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A Simple Model of Intertemporal Capital Asset Pricing and Its Implications for the Fama-French Three-Factor Model*

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Abstract

Characterizing the instantaneous investment opportunity set by the real interest rate and the maximum Sharpe ratio, a simple model of time varying investment opportunities is posited in which these two variables follow correlated Ornstein-Uhlenbeck processes, and the implications for stock and bond valuation are developed. The model suggests that the prices of certain portfolios that are related to the Fama-French HML and SMB hedge portfolios will carry information about investment opportunities, which provides a potential justification for the risk premia that have been found to be associated with these hedge portfolios. Evidence that the FF portfolios are in fact associated with variation in the investment opportunity set is found from an analysis of stock returns. Further evidence of time variation in the investment opportunity set is found by analyzing bond yields, and the time variation in investment opportunities that is identified from bond yields is shown to be associated both with the time-variation in investment opportunities that is identified from stock returns and with the returns on the Fama-French hedge portfolios. Finally, both pricing kernel and tracking portfolio approaches are used to provide estimates of the magnitude of the HML and SMB risk premia implied by our simple model.

1 Introduction

In the short run, investment opportunities depend only on the real interest rate and the slope of the capital market line, or Sharpe ratio, as in the classic Sharpe-Lintner Capital Asset Pricing Model. The slope of the capital market line depends in turn on the risk premium and volatility of the market return, and there is now strong evidence of time variation both in the equity risk premium and in market volatility, implying variation in the market Sharpe ratio, as well as in the real interest rate. Kandel and Stambaugh (1990), Whitelaw (1994, 1997), and Perez-Quiros and Timmermann (2000) all demonstrate significant cyclical variation in the market Sharpe ratio.[1] Given the evidence of time variation in short-run investment opportunities, four questions present themselves. First, how should future cash flows be valued when the investment opportunity set varies over time? Since Merton (1973) it has been clear that the empirically challenged single period CAPM is unlikely to provide reliable guidance under these circumstances although, as Cornell *et al.* (1997, p12) point out, the CAPM is the only asset pricing model that has been applied widely in practice. Secondly, is it possible that it is time variation in investment opportunities that accounts largely for the empirical failure of the single period CAPM as Merton's analysis would suggest,[2] and is it possible that the empirical success of the Fama-French three factor model[3] is related to the ability of this model to capture the risk premia associated with time variation in investment opportunities?[4] Thirdly, to what extent is variation in stock prices due to variation in investment opportunities rather

[1]Other studies that identify significant predictors of the equity risk premium include: Lintner (1975), Fama and Schwert (1977) for interest rates; Campbell and Shiller (1988) and Fama and French (1988) for dividend yield; Fama and French (1989) for term spread and junk bond yield spread; Kothari and Shanken (1999) for Book-to-Market ratio.

[2]It is possible for the single period CAPM to hold even with time-varying investment opportunities. See Rubinstein (1976) and Constantinides (1980).

[3]Daniel and Titman (1997) question the role of the FF portfolios, but Davis *et al.* (2000) confirm the original FF findings using a larger data set.

[4]Lewellen (2000, p38) remarks that "the risk factors captured by the size and B/M mimicking portfolios have not been identified. The rational pricing story will remain incomplete, and perhaps unconvincing, until we know more about the underlying risks."

than to variation in cash flow expectations?[5] Fourthly, can consideration of time variation in investment opportunities account for the finding that individual stock betas show little dependence on earnings or cash flow covariances with the market (Campbell and Mei, 1993), but are strongly related to the duration of equity cash flows (Cornell, 1999)?

In this paper, we are concerned with the first two issues: how cash flows should be valued when there is time-variation in investment opportunities, and the relation between time variation in investment opportunities and the Fama-French three factor model. First, we develop a parsimonious model of cash flow valuation that takes account of time-variation in investment opportunities and show how this can be used to value both equity claims and bonds within an integrated framework.

To address the second issue, we use the model to show that the (scaled) prices of the portfolios that Fama and French use to construct their HML and SMB hedge portfolio returns, as well as the term spread, are likely to contain information about investment opportunities. Using data on US stock returns, we find that these variables do indeed have predictive power for both the real interest rate and the Sharpe ratio. This finding is consistent with the existence of risk premia associated with loadings on these hedge portfolio returns, if risk premia are determined by a Merton (1973) type Intertemporal Capital Asset Pricing Model (ICAPM). We also estimate the model by applying a Kalman filter to the yields of pure discount Treasury bonds and find further evidence of time variation in the estimated market Sharpe ratio as well as in the real interest rate, which is further confirmed when we re-estimate the model using a time series of Sharpe ratio estimates from equity returns in addition to the Treasury bond yields. We also show that the estimates of the investment opportunity set statistics that are derived from the bond yield data are related, both to estimates of these statistics that are derived from the equity market data, and to the scaled prices of the Fama-French portfolios, and that innovations in the estimated opportunity set statistics are correlated with the returns on

[5]This issue has been examined by Campbell using a VAR framework in a series of papers including Campbell and Ammer (1993) and Campbell and Shiller (1988).

the Fama-French hedge portfolios, HML and SMB. All this evidence is consistent with the ICAPM accounting for the existence of the risk premia on these portfolios. Finally, we provide quantitative estimates of the HML and SMB risk premia that are implied by the model, using both the pricing kernel and the tracking portfolio approach. When we use the tracking protfolio approach we conclude that about 30-50% of the risk premia associated with these portfolios can be attributed to their role as investment opportunity set hedges.

Explanations that have been offered previously for the empirical success of the Fama-French three-factor model are based, first, on problems in the measurement of beta, secondly, on the ICAPM, and thirdly on the APT. Berk, Green and Naik (1999) and Gomes, Kogan, and Zhang (2000) develop models that explain the Fama-French results on the basis of problems in the measurement of beta. In these models firm betas are stochastic,[6] and there is a statistical relation between average returns, *unconditional* betas and other firm characteristics such as size and book-to-market ratio, which could be captured by a model such as the Fama-French three-factor model. The ICAPM has been suggested by Fama and French (FF) themselves as one possible reason for the premia that they find to be associated with loadings on the SMB and HML hedge portfolios that are formed on the basis of firm size and book-to-market ratio. In FF (1995) they argue that the premia, "are consistent with a multi-factor version of Merton's (1973) intertemporal asset pricing model in which size and BE/ME proxy for sensitivity to risk factors in returns." An APT interpretation has also been suggested by Fama and French who argue that "if the size and BE/ME risk factors are the results of rational pricing, they must be driven by common factors in shocks to expected earnings that are related to size and BE/ME." In contrast to the ICAPM, the APT interpretation provides an essentially single period rationale for the premia associated with these portfolios. FF find little support for the APT interpretation.[7]

[6] In these papers betas are measured with respect to the pricing kernel.

[7] However, in results not reported here, we also provide some supportive evidence for the APT inter-

Other authors have also suggested that the Fama-French portfolios may be related to the investment opportunity set and that their risk premia may therefore be justified by appeal to the ICAPM. For example, Liew and Vassalou (2000) report that annual returns on the HMB and SML hedge portfolios predict GDP growth in several countries, and Vassalou (2002) shows that a portfolio designed to track news about future GDP growth captures much of the explanatory power of the Fama French portfolios.[8]

While previous authors have suggested that scaled asset prices may have predictive power for returns,[9] we suggest in this paper that *ratios* of scaled asset prices have predictive power. We also construct a simple valuation model with time-varying riskless rates and risk premia, and relate this to cross-sectional asset pricing results,[10] and relate bond market based estimates of the Sharpe ratio and real interest rate to the FF hedge portfolio returns.

Papers that are related to our general valuation framework in allowing for time-variation in interest rates and risk premia include Ang and Liu (2001) and Bekaert and Grenadier (2000). The valuation model in this paper differs from the models presented in these papers chiefly in its parsimonious specification of the relevant state variables. In this paper, the state is completely described by the real interest rate and the instantaneous Sharpe ratio: this is a natural starting point for the choice of state variables in intertemporal asset pricing since, in a diffusion setting, these two variables provide a complete description of the instantaneous investment opportunity set.[11]

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petation, by showing that the FF portfolio returns are associated with returns on assets that are not included in the conventional measure of the (stock) market portfolio. Consistent with this, Heaton and Lucas (2000) find evidence that the inclusion of entrepreneurial income in an asset pricing model reduces the importance of the FF portfolios. See also Polk (1998), Jagannathan and Wang (1996), and Jagannathan *et al.* (1998).

[8]See also Chen (2001).

[9]See Ball (1978), Miller and Scholes (1982), Berk (1995), and Kothari and Shanken (1997).

[10]Campbell (1993) also points out that state variables that are priced in the cross-section must, in a rational model, have predictive power - in his case for future consumption.

[11]Nielsen and Vassalou (2001) demonstrate that investors hedge only against stochastic changes in the slope and the intercept of the instantaneous capital market line, which implies that only variables that forecast the real interest rate and the Sharpe ratio will be priced.

The remainder of the paper is organized as follows. In Section 2, we construct a simple valuation model with a stochastic interest rate and a stochastic Sharpe ratio. In Section 3, we specialize the model to the ICAPM and show that ratios of security prices can be used as instruments for the state variables that describe the short term investment opportunity set. In Section 4, we describe the estimation approaches and the construction of the data that are used in the subsequent empirical tests. Empirical results are reported in Section 5. Section 6 concludes.

2 Valuation with Stochastic Investment Opportunities

The value of a claim to a future cash flow depends on both the characteristics of the cash flow itself, its expected value, time to realization, and risk, and on the macro-economic environment as represented by interest rates and risk premia. Holding the risk characteristics of the cash flow constant, unanticipated changes in claim value will be driven by changes in both the expected value of the cash flow and in interest rates and risk premia. Most extant valuation models place primary emphasis on the role of cash flow related risk. However, Campbell and Ammer (1993) estimate that only about 15% of the variance of aggregate stock returns is attributable to news about future dividends. Their results further show that news about real interest rates plays a relatively minor role, leaving about 70% of the total variance of stock returns to be explained by news about future excess returns or risk premia. Fama and French (1993)[12] demonstrate that there is considerable common variation between bond and stock returns, which is also consistent with common variation in real interest rates and risk premia. In this section we construct an explicit model for the valuation of stochastic cash flows, taking account of stochastic variation in interest rates and risk premia.

[12] See also Cornell (1999).

Let V denote the value of a non-dividend paying asset. Then a pricing kernel is a random variable, m, such that $E[d(mV)] = 0$.[13] This implies that the expected return on the asset is given by:

$$\mathrm{E}\left[\frac{dV}{V}\right] = -\mathrm{E}\left[\frac{dm}{m}\right] - \mathrm{cov}\left(\frac{dm}{m}, \frac{dV}{V}\right) \tag{1}$$

Assume that the dynamics of the pricing kernel can be written as a diffusion process:

$$\frac{dm}{m} = -\mathrm{r}(X)dt - \eta(X)dz_m \tag{2}$$

where X is a vector of variables that follow a vector Markov diffusion process:

$$dX = \mu_X dt + \sigma_X dz_X \tag{3}$$

Then equations (1) and (2) imply that the expected return on the asset is given by:

$$\mathrm{E}\left[\frac{dV}{V}\right] \equiv \mu_V dt = r(X)dt + \eta(X)\rho_{Vm}\sigma_V dt \tag{4}$$

where $\rho_{Vm}dt = dz_V dz_m$, and σ_V is the volatility of the return on the asset. It follows that $r(X)$ is the riskless rate since it is the return on an asset with $\sigma_V = 0$, and $\eta(X)$ is the risk premium per unit of covariance with the pricing kernel. It is immediate from equation (4), that the Sharpe ratio for any asset V is given by $S_V \equiv (\mu_V - r)/\sigma_V = \eta\rho_{Vm}$. Recognizing that ρ_{Vm} is a correlation coefficient, it follows that η is the maximum Sharpe ratio for any asset in the market - it is the "market" Sharpe ratio. An investor's *instantaneous* investment opportunities then are fully described by the vector of the instantaneously riskless rate and the Sharpe ratio of the capital market line, $(r, \eta)'$.

[13] See Cochrane (2001) for a complete treatment.

In order to construct a tractable model, we shall simplify by identifying the vector X with $(r, \eta)'$, and assume that r and η follow simple correlated Ornstein-Uhlenbeck processes.[14] Then, since the expected returns on all securities are functions only of $(r, \eta)'$, the dynamics of the investment opportunity set are fully captured by:

$$\frac{dm}{m} = -rdt - \eta dz_m \tag{5.1}$$
$$dr = \kappa_r(\overline{r} - r)dt + \sigma_r dz_r \tag{5.2}$$
$$d\eta = \kappa_\eta(\overline{\eta} - \eta)dt + \sigma_\eta dz_\eta \tag{5.3}$$

The structure (5) implies that the riskless interest rate is stochastic, and that all risk premia are proportional to the stochastic Sharpe ratio η. To analyze the asset pricing implications of the system (5), consider a claim to a (real) cash flow, x, which is due at time T. Let the expectation at time t of the cash flow be given by $y(t) \equiv \mathrm{E}[x|\Lambda_t]$ where Λ_t is the information available at time t, and $y(t)$ follows the driftless geometric Brownian motion with constant volatility, σ_y:[15]

$$\frac{dy}{y} = \sigma_y dz_y \tag{6}$$

Letting ρ_{ij} denote the correlation between dz_i and dz_j, the value of the claim to the cash flow is given in the following theorem.

Theorem 1 *In an economy in which the investment opportunity set is described by (5),*

[14]Kim and Omberg (1996) also assume an O-U process for the Sharpe ratio. For a structural model of time variation in investment opportunities that relies on habit formation see Campbell and Cochrane (1999).

[15]The assumption of constant volatility is for convenience only. For example, as Samuelson (1965) has shown, the volatility of the expectation of a future cash flow will decrease monotonically with the time to maturity if the cash flow has a mean-reverting component.

the value at time t of a claim to a real cash flow x at time T, whose expectation, y, follows the stochastic process (6), is given by:

$$V(y,\tau,r,\eta) = \mathrm{E}_t^Q\left[x_T \exp^{-\int_t^T r(s)ds}\right] = \mathrm{E}_t^Q\left[y_T \exp^{-\int_t^T r(s)ds}\right] = yv(\tau,r,\eta) \tag{7}$$

where Q denotes the risk neutral probability measure, and

$$v(\tau,r,\eta) = \exp[A(\tau) - B(\tau)r - D(\tau)\eta] \tag{8}$$

with

$$B(\tau) = {\kappa_r}^{-1}(1-e^{-\kappa_r\tau}) \tag{9}$$

$$D(\tau) = d_1 + d_2 e^{-\kappa_\eta^*\tau} + d_3 e^{-\kappa_r\tau} \tag{10}$$

$$\begin{aligned} A(\tau) &= a_1\tau + a_2\frac{1-e^{-\kappa_r\tau}}{\kappa_r} + a_4\frac{1-e^{-\kappa_\eta^*\tau}}{\kappa_\eta^*} + a_5\frac{1-e^{-2\kappa_r\tau}}{2\kappa_r} \\ &+ a_7\frac{1-e^{-2\kappa_\eta^*\tau}}{2\kappa_\eta^*} + a_8\frac{1-e^{-(\kappa_\eta^*+\kappa_r)\tau}}{\kappa_\eta^*+\kappa_r}. \end{aligned} \tag{11}$$

where $\kappa_\eta^* \equiv \kappa_\eta + \sigma_\eta\rho_{m\eta}$, d_1, d_2, d_3 are given in equations (A16)-(A18) by setting $\rho_{mP} = \rho_{m\pi} = 0$, and $a_1, \ldots, a_8$ are given in equations (A24)-(A31) by setting σ_P, σ_π, and $\bar{\pi}$ to zero.

Theorem 1 implies that the value per unit of expected payoff of the claim is a function of the maturity τ, and the covariance with the pricing kernel, or systematic risk, $\phi_y \equiv \sigma_y\rho_{ym}$, of the underlying cash flow, as well as of the two stochastic state variables that determine the investment opportunity set, r and η.

Lemma 1 *In the "positive cash flow beta" case in which $\rho_{ym} > 0$,*[16] *$B(\tau)$ and $D(\tau)$ are positive and increase with τ, the time to maturity of the cash flow for a given systematic risk ϕ_y, provided that there is a positive risk premium for interest rate risk ($\rho_{mr} < 0$).*[17]

Lemma 1 characterizes the dependence of the state-variable sensitivities (measured by the semi-elasticities of claim value) on the cash flow maturity, τ. The longer the cash flow maturity, the more sensitive is the value of the claim to shocks in the interest rate r and risk premium η.

Following Theorem 1 and applying Ito's Lemma, the return on a claim can be written as:

$$\frac{dV}{V} = \mu(r, \eta, \tau)dt + \frac{dy}{y} - B(\tau)\sigma_r dz_r - D(\tau)\sigma_\eta dz_\eta. \tag{12}$$

The expected return is shown in the Appendix to be given by:

$$\mu \equiv \mu(r, \eta, \tau) = r + (D_\tau(\tau) + \kappa_\eta D(\tau))\eta = r + h(\tau)\eta, \tag{13}$$

where $h(\tau)$ is the asset's risk premium expressed relative to the market Sharpe ratio. The form of the risk premium expression (13) can be understood by noting that, under the assumptions we have made, the claim value can also be written as $V = \mathrm{E}\left[y_T e^{-\int_t^T (r_s + h(T-s)\eta_s)ds}\right]$. Noting that $D = -\frac{V_\eta}{V}$, differentiation of this expression with respect to η_t implies $D = \int_t^T h(T-s)e^{-\kappa_\eta(s-t)}ds$, which then leads to the expression for $h(\tau)$ in equation (13).

[16] *This is the condition for the risk premium associated with innovations in y to be positive.*

[17] *This condition for bonds that are more exposed to interest rate risk to have higher risk premia is satisfied by the empirical estimates reported in Table 4 below.*

Lemma 2 *In the positive cash flow beta case ($\rho_{ym} > 0$), if $\rho_{mr} < 0$ and $\rho_{m\eta} < 0$, then the expected return on the claim increases with the cash flow maturity;*

The restrictions imposed on the pricing kernel ($\rho_{mr} < 0$ and $\rho_{m\eta} < 0$) of the Lemma are sufficient for the risk premia associated with exposure to real interest rate and Sharpe ratio risk to be positive.[18] However, the restriction $\rho_{m\eta} < 0$ is not satisfied by the empirical estimates reported in Table 4 below, so that the direction of the effect of cash flow duration on the expected return of the cash flow claim depends on model parameters and is therefore an empirical issue.

Lemma 3 *In the "zero cash flow beta" case in which $\rho_{ym} = 0$, the value of the claim is given by $V(y, \tau, r, \eta) \equiv yv(\tau, r, \eta)$, and*

$$v(\tau, r, \eta) = \exp[A^*(\tau) - B(\tau)r - D^*(\tau)\eta] \tag{14}$$

where $A^*(\tau)$ and $D^*(\tau)$ are obtained by setting ρ_{ym} equal to zero in expressions (11) and (10).

A special case of Lemma 3 applies for a real discount bond for which $x \equiv y \equiv 1$. Then expression (14) generalizes the Vasicek (1977) model for the price of a (real) unit discount bond of maturity τ to the case in which the risk premium as well as the interest rate, is stochastic. In order to value *nominal* bonds, it is necessary to specify the stochastic process for the price level, P. We assume that the price level follows the diffusion:

$$\frac{dP}{P} = \pi dt + \sigma_P dz_P, \tag{15}$$

[18] We are assuming here that the claim values decrease with an increase in the real interest rate or Sharpe ratio.

where the volality of inflation, σ_P, is constant, while the expected rate of inflation, π, follows an Ornstein-Uhlenbeck process:

$$d\pi = \kappa_\pi(\overline{\pi} - \pi)dt + \sigma_\pi dz_\pi. \tag{16}$$

Then, noting that the real payoff of the nominal bond is $1/P_T$, the nominal price of a zero coupon bond with a face value of $\$1$ and maturity of τ, $N(P, r, \pi, \eta, \tau)$, and the corresponding real price, $n(P, r, \pi, \eta, \tau)$, are stated in the following theorem.

Theorem 2 *If the stochastic process for the price level P is as described by (15) and (16), the nominal and the real prices of a zero coupon bond with face value of $\$1$ and maturity τ, are:*

$$N(P, r, \pi, \eta, \tau) \equiv Pn(r, \pi, \eta, \tau) = \exp[\widehat{A}(\tau) - B(\tau)r - C(\tau)\pi - \widehat{D}(\tau)\eta] \tag{17}$$

where $B(\tau)$ is given in equation (9) and

$$
\begin{aligned}
C(\tau) &= \kappa_\pi^{-1}\left(1 - e^{-\kappa_\pi \tau}\right) && (18)\\
\widehat{D}(\tau) &= \hat{d}_1 + \hat{d}_2 e^{-\kappa_\eta^* \tau} + \hat{d}_3 e^{-\kappa_r \tau} + \hat{d}_4 e^{-\kappa_\pi \tau} && (19)\\
\widehat{A}(\tau) &= \hat{a}_1 \tau + \hat{a}_2 \frac{1 - e^{-\kappa_r \tau}}{\kappa_r} + \hat{a}_3 \frac{1 - e^{-\kappa_\pi \tau}}{\kappa_\pi} + \hat{a}_4 \frac{1 - e^{-\kappa_\eta^* \tau}}{\kappa_\eta^*} \\
&\quad + \hat{a}_5 \frac{1 - e^{-2\kappa_r \tau}}{2\kappa_r} + \hat{a}_6 \frac{1 - e^{-2\kappa_\pi \tau}}{2\kappa_\pi} + \hat{a}_7 \frac{1 - e^{-2\kappa_\eta^* \tau}}{2\kappa_\eta^*} \\
&\quad + \hat{a}_8 \frac{1 - e^{-(\kappa_\eta^* + \kappa_r)\tau}}{\kappa_\eta^* + \kappa_r} + \hat{a}_9 \frac{1 - e^{-(\kappa_\eta^* + \kappa_\pi)\tau}}{\kappa_\eta^* + \kappa_\pi} + \hat{a}_{10} \frac{1 - e^{-(\kappa_r + \kappa_\pi)\tau}}{\kappa_r + \kappa_\pi} && (20)
\end{aligned}
$$

$\kappa_\eta^* \equiv \kappa_\eta + \sigma_\eta \rho_{m\eta}$, and $\hat{d}_1, \ldots, \hat{d}_4$, $\hat{a}_1, \ldots, \hat{a}_{10}$ are constants whose values are given in

equations (A16)-(A19) and (A24)-(A33) by setting $\sigma_y = 0$.

In addition, the yield of the bond is given by:

$$-\frac{\ln N}{\tau} = -\frac{\widehat{A}(\tau)}{\tau} + \frac{B(\tau)}{\tau} r + \frac{C(\tau)}{\tau}\pi + \frac{\widehat{D}(\tau)}{\tau}\eta. \qquad (21)$$

Finally, Theorem 1 can also be generalized to value a security with a continuous (nominal) cash flow stream. Suppose that the growth rate of the nominal cash flow rate, X, follows an Ornstein-Uhlenbeck process so that:

$$\frac{dX}{X} = gdt + \sigma_X dz_X, \qquad (22)$$

$$dg = \kappa_g(\bar{g} - g)dt + \sigma_g dz_g, \qquad (23)$$

then the (real) value of the security at time t is given by the following theorem:

Theorem 3 *In an economy in which the investment opportunity set is described by (5), the (real) value at time* t *of a security whose (nominal) dividend follows the stochastic process (22)-(23), is given by:*

$$V(X, r, \pi, \eta, g) = \mathrm{E}^Q\left[\int_t^\infty \frac{X_s}{P_s} e^{-\int_t^s r(u)du} ds\right] = \frac{X_t}{P_t}\int_t^\infty v(s-t, r, \pi, \eta, g)ds \qquad (24)$$

where Q *denotes the risk neutral probability measure, and*

$$v(s, r, \pi, \eta, g) = \exp[\tilde{A}(s-t) - B(s-t)r - C(s-t)\pi - \tilde{D}(s-t)\eta - F(s-t)g] \quad (25)$$

with expressions for $\tilde{A}$ to F given in Appendix A.

Theorem 3 implies that the security return now depends on innovations in g, the cash flow process growth rate, as well as in r, π, and η. Expression (24) cannot, however, be further simplified, and numerical or approximation techniques must be used to value the security. To avoid this complication, we will base our empirical analysis on Theorem 1 instead of on Theorem 3.

3 Intertemporal Asset Pricing and the FF Portfolios

While the valuation model (5) explicitly allows for time-variation in the investment opportunity set, it is not equivalent to Merton's ICAPM without further specification of the covariance characteristics of the pricing kernel. For example, the valuation model will satisfy the simple CAPM if the innovation in the pricing kernel is perfectly correlated with the return on the market portfolio. A specific version of the ICAPM is obtained by specializing the pricing system (5) so that the innovation in the pricing kernel is an exact linear function of the market return and the innovations in r and η:

$$\frac{dm}{m} = -rdt - \omega\eta\zeta' dz \tag{26}$$

where $\zeta' = (\zeta_M, \zeta_\eta, \zeta_r)', dz = (dz_M, dz_\eta, dz_r)', \omega \equiv (\zeta'\Omega\zeta)^{-1/2}$, and $\Omega dt = (dz)(dz)'$, where M denotes the market portfolio.

Then, using the definition of the pricing kernel (1), and equation (26), the expected return on security i, μ_i may be written as:

$$\mu_i = r + \eta\omega\zeta'\sigma_i \tag{27}$$

where σ_i is the (3×1) vector of covariances of the security return with the market return and the innovations in the state variables, r and η. This is simply a restatement of the ICAPM.

Note that, while the state variables of the ICAPM described by equations (5) and (26), the Sharpe ratio, η, and the real interest rate, r, are not directly observable,[19] the pricing model expressed in equations (7) and (8) implies that the log of the ratio of the values of any two claims i and j can be expressed as the sum of the log ratio of the expected (real) cash flows, a time and risk-dependent constant, and linear functions of the investment opportunity set parameters r and η:

$$\ln\left(\frac{V_i}{V_j}\right) = \ln\left(\frac{y_i}{y_j}\right) + [A_i - A_j] - [B_i - B_j]\, r - [D_i - D_j]\, \eta \tag{28}$$

Moreover, equation (21) implies that the yield to maturity on a nominal bond of maturity τ, $R(\tau) \equiv -\ln N(\tau, r, \eta)/\tau$ is also a linear function of r, and η, as well as the expected rate of inflation, π. Thus, corresponding to equation (28), the yield spread between bonds with maturities τ_1 and τ_2 can be written as:

$$\begin{aligned} R(\tau_1) - R(\tau_2) \quad = \quad & \left[\frac{\widehat{A}(\tau_2)}{\tau_2} - \frac{\widehat{A}(\tau_1)}{\tau_1}\right] + \left[\frac{B(\tau_1)}{\tau_1} - \frac{B(\tau_2)}{\tau_2}\right] r \\ & + \left[\frac{C(\tau_1)}{\tau_1} - \frac{C(\tau_2)}{\tau_2}\right] \pi + \left[\frac{\widehat{D}(\tau_1)}{\tau_1} - \frac{\widehat{D}(\tau_2)}{\tau_2}\right] \eta \end{aligned} \tag{29}$$

Since the (log) value ratios are functions of the state variables, (r, η), covariances with innovations in the value ratios will correspond to covariances with linear combinations of innovations in the state variables. Therefore, under the ICAPM, covariances with innovations in the price ratios should be priced.

[19] r would be observable if short term indexed bonds were traded.

Equation (28) provides a theoretical rationale for the empirical importance of the HML and SMB hedge portfolios in the FF three-factor model since it implies a relation between the returns on these portfolios and innovations in r and η. Thus, letting H and L denote portfolios of high and low book-to-market firms, and letting R_H denote the (discrete time) return on portfolio H etc., equation (28) implies the following approximate relation between the return on HML, R_{HML}, the changes in the values of the H and L portfolios, and innovations in r and η:

$$R_{HML} \approx -(B_H - B_L)\triangle r - (C_H - C_L)\triangle \pi - (D_H - D_L)\triangle \eta + u \tag{30}$$

where $u \equiv \triangle \ln y_H - \triangle \ln y_L$ is the noise introduced by the difference between the changes in cash flow expectations for the two portfolios. Hence, if $B_H \neq B_L$, $C_H \neq C_L$, and $D_H \neq D_L$,[20] the covariance of a security return with R_{HML} will be a linear combination of its covariances with the state variable innovations $\triangle\eta$, $\triangle r$, and $\triangle r$, plus a term related to the noise component, u. Similarly, the covariance with R_{SMB} will provide a second noisy linear combination of covariances with the state variable innovations. Therefore, if the prices of portfolios of large and small firms and of high and low B/M firms show reliable predictive power for the real interest rate, r, and equity premium or Sharpe ratio, η, we should expect a cross-sectional relation between expected returns and factor loadings on the corresponding hedge portfolio returns as FF have found. In Section 5 we shall examine the predictive power of these portfolio price ratios.

[20]Since the B/M ratio is associated with growth, or the duration of firm cash flows, we should expect from Lemma 1 that firms with different B/M ratios will have different sensitivities to r and η. Moreover, Perez-Quiros and Timmermann (2000) show that portfolios of large and small firms have different sensitivities to credit conditions, so that we should expect them to have different loadings on r at least.

4 Data and Estimation

We shall adopt two distinct approaches to determining whether the Fama-French hedge portfolio returns are related to innovations in the investment opportunity set, as they must be if the risk premia associated with them are to be explained by the ICAPM. First, we shall test whether portfolio price ratios that are related to the FF hedge portfolio returns (together with the term spread) predict real riskless returns and the Sharpe ratio: these tests, while motivated by the model developed in the previous section, do not rely on the specific functional forms derived there. Secondly, we shall apply the exact model of equation (21) to bond yields and extract the time series of the state variables, r, η, and π, using a Kalman filter, and then test whether these state variable estimates are related to the Fama-French hedge portfolio returns.

The first approach is based on equations (28) and (29) which suggest that linear combinations of pairs of log value ratios or term spreads can be regarded as (noisy) instruments of the (real) investment opportunity set state variables, r and η. In the case of the log value ratios, noise is introduced by the omission of the log expected cash flow ratios, $\ln(y_i/y_j)$, and in empirical applications approximation error will be introduced by the use of prices of assets that are claims to streams of cash flows rather than to single dated cash flows. In the case of empirical term spreads, noise will be introduced if the bonds are nominal rather than real, and approximation error will be introduced if coupon bonds are used in place of the theoretically required discount bond yields. Given these errors, it is natural to think of using several price ratios or yield spreads as instruments for the state vector (r, η). Therefore we estimate the investment opportunity set variables, r and η, by regressing the realized values of the real interest rate and (normalized) market excess return on the state variable instruments and calculating the fitted values from these regressions. Since the portfolio prices (values) are non-stationary, we shall scale the prices by the book values of the portfolios; this should also alleviate the problem caused by the non-observability of the expected cash flow ratio.

The second approach to estimating the dynamics of the investment opportunity set is to employ a Kalman filter to data on inflation and bond yields using the theoretical relation (21) to estimate the unobservable state variables, r, π and η, and their dynamics. Details of the estimation are presented in Appendix C. In summary, there are n observation equations based on the yields at time t, $y_{\tau_j,t}$, on bonds with maturities τ_j, $j = 1, \cdots, n$. The observation equations are derived from equation (21) by the addition of measurement errors, ϵ_{τ_j}:

$$y_{\tau_j,t} \equiv -\frac{\ln N(t, t+\tau_j)}{\tau_j} = -\frac{\widehat{A}(t,\tau_j)}{\tau_j} + \frac{B(\tau_j)}{\tau_j} r_t + \frac{C(\tau_j)}{\tau_j}\pi_t + \frac{\widehat{D}(\tau)}{\tau}\eta_t + \epsilon_{\tau_j}(t). \quad (31)$$

The measurement errors, $\epsilon_{\tau_j}(t)$, are assumed to be serially and cross-sectionally uncorrelated, and uncorrelated with the innovations in the transition equations, and their variance is assumed to be of the form: $\sigma^2(\epsilon_{\tau_j}) = \sigma_b^2/\tau_j$ where σ_b is a parameter to be estimated. The final observation equation uses the realized rate of inflation, $\frac{P_t - P_{t-\Delta t}}{P_{t-\Delta t}}$,

$$\frac{P_t - P_{t-\Delta t}}{P_{t-\Delta t}} = \pi \Delta t + \epsilon_P(t). \quad (32)$$

The data set for the first approach consists of monthly returns on the value weighted market portfolio, and the returns and (estimated) book-to-market (B/M) ratios on four portfolios sorted according to the B/M ratio and firm size for the period from May 1953 to September 1996. Portfolios are formed at the beginning of July each year based on the B/M ratio at the end of the previous year and the firm size at the end of June. The portfolios are the Big and Small, Growth and Value portfolios which were constructed by Fama and French.[21] The Small (Big) firm portfolio contains the NYSE, AMEX and NASDAQ stocks with market equity below (above) the median of NYSE stocks in June. The Growth (Value) High (Low) B/M portfolios include the top (bottom) 30% of NYSE

[21] We are grateful to Eugene Fama and Ken French for providing us with these data.

stocks ranked according to the B/M ratio at the end of December of the previous year. Sorting stocks according to both size and B/M yields four portfolios which we denote by VB, GB, VS, GS, where G, V, B, and S stand for Growth, Value, Big and Small, respectively.

Monthly values for the B/M ratio for each portfolio are constructed from the Fama-French annual book-to-market data by a two stage process. First, the book value is assumed to be constant during the year and the Book-to-Market ratio for the beginning of July (when the portfolio composition is revised) is calculated by taking the B/M ratio at the end of December of the previous year as reported by Fama and French and updating it by dividing by the cumulative returns from January to June. Then B/M ratios for August through December are calculated by dividing the previous month's ratio by the portfolio return for the month. The second stage is to adjust the monthly figures to smooth out the "splicing error" which appears as the new B/M ratio for the portfolio is reported at the end of December. There are now two B/M ratios for each December, the one that is calculated by updating the previous year's value by the portfolio returns $(B/M)_{old}$, and the one that is calculated for the portfolio using the new balance sheet data, $(B/M)_{new}$. We replace the $(B/M)_{old}$ with $(B/M)_{new}$ for December, and update the B/M ratios for the previous eleven months by spreading the cumulative error linearly over this period.

The nominal risk free interest rate is approximated by the return on a one month Treasury Bill and the market return is the return on the value-weighted index, both taken from CRSP. The inflation rate is calculated from the Consumer Price Index. The realized real interest rate for each month is calculated by subtracting the realized rate of inflation from the riskless nominal return.

Table 1 reports summary statistics on the Market-to-Book ratios and returns for the four portfolios, as well as for the (realized) monthly real interest rate and market return. Both the mean and the variability of the Market-to-Book ratios are approximately five times as high for the two growth portfolios as for the value portfolios. The highest correlation is between the ratios of big and small value firms (0.98), while the lowest is between value small firms and growth big firms (0.87). The correlations for the returns are generally lower, with the highest for value and growth small firms and the lowest for small growth firms and big value firms. The big growth firm portfolio has the highest correlation with the market return (0.97).

State variable instruments are constructed by taking the log of the ratios of the portfolio Book-to-Market ratios which are defined as $P^B_{GV} \equiv \ln\left[\frac{(B/M)_{VB}}{(B/M)_{GB}}\right]$, $P^S_{GV} \equiv \ln\left[\frac{(B/M)_{VS}}{(B/M)_{GS}}\right]$, $P^V_{BS} \equiv \ln\left[\frac{(B/M)_{VS}}{(B/M)_{VB}}\right]$, and $P^G_{BS} \equiv \ln\left[\frac{(B/M)_{GS}}{(B/M)_{GB}}\right]$, where P^B_{GV} is the (log scaled) price ratio of growth to value for big firms; P^S_{GV} is the corresponding ratio for small firms, P^V_{BS} is the ratio of prices of big to small (value) firms, and P^G_{BS} is the ratio of prices of big to small (growth) firms. The fifth state variable proxy is TS, the term spread between the yields on the 10 and 1 year Treasury Bonds, taken from CRSP. Only three of the four log price ratios are independent; therefore P^G_{BS} is omitted from the regressions that are reported below. Table 2 reports summary statistics for the state variable proxies, and Figure 1 plots the time series of the proxies. The correlations between the state variable proxies are low, suggesting that all four proxies may be useful in predicting the investment opportunity set. The variability of the (scaled) price ratio between growth and value firms is 50% higher for large firms than for small firms, and three times as large as the price ratio between big and small value firms. Not surprisingly, all four state variable proxies are highly autocorrelated. The augmented Dickey-Fuller statistics reported in Table 2 strongly reject the null hypothesis of non-stationarity for three out of the four variables. Figure 1 shows that the growth-value ratios for big and small firms diverge for long periods of time while their mean values are quite close.

The data set for the second approach consists of monthly data on inflation and yields on eight constant maturity zero coupon U.S. treasury bonds with maturities of 3, 6 months, and 1, 2, 3, 4, 5, and 10 years for the same period from May 1953 to September 1996.[22]

[22] We thank Luis Viceira for providing the data. We use data after May 1953 because the Federal-Treasury Accord was adopted in 1952.

Table 1 reports summary statistics for the bond yield data. The sample mean of the bond yields increases slightly with maturity, while the standard deviation remains relatively constant across maturities. The inflation rate during the same sample period is calculated from the CPI and has a sample mean of 4.1% and a sample standard deviation of 1.1%.

5 Empirical Results

In Section 5.1 we show that time variation in investment opportunities is tracked by the term spread and the scaled price ratios whose innovations correspond to the returns on the Fama-French hedge portfolios. We then estimate the investment opportunity set state variables, (r, η), as the fitted variables from the regression of normalized stock and T-bill returns on the log price ratios and term spread; these fitted values are referred to as the 'return-based' estimates of the state variables since they are obtained as projections of equity returns on the state variable instruments. In Section 5.2 we show that the time series of nominal bond yields and inflation also provide strong evidence of time variation in the real interest rate and the Sharpe ratio; the filtered estimates of the state variables obtained from the yield data are referred to as the 'yield-based' estimates. Finally in Section 5.3, we show that the yield-based estimates are reflected in the prices of the Fama-French portfolios, that innovations in the yield-based estimates are correlated with the returns on the Fama-French hedge portfolios, and that the yield-based estimates are correlated with the return-based estimates. All of this evidence suggests an integrated bond-stock market in which the Fama-French hedge portfolio returns are correlated with innovations in the state variables that describe the investment opportunity set. This is consistent with the ICAPM rationale for the empirical success of the Fama-French three factor model. In Section 5.4 we consider the the quantitative implications of our model estimates for the Fama-French portfolio risk premia.

5.1 Return-based Estimates of the State Variables

We assume initially that the market volatility, σ_M, is constant. Then it follows from equation (4) that the Sharpe ratio, η, is proportional to $\mu_M - r$, the equity market risk premium.[23] Then, in order to determine whether the proposed state variable proxies (the log value ratios and the term spread) have predictive power for the investment opportunity set variables (r, η), the market excess return and real interest rate were regressed on the state variable proxies in OLS regressions, using the Newey-West (1987) adjustment for heteroscedasticity and autocorrelation to compute the standard errors. The regressions were estimated both with and without the term spread variable, TS, because, while the model of Section 2 suggests that TS is a useful state variable proxy, it does not correspond to any of the Fama-French portfolios. The results are reported in the first eight lines of Table 3.

As predicted, the proposed state variable proxies have significant predictive power for both the market excess return and the real interest rate. Both P_{GV}^{S} and P_{BS}^{V} are significant in predicting the market excess return, and all three state variable proxies are significant in the regression for the real interest rate. When TS is included in the market excess return regression, it is significant, and P_{GV}^{S} and P_{BS}^{V} remain significant, while in the real interest rate regression TS is not significant but all three state variable proxies remain significant. In order to determine whether the state variable proxies remain significant in the presence of other potential predictors whose innovations are perfectly correlated with the market return,[24] the regressions were repeated with the market dividend yield and book-to-market ratio as additional regressors.[25] While the dividend yield and book to market ratio are significant for the market excess return, the coefficients of the state

[23] Note that equation (4) implies that the conditional Sharpe ratio of any portfolio is proportional to η provided that the portfolio has constant correlation with the pricing kernel.

[24] The CAPM is indistinguishable from the ICAPM if the relevant state variables are perfectly correlated with the market return. See Chamberlain (1988).

[25] For dividend yield, see Campbell and Shiller (1988) and Fama and French (1988); for book-to-market, see Pontiff and Schall (1998) and Kothari and Shanken (1999).

variable proxies are largely unchanged as shown in lines (3)and (4) of Table 3. Neither variable is significant in the real interest rate regression. These results provide strong evidence that the portfolio price ratios whose innovations correspond to the returns on the Fama-French SMB and HML portfolios, along with the term spread, are state variables in the Merton (1973) sense in that they predict the instantaneous investment opportunity set.

To this point we have assumed that the volatility of the market return is constant, so that the expected market excess return is proportional to the Sharpe ratio which, together with the real interest rate, are the sufficient statistics for the current investment opportunity set. Since there is evidence that market volatility is not constant,[26] the analysis was repeated using two estimates of the Sharpe ratio. The first, the "realized" Sharpe ratio, was constructed as the ratio of the realized excess market return to a volatility estimate derived under a GARCH specification using a set of instruments.[27] The second "Whitelaw" Sharpe ratio was constructed as the ratio of the projected excess market return to the estimated volatility, both using the same set of instruments.

The results for the "realized" Sharpe ratio are qualitatively similar to those obtained for the market excess return: the state variable proxies P^S_{GV}, P^V_{BS} and TS remain significant in the presence of the market dividend yield and book-to-market ratio, both of which are significant. When the dependent variable is the "Whitelaw" Sharpe ratio, we again find that three state variable proxies are significant, with P^B_{GV} instead of P^V_{BS} now significant. The much higher R^2 and $t-$ratios in these regressions reflect the use of the projected instead of the realized excess return in constructing the dependent variable.

In summary, the results reported in Table 3 show that all three price-ratio state variable

[26]See French *et al.* (1987).

[27]See Whitelaw (1994, 1997). The instruments were the default premium, the dividend yield on the S & P 500 Index, and the Treasury bill rate. Two alternative approaches were also used to obtain market volatility estimates. The first approach, which used daily returns to compute monthly volatility estimates, significantly reduced the sample size since daily return data were available only from July 1962. Under the second approach, the volatility estimate is the square root of the fitted value from a regression of the squared demeaned excess returns on the set of instruments. All three approaches yield very similar results.

proxies are highly significant in predicting the real interest rate, while TS is not significant. They also show that two out of three of the price ratio state variable proxies together with TS are significant in predicting the Sharpe ratio even in the presence of the market dividend yield and book-to-market ratio. These results provide empirical support for an ICAPM-based rationale for the Fama-French finding that covariances with returns on the HML and SMB portfolios are priced. They also suggest that covariances with a portfolio whose returns are correlated with innovations in the term spread will also be priced.[28] Finally, the time series of fitted values of the real interest rate and Sharpe ratio from the regressions including TS were calculated. These are referred to as the 'return-based' estimates of the state variables since they are derived directly from the real T-bill return and market excess return. The parameters of the joint stochastic process of r and η that were estimated using the filtered values of the return-based estimates are reported in Columns (2) and (3) of Table 4 and will be discussed in Section 5.3.

5.2 Yield-based Estimates of the State Variables

In this section we report the results of using a Kalman filter to estimate the dynamics of the state variables, r and η, from data on nominal bond yields and inflation. In order to identify the process for the Sharpe ratio, η, it is necessary to impose a restriction that determines the overall favorableness of investment opportunities.[29] For purposes of identification we set $\bar{\eta}$ equal to 0.7, which is approximately the value obtained by estimating equation (5.3) using the return-based estimate of the time series of η obtained in Section 5.1.[30] To improve the efficiency of estimation, the long run means for r and π were set equal to the corresponding historical mean values. As a result, the standard

[28] Chen, Roll and Ross (1986).

[29] Equation (4) shows that the structure of risk premia is invariant up to a scalar multiplication of η and the vector of inverse security correlations with the pricing kernel with typical element, $1/\rho_{Vm}$.

[30] Mackinlay (1995) reports an average Sharpe ratio of around 0.40 for the $S\&P500$ for the period 1981-1992.

errors of all other parameters reported in Columns (1) ands (4) of Table 4 are understated because $\bar{r}$ and $\bar{\pi}$ are treated as known parameters.

The variance of the yield measurement error was assumed to be inversely proportional to maturity τ. The estimated measurement error parameter, σ_b, implies that the standard deviation of the measurement error varies from 30 basis points for the three month maturity to 5 basis points for the ten year maturity, so that the model fits the yield data quite well. The volatility of unexpected inflation, σ_P, is fixed at the CPI inflation sample volatility of 1.12% while the volatility of expected inflation, σ_π, is estimated to be around 0.6% per year. The standard deviation of the estimated real interest rate process, σ_r, is 1.9% per year, so that the real interest rate is much more volatile than the expected inflation rate. The estimated mean reversion intensity for the interest rate, κ_r, is 0.13 per year which implies a half life of about 5 years. The expected rate of inflation rate follows almost a random walk. The volatility of the Sharpe ratio process, σ_η, is 0.19 per year which compares with the imposed long run mean value of 0.70. The Wald statistic to test the null hypothesis that $\sigma_\eta = \kappa_\eta = 0$ (so that η is a constant) is highly significant so that, given the pricing model, there is strong evidence from the bond yield data that the Sharpe ratio is time varying. Finally, the t-statistics on $\rho_{\eta m}$ and ρ_{rm} strongly reject the null that the opportunity set state variables, r and η, are unpriced, providing strong evidence in favor of the ICAPM.

5.3 Evaluation of the State Variable Estimates

In the absence of model and estimation error, we should expect the yield-based estimates and return-based estimates of (r, η) to be identical.[31] Therefore in this section we first compare the two sets of state variable estimates and consider the effect of adding the

[31] Note that any subset of assets can yield estimates of η only up to a constant of proportionality. The yield-based estimates of η have been scaled by setting $\bar{\eta}$ equal to the sample mean of the return-based estimates of η.

time series of the realized (equity market) excess return normalized by its volatility as an additional observation equation for the Sharpe ratio in the Kalman filter algorithm used in the previous section in order to obtain 'hybrid" estimates. We then examine whether the yield-based estimates are reflected in the prices of the Fama-French portfolios, and whether innovations in the yield-based state variable estimates are correlated with the returns on the Fama-French hedge portfolios.

Figure 2 plots the time series of the yield-based and return-based estimates of the real interest rate. There is only limited correspondence between the two series, the yield-based estimate being much more variable than the return-based estimate. The correlation between the levels of the two series of estimates is -0.07, which is not supportive of our conjecture; however the correlation between the monthly innovations in the series is slightly positive (0.08). Figure 3 plots the time series of the yield-based and return-based estimates of the Sharpe ratio.[32] These results are much more encouraging. The correlation between the levels of the two series of estimates is 0.56, and the correlation between their monthly innovations is 0.31. Table 5, which reports the results of simple regressions of the return-based estimates on the yield-based estimates, confirms that the relation between the two Sharpe ratio (η) estimates is much stronger than that between the two real interest rate (r) estimates. In view of the radically different approaches and data sets (bond *yields* on the one hand, and equity and bill *returns* on the other) used to generate these two sets of estimates, the correspondence between them is highly encouraging.

"Hybrid" model estimates were obtained by adding an additional observation equation for η to the Kalman filter that was used to obtain the yield-based estimates:

$$\hat{\eta}_{return}(t) = a + b\eta(t) + \epsilon_{\eta}(t), \tag{33}$$

[32] The return based estimate is the predicted value using "Whitelaw" estimate of the Sharpe ratio. See line 14 of Table 3.

where $\hat{\eta}_{return}$ is the fitted value of η from the regression of $SH_{Whitelaw}$ (SHW) on the price ratio state variable proxies and TS reported in line 14 of Table 3. The "hybrid" parameter estimates reported in Column (4) of Table 4 are little changed by this additional observation equation from the yield-based estimates in Column (1) . Moreover, the estimates of $a(0.13)$ and $b(0.73)$ are quite close to their theoretical values of zero and one although it is possible to reject the null that they are equal to their theoretical values. This provides encouraging support for our simple valuation approach, given that $\hat{\eta}_{eturn}$ is estimated from *realized stock returns* while the other eight observation equations for η are based on *bond yields*.

The shaded areas in the Figures 2 and 3 correspond to periods of U.S. recession as determined by the National Bureau of Economic Research.[33] For the yield-based estimates, the correlation between r and η is about -0.22, while the correlation is about -0.48 between the return-based estimates of r and η. The recessions are generally associated with a declining real interest rate but increasing Sharpe Ratio. Whitelaw (1997) and Perez-Quiros and Timmermann (2000) have found similar cyclical patterns in the Sharpe ratio. Their approach to estimation of the Sharpe ratio is similar to that employed to obtain our return-based estimates except that, instead of using Fama-French portfolio price ratios as predictors, they use the more conventional dividend yield, default spread, and yield variables.[34]

The return-based estimates for the Sharpe ratio and the real interest rate were used to estimate the parameters of the corresponding Ornstein-Uhlenbeck processes and the results are reported in Columns (2) and (3) of Table 4, while the parameter estimates obtained from the bond yield data using the Kalman filter are reported in Column (1). The estimated mean reversion coefficients for the Sharpe ratio and the real interest rate using the return-based estimates of η and r are about 0.91 (SHW) and 0.44: the corresponding half-lives

[33]The period of recession is measured from peak to trough.

[34]Fama and French (1989) have also documented common variation in expected returns on bonds and stocks that is related to business conditions.

are about 0.76 and 1.58 years. In contrast, for the much smoother series of the Sharpe ratio and the real interest rate obtained from the bond yield data, the mean reversion coefficients are 0.15 and 0.13, implying half-lives of 4.62 and 5.33 years. The estimated long term (annualized) Sharpe ratio is about 0.70 for the return-based estimates.[35]

As a further check on the relation between the yield-based estimates of the state variables and the FF hedge portfolios, the log portfolio price ratios were regressed on the yield-based estimates of r, η and π, and the results are reported in Panel A of Table 6. All four price ratios are negatively related to the estimated real interest rate. For the "growth-value" ratios this is consistent with Lemma 1: high real interest rates depress the prices of growth firms relative to value firms. High real interest rates also appear to depress the prices of big firms relative to small firms. The Sharpe ratio has a significant positive effect on the "growth-value" price ratio for small firms, and significant negative effects on the "big-small" price ratio for both value and growth firms, but insignificant negative impact on the "growth-value" price ratio for big firms. The direction of the effect implied by the model developed in Section 2 depends on both the relative durations of the firm cash flows and their correlations with the pricing kernel, ρ_{ym}. Expected inflation also has a significant effect on two of the price ratios which suggests that this variable has information about the relative real cash flows of the different portfolios. In summary, there is strong evidence that the portfolio price ratios, which are the basis of the Fama-French hedge portfolio returns, are related to the state variables which determine the investment opportunity set.

If the ICAPM is to provide an explanation for the risk premia on the FF hedge portfolios it must be the case that innovations in the investment opportunity set state variables are correlated with the returns on these portfolios. We have already seen evidence of this in that the equity market state variable proxies (the price ratios of the portfolios corresponding to the FF hedge portfolio returns) do have predictive power for the investment

[35]For the bond yield estimates $\bar{\eta}$ was set at 0.7.

opportunity set. As a further test of this we examine whether the innovations in the yield-based state variable estimates are correlated with the returns on the FF portfolios.[36] To this end we calculate the innovations, $\hat{\epsilon_r}$, $\hat{\epsilon_\pi}$ and $\hat{\epsilon_\eta}$, in the yield-based state variable estimates using the parameter values reported in Table 4, and then regress these estimated innovations on the returns on the three FF portfolios. The results are reported in Panel B of Table 6. The innovation in r is significantly related to the excess return on the market portfolio and to the return on the HML portfolio. The innovation in η is significantly related to the returns on both the SMB and HML portfolios. Once again, these results point to a link between the FF portfolio returns and innovations in the investment opportunity set. Given the ICAPM, we should expect to find risk premia associated with loadings on these portfolio returns. In the next section, we provide some simple calculations of the risk premia on the Fama-French SMB and HML portfolios that are implied by the parameter estimates reported in Table 4.

5.4 Implications for Fama-French Portfolio Risk Premia

We consider two separate approaches to estimation of risk premia on the HML and SMB portfolios that are implied by our simple ICAPM. The first, "pricing kernel", approach exploits the fact that risk premia[37] are equal to the covariances between portfolio returns and the pricing kernel as shown in equation (4). Under this approach, the unconditional risk premium for asset i is equal to $\bar{\eta}\rho_{im}\sigma_i$. Under the simplified version of the ICAPM described in Section 3, the pricing kernel is a linear function of the excess return on the market portfolio and the innovations in the state variables r and η. Then

[36] Since the return-based state variable estimates are linear functions of the price ratios, their innovations are correlated with the returns on the FF portfolios by construction.

[37] Note that since HML and SMB are arbitrage portfolios, their expected returns are equal to their risk premia.

the unconditional risk premium on asset i is

$$\bar{\eta}\left(b_{iM}\rho_{mM}\sigma_M + b_{ir}\rho_{mr}\sigma_r + b_{i\eta}\rho_{m\eta}\sigma_\eta\right) \tag{34}$$

where b_{iM} etc. are coefficients from the regression of asset returns on market excess returns and state variable innovations:

$$R_i = b_{i0} + b_{iM}\left(R_M - R_f\right) + b_{i\eta}\Delta\eta + b_{ir}\Delta r + \epsilon_i. \tag{35}$$

Table 7 reports estimates of equation (35) for HML and SMB portfolio returns. The coefficient on $\Delta\eta$ is positive and significant for both HML and SMB. The coefficient on Δr is negative and significant for HML but positive and insignificant for SMB. Given that two of the four FF price ratios are significantly related to yield-based estimates of π as reported in Panel A Table 6, we also estimate an extended version of equation (35) by including $\Delta\pi$ as an additional state variable. The results in Table 7 (Rows (2) and (4)) indicate that $\Delta\pi$ is not significantly related to R_{HML} but is significantly and positively related to R_{SMB}.

The second and third columns of Panel B Table 9 report the "pricing kernel" estimates of the portfolio risk premia with and without π as an additional state variable. These are constructed from expression (34) using the estimates of b_{iM} etc. that are reported in Table 7: the market risk premium $(\bar{\eta}\rho_{mM}\sigma_M)$ was taken as the sample mean excess return on the CRSP value-weighted index (7.07% p.a.); $\bar{\eta}$ was taken as 0.7 and the other parameter values were taken from Table 4. It is apparent that these pricing kernel estimates of the risk premia fall far short of the historical premia which are reported in the first column, and the inclusion of $\Delta\pi$ does not have any significant effect on the risk premium estimates. It seems likely that one reason for the failure of the pricing kernel approach is error in the estimates of the innovations in r, π and η, which would bias the estimated factor loadings reported in Table 7, and therefore, our estimates of the portfolio risk premia.

The second approach deals with the errors-in-variable problem by using returns on "tracking portfolios" as instrumental variables. Following Breeden, Gibbons, and Litzenberger (1989) and Lamont (2001), we first construct "tracking" portfolios which have maximal correlations with the estimated innovations in r, π, and η, and then use the returns on these portfolios as instruments for Δr, $\Delta\pi$ and $\Delta\eta$. This allows us to estimate both the sensitivity of the FF portfolios to innovations in the state variables and to estimate the risk premia on the portfolios that are most highly correlated with the state variable innovations.[38] Tracking portfolios are constructed by estimating a set of tracking portfolio formation regressions in which the innovations in the state variables are regressed on the excess returns on a set of base assets and a set of variables to control for the expected excess returns on the base assets. The six size and value sorted portfolios of Fama and French were used as the base assets. The model implies that η_{t-1} is a sufficient control variable. However, since η is measured with error, tracking portfolios were constructed both with and without η as a control.

It follows from the properties of tracking portfolios that asset risk premia are related to the expected returns on the tracking portfolios by:

$$\mathrm{E}\left[R_i\right] = b_{iM}^{*}\mathrm{E}\left[R_M - R_f\right] + b_{i\eta}^{*}\mathrm{E}\left[R_\eta\right] + b_{ir}^{*}\mathrm{E}\left[R_r\right], \tag{36}$$

where R_η and R_r are the returns on the tracking portfolios, and b_{iM}^{*} etc. are the coefficients from the regressions of asset i returns on the market and tracking portfolio returns.

The tracking portfolio formation regressions that were used to determine the composition of the tracking portfolios are reported in Table 8 Panel A. For both Δr and $\Delta\eta$, the coefficients of the base assets are jointly significant at better than the 1% level, and the inclusion of η as a control variable has little effect on the coefficients. In addition, we

[38] Vassalou (2002) uses a similar approach.

also regress innovations in π on the six base assets, and the results are reported in columns (5) and (6) in Table 8 Panel A. The coefficients are not jointly or individually significant. Although the control variable η_{t-1} enters significantly, the coefficients in front of the six base assets have only small changes. Therefore, the tracking portfolios were constructed using the coefficients reported in columns (1), (3) and (5).

Panel B of Table 8 reports estimates of b^*_{iM} etc. from regressions of the HML and SMB portfolio returns on the market excess return and the tracking portfolio returns. The regressions explain around 72% of the HML and SMB portfolio returns without R_π, the return on the inflation tracking portfolio return, and they explain about 82% and 84% of the HML and SMB returns with R_π included. The coefficients of all the tracking portfolio returns are highly significant, and their signs correspond to those in regression (35) reported in Table 7.

The estimated risk premia of the tracking portfolios and the market portfolio together with their t-statistics are reported in Panel A of Table 9. The estimated market risk premium is 7.1% per year, and the η tracking portfolio has an estimated risk premium of 7.3% per year. The estimated r and π tracking portfolio risk premia, -0.44% and -0.03% respectively, are negative and much smaller in magnitude. The estimated risk premia for the market portfolio and for the r and η tracking portfolios are highly significant while that for the π tracking portfolio returns is not significant.

The coefficients from the regressions of the HML and SMB portfolio returns on the market excess return and the tracking portfolio returns, together with the sample means of the tracking portfolio returns, were used to calculate conditional estimates of the risk premia given by equation (36), which are reported in the fourth and fifth columns of Panel B Table 9.[39] Now the simplified ICAPM predicts risk premia of 5.09% for HML and 2.02% for SMB, which compare with the sample mean estimates of these risk premia of

[39] The risk premium estimates are conditional on risk premium estimates for the tracking portfolios.

10.32% and 7.08%, respectively.[40]

Thus, the tracking portfolio results imply that about 50% (30%) of the sample estimates of the risk premia on HML (SMB) can be attributed to their covariation with innovations in the ICAPM state variables, r and η. Although these results still leave a significant part of risk premia on HML and SMB unexplained, they are quite encouraging given the simplified nature of our ICAPM and the fact that the state variables were identified using only data on bond yields. Moreover, as mentioned in footnote 7, there is evidence that returns on these portfolios are related to the returns on extra-market components of wealth such as human capital and investments in unincorporated businesses. Therefore, it is possible that the part of their risk premia which is not accounted for by ICAPM type risk premia could be due to their role as proxies for the return on components of aggregate wealth.[41]

6 Conclusion

In this paper we have developed a simple model of asset valuation for a setting in which real interest rates and risk premia vary stochastically. The model implies, first, that the ratios of the prices of Fama-French size and value portfolios, as well as the term spreads, will carry information about the real interest rate and the Sharpe ratio. This provides a justification in the context of the ICAPM for the risk premia that Fama and French have found to be associated with the HML and SMB portfolio returns. We find

[40] It is possible that there is a risk premium associated with innovations in π since this variable will be associated with returns on the component of wealth which is fixed in nominal terms and which is not captured by the return on the (equity) market portfolio. However, allowing for a risk premium associated with this variable has an insignificant effect on both the pricing kernel and tracking portfolio estimates of the risk premia associated with HML and SMB.

[41] Jagannathan and Wang (1996) report that the performance of (a conditional specification of) a simple CAPM is significantly improved by including estimates of the return to human capital. Jagannathan *et al.* (1998) find similar results for Japan, and report that the inclusion of human capital can account for the risk premium associated with SMB (but not with HML).

strong empirical evidence that the FF portfolios do predict the real interest rate and the Sharpe ratio. The model also implies that zero-coupon bond yields are linearly related to the state variables r and η, and data on bond yields are used to provide a second set of estimates of the state variables. We find that these yield-based estimates are related to the first set of estimates, as well as to the returns on the Fama-French portfolios. Both sets of estimates of the Sharpe ratio display strong cyclical variation, rising during recessions and falling during expansions. The yield-based estimates were used to calculate the risk premia on HML and SMB portfolio using both the pricing kernel and the tracking portfolio approaches. We find that at least 30-50% of the historical risk premia on SMB and HML portfolio can be attributed to their association with the simple ICAPM state variables.

Appendix

A. Proof of Theorems 1 and 2

The real part of the economy is described by the processes for the real pricing kernel, the real interest rate, and the maximum Sharpe ratio (5.1)-(5.3), while the nominal part of the economy is described by the processes for the price level and the expected inflation rate (15)-(16). Under the risk neutral probability measure Q, we can write these processes as:

$$\begin{aligned} dr &= \kappa_r(\bar{r}-r)dt - \sigma_r\rho_{mr}\eta dt + \sigma_r dz_r^Q && \text{(A1)} \\ d\pi &= \kappa_\pi(\bar{\pi}-\pi)dt - \sigma_\pi\rho_{m\pi}\eta dt + \sigma_\pi dz_\pi^Q && \text{(A2)} \\ d\eta &= \kappa_\eta^*(\bar{\eta}^*-\eta)dt + \sigma_\eta dz_\eta^Q && \text{(A3)} \end{aligned}$$

where $\kappa_\eta^* = \kappa_\eta + \sigma_\eta\rho_{m\eta}$ and $\bar{\eta}^* = \frac{\kappa_\eta\bar{\eta}}{\kappa_\eta^*}$.

Let y, whose stochastic process is given by (6), denote the expectation of a nominal cash flow at a future date T, X_T. The process for $\xi \equiv y/P$, the deflated expectation of the nominal cash flow, under the risk neutral probability measure can be written as:

$$\frac{d\xi}{\xi} = \left[-\pi - \sigma_y\sigma_P\rho_{yP} + \sigma_P^2 - \eta(\sigma_y\rho_{ym} - \sigma_P\rho_{Pm})\right]dt + \sigma_y dz_y^Q - \sigma_P dz_P^Q. \quad \text{(A4)}$$

The real value at time t of the claim to the nominal cash flow at time T, X_T, is given by expected discounted value of the real cash flow under Q:

$$\begin{aligned} V(\xi,r,\pi,\eta,T-t) &= \mathrm{E}_t^Q\left[\frac{X_T}{P_T}\exp^{-\int_t^T r(s)ds}\right] = \mathrm{E}_t^Q\left[\frac{y_T}{P_T}\exp^{-\int_t^T r(s)ds}\right] \\ &= \mathrm{E}_t^Q\left[\xi_T\exp^{-\int_t^T r(s)ds}\right] && \text{(A5)} \end{aligned}$$

Using equation (A4), we have

$$\begin{aligned}\xi_T &= \xi_t \exp\left\{\left(-\frac{1}{2}\sigma_y^2+\frac{1}{2}\sigma_P^2\right)(T-t)-(\sigma_y\rho_{ym}-\sigma_P\rho_{Pm})\int_t^T \eta(s)ds\right.\\ &- \left.\int_t^T \pi(s)ds+\sigma_y\int_t^T dz_y^Q-\sigma_P\int_t^T dz_P^Q\right\}. \qquad \text{(A6)}\end{aligned}$$

A tedious calculation from equations (A1), (A2), and (A3) gives us the following results:

$$\begin{aligned}\int_t^T \eta(s)ds &= \eta_t\frac{1-e^{-\kappa_\eta^*(T-t)}}{\kappa_\eta^*}+\bar{\eta}^*\left[T-t-\frac{1-e^{-\kappa_\eta^*(T-t)}}{\kappa_\eta^*}\right]\\ &+ \sigma_\eta\int_t^T\frac{1-e^{-\kappa_\eta^*(T-s)}}{\kappa_\eta^*}dz_\eta^Q(s), \qquad \text{(A7)}\end{aligned}$$

$$\begin{aligned}\int_t^T \pi(s)ds &= \pi_t\frac{1-e^{-\kappa_\pi(T-t)}}{\kappa_\pi}+\left(\bar{\pi}-\frac{\sigma_\pi\rho_{m\pi}\bar{\eta}^*}{\kappa_\pi}\right)\left[T-t-\frac{1-e^{-\kappa_\pi(T-t)}}{\kappa_\pi}\right]\\ &+ \left(\frac{\sigma_\pi\rho_{m\pi}\eta_t}{\kappa_\eta^*-\kappa_\pi}-\frac{\sigma_\pi\rho_{m\pi}\bar{\eta}^*}{\kappa_\eta^*-\kappa_\pi}\right)\left[\frac{1-e^{-\kappa_\eta^*(T-t)}}{\kappa_\eta^*}-\frac{1-e^{-\kappa_\pi(T-t)}}{\kappa_\pi}\right]\\ &+ \frac{\sigma_\pi\rho_{m\pi}\sigma_\eta}{\kappa_\eta^*-\kappa_\pi}\int_t^T\left[\frac{1-e^{-\kappa_\eta^*(T-s)}}{\kappa_\eta^*}-\frac{1-e^{-\kappa_\pi(T-s)}}{\kappa_\pi}\right]dz_\eta^Q(s)\\ &+ \sigma_\pi\int_t^T\frac{1-e^{-\kappa_\pi(T-s)}}{\kappa_\pi}dz_\pi^Q(s) \qquad \text{(A8)}\end{aligned}$$

and

$$\begin{aligned}
\int_t^T r(s)ds &= r_t \frac{1-e^{-\kappa_r(T-t)}}{\kappa_r} + \left(\bar{r} - \frac{\sigma_r \rho_{mr} \bar{\eta}^*}{\kappa_r}\right)\left[T - t - \frac{1-e^{-\kappa_r(T-t)}}{\kappa_r}\right] \\
&+ \left(\frac{\sigma_r \rho_{mr} \eta_t}{\kappa_\eta^* - \kappa_r} - \frac{\sigma_r \rho_{mr} \bar{\eta}^*}{\kappa_\eta^* - \kappa_r}\right)\left[\frac{1-e^{-\kappa_\eta^*(T-t)}}{\kappa_\eta^*} - \frac{1-e^{-\kappa_r(T-t)}}{\kappa_r}\right] \\
&+ \frac{\sigma_r \rho_{mr} \sigma_\eta}{\kappa_\eta^* - \kappa_r} \int_t^T \left[\frac{1-e^{-\kappa_\eta^*(T-s)}}{\kappa_\eta^*} - \frac{1-e^{-\kappa_r(T-s)}}{\kappa_r}\right] dz_\eta^Q(s) \\
&+ \sigma_r \int_t^T \frac{1-e^{-\kappa_r(T-s)}}{\kappa_r} dz_r^Q(s) \qquad \text{(A9)}
\end{aligned}$$

Substituting equations (A6)-(A9) into equation (A5) yields

$$V(\xi, r, \pi, \eta, T-t) = \xi_t G \mathrm{E}_t^Q \left[\exp^{\psi}\right], \qquad \text{(A10)}$$

where G is given by

$$G = \exp\left\{E(\tau) - B(\tau) r_t - C(\tau)\pi_t - D(\tau)\eta_t\right\} \qquad \text{(A11)}$$

and

$$\begin{aligned}
B(\tau) &= \frac{1-e^{-\kappa_r(T-t)}}{\kappa_r} \qquad \text{(A12)} \\
C(\tau) &= \frac{1-e^{-\kappa_\pi(T-t)}}{\kappa_\pi} \qquad \text{(A13)} \\
D(\tau) &= d_1 + d_2 e^{-\kappa_\eta^* \tau} + d_3 e^{-\kappa_r \tau} + d_4 e^{-\kappa_\pi \tau} \qquad \text{(A14)} \\
E(\tau) &= \left(-\frac{1}{2}\sigma_y^2 + \frac{1}{2}\sigma_P^2 - \bar{r} - \bar{\pi} - d_1 \kappa_\eta^* \bar{\eta}^*\right)\tau + \left(\bar{r} - d_3 \kappa_\eta^* \bar{\eta}^*\right) B(\tau) \\
&+ \left(\bar{\pi} - d_4 \kappa_\eta^* \bar{\eta}^*\right) C(\tau) - d_2 \kappa_\eta^* \bar{\eta}^* d(\tau) \qquad \text{(A15)}
\end{aligned}$$

with $d(\tau) = \left(1 - e^{-\kappa_\eta^*(T-t)}\right) / \kappa_\eta^*$, and finally

$$
\begin{aligned}
d_1 &= -\frac{\sigma_P \rho_{mP} - \sigma_y \rho_{my}}{\kappa_\eta^*} - \frac{\sigma_r \rho_{mr}}{\kappa_r \ \kappa_\eta^*} - \frac{\sigma_\pi \rho_{m\pi}}{\kappa_\pi \kappa_\eta^*} && \text{(A16)} \\
d_2 &= -\frac{\sigma_y \rho_{my}}{\kappa_\eta^*} - \frac{\sigma_r \rho_{mr}}{(\kappa_\eta^* - \kappa_r)\kappa_\eta^*} - \frac{\sigma_\pi \rho_{m\pi}}{(\kappa_\eta^* - \kappa_\pi)\kappa_\eta^*} && \text{(A17)} \\
&= -d_1 - d_3 - d_4 \\
d_3 &= \frac{\sigma_r \rho_{mr}}{(\kappa_\eta^* - \kappa_r)\kappa_r} && \text{(A18)} \\
d_4 &= \frac{\sigma_\pi \rho_{m\pi}}{(\kappa_\eta^* - \kappa_\pi)\kappa_\pi} && \text{(A19)}
\end{aligned}
$$

The stochastic variable ψ is a linear function of the Brownian motions:

$$
\begin{aligned}
\psi &= \sigma_\eta \int_t^T \left[d_2 \left(1 - e^{-\kappa_\eta^*(T-s)}\right) + d_3 \left(1 - e^{-\kappa_r(T-s)}\right) + d_4 \left(1 - e^{-\kappa_\pi(T-s)}\right) \right] dz_\eta^*(s) \\
&- \frac{\sigma_r}{\kappa_r} \int_t^T \left(1 - e^{-\kappa_r(T-s)}\right) dz_r^*(s) - \frac{\sigma_\pi}{\kappa_\pi} \int_t^T \left(1 - e^{-\kappa_\pi(T-s)}\right) dz_\pi^*(s) \\
&+ \sigma_y \int_t^T dz_y^*(s) - \sigma_P \int_t^T dz_P^*(s). && \text{(A20)}
\end{aligned}
$$

Since ψ is normally distributed with mean zero, V is given by

$$
V(\xi, r, \pi, \eta, T-t) = \xi_t G_1 \exp\left\{ \frac{1}{2} \mathrm{Var}_t(\psi) \right\} \qquad \text{(A21)}
$$

Calculating $\mathrm{Var}_t(\psi)$ and collecting terms, we get that

$$
V(\xi, r, \pi, \eta, T-t) = \xi_t \exp\left\{ A(\tau) - B(\tau) r_t - C(\tau)\pi_t - D(\tau)\eta_t \right\} \qquad \text{(A22)}
$$

where

$$
\begin{aligned}
A(\tau) &= a_1\tau + a_2\frac{1-e^{-\kappa_r\tau}}{\kappa_r} + a_3\frac{1-e^{-\kappa_\pi\tau}}{\kappa_\pi} + a_4\frac{1-e^{-\kappa_\eta^*\tau}}{\kappa_\eta^*} \\
&\quad + a_5\frac{1-e^{-2\kappa_r\tau}}{2\kappa_r} + a_6\frac{1-e^{-2\kappa_\pi\tau}}{2\kappa_\pi} + a_7\frac{1-e^{-2\kappa_\eta^*\tau}}{2\kappa_\eta^*} \\
&\quad + a_8\frac{1-e^{-(\kappa_\eta^*+\kappa_r)\tau}}{\kappa_\eta^*+\kappa_r} + a_9\frac{1-e^{-(\kappa_\eta^*+\kappa_\pi)\tau}}{\kappa_\eta^*+\kappa_\pi} + a_{10}\frac{1-e^{-(\kappa_r+\kappa_\pi)\tau}}{\kappa_r+\kappa_\pi}. \qquad \text{(A23)}
\end{aligned}
$$

Define $a_0 \equiv \frac{\sigma_{r\eta}}{\kappa_r} + \frac{\sigma_{\pi\eta}}{\kappa_\pi} + \sigma_{P\eta} - \sigma_{y\eta} - \kappa_\eta^*\bar{\eta}^*$, $\bar{r}^* \equiv \bar{r} - \frac{\sigma_{Pr}-\sigma_{yr}}{\kappa_r}$, and $\bar{\pi}^* \equiv \bar{\pi} - \frac{\sigma_{P\pi}-\sigma_{y\pi}}{\kappa_\pi}$, then $a_1, \dots, a_{10}$ are expressed as

$$
\begin{aligned}
a_1 &= \sigma_P^2 - \sigma_{yP} + \frac{\sigma_r^2}{2\kappa_r^2} + \frac{\sigma_\pi^2}{2\kappa_\pi^2} + \frac{\sigma_{r\pi}}{\kappa_r\kappa_\pi} + \frac{\sigma_\eta^2}{2}d_1^2 - \bar{r}^* - \bar{\pi}^* + a_0 d_1 && \text{(A24)} \\
a_2 &= \bar{r}^* - \frac{\sigma_r^2}{\kappa_r^2} - \frac{\sigma_{r\pi}}{\kappa_r\kappa_\pi} - \frac{\sigma_{r\eta}}{\kappa_r}d_1 + a_0 d_3 + \sigma_\eta^2 d_1 d_3 && \text{(A25)} \\
a_3 &= \bar{\pi}^* - \frac{\sigma_\pi^2}{\kappa_\pi^2} - \frac{\sigma_{r\pi}}{\kappa_r\kappa_\pi} - \frac{\sigma_{\pi\eta}}{\kappa_\pi}d_1 + a_0 d_4 + \sigma_\eta^2 d_1 d_4 && \text{(A26)} \\
a_4 &= a_0 d_2 + \sigma_\eta^2 d_1 d_2 && \text{(A27)} \\
a_5 &= \frac{\sigma_r^2}{2\kappa_r^2} + \frac{\sigma_\eta^2}{2}d_3^2 - \frac{\sigma_{r\eta}}{\kappa_r}d_3 && \text{(A28)} \\
a_6 &= \frac{\sigma_\pi^2}{2\kappa_\pi^2} + \frac{\sigma_\eta^2}{2}d_4^2 - \frac{\sigma_{\pi\eta}}{\kappa_\pi}d_4 && \text{(A29)} \\
a_7 &= \frac{\sigma_\eta^2}{2}d_2^2 && \text{(A30)} \\
a_8 &= -\frac{\sigma_{r\eta}}{\kappa_r}d_2 + \sigma_\eta^2 d_2 d_3 && \text{(A31)} \\
a_9 &= -\frac{\sigma_{\pi\eta}}{\kappa_\pi}d_2 + \sigma_\eta^2 d_2 d_4 && \text{(A32)} \\
a_{10} &= \frac{\sigma_{r\pi}}{\kappa_r\kappa_\pi} - \frac{\sigma_{\pi\eta}}{\kappa_\pi}d_3 - \frac{\sigma_{r\eta}}{\kappa_r}d_4 + \sigma_\eta^2 d_3 d_4 && \text{(A33)}
\end{aligned}
$$

Theorems 1 and 2 follow as special cases of equation (A22). Theorem 1 is obtained by setting σ_P and the parameters in the expected inflation process (A2) to zero. Theorem 2 is obtained by setting σ_y to zero.

Theorem 3 is more complicated because of the additional state variable g. Using the same approach as above, we can derive that an equity value at time t is given by

$$V(X,r,\pi,\eta,g) = \mathrm{E}^{Q}\left[\int_t^{\infty}\frac{X_s}{P_s}e^{-\int_t^s r(u)du}ds\right] = \frac{X_t}{P_t}\int_t^{\infty} v(s-t,r,\pi,\eta,g)ds \quad \text{(A34)}$$

where Q denotes the risk neutral probability measure, and

$$v(s,r,\pi,\eta,g) = \exp[\tilde{A}(s-t) - B(s-t)r - C(s-t)\pi - \tilde{D}(s-t)\eta - F(s-t)g] \quad \text{(A35)}$$

where

$$B(s-t) = \kappa_r^{-1}\left(1-e^{-\kappa_r(s-t)}\right) \quad \text{(A36)}$$

$$C(s-t) = \kappa_\pi^{-1}\left(1-e^{-\kappa_\pi(s-t)}\right) \quad \text{(A37)}$$

$$\tilde{D}((s-t)) = \tilde{d}_1 + \tilde{d}_2 e^{-\kappa_\eta^*(s-t)} + \tilde{d}_3 e^{-\kappa_r(s-t)} + \tilde{d}_4 e^{-\kappa_\pi(s-t)} + \tilde{d}_5 e^{-\kappa_g(s-t)} \quad \text{(A38)}$$

$$\begin{aligned}\tilde{A}((s-t)) = {} & \tilde{a}_1(s-t) + \tilde{a}_2\frac{1-e^{-\kappa_r(s-t)}}{\kappa_r} + \tilde{a}_3\frac{1-e^{-\kappa_\pi(s-t)}}{\kappa_\pi} + \tilde{a}_4\frac{1-e^{-\kappa_\eta^*(s-t)}}{\kappa_\eta^*} \\ & +\tilde{a}_5\frac{1-e^{-2\kappa_r(s-t)}}{2\kappa_r} + \tilde{a}_6\frac{1-e^{-2\kappa_\pi(s-t)}}{2\kappa_\pi} + \tilde{a}_7\frac{1-e^{-2\kappa_\eta^*(s-t)}}{2\kappa_\eta^*} \\ & +\tilde{a}_8\frac{1-e^{-(\kappa_\eta^*+\kappa_r)(s-t)}}{\kappa_\eta^*+\kappa_r} + \tilde{a}_9\frac{1-e^{-(\kappa_\eta^*+\kappa_\pi)(s-t)}}{\kappa_\eta^*+\kappa_\pi} + \tilde{a}_{10}\frac{1-e^{-(\kappa_r+\kappa_\pi)(s-t)}}{\kappa_r+\kappa_\pi} \\ & +\tilde{a}_{11}\frac{1-e^{-(\kappa_\eta^*+\kappa_g)(s-t)}}{\kappa_\eta^*+\kappa_g} + \tilde{a}_{12}\frac{1-e^{-(\kappa_g+\kappa_\pi)(s-t)}}{\kappa_g+\kappa_\pi} + \tilde{a}_{13}\frac{1-e^{-(\kappa_r+\kappa_g)(s-t)}}{\kappa_r+\kappa_g} \\ & +\tilde{a}_{14}\frac{1-e^{-\kappa_g(s-t)}}{\kappa_g} + \tilde{a}_5\frac{1-e^{-2\kappa_g(s-t)}}{2\kappa_g} \end{aligned} \quad \text{(A39)}$$

$\kappa_\eta^* \equiv \kappa_\eta + \sigma_\eta\rho_{m\eta}$, and $\tilde{d}_1,\ldots,\tilde{d}_4$, $\tilde{a}_1,\ldots,\tilde{a}_{10}$ are constants whose values are available upon request.

B. Proof of Lemmas 1 and 2

Proof of Lemma 1:

It is immediate that $B(\tau)$, $\frac{\partial B(\tau)}{\partial \tau} > 0$. From equations (A14) and (A16-A19), we note that

$$D(0) \;=\; d_1 + d_2 + d_3 = 0$$

Taking the derivative of (A14) with respect to τ,

$$D_\tau \;=\; \sigma_y \rho_{ym} e^{-\kappa_\eta^* \tau} + \frac{\sigma_r \rho_{mr}}{\kappa_\eta^* - \kappa_r}(e^{-\kappa_\eta^* \tau} - e^{-\kappa_r \tau}). \tag{B1}$$

Then the assumptions $\rho_{ym} > 0$ and $\rho_{mr} < 0$ imply that $D_\tau \geq 0$, so that $D(\tau) \geq 0 \; \forall \tau$.

Proof of Lemma 2:

Applying Ito's lemma to the V function from Theorem 1, the expected return on the claim can be written as:

$$\begin{aligned}
\mu \equiv \mu(r, \eta, \tau) \;&=\; -A_\tau + rB_\tau + \eta D_\tau + \frac{1}{2} D^2 \sigma_\eta^2 + BD\rho_{\eta r}\sigma_\eta \sigma_r + \frac{1}{2} B^2 \sigma_r^2 \\
&\quad -D\rho_{y\eta}\sigma_y\sigma_\eta - B\rho_{yr}\sigma_y\sigma_r - D\kappa_\eta(\bar{\eta} - \eta) - B\kappa_r(\bar{r} - r) \\
&=\; D_\tau \eta + B_\tau r + D\kappa_\eta \eta + B\kappa_r r \\
&=\; r + (D_\tau + D\kappa_\eta)\eta,
\end{aligned} \tag{B2}$$

then

$$\begin{aligned}
\frac{\partial \mu}{\partial \tau} \;&=\; (D_{\tau\tau} + D_\tau \kappa_\eta)\eta \\
&=\; \eta \left[\sigma_y \rho_{ym} \kappa_\eta - \sigma_r \rho_{mr} e^{-\kappa_r \tau} + \frac{\sigma_r \rho_{mr} \sigma_\eta \rho_{m\eta}}{\kappa_r - \kappa_\eta^*}(e^{-\kappa_\eta^* \tau} - e^{-\kappa_r \tau}) \right],
\end{aligned} \tag{B3}$$

form which, it follows that $\frac{\partial \mu}{\partial \tau} \geq 0$ when $\rho_{mr}, \rho_{m\eta} \leq 0$ and $\rho_{ym} \geq 0$.

Lemma 3 is straightforward and the proof is omitted.

C. Details of Kalman Filter

The yield-based estimates of the state variable dynamics are derived by applying a Kalman filter to data on bond yields and inflation using equations (31) and (32). The transition equations for the state variables, r, π and η are derived by discretizing equations (5.2), (5.3), and (16):

$$\begin{pmatrix} r_t \\ \pi_t \\ \eta_t \end{pmatrix} = \begin{pmatrix} e^{-\kappa_r \Delta t} & 0 & 0 \\ 0 & e^{-\kappa_\pi \Delta t} & 0 \\ 0 & 0 & e^{-\kappa_\eta \Delta t} \end{pmatrix} \begin{pmatrix} r_{t-\Delta t} \\ \pi_{t-\Delta t} \\ \eta_{t-\Delta t} \end{pmatrix} + \begin{pmatrix} \bar{r}\left[1 - e^{-\kappa_r \Delta t}\right] \\ \bar{\pi}\left[1 - e^{-\kappa_\pi \Delta t}\right] \\ \bar{\eta}\left[1 - e^{-\kappa_\eta \Delta t}\right] \end{pmatrix} + \begin{pmatrix} \epsilon_r(t) \\ \epsilon_\pi(t) \\ \epsilon_\eta(t) \end{pmatrix}, \quad \text{(C1)}$$

where the vector of innovations is related to the standard Brownian motions, dz_r, dz_π and dz_η, by

$$\begin{pmatrix} \epsilon_r(t) \\ \epsilon_\pi(t) \\ \epsilon_\eta(t) \end{pmatrix} = \begin{pmatrix} \sigma_r e^{-\kappa_r (t-\Delta t)} \int_{t-\Delta t}^{t} e^{-\kappa_r \tau} dz_r(\tau) \\ \sigma_\pi e^{-\kappa_\pi (t-\Delta t)} \int_{t-\Delta t}^{t} e^{-\kappa_\pi \tau} dz_\pi(\tau) \\ \sigma_\eta e^{-\kappa_\eta (t-\Delta t)} \int_{t-\Delta t}^{t} e^{-\kappa_\eta \tau} dz_\eta(\tau) \end{pmatrix}, \quad \text{(C2)}$$

and the variance-covariance matrix of the innovations is

$$Q = \begin{pmatrix} \frac{\sigma_r^2}{2\kappa_r}\left[1 - e^{-2\kappa_r \Delta t}\right] & \frac{\sigma_r \sigma_\pi \rho_{r\pi}}{\kappa_r + \kappa_\pi}\left[1 - e^{-(\kappa_r + \kappa_\pi)\Delta t}\right] & \frac{\sigma_r \sigma_\eta \rho_{r\eta}}{\kappa_r + \kappa_\eta}\left[1 - e^{-(\kappa_r + \kappa_\eta)\Delta t}\right] \\ \frac{\sigma_r \sigma_\pi \rho_{r\pi}}{\kappa_r + \kappa_\pi}\left[1 - e^{-(\kappa_r + \kappa_\pi)\Delta t}\right] & \frac{\sigma_\pi^2}{2\kappa_\pi}\left[1 - e^{-2\kappa_\pi \Delta t}\right] & \frac{\sigma_\pi \sigma_\eta \rho_{\pi\eta}}{\kappa_\pi + \kappa_\eta}\left[1 - e^{-(\kappa_\pi + \kappa_\eta)\Delta t}\right] \\ \frac{\sigma_r \sigma_\eta \rho_{r\eta}}{\kappa_r + \kappa_\eta}\left[1 - e^{-(\kappa_r + \kappa_\eta)\Delta t}\right] & \frac{\sigma_\pi \sigma_\eta \rho_{\pi\eta}}{\kappa_\pi + \kappa_\eta}\left[1 - e^{-(\kappa_\pi + \kappa_\eta)\Delta t}\right] & \frac{\sigma_\eta^2}{2\kappa_\eta}\left[1 - e^{-2\kappa_\eta \Delta t}\right] \end{pmatrix}. \quad \text{(C3)}$$

The first n observation equations assume that the observed yields at time t, $y_{\tau_j,t}$, on bonds with maturities τ_j, $j = 1, \cdots, n$, are given by equation (31) plus a measurement error terms, ϵ_{τ_j}:

$$y_{\tau_j,t} \equiv -\frac{\ln V(t, t+\tau_j)}{\tau_j} = -\frac{A(t,\tau_j)}{\tau_j} + \frac{B(\tau_j)}{\tau_j} r_t + \frac{C(\tau_j)}{\tau_j} \pi_t + \frac{D(\tau_j)}{\tau_j} \eta + \epsilon_{\tau_j}(t). \quad \text{(C4)}$$

The measurement errors, $\epsilon_{\tau_j}(t)$, are assumed to be serially and cross-sectionally uncorrelated and are uncorrelated with the innovations in the transition equations.

The $n+1$ observation equation uses the realized rate of inflation:

$$\frac{P_t - P_{t-\Delta t}}{P_{t-\Delta t}} = \pi \Delta t + \epsilon_P(t), \quad \text{(C5)}$$

where $\epsilon_P = \sigma_P \int_{t-\Delta t}^{t} dz_P$ with variance $\sigma_P^2 \Delta t$, and is assumed to be uncorrelated with the yield measurement errors and the innovations in the transition equation.

The hybrid estimates discussed in Section 5.3 include an additional observation equation on the Sharpe ratio of the form:

$$\hat{\eta}_{return}(t) = a + b\eta(t) + \epsilon_\eta(t) \quad \text{(C6)}$$

where $\hat{\eta}_{return}$ is the estimate of η based on equity returns.

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Table 1

Summary Statistics on Four Fama-French Portfolios and Bond Yields

Returns are in percent per month. r_f, the real interest rate, is obtained by subtractng the realized rate of inflation from the one month T-bill rate. r_M is the excess return on the CRSP value-weighted index. Portfolio GS is the Fama-French low book-to-market portfolio of small firms; VS the high book-to-market portfolio of small firms; GB the low book-to-market portfolio of big firms; VB the high book-to-market portfolio of big firms. The sample is from January 1950 to December 1999. The bond data are monthly constant maturity zero coupon U.S. Treasury yields for the period from March 1953 to September 1996. Inflation is calculated using CPI data for the same sample period.

A.Distribution of Returns and Market to Book (M/B) ratios

	M/B				Return					
Portfolio	GS	VS	GB	VB	GS	VS	GB	VB	r_f	r_M
Mean	2.89	0.62	3.07	0.69	1.06	1.49	1.11	1.34	0.08	0.71
Minimum	1.15	0.25	1.47	0.28	-32.09	-27.98	-23.07	-18.95	-1.5	-22.83
Median	2.86	0.58	2.88	0.68	1.25	1.83	1.35	1.57	0.11	1.00
Maximum	5.82	1.20	6.28	1.18	24.94	29.60	21.43	20.95	1.12	16.00
Std.Dev	0.98	0.19	1.03	0.20	6.05	5.07	4.45	4.39	0.29	4.17
Autocorrelation	0.97	0.98	0.99	0.99	0.17	0.17	0.05	0.04	0.48	0.06

B. Correlations

Market to Book Ratios

	GS	VS	GB	VB
GS	1			
VS	0.96	1		
GB	0.89	0.87	1	
VB	0.95	0.98	0.89	1

Portfolio and Market Returns

	GS	VS	GB	VB	r_M
GS	1				
VS	0.89	1			
GB	0.83	0.74	1		
VB	0.75	0.88	0.78	1	
r_M	0.88	0.84	0.97	0.87	1

C. Bond Yields and Inflation (% per year)

Bond Maturity (years)	0.25	0.5	1	2	3	4	5	10	Inflation
Mean	5.62	5.85	6.06	6.29	6.44	6.55	6.63	6.84	4.10
Std. Dev.	2.96	3.00	2.98	2.91	2.87	2.84	2.82	2.78	1.12

Table 2

Summary Statistics for State Variable Instruments

P_{GV}^B is the (log of the) ratio of the market-to-book ratios for large growth and value firms; P_{GV}^S is the (log of the) ratio of the market-to-book ratios for small growth and value firms; P_{BS}^V is the (log of the) ratio of the market-to-book ratios for value big and small firms; TS is the term spread as measured by the difference between the yields on the 10 and 1 year Treasury Bonds (% per year). The sample period is from May 1953 to September 1996.

A.Distribution of log of Market to Book ratios and Term Spread				
	P_{GV}^B	P_{GV}^S	P_{BS}^V	TS
Mean	1.471	1.531	0.114	0.662
Minimum	1.157	1.270	-0.018	-3.856
Median	1.501	1.530	0.105	0.714
Maximum	1.800	1.802	0.270	3.268
Std.Dev	0.151	0.108	0.055	0.948
Autocorrelation	0.985	0.959	0.908	0.923

B. Correlations				
	P_{GV}^B	P_{GV}^S	P_{BS}^V	TS
P_{GV}^B	1			
P_{GV}^S	0.373	1		
P_{BS}^V	0.196	0.363	1	
TS	0.054	0.300	0.164	1

C. Unit Root Test	
	ADF Statistic
P_{GV}^B	-2.06
P_{GV}^S	-3.14^*
P_{BS}^V	-4.81^{**}
TS	-3.35^*

** Significant at the 1% level
* Significant at the 5% level

Table 3

State Variable Predictive Regressions

Monthly regressions of the realized market excess return, r_M, real interest rate, r_f, Sharpe ratio constructed by following Whitelaw's specification, $SH_{whitelaw}$, and Sharpe ratio constructed by using the realized excess market returns scaled by the fitted volatility, $SH_{realized}$, on state variable proxies: P^B_{GV}, the (log of the) ratio of the market-to-book ratios for large growth and value firms; P^S_{GV}, the (log of the) ratio of the market-to-book ratios for small growth and value firms; P^V_{BS}, the (log of the) ratio of the market-to-book ratios for value big and small firms; TS, the term spread measured by the difference between the yields of 10 year and 1 year Treasury Bonds (% per year), log of the dividend yields on S& P 500 Index, SP_{Divyld}, and log of the Book_to_market ratio of S& P 500 Index, SP_BM. The sample period is from May 1953 to September 1996. *t*-ratios adjusting for HAC errors are in parentheses.

	Dependent Variables	Const.	$P^B_{GV}(-1)$	$P^S_{GV}(-1)$	$P^V_{BS}(-1)$	$TS(-1)$	$SP\_Divyld(-1)$	$SP\_BM(-1)$	R^2
(1)	r_M	7.336	-0.511	-4.600	9.200				0.01
		(2.34)	(0.32)	(2.10)	(2.48)				
(2)	r_M	9.580	-0.158	-6.678	8.285	0.791			0.04
		(3.11)	(0.11)	(3.02)	(2.27)	(3.78)			
(3)	r_M	13.639	0.400	-5.437	9.206	0.830	2.077		0.05
		(3.83)	(0.28)	(2.32)	(2.46)	(3.87)	(1.95)		
(4)	r_M	8.911	-0.185	-5.652	9.422	0.899		1.554	0.05
		(2.77)	(0.14)	(2.47)	(2.50)	(4.14)		(1.99)	
(5)	r_f	-0.369	-0.634	0.968	-0.742				0.18
		(1.23)	(5.13)	(5.41)	(3.33)				
(6)	r_f	-0.337	-0.629	0.939	-0.755	0.011			0.18
		(1.15)	(5.18)	(5.25)	(3.27)	(0.72)			
(7)	r_f	-0.286	-0.622	0.954	-0.744	0.012	0.026		0.18
		(0.87)	(5.14)	(5.06)	(3.23)	(0.76)	(0.33)		
(8)	r_f	-0.349	-0.630	0.957	-0.735	0.013		0.028	0.18
		(1.17)	(5.19)	(5.23)	(3.21)	(0.85)		(0.51)	
(9)	$SH_{realized}$	1.458	0.067	-1.065	2.203				0.01
		(1.86)	(0.17)	(1.97)	(2.40)				
(10)	$SH_{realized}$	2.039	0.159	-1.604	1.966	0.205			0.04
		(2.64)	(0.47)	(2.87)	(2.16)	(3.92)			
(11)	$SH_{realized}$	3.245	0.325	-1.235	2.239	0.217	0.617		0.05
		(3.59)	(0.92)	(2.18)	(2.42)	(4.09)	(2.40)		
(12)	$SH_{realized}$	1.847	0.151	-1.309	2.293	0.236		0.447	0.05
		(2.30)	(0.46)	(2.34)	(2.45)	(4.37)		(2.36)	
(13)	$SH_{whitelaw}$	-0.015	0.626	-0.439	0.292				0.10
		(0.05)	(3.16)	(1.82)	(0.76)				
(14)	$SH_{whitelaw}$	0.422	0.695	-0.843	-0.471	0.154			0.34
		(1.42)	(4.22)	(3.41)	(1.41)	(6.34)			
(15)	$SH_{whitelaw}$	1.750	0.877	-0.437	-0.169	0.167	0.680		0.58
		(5.07)	(6.31)	(2.21)	(0.65)	(7.85)	(7.10)		
(16)	$SH_{whitelaw}$	0.212	0.686	-0.521	-0.114	0.188		0.488	0.56
		(0.77)	(5.01)	(2.63)	(0.44)	(8.08)		(6.95)	

Table 4

Return Based, Yield Based, and Hybrid Parameter Estimates

This table reports estimates of the parameters of the stochastic process of the investment opportunity set. The yield-based estimates (Column (1)) are derived from the Kalman filter applied to the inflation and bond yield data with $\bar{r} = 1.24\%$, $\bar{\pi} = 4.1\%$ and $\bar{\eta} = 0.7$. $\bar{r}$ and $\bar{\pi}$ are the sample means, and $\bar{\eta}$ is slightly above the returned-based estimate of this parameter. For yield-based estimates, the Wald test is for $H_0 : \kappa_\eta = \sigma_\eta = 0$. Asymptotic t-ratios are in parenthesis.
The return-based estimates (Columns (2) and (3)) are obtained from the non-linear least squares regressions of the following discretized version of equations (5.2) and (5.3):

$$\begin{aligned} r_t &= \bar{r}\left(1 - e^{-\kappa_r \Delta t}\right) + e^{-\kappa_r \Delta t} r_{t-\Delta t} + \epsilon_{r,t}, \\ \eta_t &= \bar{\eta}\left(1 - e^{-\kappa_\eta \Delta t}\right) + e^{-\kappa_\eta \Delta t} \eta_{t-\Delta t} + \epsilon_{\eta,t}, \end{aligned}$$

where r and η are the fitted state variables from the regressions in Table 3. Results related to η in column (2) apply to $SH_{Whitelaw}$ (Row (14) of Table 3) while results in column (3) apply to $SH_{realized}$ (row (10) of Table 3). Results related to r apply to r_f (row (5) of Table 3).
The hybrid estimates (column (4)) are derived in a similar way as the yield-based estimates except that $SH_{Whitelaw}$ is used to provide an observation equation in addition to those provided by the data on inflation and bond yields.

	(1) Yield Based Estimates	(2) Return Based Estimate (SHR)	(3) Return Based Estimate (SHW)	(4) Hybrid Estimate
σ_b	0.15%			0.12%
	(82.58)			(87.84)
σ_r	1.94%	1.27%		1.58%
	(26.91)	n.a.		(36.13)
σ_π	0.62%			1.14%
	(10.42)			(21.65)
σ_η	0.19	1.23	0.74	0.16
	(31.44)	n.a.	n.a.	(25.03)
κ_r	0.13	0.44		0.13
	(69.06)	(2.98)		(88.85)
κ_π	0.00			0.00
	(0.79)			(0.76)
κ_η	0.15	1.36	0.91	0.19
	(6.74)	(5.10)	(4.31)	(8.30)

Table 4 (continued)

	(1) Yield Based Estimates	(2) Return Based Estimate (SHR)	(3) Return Based Estimate (SHW)	(4) Hybrid Estimate
$\rho_{r\pi}$	-0.07			-0.08
	(0.78)			(1.58)
$\rho_{r\eta}$	-0.17	-0.05		-0.20
	(1.82)	n.a.		(2.57)
ρ_{rm}	-0.65			-0.70
	(14.69)			(10.35)
$\rho_{\pi\eta}$	-0.17			-0.21
	(2.06)			(3.83)
$\rho_{\pi m}$	0.25			0.17
	(3.03)			(4.39)
$\rho_{\eta m}$	0.72			0.76
	(11.60)			(22.46)
ρ_{Pm}	0.23			0.19
	(2.53)			(6.49)
$\bar{r}$	1.50%	1.26%		1.50%
	n.a.	(2.81)		n.a.
$\bar{\pi}$	4.10%			4.10%
	n.a.			n.a.
$\bar{\eta}$	0.70	0.61	0.68	0.70
	n.a.	(4.19)	(5.31)	n.a.
σ_P	1.12%			1.12%
	n.a.			n.a.
a_{SH}				0.13
				(3.76)
b_{SH}				0.73
				(18.37)
σ_{SH}				30.06%
				(39.72)
ML	26,617.71			25,158.62
$Wald$	1,072.82			750.73

Table 5

Relation Between Return-Based and Yield-Based State Variables Estimates

η_{SHR} is the return-based estimate of the Sharpe ratio using the realized equity premium as the input while η_{SHW} is the return-based estimate of the Sharpe ratio using the Whitelaw-fitted equity premium as the input. r_{yield} and η_{yield} are the corresponding yield-based estimates. r_p and $r_{p,ts}$ are the return-based estimates of the real interest rate using the FF price ratios or the FF price ratios and the term spread as proxies. *t*-ratios adjusting for HAC errors are in parentheses..

	Dependent Variable	$Constant$	η_{yield}	r_{yield}	Adj. R^2	$S.E.$
(1)	η_{SHR}	0.33	0.56		0.31	0.47
		(5.1)	(6.8)			
(2)	η_{SHW}	0.22	0.60		0.19	0.71
		(2.3)	(5.1)			
(3)	r_p	0.01		-0.04	-0.00	0.01
		(5.3)		(0.5)		
(4)	$r_{p,ts}$	0.01		-0.07	0.005	0.01
		(5.8)		(0.9)		

Table 6

FF Portfolio Price Ratios, Returns, and Yield-Based State Variable Estimates

Panel A reports regressions of FF portfolio price ratios on the yield-based state variable estimates. Panel B reports regressions of innovations in r, π, and η on the Fama-French portfolio returns. *t*-ratios, reported in parentheses, are calculated using the Newey-West adjustment for heterogeneity and serial correlation.

A. FF Portfolio Price Ratios vs. Yield Based State Variable Estimates

	Dependent Variables	Const.	r_{yield}	π_{yield}	η_{yield}	Adj. R^2	S.E.
(1)	P^B_{GV}	1.63	-1.22	-3.24	-0.02	0.34	0.12
		(54.2)	(2.3)	(10.0)	(0.6)		
(2)	P^S_{GV}	1.52	-1.01	0.10	0.04	0.08	0.10
		(47.4)	(1.8)	(0.2)	(2.0)		
(3)	P^V_{BS}	0.16	-1.33	-0.24	-0.01	0.14	0.05
		(11.9)	(4.6)	(1.5)	(2.0)		
(4)	P^G_{BS}	0.27	-1.54	-3.59	-0.07	0.42	0.12
		(9.7)	(2.9)	(11.5)	(3.4)		

B. Innovations in the Yield Based State Variable Estimates vs. FF Portfolio Returns

	Dependent Variables	$Const.$	$r_m - r_f$	SMB	HML	Adj. R^2	S.E.
(1)	$\hat{\epsilon}_r$	0.001	-0.028	0.010	-0.038	0.04	0.006
		(1.7)	(3.7)	(1.0)	(2.6)		
(2)	$\hat{\epsilon}_\pi$	-0.000	-0.006	0.005	0.008	0.01	0.003
		(0.0)	(1.7)	(1.1)	(1.4)		
(3)	$\hat{\epsilon}_\eta$	-0.010	0.123	0.533	0.726	0.04	0.103
		(1.6)	(0.8)	(2.8)	(3.4)		

Table 7

Regressions of FF portfolio returns on market return and estimated state variable innovations

This table reports regressions of HML and SMB monthly portfolio returns on market excess returns and innovations in the yield-based estimates of the state variables r, π, and η for the period May 1953 to September 1996. The *t*-ratios, reported in parentheses, are calculated using the Newey-West adjustment for heterogeneity and serial correlation.

	Dep. Variable	$Const.$	$R_m - R_f$	$\Delta\eta$	Δr	$\Delta\pi$	$\bar{R}^2$
(1)	R_{HML}	0.97%	-0.19	2.88	-47.72		12.95%
		(8.33)	(5.80)	(2.54)	(2.34)		
(2)	R_{HML}	0.97%	-0.19	3.05	-40.67	30.06	12.85%
		(8.31)	(5.67)	(2.55)	(1.81)	(0.67)	
(3)	R_{SMB}	0.49%	0.18	3.44	30.37		8.95%
		(3.83)	(4.59)	(3.16)	(1.59)		
(4)	R_{SMB}	0.47%	0.19	4.20	63.64	141.83	10.15%
		(3.76)	(4.85)	(3.68)	(2.87)	(2.91)	

Table 8

Tracking Portfolios for Innovations in r and η

Panel A reports two sets of estimates of tracking portfolios for the yield-based estimates of the innovations in r, π, and η for the sample period from May 1953 to September 1996. Variables XSG through XBV are excess returns on six size and book-to-market sorted portfolios formed by Fama and French. r, π, and η are the yield-based estimates of the real interest rate and the maximum Sharpe ratio. The coefficients reported in Panel A are multiplied by 1,000.
Panel B reports the regressions of HML and SMB portfolio returns on the market risk premium, $R_M - R_f$, and the returns on the r, π, and η tracking portfolios, R_r, R_π, and R_η.
The *t*-ratios, reported in parentheses, are calculated using the Newey-West adjustment for heterogeneity and serial correlation.

A. Tracking Portfolio Formation Regressions

	(1)	(2)	(3)	(4)	(5)	(6)
	Δr	Δr	$\Delta\eta$	$\Delta\eta$	$\Delta\pi$	$\Delta\pi$
Const.	5.15	-2.67	-63.82	126.42	1.17	6.58
	(1.84)	(0.61)	(1.01)	(0.83)	(0.78)	(2.57)
XSG	4.90	5.27	-79.79	-92.70	0.16	-0.34
	(2.95)	(3.13)	(2.42)	(2.92)	(0.15)	(0.34)
XSN	-4.29	-4.71	96.66	109.91	0.27	0.78
	(1.30)	(1.42)	(1.66)	(1.89)	(0.17)	(0.51)
XSV	-0.59	-0.55	56.65	56.22	-0.02	-0.03
	(0.23)	(0.22)	(1.42)	(1.38)	(0.01)	(0.02)
XBG	-0.44	-0.58	10.41	15.96	-1.21	-0.98
	(0.29)	(0.38)	(0.45)	(0.74)	(1.52)	(1.29)
XBN	-2.85	-2.74	-72.08	-72.68	-0.14	-0.15
	(1.53)	(1.48)	(2.16)	(2.16)	(0.14)	(0.15)
XBV	-0.10	-0.09	3.06	2.56	0.23	0.20
	(0.06)	(0.06)	(0.10)	(0.08)	(0.28)	(0.25)
η_{-1}		11.29		-332.54		-9.08
		(1.80)		(1.58)		(3.08)
$\bar{R}^2$	3.77%	3.86%	4.31%	5.81%	0.67%	4.52%

B. Regressions of FF portfolio returns on market and tracking portfolio returns

	(1)	(2)	(3)	(4)
	R_{HML}	R_{HML}	R_{SMB}	R_{SMB}
Const.	0.43%	0.49%	0.42%	0.49%
	(6.72)	(9.49)	(5.48)	(9.37)
$R_m - R_f$	-0.47	-0.14	0.42	0.82
	(23.06)	(5.88)	(13.11)	(31.62)
R_η	35.45	11.01	85.80	56.19
	(10.63)	(3.41)	(21.41)	(16.46)
R_r	-1318.17	-1366.83	1653.57	1594.62
	(19.64)	(25.11)	(20.53)	(28.02)
R_π		4586.86		5557.31
		(20.66)		(16.91)
$\bar{R}^2$	71.88%	81.84%	71.89%	84.30%

Table 9

Risk Premium Estimates for the HML and SMB Portfolios

Panel A of the table reports the risk premia and its standard deviation and the t-statistics of the zero-investment market (long in the market portfolio and short the same amount in the riskless asset) r, π and η factor-tracking portfolios. The sample period is from June 1953 to September 1996 with 521 observations. Panel B of the table reports for HML and SMB the sample mean returns, the ICAPM pricing kernel estimates of the risk premia with and without π as the additional state variable, and the ICAPM tracking portfolio estimates of the risk premia with and without π as an additional state variable.

Panel A: Summary Statistics of the r, π, and η Tracking Portfolios

		Zero-Invesment Market Portfolio	r Tracking Portfolio	η Tracking Portfolio	π Tracking Portfolio
(1)	Risk Premium	7.07%	-0.44%	7.30%	-0.03%
(2)	Standard Deviation	4.18%	0.13%	2.44%	0.04%
(3)	t-Statistics	3.22	6.28	5.69	1.67

Panel B: Risk Premia for the HML and SMB Portfolios

		Sample Mean	Pricing Kernel (without π)	Pricing Kernel (with π)	Tracking Portfolio (without π)	Tracking Portfolio (with π)
(1)	HML	10.32%	-0.65%	-0.65%	5.09%	4.37%
(2)	SMB	7.08%	1.31%	1.35%	2.02%	1.13%

Figure 1
Time Series of State Variable Proxies

P^B_{GV} is the (log of the) ratio of the market-to-book ratios for large growth and value firms; P^S_{GV} is the (log of the) ratio of the market-to-book ratios for small growth and value firms; P^V_{BS} is the (log of the) ratio of the market-to-book ratios for value big and small firms; TS is the term spread as measured by the difference between the yields on the 10 and 1 year Treasury Bonds (% per year). The log price ratios are plotted off the left Y-Axis and TS is plotted off the right Y-Axis.

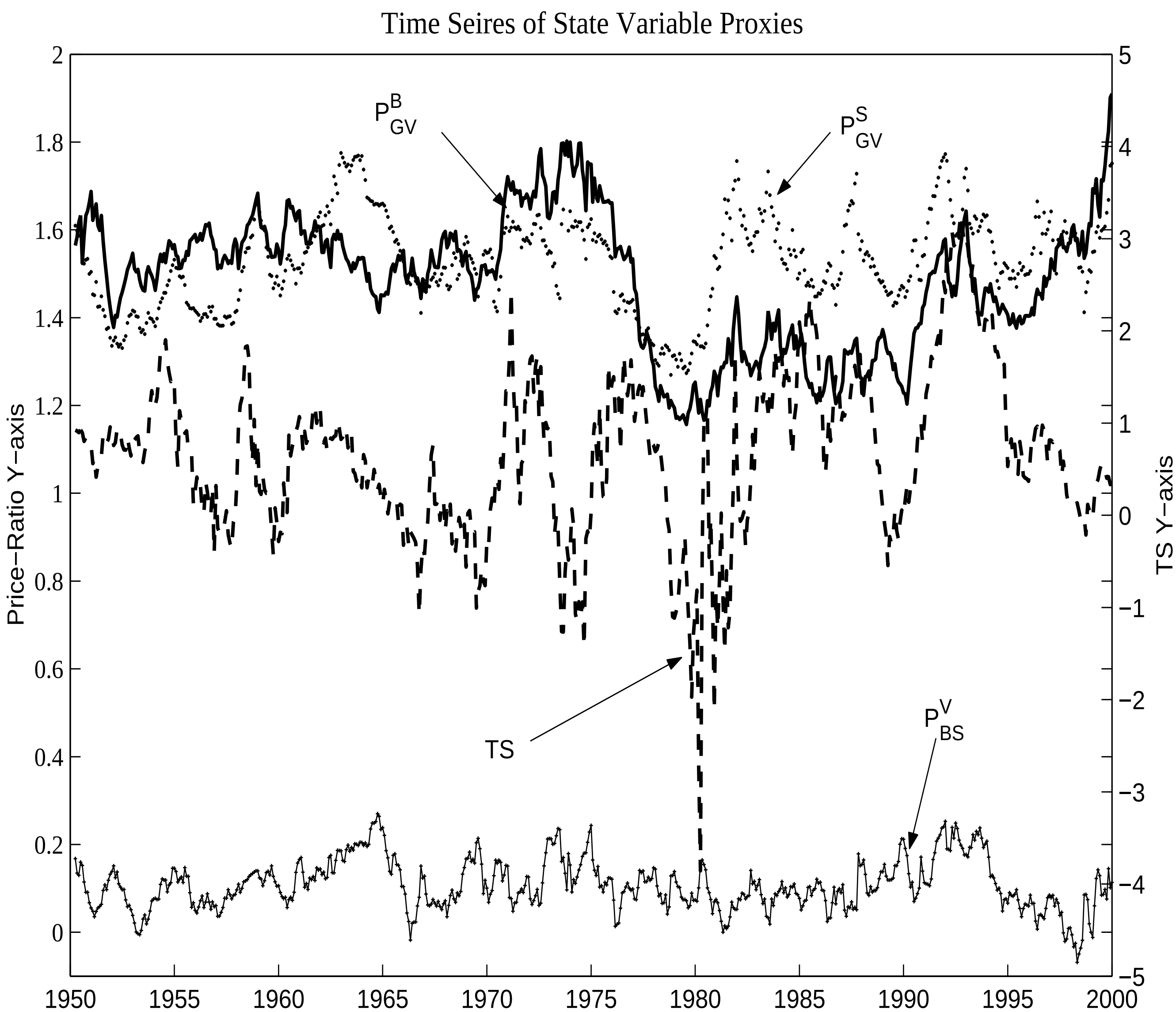

Figure 2
Time Series of Real Interest Rate Estimates

The figure plots two estimated real interest rate series from May 1953 to September 1996: the yield-based estimates are filtered out from the bond yield and inflation data and the return-based estimates are fitted values from predictive regressions of real bill returns on the log of ratio of the market-to-book ratios. Shaded area indicates periods of U.S. recessions.

Return based estimates - solid line; Yield based estimates - dashed line.

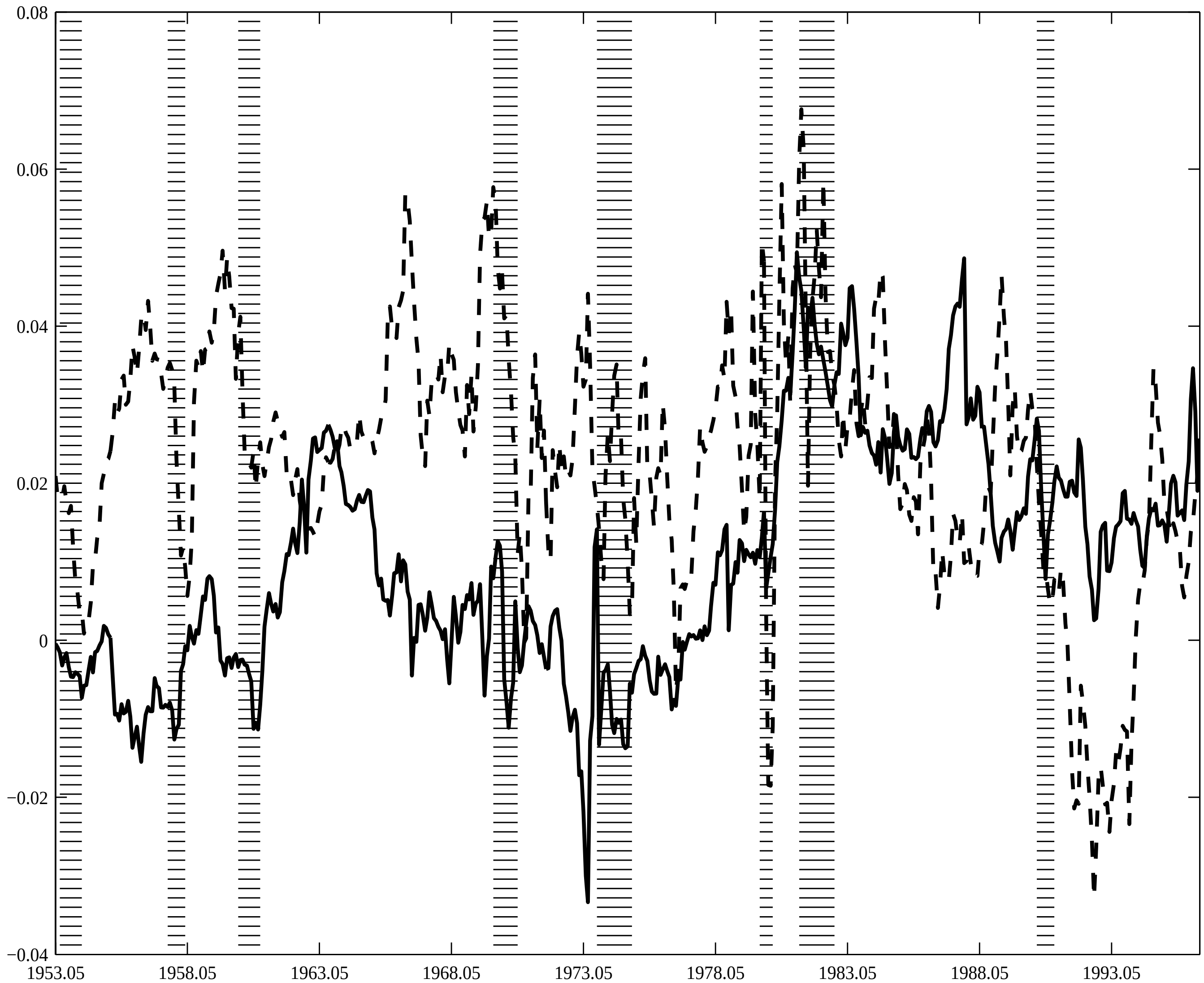

Figure 3
Time Series of Sharpe Ratio Estimates

The figure plots two estimated Sharpe ratio series from May 1953 to September 1996: the yield-based estimates are filtered out from the bond yield and inflation data and the return-based estimates are fitted values from predictive regressions of market excess return. Shaded area indicates periods of U.S. recessions.

Return based estimates - solid line; Yield based estimates - dashed line.