distance between the two set of beliefs is the "volatility ratio" defined as:

$$\frac{\frac{\sigma_g}{\sqrt{2}\rho}}{\widehat{\sigma_f}^A}$$

The numerator $\left(\frac{\sigma_g}{\sqrt{2}\rho}\right)$ is the standard deviation of the stationary distribution of the difference in beliefs. The denominator is the average standard deviation of the posterior belief distribution of the rational agent A. A volatility ratio of 1 *implies that "on average"* agent B's beliefs about the posterior mean of f *are within one standard deviation* of the beliefs of agent A. Therefore, the volatility ratio provides a convenient way to translate values of ϕ into statements about the average distance between the beliefs of the two types of agents.

Figure 1 contains a mapping from levels of ϕ to volatility ratios. Levels of ϕ close to 0.5 imply fairly low volatility ratios. As ϕ approaches 0.9 the corresponding ratio reaches 1.4. The mapping becomes fairly nonlinear as one approaches 1. It will be useful to refer to this mapping between ϕ and the volatility ratio throughout.

Figure 2 presents an attempt to measure the extent of interaction between investment and mispricing in the following sense: The model presented in this paper nests the models of Abel and Eberly (1994,1997) and Scheinkman and Xiong (2003) as special cases. One would like to know how much can be gained by their

in (expected) absolute returns, which will vary considerably, because of the time varying influence of irrational traders. It merely means that r will be an upper bound to the expected return.

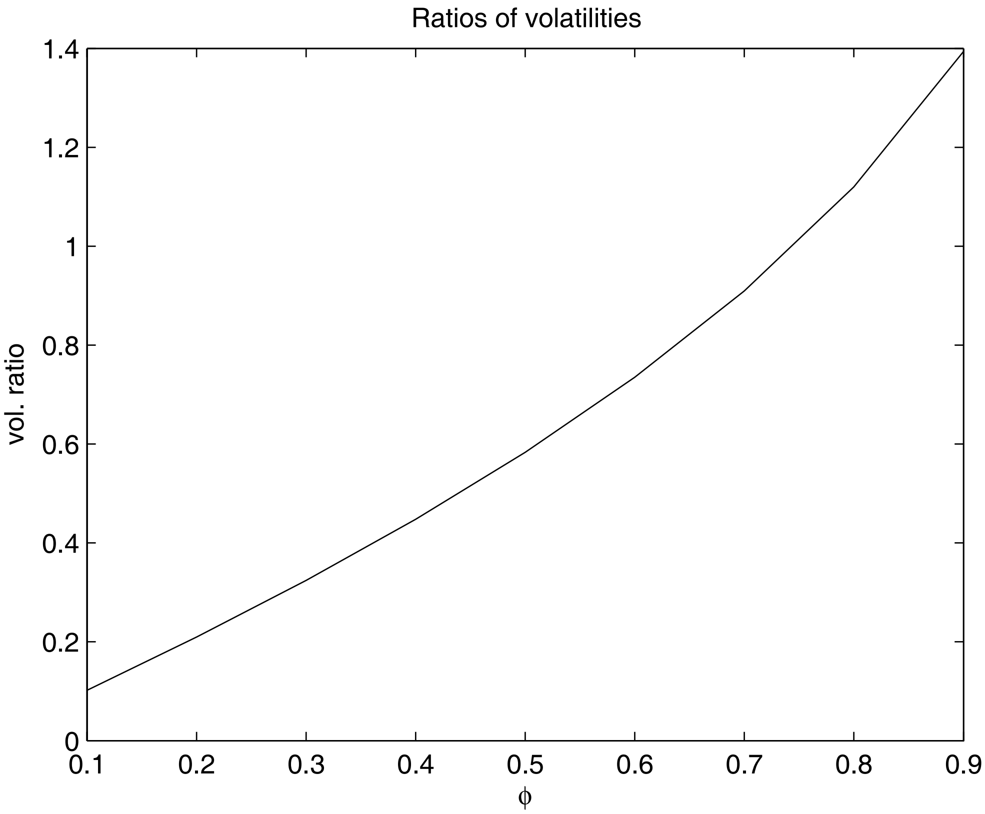

Figure 1: Volatility ratio and ϕ.

interaction. Figure 2 plots marginal and average q for various levels of ϕ and different assumptions about speculative trading.

It is most instructive to start with the case where all market participants are of the type A. Then ϕ may vary but there are no speculative components in the price or marginal q. In this case one obtains the model of Abel and Eberly (1994). Comparing the difference between marginal q and average q amounts to comparing the value of a "tree" that cannot adjust its capital stock to a company that can adjust its capital stock and hence has "growth options". Throughout, when I speak of marginal or average q, I evaluate all quantities at the stationary means of $\widehat{f}_t^A$ and g_t which are given by $\overline{f}$ and 0 respectively. The capital stock is also evaluated at the stationary mean of K_t, assuming all market participants are of the same type and therefore there is no speculation.

By construction, *in the absence of speculation*, marginal q is equal to 1[39] (irrespective of ϕ) . Average q

[39] Since $\overline{f} = r + \delta$

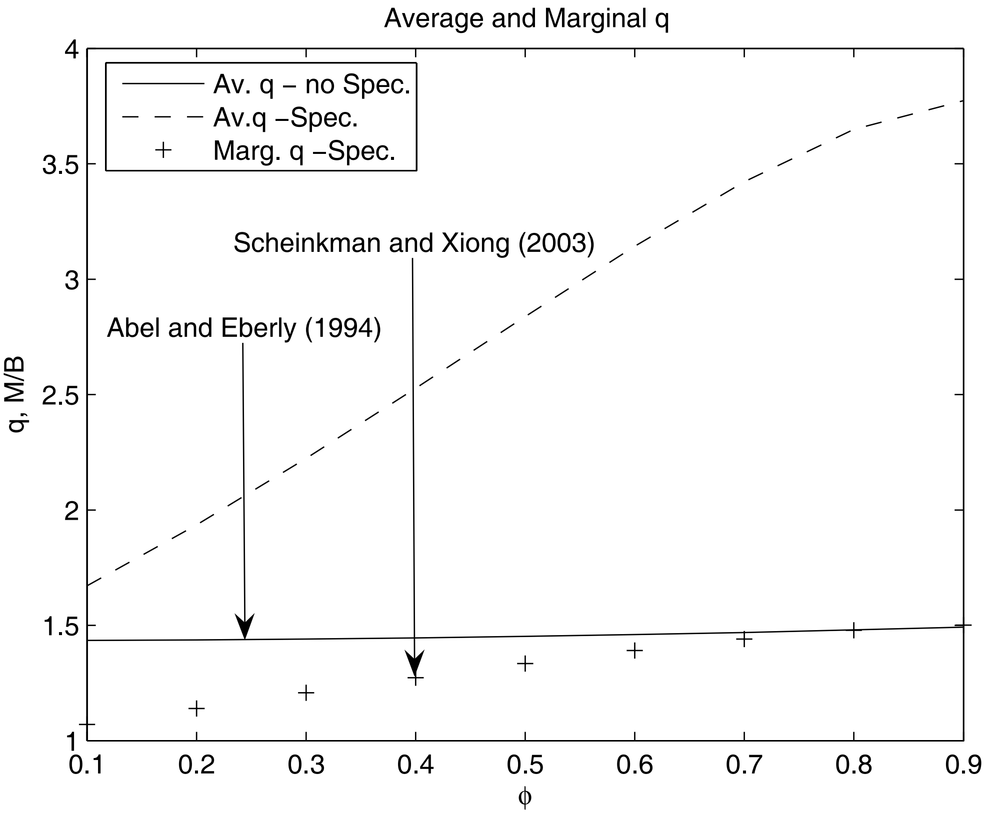

Figure 2: Average and marginal q, in models with and without speculation.

in the absence of speculation is given by the line 'Av-q -No Spec.' and is about 1.5 for most values of ϕ.[40] Hence the value of a firm that has the "option" to invest, compared to a simple "tree", that has no option to invest, is about 50% higher *in the absence of speculation.* The exact numbers are given in Figure 2 and are labeled 'Av-q -No Spec.'.

The next natural exercise is to compare the value of two "trees" that cannot invest. The "trees" are identical except that in the one case there is speculation, and in the other case there is none. This amounts to comparing "marginal q" when there is speculation to marginal "q" when there is no speculation. In the latter case marginal q is identically 1. The exact numbers for marginal q in the presence of speculation are given in Figure 2 and are labeled 'Marg-q -Spec.'. Effectively, this comparison is identical to the comparisons performed in Scheinkman and Xiong (2003) where they compare values of "trees" in the absence of investment.[41]. The effect of speculation raises the value of marginal q between 10% and 50%, depending on

[40](Notice that average q rises somewhat with ϕ even though there is no speculation and all agents have the same beliefs. This is because ϕ affects somewhat the variance of the beliefs of agent A)

[41]It is important to note that compared to Scheinkman and Xiong (2003) the numbers reported here for the comparison

ϕ.

The above two comparisons effectively present two special cases of the model, since *either speculation or investment were excluded*. This was instructive because it helped uncover the models of Abel and Eberly (1994) and Scheinkman and Xiong (2003) as special cases. Once both forces are allowed (i.e. both investment and speculation) then the numbers for average q increase by several orders of magnitude, as is demonstrated by the line 'Average q-Spec.'.

This occurs because the two forces interact: speculation increases the need for investment so that the ability to adjust the capital stock (as captured by growth options) becomes more valuable. Moreover, the growth options themselves depend on the f_t which is the parameter that agents disagree about, thus speculative components are built into the growth options as well (see the appendix for details). As such this second effect reinforces the first. The result is a significantly increased level of average q, reflecting the increased value of the growth options.

To recap, the model appears to be able to produce fairly strong interactions between investment and speculation. The amplification works through growth options that seem to become more important when there is speculation in the market. This appears to be a key prediction of the model and a potential reson for the very large levels of q (or market-to-book in finance jargon) that one observes during speculative episodes:

Figure 3 reports results on the ability of the model to produce both reasonable book-to-market ratios and predictability. I simulated paths of 2300 companies over 27 years in order to repeat the experimental setting of Fama and French (1992). In doing so, I assumed that all companies are identical, except for ϕ[42].The returns of these companies and the book-to-market ratios were simulated under the assumption that for 75% of the companies there is no disagreement ($\phi = 0$) whereas for 25% of these companies agents disagree with $\phi = 0.9$. I then calculated equal weighted returns for 10 portfolios formed on book-to-market as described in Fama and French (1992). Figure 3 plots the resulting returns and compares them to the results reported in Fama and French (1992)[43]. I focused only on the portfolios with the 5 lowest B/M ratios since this paper is concerned with overpricing.

The circles in Figure 3 depict simulation results, while the crosses reproduce the Fama and French (1992) data. The important result of this figure is that for both the simulations and the data the scatter of points is aligned *on a line with similar slope*. Fama-MacBeth regressions produce regression coefficients of roughly 0.44 compared to 0.5 reported in Fama and French (1992).

between the two levels of marginal q are smaller. This is due mostly to the fact that in the present paper there is depreciation that makes the "effective" interest rate $r + \delta = 0.13$.

[42] A number of initial years (prior to the 37 that form the simulation study) was dropped to make sure that initial capital stocks, fundamentals and disagreement are drown from the stationary distribution.

[43] To compare the results I subtracted a 7.2% annual (or 0.6% monthly) from the results in Fama anf French (1992) in order to compute real returns

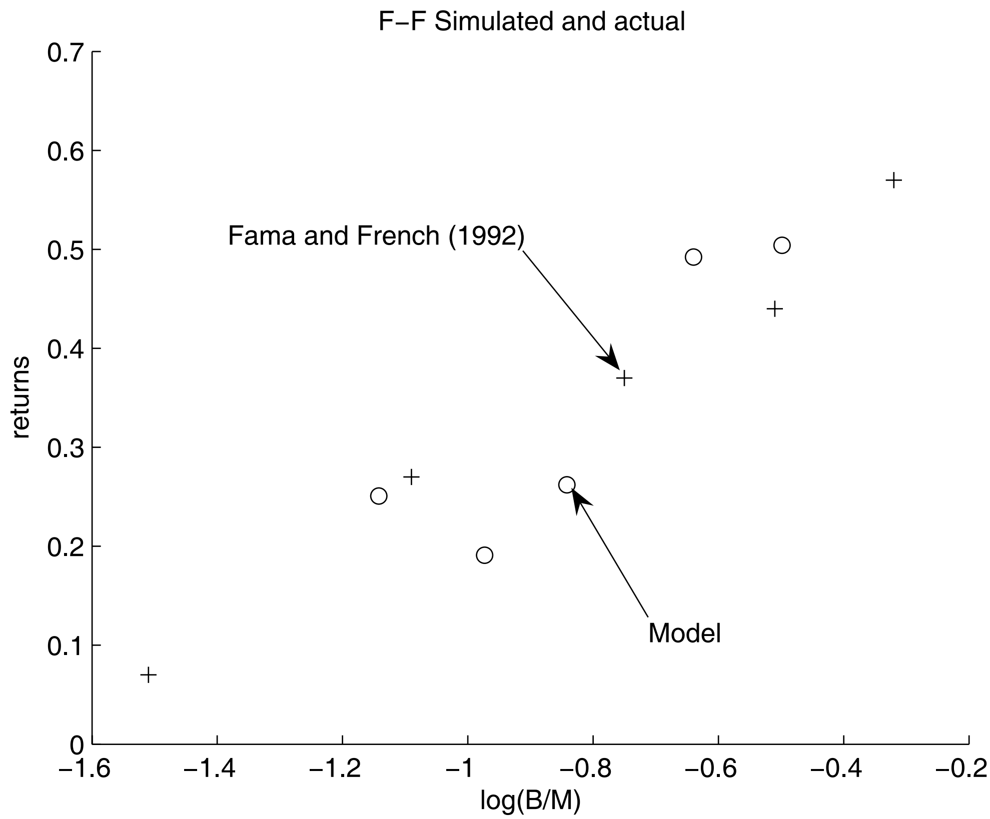

Figure 3: Simulated returns and actual returns on portfolios sorted on B/M.

It is also interesting to note that the simulated data produce a distribution of book-to-market ratios that are not very far away from the numbers reported in Fama and French (1992). For the portfolios with the lowest log(b/m) in Fama and French (1992)[44] the log(b/m) is around -1.5, whereas in the simulations the respective number is around -1.2.

These results suggest that the present model can produce degrees of predictability very similar to the ones observed in the data. Perhaps a more interesing conclusion is that the present model needs to assume that only a small number of companies needs to be overpriced in order to explain the data. However, the disagreement in these companies needs to be relatively large: $\phi = 0.9$ inplies a volatility ratio of 1.4 which seems somewhat large.

[44] I ignored portfolio 1B in their study and focused on portfolio 1A in order to avoid outliers.

6 Conclusion

In this paper I develop a framework to study the interactions between speculative trading and investment. In the model agents have heterogenous beliefs, and are prevented from short-selling. Thus, speculative components emerge endogenously. If firms maximize share prices (as is typically assumed in neoclassical investment theory), then investment reacts to speculative overpricing. Marginal q will endogenously contain a speculative bubble and hence will be higher than the appropriately discounted marginal product of capital under *any* market participant's beliefs. This speculative component will be higher, when belief heterogeneity increases. This may help explain why the investment behavior of young (high tech companies for example) is particularly affected during speculative episodes: The marginal product of capital in such untested ventures is likely to leave room for disagreement.

Somewhat surprisingly, the level of investment arising from share price maximization is efficient. A central planner that cannot alter the underlying asset structure and the associated trading constraints will effectively replicate the market outcome when choosing investment. This is a surprising result since it suggests that asset price bubbles do not necessarily justify policy intervention.

The link between speculation and investment is likely to break down if investment is controlled by shareholders lacking perfect market access. I model imperfect access as a cost these shareholders must pay in order to access markets. This may happen for a variety of reasons: capital gains taxes, losses of private benefits associated with control, nonlinear price pressure etc. If these costs become very large then the optimal investment policy converges to the optimal policy one would obtain in the complete absence of speculation ("long-termism"). In this paper long-termism and short-termism are not mutually exclusive alternatives, but present the extreme limits of a continuum of possibilities, depending on the cost of access to financial markets by controlling shareholders.

Beyond providing a new formalization of these notions, I show that endogeneizing investor horizons allows one to address very practical issues. For instance, I show that a capital gains tax appears to be particularly well suited to prevent the "pass-through" of speculative components to investment.

According to the model , investment might also help distinguish between rational and behavioral theories of so-called asset pricing "anomalies", since it can act as a "filter" of speculative components. I argue that both approaches will yield similar predictions if investment is affected by speculation, since they will both predict that investment should co-move with expected returns. The benchmark (short-termist) case of the present model is consistent with most of the facts that have been put forth as evidence in favor of rational asset pricing theories, even though it is a behavioral model. This is a negative result, since it suggests that investment *might not* "filter out" speculative components. On the positive side however, the model suggests that the investment behavior of firms with large "long-termist" controlling shareholders is unlikely to be affected by speculation, and hence can be used to disentangle rational variation in risk premia from speculative components.

Another finding of the paper is that investment tends to considerably amplify the effects of speculation on asset prices. The ability to invest becomes particularly valuable during a speculative episode, thus growth options increase significantly in value. This may help explain why the most commonly used measures of growth options (like market-to-book, price-to-earnings etc.) tend to reach record high levels during speculative episodes.

There are several directions that are not explored in this paper. A first major issue concerns the financing side in a speculative market. As Panageas (2005) shows, the Modigliani Miller (MM) Theorem is not violated in the setup of this paper. Farhi and Panageas (2005) introduce endogenous financing constraints to obtain a violation of the MM Theorem and show that financing constraints can significantly alter the welfare implications of the model. In their model, speculation acts like a substitute for collateral which can help alleviate contractual inefficiencies arising from the financing constraint. They conclude that even a *paternalistic* social planner, might find welfare improvements in the presence of speculation.

A second issue is whether the theory developed above could be used to derive new empirical tests. Panageas (2005) extends this theory to allow for linear homogenous adjustment costs, and shows that marginal and average q are equated. Unfortunately, one cannot solve explicitly for the price in that case. However the usual relation between investment and average q continues to hold, despite the speculative components. Panageas (2005) derives appropriate Euler and first order conditions that are implied by the present framework and uses them to test short-termism vs. long-termism in the context of a "natural experiment" in the 1920's.

A third question concerns the implications of this theory for executive compensation. This paper analyzed investment in a dynamic q-theoretic environment, under the assumption that investment policy is determined in the best interest of some group of shareholders. In reality investment decisions are made by the management of a firm.. Bolton, Scheinkman and Xiong (2003) analyze the implications of these considerations. Their analysis provides a potential explanation for the observed compensation schemes which were widespread during the speculative episode of the late 1990s.

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A Appendix: Proofs

A.1 Proofs for section 2.2

The essential difficulty in proving proposition 1 is the nonlinearity introduced by the square root in (2). It is well known in the literature on filtering that problems of these kind do not admit a finite dimensional characterization. A particularly popular approach in the engineering literature to deal with this difficulty is to use the so-called "extended Kalman Filter", which is proposed in Jazwinski (1970)[45]. The key idea behind this filter is to focus on the first two moments of the posterior distribution and ignore the feedback from moments higher than 2.

Applying the extended Kalman filter to the filtering problem of agent A (the formulas are similar for agent B) one obtains the system of equations:

$$\begin{aligned} d\widehat{f}_t^A &= -\lambda\left(\widehat{f}_t^A - \overline{f}\right)dt + \frac{\phi\sigma_s\sigma\sqrt{\frac{\widehat{f_t}^A}{\overline{f}}} + \gamma_t^A}{\sigma_s^2}\left(ds^A - \widehat{f}_t^A dt\right) \\ &\quad + \frac{\gamma_t^A}{\sigma_s^2}\left(ds^B - \widehat{f}_t^A dt\right) + \frac{\gamma_t^A}{\sigma_D^2}\left(\frac{dD_t}{K_t} - \widehat{f}_t^A dt\right) \end{aligned} \tag{32}$$

$$\frac{d\gamma_t^A}{dt} = -2\left(\lambda + \phi\frac{\sigma}{\sigma_s}\sqrt{\frac{\widehat{f_t^A}}{\overline{f}}}\right)\gamma_t^A + (1-\phi^2)\sigma^2\frac{\widehat{f_t^A}}{\overline{f}} - \left(\gamma_t^A\right)^2\left(\frac{2}{\sigma_s^2} + \frac{1}{\sigma_D^2}\right) \tag{33}$$

where $\widehat{f_t^A}$ denotes the posterior mean of agent A about f_t and γ_t^A the posterior variance.

One interesting observation is the extended Kalman filter and the regular Kalman filter coincide if one sets $\widehat{f_t^A} = \overline{f}$. One can easily check that in this case equation (32) coincides with the corresponding equation in Scheinkman and Xiong (2003), while setting $\frac{d\gamma_t^A}{dt} = 0$ in equation (33) (so as to obtain the stationary variance of beliefs) and solving for γ_t^A gives the same expression as in Scheinkman and Xiong (2003).

Assuming that λ is relatively large and σ_f relatively small, then $\sqrt{\frac{\widehat{f_t^A}}{\overline{f}}} \sim 1$ and one can rewrite (33) as:

$$\frac{d\gamma_t^A}{dt} = -2\left(\lambda + \phi\frac{\sigma}{\sigma_s}\right)\sqrt{\frac{\widehat{f_t^A}}{\overline{f}}}\gamma_t^A + (1-\phi^2)\sigma^2\frac{\widehat{f_t^A}}{\overline{f}} - \left(\gamma_t^A\right)^2\left(\frac{2}{\sigma_s^2} + \frac{1}{\sigma_D^2}\right) \tag{34}$$

Equation (34) is an ordinary differential equation for a given path of $\widehat{f_t^A}$. One can solve it explicitly. However, one can get some insights on certain properties of the solution by "integrating" forward to obtain:

$$\gamma_t^A = \gamma_{t_0}^A e^{-2\int_{t_0}^t w_s ds} + \int_{t_0}^t e^{-2\left(\int_\xi^t w_s ds\right)}\left[(1-\phi^2)\left(\frac{\sigma^2}{\overline{f}}\right)\widehat{f_\xi^A} - \left(\frac{2}{\sigma_s^2} + \frac{1}{\sigma_D^2}\right)\gamma_\xi^2\right]d\xi$$

[45] Unfortuanately, this filter does not make a claim to approximate the optimal non-linear filter, even though in applications it seems to have very good performance. Recent literature discuss the properties and the efficiency of this filter for "small" noise expansions.

where:

$$w_t = \left(\lambda + \phi \frac{\sigma}{\sigma_s}\right) \sqrt{\frac{\widehat{f_t^A}}{\overline{f}}}$$

As $t_0 \to -\infty$ one obtains

$$\gamma_t^A = \int_{-\infty}^{t} e^{-2\left(\int_\xi^t w_s ds\right)} \left[(1-\phi^2) \left(\frac{\sigma^2}{\overline{f}}\right) \widehat{f_\xi^A} - \left(\frac{2}{\sigma_s^2} + \frac{1}{\sigma_D^2}\right) \left(\gamma_\xi^A\right)^2 \right] d\xi$$

In other words γ_t is an infinite moving average of past values of $\widehat{f}_t$ and γ_t^2. As long as mean reversion is strong (high λ, small σ_f) only the most recent values of $\widehat{f}_t$ and γ_t^2 will "matter", and this motivates the approximation:

$$\gamma_t^A = \frac{(1-\phi^2)\left(\frac{\sigma^2}{\overline{f}}\right) \widehat{f_t^A} - \left(\frac{2}{\sigma_s^2} + \frac{1}{\sigma_D^2}\right) \left(\gamma_t^A\right)^2}{2\left(\lambda + \phi \frac{\sigma}{\sigma_s}\right) \sqrt{\frac{\widehat{f_t^A}}{\overline{f}}}}$$

which can be solved to yield:

$$\gamma_t^A = \sqrt{\frac{\widehat{f_t^A}}{\overline{f}}} \widetilde{\gamma} \tag{35}$$

where:

$$\widetilde{\gamma} = \frac{\sqrt{\left(\lambda + \phi \frac{\sigma}{\sigma_s}\right)^2 + (1-\phi^2)\left(2\frac{\sigma^2}{\sigma_s^2} + \frac{\sigma^2}{\sigma_D^2}\right)} - \left(\lambda + \phi \frac{\sigma}{\sigma_s}\right)}{\frac{2}{\sigma_s^2} + \frac{1}{\sigma_D^2}}$$

Replacing (35) into (32) leads to the approximate belief processes:

$$d\widehat{f_t^A} = -\lambda \left(\widehat{f_t^A} - \overline{f}\right) dt + \tag{36}$$

$$\sqrt{\frac{\widehat{f_t^A}}{\overline{f}}} \left[\frac{\phi \sigma_s \sigma + \widetilde{\gamma}}{\sigma_s^2} \left(ds^A - \widehat{f_t^A} dt\right) + \frac{\widetilde{\gamma}}{\sigma_s^2} \left(ds^B - \widehat{f_t^A} dt\right) + \frac{\widetilde{\gamma}}{\sigma_D^2} \left(\frac{dD_t}{K_t} - \widehat{f_t^A} dt\right) \right] \tag{37}$$

$$d\widehat{f_t^B} = -\lambda \left(\widehat{f_t^B} - \overline{f}\right) dt + \tag{38}$$

$$\sqrt{\frac{\widehat{f_t^B}}{\overline{f}}} \left[\frac{\phi \sigma_s \sigma + \widetilde{\gamma}}{\sigma_s^2} \left(ds^B - \widehat{f_t^B} dt\right) + \frac{\widetilde{\gamma}}{\sigma_s^2} \left(ds^A - \widehat{f_t^B} dt\right) + \frac{\widetilde{\gamma}}{\sigma_D^2} \left(\frac{dD_t}{K_t} - \widehat{f_t^B} dt\right) \right] \tag{39}$$

Since $\frac{ds^A - \widehat{f_t^A} dt}{\sigma_s}$, $\frac{ds^B - \widehat{f_t^A} dt}{\sigma_s}$, $\frac{\frac{dD_t}{K_t} - \widehat{f_t^A} dt}{\sigma_D}$ are (standard) Brownian motions in the mind of agents of type A, it will be convenient to define the "total volatility" of $\widehat{f_t^A}$ (or $\widehat{f_t^B}$) by

$$\sigma_f = \sqrt{\left(\frac{\phi \sigma_s \sigma + \widetilde{\gamma}}{\sigma_s}\right)^2 + \left(\frac{\widetilde{\gamma}}{\sigma_s}\right)^2 + \left(\frac{\widetilde{\gamma}}{\sigma_D}\right)^2}$$

which leads to formulas (5) and (4). It is quite interesting to notice that the approximate filters derived are very close to the ones obtained in Scheinkman and Xiong (2003): except for the presence of the square root in (36) and (38), the updating equations are identical to the respective equations in Scheinkman and Xiong (2003).

This is a key observation: As long as mean reversion is strong enoungh, the ratio $\sqrt{\frac{\widehat{f_t^A}}{f}}, \sqrt{\frac{\widehat{f_t^B}}{f}}$ will not differ significantly from 1[46]. Setting $\sqrt{\frac{\widehat{f_t^A}}{f}} = 1, \sqrt{\frac{\widehat{f_t^B}}{f}} = 1$, one can then replicate the arguments in Scheinkman and Xiong (2003) to arrive at the dynamics of the disagreement process g_t^A :

$$dg_t^A \equiv d\widehat{f_t^B} - d\widehat{f_t^A} \simeq -\rho g_t^A dt + \sigma_g dW_g^A \tag{40}$$

where ρ, σ_g are given by:

$$\begin{aligned} \rho &= \sqrt{\left(\lambda + \phi\frac{\sigma}{\sigma_s}\right)^2 + (1-\phi^2)\sigma^2\left(\frac{2}{\sigma_s^2} + \frac{1}{\sigma_D^2}\right)} \\ \sigma_g &= \sqrt{2}\phi\sigma \end{aligned}$$

To compute the covariance between the two processes note that[47]:

$$Var(dg_t^A) = Cov(d\widehat{f_t^B} - d\widehat{f_t^A}, dg_t^A) = Cov(d\widehat{f_t^B}, dg_t^A) - Cov(d\widehat{f_t^A}, dg_t^A)$$

Since

$$dg_t^A = -dg_t^B$$

one gets:

$$Cov(d\widehat{f_t^B}, dg_t^A) = -Cov(d\widehat{f_t^B}, dg_t^B)$$

and since the model is symmetric:

$$Cov(d\widehat{f_t^B}, dg_t^B) = Cov(d\widehat{f_t^A}, dg_t^A)$$

Hence:

$$Var(dg_t^A) = -2Cov(d\widehat{f_t^A}, dg_t^A) \tag{41}$$

or:

$$Cov(d\widehat{f_t^A}, dg_t^A) = -\phi^2\sigma^2 dt$$

Note that the calculations made for the covariance did not rely on anything else except for the symmetry in the model. Hence equation (41) is true for both the exact and the approximate belief model.

Eventually, all of the above arguments are meant to give a rough justification of why in the presence of reasonably strong mean reversion in f_t one can proceed with the above approximations. How well this approximation is likely to perform will depend on the application at hand and the parameters used.

[46] One could derive an alternative approximation to this disagreement process by subtracting df_t^A from df_t^B and then approximating all terms to the first order. Such an approximation would yield something close to the OU process used here for reasonably small ϕ. For simplicity I chose the approximate OU process described in the beginning of this section to be able to compare the results to Scheinkman and Xiong (2003).

[47] I am indebted to Bernard Dumas for providing me with notes that provided the correct calculation of the covariance in the Scheinkman and Xiong (2003) model.

Figure 4 demonstrates the performance of these approximations for the quantitative calibration in section 5 for 400 months of data. The top left figure compares the solution to (33) (obtained by an Euler Scheme) with (35). There are two observations about the top left panel. First, the two volatilities co-move quite closely and second the posterior standard deviation (captured by γ_t^A) does not vary too much.

These two observations help understand the next three panels. The top right panel is depicting the exact solution to the extended Kalman filter obtained by solving the two dimensional system (32) and (33) and the approximate filter obtained using (36).

The two processes basically cannot be disentangled from each other, since they practically coincide. The bottom left panel depicts the performance of the extended Kalman Filter against the actual process f_t. It is easy to see that the extended Kalman Filter performs well in "recovering" the path of f_t. Finally, the bottom left panel depicts the difference in beliefs between agents A and B obtained from the approximate equation (40). Once again, the approximation is sufficiently good that one cannot disentangle the two processes, since they are practically identical. From these simulations it can be reasonably claimed that the approximation used is sufficiently accurate for all practical purposes.

A.2 Proofs for section 3.1

Proof. (*Proposition 2)* I use a standard verification argument to verify that (9) provides the solution to (7). To start, conjecture a value function of the form:

$$P\left(\widehat{f}_t^A, K_t\right) = q^F\left(\widehat{f}_t^A\right) K_t + u^F\left(\widehat{f}_t^A\right) \tag{42}$$

and substitute this conjecture into the Hamilton Jacobi Bellman equation:

$$\max_{i_t}\left[\frac{1}{2}\sigma_f^2\frac{\widehat{f}_t^A}{\overline{f}}P_{ff} - \lambda(\widehat{f}_t^A - \overline{f})P_f + P_K\left(-\delta K + i_t\right) - rP + \widehat{f}_t^A K_t - pi_t - \frac{\chi}{2}(i_t^2)\right] = 0 \tag{43}$$

Solving for i_t gives:

$$i_t = \frac{1}{\chi}\left(P_K - p\right)$$

Plugging this optimal i_t back into (43) it is straightforward to check that (42) satisfies (43) if and only if the functions $q^F\left(\widehat{f}_t^A\right)$ and $u^F\left(\widehat{f}_t^A\right)$ solve the ordinary differential equations:

$$\frac{1}{2}\sigma_f^2\frac{\widehat{f}_t^A}{\overline{f}}q_{ff}^F - \lambda(\widehat{f}_t^A - \overline{f})q_f^F - (r+\delta)q^F + \widehat{f}_t^A = 0 \tag{44}$$

$$\frac{1}{2}\sigma_f^2\frac{\widehat{f}_t^A}{\overline{f}}u_{ff}^F - \lambda(\widehat{f}_t^A - \overline{f})u_f^F - ru^F + \frac{\left(q^F - p\right)^2}{2\chi} = 0 \tag{45}$$

The solution to equation (44) is[48]:

$$q^F\left(\widehat{f}_t^A\right) = \frac{\overline{f}}{r+\delta} + \frac{\widehat{f}_t^A - \overline{f}}{r+\delta+\lambda} \tag{46}$$

[48] In this paper only particular solutions of ODE's will be considered. Economically this means that "rational bubbles" will not be allowed, i.e. terms that grow unboundedly in expectation at the riskless rate. See Abel and Eberly (1997) on this point.

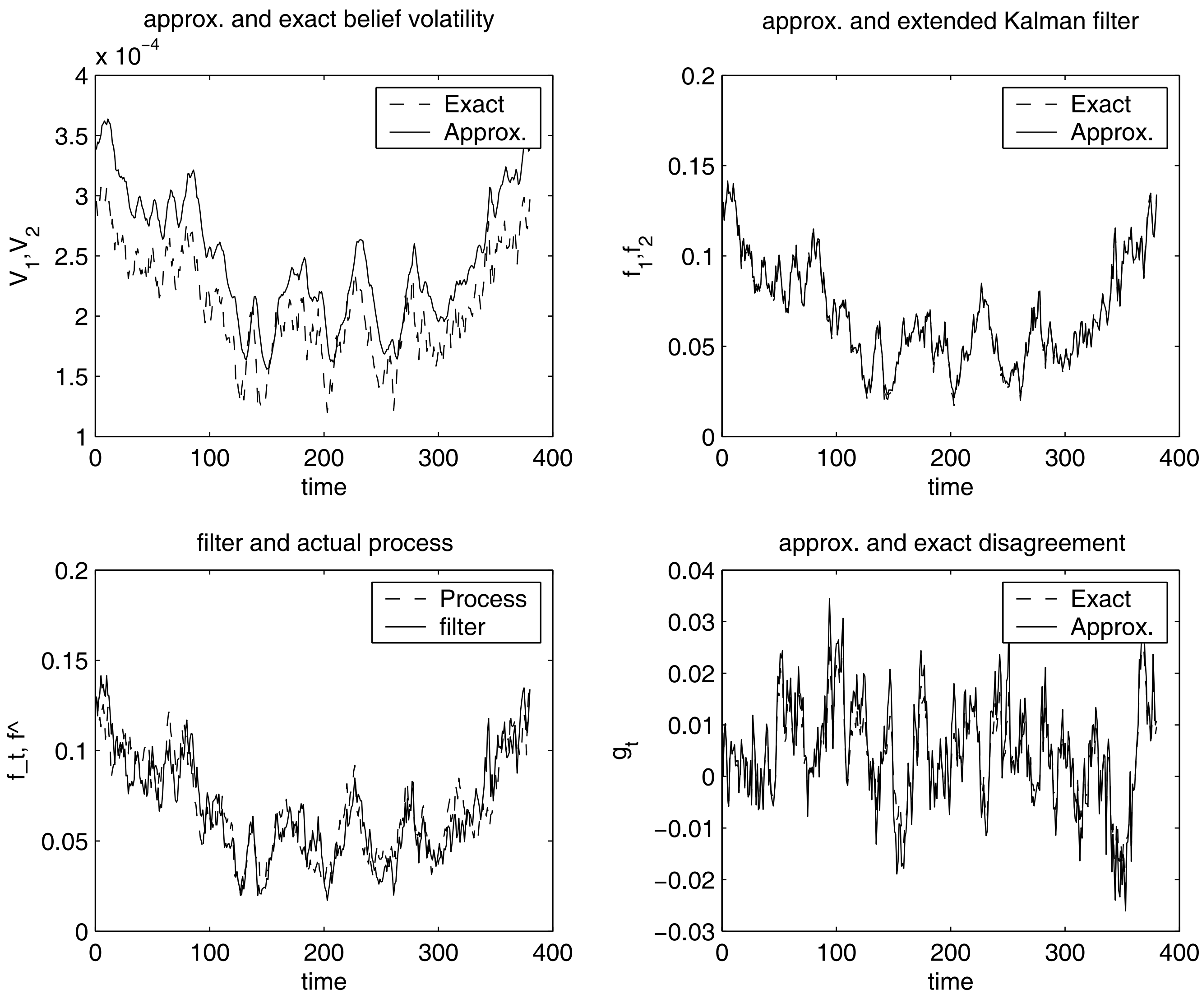

Figure 4: Performance of the Approximate Filter.

whereas the solution to $u^F\left(\widehat{f}_t^A\right)$ is given as

$$u^F(\widehat{f}_t^A) = C_1\left(\widehat{f}_t^A - \overline{f}\right)^2 + C_2\left(\widehat{f}_t^A - \overline{f}\right) + C_3$$

with:

$$\begin{aligned}
C_1 &= \frac{1}{\chi}\frac{1}{r+2\lambda}\frac{1}{2}\left(\frac{1}{r+\delta+\lambda}\right)^2 && (47)\\
C_2 &= \frac{1}{\chi}\frac{1}{(r+\lambda)}\frac{1}{(r+\delta+\lambda)}\left[\left(\frac{\overline{f}(1-\widetilde{p})}{r+\delta}\right)+\frac{\sigma_f^2}{2}\frac{1}{\overline{f}}\frac{1}{r+2\lambda}\frac{1}{r+\delta+\lambda}\right] && (48)\\
C_3 &= \frac{1}{r}\left[\frac{1}{2\chi}\left(\frac{\overline{f}(1-\widetilde{p})}{r+\delta}\right)^2+\sigma_f^2 C_1\right] && (49)
\end{aligned}$$

where $\widetilde{p} = p\frac{r+\delta}{\overline{f}}$.It is also interesting to note that the Feynman Kac Theorem (see Oksendal 1998) implies that:

$$u^F(\widehat{f}_t^A) = \frac{1}{2\chi}E^A\left(\int_t^\infty e^{-r(s-t)}\left(P_K(s)-p\right)^2 ds\right)$$

since u^F solves (45). ■

Proof. *(Lemma 1)* The derivative of u^F w.r.t $\widehat{f}_t^A$ is given by:

$$2C_1\left(\widehat{f}_t^A - \overline{f}\right) + C_2$$

Since $\widehat{f}_t^A$ will always be positive it suffices to check that:

$$-2C_1\overline{f} + C_2 > 0$$

Finally, use the definitions of C_1, C_2 to obtain:

$$-2C_1\overline{f} + C_2 > 0 \Leftrightarrow \widetilde{p} < 1 - \frac{r+\delta}{r+\delta+\lambda}\frac{1}{r+2\lambda}\left(r+\lambda-\frac{\sigma_f^2}{2\overline{f}^2}\right)$$

By assumption (12)

$$\widetilde{p} < 1 - \frac{r+\delta}{r+\delta+\lambda} < 1 - \frac{r+\delta}{r+\delta+\lambda}\frac{1}{r+2\lambda}\left(r+\lambda-\frac{\sigma_f^2}{2\overline{f}^2}\right)$$

since

$$\frac{1}{r+2\lambda}\left(r+\lambda-\frac{\sigma_f^2}{2\overline{f}^2}\right) < 1$$

and the proof is complete. ■

A.3 Proofs for section 3.2

The goal of this section is to prove proposition 3. This involves several steps which are summarized here: Lemmas 2 and 3 along with definition 3 establish some preliminary results. They discuss solutions to certain ordinary differential equations that will show up repeatedly in the proofs. The next step is to conjecture an optimal investment policy whose optimality is verified at the end of the proof. This is done in conjecture 1. A useful intermediate step is to derive the value of the firm to an investor who will never resell her stock, but takes the investment policy given in conjecture 1 as exogenously given. The resulting value is computed in proposition 5 and is used in conjecture 2 to arrive at a conjecture about the equilibrium pricing function. The conjectured equilibrium pricing function decomposes the price into the value to an infinite horizon investor (obtained in proposition 5) and a speculative premium which is shown to be the solution to a specific optimal stopping problem in Lemma 4. Using continuity and smooth pasting proposition 6 shows that the solution to this optimal stopping problem indeed has the form conjectured in conjecture 2. To complete the verification that conjecture 2 provides the actual pricing function one needs proposition 7 which effectively establishes that an agent sells the stock if and only if she perceives that its expected return is less than the interest rate. The proof is concluded by using the conjectured pricing function to show 1) that the conjectured optimal investment policy is optimal given that equilibrium price, 2) that an agent of type A will find it optimal to resell the stock immediately when $\widehat{f}_t^A < \widehat{f}_t^B$ and 3) that her reservation value coincides with the equilibrium price. This is achieved in the proof of proposition 3. The equivalent results for agent B are perfectly symmetric and are omitted.)

Lemma 2 *Consider the linear second order ordinary differential equation (ODE):*

$$\frac{\sigma_g^2}{2}y'' - \rho x y' - (r+\delta)y = 0 \tag{50}$$

Then there are two linearly independent solutions to this ODE and are given by

$$\begin{aligned} y_1(x) &= \begin{cases} U\left(\frac{r+\delta}{2\rho}, \frac{1}{2}, \frac{\rho}{\sigma_g^2}x^2\right) & \text{if } x \leq 0 \\ \frac{2\pi}{\Gamma\left(\frac{1}{2}+\frac{r+\delta}{2\rho}\right)\Gamma\left(\frac{1}{2}\right)} M\left(\frac{r+\delta}{2\rho}, \frac{1}{2}, \frac{\rho}{\sigma_g^2}x^2\right) - U\left(\frac{r+\delta}{2\rho}, \frac{1}{2}, \frac{\rho}{\sigma_g^2}x^2\right) & \text{if } x > 0 \end{cases} \\ y_2(x) &= y_1(-x) \end{aligned}$$

where $U()$ and $M()$ are Kummer's U and M functions[49]. $y_1(x)$ *is positive, increasing* $(y_1'(x) > 0)$, *convex* $(y_1''(x) > 0)$ *and satisfies* $\lim_{x\to-\infty} y_1(x) = 0, \lim_{x\to+\infty} y_1(x) = \infty$. *Accordingly,* $y_2(x)$ *is positive, decreasing and satisfies:*$\lim_{x\to-\infty} y_1(x) = \infty, \lim_{x\to+\infty} y_1(x) = 0$. *Moreover any positive solution that satisfies equation (50) and* $\lim_{x\to-\infty} y(x) = 0$ *is given by:* $\beta y_1(x)$ *where* $\beta > 0$ *an arbitrary constant. Similarly any solution to (50) that is positive and satisfies:* $\lim_{x\to\infty} y(x) = 0$ *is given by* $\beta y_2(x)$ *where* $\beta > 0$ *is an arbitrary constant.*

Proof. *Lemma* 2. The proof effectively replicates arguments in Scheinkman and Xiong (2003) and therefore large portions are omitted. If $v(z)$ is a solution to:

$$z v''(z) + \left(\frac{1}{2} - z\right) v'(z) - \frac{r+\delta}{2\rho} v(z) = 0 \tag{51}$$

[49] These functions are described in Abramowitz and Stegun (1965) p.504.

then $y(x) = v\left(\frac{\rho}{\sigma_g^2}x^2\right)$ satisfies (50). The general solution to equation (51) is given by[50]:

$$v(z) = \alpha M\left(\frac{r+\delta}{2\rho}, \frac{1}{2}, z\right) + \beta U\left(\frac{r+\delta}{2\rho}, \frac{1}{2}, z\right)$$

where the functions $M()$ and $U()$ are given in terms of their power series expansion in 13.1.2. and 13.1.3. of Abramowitz and Stegun (1964). The properties $y_1 > 0$, $y_1' > 0$, $y_1'' > 0$, $\lim_{x\to-\infty} y_1(x) = 0, \lim_{x\to+\infty} y_1(x) = \infty$ can be established as in Scheinkman and Xiong (2003). To establish independence, it remains to show that the Wronskian of the two solutions $(y_1y_2' - y_1'y_2)$ is different from 0 everywhere. This is immediate since $y_1(x), y_2(x) > 0$ and $y_1'(x) > 0, y_2'(x) < 0$. ■

Lemma 3 *Consider the linear (inhomogenous) second order ODE:*

$$\frac{\sigma_g^2}{2}y'' - \rho x y' - (r+\delta)y = -f(x) \tag{52}$$

Using y_1, y_2 from Lemma 2 the general solution to (52) is given as:

$$y(x) = \left[\int_x^{+\infty}\left(\frac{\frac{2}{\sigma_g^2}f(z)y_2(z)}{y_1'(z)y_2(z) - y_1(z)y_2'(z)}\right)dz + C_1\right]y_1(x) + \left[\int_{-\infty}^{x}\left(\frac{\frac{2}{\sigma_g^2}f(z)y_1(z)}{y_1'(z)y_2(z) - y_1(z)y_2'(z)}\right)dz + C_2\right]y_2(x)$$

provided that the above integrals exist. Moreover the derivative $y'(x)$ is given as:

$$y'(x) = \left[\int_x^{+\infty}\left(\frac{\frac{2}{\sigma_g^2}f(z)y_2(z)}{y_1'(z)y_2(z) - y_1(z)y_2'(z)}\right)dz + C_1\right]y_1'(x) + \left[\int_{-\infty}^{x}\left(\frac{\frac{2}{\sigma_g^2}f(z)y_1(z)}{y_1'(z)y_2(z) - y_1(z)y_2'(z)}\right)dz + C_2\right]y_2'(x)$$

Proof. *Lemma* (3)The proof is a basic variations of parameters argument and is omitted. For details see e.g. Section 9.3. in Rainville, Bedient and Bedient (1997). ■

By setting $C_1 = C_2 = 0$ in the above Lemma one gets the so called particular solution to (52), which will depend on σ, ρ, r and the specific functional form of $f(x)$.The following definition will be useful later on:

Definition 3 *For any given $f(x)$, $\sigma_g, \rho, r+\delta$ let:*

$$Z(x; f(x), \sigma_g, \rho, r+\delta) = y(x; f(x), \sigma_g, \rho, r+\delta, C_1 = C_2 = 0)$$

These two Lemmas will be used repeatedly in the proof.

The first step is to make a guess on the form of optimal investment. In particular suppose that the firm's investment policy is given by:

[50] See. Abramowitz and Stegun (1965) p. 504

Conjecture 1 *The optimal investment policy in equilibrium is given as:*

$$i_t = \frac{1}{\chi}\left(\frac{\overline{f}(1-\widetilde{p})}{r+\delta} + \frac{\widehat{f}_t^A - \overline{f}}{r+\delta+\lambda} + 1\{g_t^A > 0\}\frac{g_t^A}{r+\delta+\lambda} + \beta y_1(-\left|g_t^A\right|)\right) \tag{53}$$

where β is a constant that can be determined as:

$$\beta = \frac{1}{2(r+\delta+\lambda)y_1'(0)} \tag{54}$$

and y_1 is the function described in Lemma 2.

The next step will be to determine the equilibrium prices, stopping times for agents etc. *conditional* on the investment policy (53).

To do this it is easiest to start by computing the "infinite" horizon value of the company to an investor of type A conditional on the policy (53). One can focus without loss of generality on the determination of the reservation price for agent A, since the problem for agent B is symmetric. Formally, the goal will be to determine the value of:

$$V(K_t, \widehat{f}_t^A, g_t^A) = E_t^A\left[\int_t^\infty e^{-r(s-t)}\left(\widehat{f}_s^A K_s - p i_s - \frac{\chi}{2}(i_s^2)\right)ds\right] \tag{55}$$

This function captures the value of the asset to an "infinite" horizon investor of type A who takes the conjectured investment policy (53) as given.

Proposition 5 *The solution to (55) is given by:*

$$\begin{aligned} V\left(K_t, \widehat{f}_t^A, g_t^A\right) &= \left[\frac{\overline{f}}{r+\delta} + \frac{\widehat{f}_t^A - \overline{f}}{r+\delta+\lambda}\right]K_t + \\ &+ \left(C_1\left(\widehat{f}_t^A - \overline{f}\right)^2 + C_2\left(\widehat{f}_t^A - \overline{f}\right) + C_3\right) \\ &+ u(g_t^A) \end{aligned}$$

where $u(g_t^A) < 0$ and C_1, C_2, C_3 are the same constants as in Proposition 2.

Proof. *(Proposition 5)* According to the Feynman Kac Theorem[51] the solution $V\left(K_t, \widehat{f}_t^A, g_t^A\right)$ to (55) must satisfy the partial differential equation:

$$\mathcal{A}V + \widehat{f}_t^A K_t - i_t\left(p + \frac{\chi}{2}i_t\right) = 0 \tag{56}$$

where $\mathcal{A}$ is the infinitesimal operator given by:

$$\mathcal{A}V = \frac{\sigma_f^2}{2}\frac{\widehat{f}_t^A}{\overline{f}}V_{ff} + \frac{\sigma_g^2}{2}V_{gg} - \frac{\sigma_g^2}{2}V_{fg} - \lambda(\widehat{f}_t^A - \overline{f})V_f - \rho g_t^A V_g + V_k\left(-\delta K_t + i_t\right) - rV$$

[51] See Oksendal (1998).

(Recall from (41) that $cov(d\widehat{f}_t^A, dg_t^A) = -0.5var(dg_t^A)$.) Next, conjecture a solution of the form:

$$V = h\left(\widehat{f}_t^A\right)K_t + z\left(\widehat{f}_t^A, g_t^A\right)$$

and substitute this conjecture back into (56). This gives conditions that $h()$ and $z()$ have to satisfy in order to satisfy (56). $h()$ has to satisfy:

$$\frac{\sigma_f^2}{2}\frac{\widehat{f}_t^A}{\overline{f}}h_{ff} - \lambda(\widehat{f}_t^A - \overline{f})h_f - (r+\delta)h + \widehat{f}_t^A = 0$$

A particular solution is given by[52]:

$$h\left(\widehat{f}_t^A\right) = \frac{\overline{f}}{r+\delta} + \frac{\widehat{f}_t^A - \overline{f}}{r+\delta+\lambda}$$

while $z\left(\widehat{f}_t^A, g_t^A\right)$ solves the partial differential equation:

$$\frac{\sigma_f^2}{2}\frac{\widehat{f}_t^A}{\overline{f}}z_{ff} + \frac{\sigma_g^2}{2}z_{gg} - \frac{\sigma_g^2}{2}z_{fg} - \lambda(\widehat{f}_t^A - \overline{f})z_f - \rho g_t^A z_g - rz + h(\widehat{f}_t^A)i_t - i_t\left(p + \frac{\chi}{2}i_t\right) = 0 \tag{57}$$

Now rewrite the last two terms on the left hand side of (57) using Conjecture 1 to obtain:

$$h(\widehat{f}_t^A)i_t - i_t\left(p + \frac{\chi}{2}i_t\right) = \frac{1}{2\chi}\left(\frac{\overline{f}}{r+\delta} + \frac{\widehat{f}_t^A - \overline{f}}{r+\delta+\lambda} - p\right)^2 - \frac{1}{2\chi}\left(\widetilde{b}(g_t^A)\right)^2 \tag{58}$$

where

$$\widetilde{b}(g_t^A) = \beta y_1(-\left|g_t^A\right|) + 1\{g_t^A > 0\}\frac{g_t^A}{r+\delta+\lambda}$$

(58) implies that the last two terms on the left hand side of (57) are additively separable in terms involving $\widehat{f}_t^A$ and g_t^A. Thus one can reduce the solution to the PDE (57) to two ordinary differential equations $z_1(\widehat{f}_t^A), u(g_t^A)$ that satisfy:

$$\frac{\sigma_f^2}{2}\frac{\widehat{f}_t^A}{\overline{f}}z_{1ff} - \lambda(\widehat{f}_t^A - \overline{f})z_{1f} - rz_1 + \frac{1}{2\chi}\left(\frac{\overline{f}}{r+\delta} + \frac{\widehat{f}_t^A - \overline{f}}{r+\delta+\lambda} - p\right)^2 = 0 \tag{59}$$

$$\frac{\sigma_g^2}{2}u_{gg} - \rho g u_g - ru - \frac{1}{2\chi}\left(\widetilde{b}(g_t^A)\right)^2 = 0 \tag{60}$$

$z_1(\widehat{f}_t^A)$ solves the exact same ODE as $u^F\left(\widehat{f}_t^A\right)$ in Proposition (2) and thus it will have the same solution:

$$z_1(\widehat{f}_t^A) = C_1\left(\widehat{f}_t^A - \overline{f}\right)^2 + C_2\left(\widehat{f}_t^A - \overline{f}\right) + C_3$$

for the same constants as in Proposition 2. Finally, one can use the results in Lemma 3 to construct the solution to (60). It is given by:

$$u(g_t^A) = Z\left(g_t^A; -\frac{1}{2\chi}\left(\widetilde{b}(g_t^A)\right)^2, \sigma_g, \rho, r\right) < 0$$

■

With an expression for the value of the asset to an agent who does not intend to resell it ever in the future, one can proceed to guess an equilibrium pricing function and an optimal stopping policy. An informed "guess" is that

[52] Obviously there are other solutions that "explode" at the rate r but we will only be interested in bounded solutions in this paper.

the optimal stopping policy is of a particularly simple form: Agent A should sell once $\widehat{f}_t^A < \widehat{f}_t^B$ and agent B should sell once $\widehat{f}_t^B < \widehat{f}_t^A$. This is the case because there are no transactions costs in this model. Accordingly, each agent sells the asset once she stops having the most optimistic beliefs in the market. In particular, using the function V, conjecture that the equilibrium price will have the following form:

Conjecture 2 *Using the function V determined in proposition 5, there exist a function $s(K_t, \widehat{f}_t^A, g_t^A)$ so that equilibrium is given by:*

$$P(K_t,\widehat{f}_t^A, g_t^A) = V\left(K_t, \widehat{f}_t^A, g_t^A\right) + s(K_t,\widehat{f}_t^A, g_t^A) \tag{61}$$

whenever $g_t^A < 0$ and by[53]*:*

$$P(K_t,\widehat{f}_t^A + g_t^A, -g_t^A) = P(K_t,\widehat{f}_t^B, g_t^B) = V\left(K_t, \widehat{f}_t^B, g_t^B\right) + s(K_t,\widehat{f}_t^B, g_t^B) \tag{62}$$

whenever $g_t^A \geq 0$. (The first equality in (62) follows fron the definition of $\widehat{f}_t^B, g_t^B$). Finally, $s(K_t, \widehat{f}_t^A, g_t^A)$ can be expressed as

$$s(K_t,\widehat{f}_t^A, g_t^A) = \beta y_1(g_t^A)K_t + n(g_t^A)\left(\widehat{f}_t^A - \overline{f}\right) + v(g_t^A) \tag{63}$$

for some appropriate functions $n(\cdot), v(\cdot)$.

The rest of the proof is devoted to verifying this conjecture and constructing appropriate functions, so that the conjecture is true. It is interesting to note that the conjectured price decomposes the price into a fundamental and a speculative component. The fundamental component corresponds to the "infinite horizon" valuation of the most optimistic agent. The function $s(K_t, \widehat{f}_t^A, g_t^A)$ is the speculative component in the price. The next Lemma gives a characterization of $s(K_t, \widehat{f}_t^A, g_t^A)$ that will prove very useful.

Lemma 4 *The pricing function in Conjecture 2 holds true if and only if:*

$$s(K_t, \widehat{f}_t^A, g_t^A) = \sup_\tau E e^{-r\tau}\left[\left(\frac{g_\tau^A}{r+\delta+\lambda} + \beta y_1(-g_\tau^A)\right)K_\tau + w(\widehat{f}_\tau^A, g_\tau^A)\right] \tag{64}$$

for $w(\widehat{f}_\tau^A, g_\tau^A)$ given by:

$$w(\widehat{f}_\tau^A, g_\tau^A) = \left[C_2 + n(-g_\tau^A)\right]g_\tau^A + C_1\left(g_\tau^A\right)^2 + u(-g_\tau^A) - u(g_\tau^A) + \left[n(-g_\tau^A) + g_\tau^A 2C_1\right]\left(\widehat{f}_\tau^A - \overline{f}\right) + v(-g_\tau^A)$$

Proof. *(Lemma 4)* The argument is similar to the one given in Scheinkman and Xiong (2003). Using (14) and (61) one obtains that:

$$P(K_t, \widehat{f}_t^A, g_t^A) =$$

[53]The notation $P(K_t, \widehat{f}_t^A + g_t^A, -g_t^A)$ means that the function $P(x_1, x_2, x_3)$ which has three arguments x_1, x_2 and x_3 should be evaluated at $x_1 = K_t$, $x_2 = \widehat{f}_t^A + g_t^A$, $x_3 = -g_t^A$.

$$
\begin{aligned}
&= \sup_{\tau} E\left[\int_0^{\tau} e^{-rt}\left(\widehat{f}_t^A K_t - pi_t - \frac{\chi}{2} i_t^2\right) dt + e^{-r\tau} P(K_\tau, \widehat{f}_\tau^B, g_\tau^B)\right] = \\
&= \sup_{\tau} E\left[\int_0^{\tau} e^{-rt}\left(\widehat{f}_t^A K_t - pi_t - \frac{\chi}{2} i_t^2\right) dt + e^{-r\tau}\left[V\left(K_\tau, \widehat{f}_\tau^B, g_\tau^B\right) + \beta y_1(g_\tau^B) K_\tau + n(g_\tau^B)\left(\widehat{f}_\tau^B - \overline{f}\right) + v(g_\tau^B)\right]\right] = \\
&= \sup_{\tau} E\left[\begin{array}{c} \int_0^{\infty} e^{-rt}\left(\widehat{f}_t^A K_t - pi_t - \frac{\chi}{2} i_t^2\right) dt + \\ e^{-r\tau}\left[V\left(K_\tau, \widehat{f}_\tau^B, g_\tau^B\right) - V\left(K_\tau, \widehat{f}_\tau^A, g_\tau^A\right) + \beta y_1(g_\tau^B) K_\tau + n(g_\tau^B)\left(\widehat{f}_\tau^B - \overline{f}\right) + v(g_\tau^B)\right] \end{array}\right] = \\
&= V\left(K_t, \widehat{f}_t^A, g_t^A\right) + \sup_{\tau} E e^{-r\tau}\left[\begin{array}{c} \left(\frac{g_\tau^A}{r+\delta+\lambda} + \beta y_1(g_\tau^B)\right) K_\tau + C_2 g_\tau^A + u(g_\tau^B) - u(g_\tau^A) + \\ C_1\left[\left(\widehat{f}_\tau^B - \overline{f}\right)^2 - \left(\widehat{f}_\tau^A - \overline{f}\right)^2\right] + n(g_\tau^B)\left(\widehat{f}_\tau^B - \overline{f}\right) + v(g_\tau^B) \end{array}\right] = \\
&= V^A + \sup_{\tau} E e^{-r\tau}\left[\begin{array}{c} \left(\frac{g_\tau^A}{r+\delta+\lambda} + \beta y_1(-g_\tau^A)\right) K_\tau + \left[C_2 + n(-g_\tau^A)\right] g_\tau^A + C_1\left(g_\tau^A\right)^2 + \\ + u(-g_\tau^A) - u(g_\tau^A) + \left[n(-g_\tau^A) + g_\tau^A 2C_1\right]\left(\widehat{f}_\tau^A - \overline{f}\right) + v(-g_\tau^A) \end{array}\right]
\end{aligned}
$$

where the third to last line follows from (55), the next to last line follows from Proposition 5 and (63) and the last line follows from the identities:

$$
\begin{aligned}
\widehat{f}_\tau^B &= \widehat{f}_\tau^A + g_\tau^A \\
g_\tau^B &= -g_\tau^A
\end{aligned}
$$

Defining the function $w(\widehat{f}_\tau^A, g_\tau)$ as:

$$
\begin{aligned}
w(\widehat{f}_\tau^A, g_\tau) &= \left[C_2 + n(-g_\tau^A)\right] g_\tau^A + C_1\left(g_\tau^A\right)^2 + u(-g_\tau^A) - u(g_\tau^A) + \\
&\quad + \left[n(-g_\tau^A) + g_\tau^A 2C_1\right]\left(\widehat{f}_\tau^A - \overline{f}\right) + v(-g_\tau^A)
\end{aligned}
$$

concludes the proof. ■

In light of Lemma 4 it suffices to show that (63) is right in order to verify conjecture 2. In particular one needs to establish that there exist appropriate functions $n(\cdot)$, $v(\cdot)$, and an appropriate constant β such that:

$$
\beta y_1(g_t^A) K_t + n(g_t^A)\left(\widehat{f}_t^A - \overline{f}\right) + v(g_t^A) = \sup_{\tau} E e^{-r\tau}\left[\left(\frac{g_\tau^A}{r+\delta+\lambda} + \beta y_1(-g_\tau^A)\right) K_\tau + w(\widehat{f}_\tau^A, g_\tau^A)\right] \tag{65}
$$

In other words it remains to establish the existence of functions $n(g_t^A), v(g_t^A)$ and a constant β so that the Value function of the optimal stopping problem on the right hand side has the form on the left hand side inside the continuation region, i.e. inside the region where agent A finds it optimal to hold the asset. Moreover, to verify the conjecture one also needs to show that it will be optimal for agent A to resell the asset once $g_t^A \geq 0$. The right hand side of (65) is a three dimensional optimal stopping problem (in $K_t, \widehat{f}_t^A, g_t^A$) and in general there is no method to solve such problems analytically. This is in contrast to one dimensional optimal stopping problems where continuity along with smooth pasting is enough to determine the stopping region and the associated value function in most cases. Fortunately, the simple form of the conjectured continuation and stopping region allows one to solve this problem as is demonstrated in the next proposition:

Proposition 6 *There exist functions $n(g_t^A), v(g_t^A)$ and a constant β such that the function:*

$$s(K_t, \widehat{f}_t^A, g_t^A) = \begin{cases} \beta y_1(g_t^A)K_t + n(g_t^A)\left(\widehat{f}_t^A - \overline{f}\right) + v(g_t^A) \text{ if } g_t^A < 0 \\ \left(\frac{g_t^A}{r+\delta+\lambda} + \beta y_1(-g_t^A)\right) K_\tau + w(\widehat{f}_t^A, g_t^A) \text{ if } g_t^A \geq 0 \end{cases} \tag{66}$$

satisfies

$$\frac{\sigma_f^2}{2}\frac{\widehat{f}_t^A}{\overline{f}}s_{ff} + \frac{\sigma_g^2}{2}s_{gg} - \frac{\sigma_g^2}{2}s_{fg} - \lambda(\widehat{f}_t^A - \overline{f})s_f - \rho g_t^A s_g + s_K\left[-\delta K + \frac{1}{\chi}\left(\frac{\overline{f}(1-\widetilde{p})}{r+\delta} + \frac{\widehat{f}_t^A - \overline{f}}{r+\delta+\lambda} + \beta y_1(g_t^A)\right)\right] - rs = 0$$

if $g_t^A < 0$, is twice continuously differentiable in the region $g_t^A > 0$ and in the region $g_t^A < 0$ and once cont. differentiable everywhere. The constant β is given by:

$$\beta = \frac{1}{2(r+\delta+\lambda)}\frac{1}{y_1'(0)}$$

and the functions $n(\cdot)$ and $v(\cdot)$ are given in the proof.

Proof. *(Proposition 6)* The first step is to construct the Value function under the assumption that both the conjecture for the optimal stopping region and the equilibrium investment strategy is correct. Since for $g_t^A < 0$ the conjectured optimal strategy is to hold the asset, one can formulate a necessary condition for the value function s of the optimal stopping problem on the right hand side of (65). Namely, it has to be the case that inside this region $g_t^A < 0$:

$$\frac{\sigma_f^2}{2}\frac{\widehat{f}_t^A}{\overline{f}}s_{ff} + \frac{\sigma_g^2}{2}s_{gg} - \frac{\sigma_g^2}{2}s_{fg} - \lambda(\widehat{f}_t^A - \overline{f})s_f - \rho g_t^A s_g + s_K\left[-\delta K + \frac{1}{\chi}\left(\frac{\overline{f}(1-\widetilde{p})}{r+\delta} + \frac{\widehat{f}_t^A - \overline{f}}{r+\delta+\lambda} + \beta y_1(g_t^A)\right)\right] - rs = 0 \tag{67}$$

An informed guess is that this PDE has a solution of the form:

$$\beta y_1(g_t^A)K_t + \zeta(\widehat{f}_t^A, g_t^A)$$

Plugging this conjecture into (67) one gets the set of equations:

$$\frac{\sigma_g^2}{2}y_{1gg} - \rho g_t^A y_{1g} - (r+\delta)y_1 = 0 \tag{68}$$

$$\frac{\sigma_f^2}{2}\frac{\widehat{f}^A}{\overline{f}}\zeta_{ff} + \frac{\sigma_g^2}{2}\zeta_{gg} - \frac{\sigma_g^2}{2}\zeta_{fg} - -\lambda(\widehat{f}_t^A - \overline{f})\zeta_f - \rho g_t^A \zeta_g + \beta y_1(g_t^A)i_t - r\zeta = 0 \tag{69}$$

where:

$$i_t = \frac{1}{\chi}\left(\frac{\overline{f}(1-\widetilde{p})}{r+\delta} + \frac{\widehat{f}_t^A - \overline{f}}{r+\delta+\lambda} + \beta y_1(g_t^A)\right)$$

It is immediate that the function $y_1(g_t^A)$ constructed in Lemma 2 satisfies (68) by construction[54]. One can determine a solution to equation (69) by postulating that the solution u is given by:

$$\zeta(\widehat{f}_t^A, g_t^A) = v(g_t^A) + n(g_t^A)\left(\widehat{f}_t^A - \overline{f}\right)$$

[54] Moreover it is the only solution that vanishes as $g_t^A \to -\infty$

and upon substituting this conjecture into (69) it is easy to see that $v(g_t^A)$ and $n(g_t^A)$ have to satisfy the two ordinary differential equations:

$$\begin{aligned} \frac{\sigma_g^2}{2}v_{gg} - \rho g_t^A v_g - rv + \frac{1}{\chi}\left(\beta y_1(g_t^A)\left(\frac{\overline{f}(1-\widetilde{p})}{r+\delta} + \beta y_1(g_t^A)\right)\right) - \frac{\sigma_g^2}{2}n_g &= 0 \qquad (70)\\ \frac{\sigma_g^2}{2}n_{gg} - \rho g_t^A n_g - (r+\lambda)\,n(g_t^A) + \frac{1}{\chi}\left(\frac{\beta}{r+\delta+\lambda}\right)y_1(g_t^A) &= 0 \qquad (71)\end{aligned}$$

In the $g_t^A < 0$ region, the general solution to (70) and (71) is given by:

$$v(g_t^A) = c_1 y_1^{(r)}(g_t^A) + c_2 y_2^{(r)}(g_t^A) + v_P(g_t^A) \qquad (72)$$

$$n(g_t^A) = \widetilde{c_1} y_1^{(r+\lambda)}(g_t^A) + \widetilde{c_2} y_2^{(r+\lambda)}(g_t^A) + n_P(g_t^A) \qquad (73)$$

where $v_P(g_t^A), n_P(g_t^A)$ are the particular solutions to the above equations obtained by Lemma 3[55]:

$$\begin{aligned} v_P(g_t^A) &= Z\left[g_t^A; \frac{1}{\chi}\left(\beta y_1(-|g_t^A|)\left(\frac{\overline{f}(1-\widetilde{p})}{r+\delta} + \beta y_1(-|g_t^A|)\right)\right) - \frac{\sigma_g^2}{2}n_g\left(-|g_t^A|\right), \sigma_g, \rho, r\right]\\ n_P(g_t^A) &= Z\left[g_t^A; \frac{1}{\chi}\left(\frac{\beta}{r+\delta+\lambda}\right)y_1(-|g_t^A|), \sigma_g, \rho, r+\lambda\right]\end{aligned}$$

and $y_1^{(x)}(g_t^A), y_2^{(x)}(g_t^A)$ are defined in an identical way to $y_1(g_t^A)$ and $y_2(g_t^A)$ of Lemma 2 with the only exception that $r+\delta$ is replaced by x. It is also clear that since $y_1(-\left|g_t^A\right|)$,$n_g\left(-|g_t^A|\right)$ are bounded functions[56], the above functions are finite. Moreover, it is easy to check that the particular solutions to the above equations satisfy $v_P'(0) = 0$ and $n_P'(0) = 0$.[57] Finally, to keep only solutions that do not explode as $g_t^A \to -\infty$, set $c_2 = \widetilde{c_2} = 0$.

Observe that $s(\cdot)$ is of the form posited in the left hand side of equation (65). To conclude, it remains to determine the constants $\beta, c_1, \widetilde{c_1}$ in such a way that the resulting value function for the optimal stopping problem (65) is both continuous and cont. differentiable everywhere. For $g_t^A > 0$ the conjecture is that agent A resells to agent B, so that the value function for this case is given by the value of "immediate exercise" i.e.

$$s(K_t, \widehat{f}_t^A, g_t^A) = \left(\frac{g_t^A}{r+\delta+\lambda} + \beta y_1(-g_t^A)\right)K_t + w(\widehat{f}_t^A, g_t^A) \text{ if } g_t^A > 0$$

In each of the two regions ($g_t^A < 0, g_t^A > 0$) the function $s(\cdot)$ is twice cont. differentiable, accordingly continuity and differentiability only needs to be enforced at $g_t^A = 0$. The left limit of $s(\cdot)$ at $g_t^A = 0$ is given by:

$$\beta y_1(0)K_t + v(0) + n(0)\left(\widehat{f}_t^A - \overline{f}\right)$$

[55] Note that in the $g_t^A < 0$ region: $g_t^A = -|g_t^A|$

[56] To see this, proceed as in Scheinkman and Xiong (2003) to establish that the limits of these two functions at $-\infty$ are 0 and that they are increasing functions. This establishes that $y_1(-\left|g_t^A\right|)$,$n_g\left(-|g_t^A|\right)$ are bounded between $(0, y_1(0))$ and $(0, n_g(0))$ respectively.

[57] Since they are symmetric around 0 and continuously differentiable.

whereas the right limit is obtained by evaluating $\left(\frac{g_t^A}{r+\delta+\lambda}+\beta y_1(-g_t^A)\right)K_t+w(\widehat{f}_t^A,g_t^A)$ around $g_t^A=0$. This yields after obvious simplifications:

$$\beta y_1(0)K_t+v(0)+n(0)\left(\widehat{f}_t^A-\overline{f}\right)$$

so that continuity is immediately satisfied. It is straightforward to also verify that the derivatives w.r.t. $\widehat{f}_t^A$ and K_t are continuous at $g_t^A=0$.Hence it remains to enforce continuity of the derivative w.r.t. g_t^A. To this end compute left and right derivatives of (66) (w.r.t. g_t^A) at $g_t^A=0$ and using (72) and (73) along with the fact that $v_P'(0)=0$ and $n_P'(0)=0$, require that the derivative is continuous for any $K_t,\widehat{f}_t^A,g_t^A$ to obtain the conditions

$$\begin{aligned}
\beta y_1'(0) &= \frac{1}{r+\delta+\lambda}-\beta y_1'(0)\\
\widetilde{c_1}y_1^{(r+\lambda)'}(0) &= -\widetilde{c_1}y_1^{(r+\lambda)'}(0)+2C_1\\
2c_1y_1^{(r)\prime}(0) &= -2u'(0)+C_2+n(0)
\end{aligned}$$

which implies that:

$$\begin{aligned}
\beta &= \frac{1}{2(r+\delta+\lambda)}\frac{1}{y_1'(0)}\\
\widetilde{c_1} &= \frac{C_1}{y_1^{(r+\lambda)\prime}(0)}\\
c_1 &= \frac{-2u'(0)+C_2+n(0)}{2y_1^{(r)\prime}(0)}
\end{aligned}$$

With these constants and Proposition 6 along with conjecture 2 and Proposition 5 completes the construction of the candidate price function. Moreover, it is clear that it is continuously differentiable everywhere and twice continuously differentiable except possibly at $g_t^A=0$. ■

To finalize the proof one needs the next proposition:

Proposition 7 *Suppose condition (12) holds and let*

$$\mathcal{L}P=\max_i\left(\mathcal{A}P+\widehat{f}_t^AK_t-i\left(p+\frac{\chi}{2}i\right)\right)\leq 0 \tag{74}$$

where the operator $\mathcal{A}P$ *is defined as*

$$\mathcal{A}P=\frac{1}{2}\frac{\sigma_f^2}{\overline{f}}\widehat{f}_t^AP_{\widehat{f}_t^A\widehat{f}_t^A}-\lambda(\widehat{f}_t^A-\overline{f})P_{\widehat{f}_t^A}+\frac{1}{2}\sigma_g^2P_{g^Ag^A}-\frac{1}{2}\sigma_g^2P_{\widehat{f}_t^Ag^A}-\rho g_t^AP_{g^A}+P_K(-\delta K_t+i_t)-rP$$

Then for $g_t^A\geq 0$:

$$\mathcal{L}P\leq 0$$

Proof. *Proposition (7)*

When $g_t^A \geq 0$, the price according to the maintained conjecture (2) is given by:

$$P(K_t, \widehat{f_t^B}, g_t^B) = V\left(K_t, \widehat{f_t^B}, g_t^B\right) + s\left(K_t, \widehat{f_t^B}, g_t^B\right)$$

Agents of type B hold the shares of the company and according to the perfect competition assumption, these agents perceive that the expected return on the stock is equal to rdt according to their subjective beliefs. Therefore:

$$\max_i \left\{ \frac{1}{2}\frac{\sigma_f^2}{\overline{f}} \widehat{f}_t^B P_{\widehat{f}_t^B \widehat{f}_t^B} - \lambda(\widehat{f}_t^B - \overline{f}) P_{\widehat{f}_t^B} + \frac{1}{2}\sigma_g^2 P_{g^B g^B} - \frac{1}{2}\sigma_g^2 P_{f^B g^B} - \rho g_t^B P_{g^B} + P_K(-\delta K_t + i_t) - rP + \widehat{f}_t^B K_t - i\left(p + \frac{\chi}{2} i\right) \right\} = 0 \tag{75}$$

This fact is a trivial consequence of the perfect competition assumption, and it can be shown directly by using the definition of the equilibrium price. To avoid confusion due to excess notation, it is important to note that P_{ff} (and similarly for all other derivatives) denotes the second partial derivative of the price function $P(K_t, \widehat{f}_t^B, g_t^B)$ w.r.t. its second argument $\left(\widehat{f}_t^B\right)$ evaluated at $\widehat{f}_t^B$. One can further rewrite (75) as $P(K_t, \widehat{f}_t^B, g_t^B) = P(K_t, \widehat{f}_t^A + g_t^A, -g_t^A)$ by using the following two identities:

$$\widehat{f}_t^B = \widehat{f}_t^A + g_t^A \tag{76}$$

$$g_t^B = -g_t^A \tag{77}$$

It will greatly simplify notation to denote P_1 as the (partial) derivative of P w.r.t. its first argument, P_2 as the partial derivative w.r.t. the second argument etc. and rewrite (75) as:

$$\max_i \left\{ \frac{1}{2}\frac{\sigma_f^2}{\overline{f}} \widehat{f}_t^B P_{22} - \lambda(\widehat{f}_t^B - \overline{f}) P_2 + \frac{1}{2}\sigma_g^2 P_{33} - \frac{1}{2}\sigma_g^2 P_{23} - \rho g_t^B P_3 + P_1(-\delta K_t + i_t) - rP + \widehat{f}_t^B K_t - i\left(p + \frac{\chi}{2} i\right) \right\} = 0 \tag{78}$$

Now note that $P(K_t, \widehat{f}_t^B, g_t^B) = P(K_t, \widehat{f}_t^A + g_t^A, -g_t^A)$ along with (76), (77) imply that:

$$\begin{aligned} P_{g^A} &= -P_3 + P_2 \\ P_{g^A g^A} &= P_{33} - 2P_{23} + P_{22} \\ P_{g^A \widehat{f}_t^A} &= -P_{32} + P_{22} \\ P_{\widehat{f}_t^A} &= P_2 \\ P_{\widehat{f}_t^A \widehat{f}_t^A} &= P_{22} \end{aligned}$$

Thus one can rewrite (74) as:

$$\frac{1}{2}\frac{\sigma_f^2}{\overline{f}}\left(\widehat{f}_t^B - g_t^A\right) P_{22} - \lambda(\widehat{f}_t^B - g_t^A - \overline{f}) P_2 + \frac{1}{2}\sigma_g^2 (P_{33} - 2P_{23} + P_{22}) - \frac{1}{2}\sigma_g^2(-P_{32} + P_{22}) - \rho\left(-g_t^B\right)(-P_3 + P_2)$$
$$-rP + \left(\widehat{f}_t^B - g_t^A\right) K_t + \max_i \left\{ P_1(-\delta K_t + i_t) - i\left(p + \frac{\chi}{2} i\right)\right\}$$

Using (78) and making some obvious cancellations, one can rewrite the above expression as:

$$-g_t^A\left[\frac{1}{2}\frac{\sigma_f^2}{\overline{f}}P_{22}+(\rho-\lambda)P_2+K_t\right]$$

Hence to complete the proof, it remains to show that:

$$P_2>0, P_{22}>0, K_t>0$$

The fact that $P_2>0$ is immediate by Lemma 1, and the fact $n(\cdot)\geq 0$, which can be established by arguments similar to Scheinkman and Xiong (2003). $P_{22}=V_{22}=C_1>0$. And K_t is guaranteed to be positive since condition (12) guarantees that investment is always positive. ■

Combining all of the above results, one gets:

Proof. *(Proposition 3)* By Proposition 5, Proposition 7 and equation (56) it is straightforward to verify that the conjecture 2 for $P(\widehat{f}_t^A, g_t^A, K_t)$ satisfies the following properties:

$$\mathcal{L}P \;=0 \quad \text{if } g_t^A<0 \tag{79}$$

$$\mathcal{L}P \;\leq 0 \quad \text{if } g_t^A\geq 0 \tag{80}$$

where:

$$\mathcal{L}P=\max_i\left(\mathcal{A}P+fK-i\left(p+\frac{\chi}{2}i\right)\right)\leq 0 \tag{81}$$

and

$$\mathcal{A}P=\frac{1}{2}\frac{\sigma_f^2}{\overline{f}}\widehat{f}_t^A P_{ff}-\lambda(\widehat{f}_t^A-\overline{f})P_f+\frac{1}{2}\sigma_g^2 P_{gg}-\frac{1}{2}\sigma_g^2 P_{gf}-\rho g P_g+P_K(-\delta K_t+i_t)-rP$$

Notice also that the investment strategy that maximizes (81) satisfies:

$$i_t=\frac{1}{\chi}\left(P_K-p\right)=\frac{1}{\chi}\left(\frac{\overline{f}(1-\widetilde{p})}{r+\delta}+\frac{\widehat{f}_t^A-\overline{f}}{r+\delta+\lambda}+1\{g_t^A>0\}\frac{g_t^A}{r+\delta+\lambda}+\beta y_1(-\left|g_t^A\right|)\right)$$

which coincides with the conjectured investment strategy. Moreover $P(\widehat{f}_t^A, g_t^A, K_t)$ is once cont. differentiable everywhere and also twice cont. differentiable except at $g_t^A=0$. To verify optimality, consider now any policy i_t and any stopping time τ. Then Ito's Lemma implies:

$$e^{-r(\tau-t)}P_\tau=P_t+\int_t^\tau e^{-r(s-t)}\mathcal{A}Pds+\int_t^\tau dM_s$$

where dM_t is a local martingale. Since $P_t\geq 0$ for all t, one can conclude that $E^A\left(\int_0^\tau dM_t\right)\leq 0$ and thus:

$$E^A\left(e^{-r(\tau-t)}P_\tau\right)\leq P_t+E^A\left[\int_t^\tau e^{-r(s-t)}\left(\mathcal{A}P+\widehat{f}_s^A K_s-i_s\left(p+\frac{\chi}{2}i_s\right)ds\right)\right]-E^A\left[\int_t^\tau e^{-r(s-t)}\left(\widehat{f}_s^A K_s-i_s\left(p+\frac{\chi}{2}i_s\right)ds\right)\right]$$

Then the following set of inequalities follows

$$\begin{aligned} P_t &\geq P_t+E^A\left[\int_t^\tau e^{-r(s-t)}\mathcal{L}P_s ds\right] \geq P_t+E^A\left[\int_t^\tau e^{-r(s-t)}\left(\mathcal{A}P+\widehat{f}_s^A K_s - i_s\left(p+\frac{\chi}{2}i_s\right)\right)ds\right] \\ &\geq E^A\left(e^{-r(\tau-t)}P_\tau\right)+E^A\left[\int_t^\tau e^{-r(s-t)}\left(\widehat{f}_s^A K_s - i_s\left(p+\frac{\chi}{2}i_s\right)\right)ds\right] \end{aligned}$$

Thus there is no set of investment policy / stopping policies that can yield more than P_t. Moreover, the conjectured investment and stopping policies turn the above inequalities into equalities and hence must be optimal. In other words whenever $g_t^A < 0$, it is indeed optimal for agent A to hold the asset until $g_t^A \geq 0$ and to invest according to the conjectured investment strategy. Moreover, the conjectured price of the asset coincides with the "reservation value" of agent A whenever $g_t^A < 0$, since she is the "pivotal" agent in those states. The case for $g_t^A > 0$ is perfectly symmetric, with the only exception that agent B is the "pivotal" agent. This shows that given the conjectured equilibrium price the conjectured investment and reselling strategies are optimal and that the conjectured equilibrium prices coincide with the reservation values of the pivotal agent. This concludes the proof. ■

A.4 Proofs for section 4

Proof. *(Proposition 4)*Let

$$dG_t = d(D_t + P_t)$$

denote the total gains process from investing in the stock. The intertemporal budget constraint of agent x is:

$$\frac{dW_t^x}{W_t^x} = \pi_t^x \frac{dG_t}{P_t} + (1-\pi_t^x)rdt - \frac{dC_t}{W_t^x}$$

The short selling constraint implies $\pi_t^x \geq 0$. By the definition of the equilibrium price (24), the expected return to the stock market from the perspective of someone who is holding stock has to be equal to the interest rate. Therefore $\pi_t^x > 0$ implies that:

$$\frac{dG_t}{P_t} = rdt + dM_t$$

where dM_t is a martingale difference process under the beliefs of agent x. If $\pi_t^x = 0$ and the agent is not holding stock, dM_t need not be a martingale difference according to her beliefs. However in those cases the dynamics of her budget constraint are given simply by:

$$\frac{dW_t^x}{W_t^x} = rdt - \frac{dC_t}{W_t^x}$$

and hence dM_t does not even enter the dynamics of the budget constraint. Note that these results hold true for any investment policy of the firm. In sum, the intertemporal budget constraint under agent $x's$ beliefs can be written as:

$$\frac{dW_t^x}{W_t^x} = rdt - \frac{dC_t}{W_t^x} + \pi_t^x dM_t$$

Integrating and imposing a transversality condition gives:

$$E^x \int_t^\infty e^{-r(s-t)} dC_s = W_0^x$$

since[58]:

$$E^x \int_t^\infty e^{-r(s-t)} \pi_s^x dM_s = 0$$

But:

$$E^x \int_t^\infty e^{-r(s-t)} dC_s = W_t^x = N_B^x P_t^B + N_S^x P_t$$

where N_B^x is agent $x's$ endowment of bonds and P_t^B is the price of the bond (which is obviously not affected by the investment policy of the firm, since r is exogenously given). N_S^x is the agent's endowment of stock and P_t is the price of the stock. Thus each agent's welfare is an affine function of the price of the stock (P_t), with $N_S^x \geq 0$. ■

[58] Assume also the technical restriction on the portfolios:

$$E^i \int_0^\infty e^{-rs} \left(\pi_s^i\right)^2 ds < \infty$$

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