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Externalities and Renegotiations in Three-Player Coalitional Bargaining

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Abstract

We study strategic three-player coalitional bargaining problems in an environment with externalities where contracts forming coalitions can be written and renegotiated. The theory yields a unique stationary subgame perfect Nash equilibrium outcome (the coalitional bargaining value). This solution has an intuitive economic interpretation using credible outside options, and it can either be the Nash bargaining solution, for games where the worth of all pairwise coalition is less than a third of the grand coalition value; the Shapley value, for games where the sum of the values created by all pairwise coalitions is greater than the grand coalition value; or the nucleolus, for games where only the 'natural coalition' among two 'natural partners' creates significant value, and those where only the two pairwise coalitions including a 'pivotal player' create significant value.

JEL: C71, C72, C78, D62. KEYWORDS: Coalitional bargaining, uniqueness, externalities, conditional and unconditional offers.

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1. Introduction

This paper studies multilateral negotiations in an economic context with *externalities* where binding agreements forming coalitions can be written and *renegotiated*. These features prevail in a variety of economic and social situations. For example, in mergers and acquisitions, the merger of two firms (a binding agreement) may create positive or negative externalities for other firms in the industry, and firms often expand through a process of sequential mergers or acquisitions negotiations. Other important problems, such as the formation of labor unions and coalitional governments, and trading in an exchange economy also share many of these features. What coalitions are expected to form? What are the expected values of players and coalitions? Our goal in this paper is to propose a strategic or non-cooperative model of negotiations that incorporates externalities and renegotiations, and to thoroughly analyze the solution that arises from this model.

The search of solutions for coalitional bargaining games is an old problem that, since the publication of von Neumann and Morgenstern's (1944) seminal book, have attracted the interest of many researchers. In one branch of the literature, cooperative game theory using axioms and intuitive normative concepts have developed numerous solution concepts to the problem, such as the Nash bargaining solution (Nash (1950, 1953)), the Shapley value (Shapley (1953)), the nucleolus (Schmeidler (1969)), the core, among many others (see survey in Maschler (1992)). Another branch of the literature, pursued the so called Nash program of establishing the non-cooperative foundations supporting the cooperative approach. Along this line is the work of Rubinstein (1982) and Binmore, Rubinstein and Wolinsky (1986) implementing the Nash bargaining solution, Gul (1989) and Hart and Mas-Colell (1996) implementing the Shapley value, Serrano (1993) implementing the nucleolus, and Perry and Reny (1994) implementing the core. Unfortunately, most of the cooperative and non-cooperative studies do not apply to problems where there are externalities and renegotiations.[1]

While two-person bargaining problems are well-known, and most solution concepts make similar predictions for the outcome (an equal split of the surplus), coalitional

[1]Myerson (1977) and Lucas and Thrall (1963) study cooperative games in partition function form. Ray and Vohra (1999), Jehiel and Moldovanu (1995), Bloch (1996), and Yi (1997) use the non-cooperative approach and allow for externalities but not renegotiations.

bargaining problems with three or more players are more complex due to the possibility of coalition formation and renegotiations: on one hand, for situations without externalities, the various existing solution concepts make significantly different predictions for the outcome; on the other hand, for situations with externalities, there is a lack of solutions. What is the appropriate solution to apply to a particular multilateral problem?

We consider in this paper economies in which there are only three players.[2] Each player own a tradeable resource (or property right), and an exogenous set of parameters describes the gains of trade in the economy. The parameters of the game determine the value of all resources for all their possible combinations or coalitions, and coalitions may create positive or negative externalities. The negotiation game evolves with players making offers to acquire resources followed by players that have received offers making their response. We allow for the possibility of coalitions, once formed, to remain negotiating until all gains from trade have been exploited. Also, during the negotiation, players may make offers that can be both *conditional* or *unconditional* to acceptances (for example, an offer to players A and B could be conditional on the acceptance of A and unconditional on the acceptance of B, or conditional on the acceptances of both players, or unconditional to acceptances).

We show that the model has a *unique* stationary subgame perfect Nash equilibrium outcome–named the *coalitional bargaining value*–and show that this solution is Pareto efficient and continuous on the parameters of the game. Existence, uniqueness, and Pareto efficiency results in non-cooperative coalitional bargaining models are novel and surprising in light of the results in the literature.[3]

In addition, we show that the parameter space is divided into distinct convex conical regions, and in each region the coalitional bargaining value is a linear function of the parameters of the game. In the limit, as the interval between offers shrinks to zero, the different regions into which the parameter space is divided collapses into four regions (eight including all permutations). For games without externalities, in

[2]The general case with n players can be analyzed with a similar framework (see Gomes (2001)). The main advantage of restricting the analysis to three-player games is that a complete characterization of the solution is possible, while in n-player games we can only obtain some general properties of the solution.

[3]For example, in the non-cooperative bargaining models of Selten (1981) and Moldovanu (1992), there can be multiple stationary solutions, and in Chartterjee et al. (1993), Seidmann and Winter (1998), Okada (1996), and Ray and Vohra (1999) the solutions are not Pareto efficient.

one of the regions the coalitional bargaining value coincides with the Nash bargaining solution, in another region with the Shapley value, and in the other regions with the nucleolus. Therefore this paper offers a criteria to select a specific cooperative solution for all games without externalities.

We also propose a simple criteria to deal with externalities: measure the worth of a pairwise coalition by the value that it creates plus the amount of negative externalities (or minus the amount of positive externalities) that it creates for the excluded player. As we discuss next, each of the four relevant types of games can be described in terms of the relative worth (measured accounting for externalities) of pairwise coalitions. Furthermore, the strategies employed by players in each region have an intuitive economic interpretation in terms of credible outside options (see also Sutton (1986)), that serves to enhance our understanding of the related cooperative solution concepts, and to extend them to environments with externalities.

First, the coalitional bargaining value is equal to the Nash bargaining solution (equal split of the surplus) for games where the worth of all pairwise coalition is less than a third of the grand coalition value (for example, unanimous bargaining games). In this region no player is able to demand more than an equal share of surplus because the outside option of forming a pairwise coalition is not credible. An interesting comparative statics implication is that even if a player is relatively stronger than others, but not too stronger, she should also get the same payoff as the other players.

Second, the coalitional bargaining value coincides with the Shapley value for games where the sum of the values created by all pairwise coalitions is greater than the grand coalition value (for example, one-seller two-buyer market games, zero-sum games, and majority voting games). In these games, there is an advantage from being the proposer (first mover advantage) and a disadvantage from being excluded from a pairwise coalition. The Shapley value arises as the equilibrium in problems where the formation of all pairwise coalitions are credible options available to players. Note that this is the opposite of what happens in the Nash bargaining solution case where no pairwise coalition can credibly form.

Finally, there are two novel cases in which the coalitional bargaining value coincides with the nucleolus: games where only the 'natural coalition' among two 'natural partners' creates significant value, and games where only the two pairwise coalitions including a 'pivotal player' create significant value. In the first case, the player excluded from the natural coalition agrees with a payoff lower than an equal split of the surplus, and the natural partners equally split the gains from forming the natural coalition–an outcome that is driven by the fact that only the natural coalition can credibly form in equilibrium. In the second case, both non-pivotal players agree to form a coalition with the pivotal player receiving a payoff lower than an equal split of the surplus–an outcome that is driven by the fact that only the pairwise coalitions including the pivotal player can credibly form. We illustrate these two new cases with a mergers and acquisitions example in an oligopolistic industry in which there are two natural merger partners, and an example of collective bargaining for wages where two unions are better off bargaining separately with the firm rather than forming a larger union to collectively bargain for wages.

We also discuss the connection between the coalitional bargaining value and the core. This relationship can be illustrated with a situation where there is one seller of a good and two buyers with an equal valuation for the good. This game has a unique core allocation where the seller extracts all the surplus from the buyers, which is significantly different than the coalitional bargaining value prediction–the Shapley value. But isn't the core a more reasonable prediction for this game? We believe that the outcome of this game should be very different than the core's prediction. The main reason why the seller can't extract the entire surplus from the buyers is that both buyers have the option of forming a cartel to bid for the good, rather than launching a bidding war. Thus, the seller rather than auctioning the good, prefers to negotiate with one buyer, leaving the second buyer with nothing. Because all agreements are binding, after any deal is closed (i.e, either a buyers' cartel is formed or the good is sold) there is no way for the player left out to undue the deal offering a slightly better proposal (which is exactly the story motivating the core). In equilibrium, any of three deals (sale of the good to one of the two buyers and formation of a buyers' cartel) may occur leading to an expected outcome equal to the Shapley value.

Our model is different from other coalitional bargaining models for several reasons. In the strategic models of Selten (1981), Chatterjee et al. (1993), Ray and Vohra (1999), and Hart and Mas-Colell (1996), once a coalition reaches an agreement, it cannot be further renegotiated and the coalition leaves the game (renegotiations are also considered in Seidmann and Winter (1998) and Gul (1989)). Externalities in coalitional bargaining are addressed by only a few studies including, in a setting very similar to ours, Ray and Vohra (1999), and also Jehiel and Moldovanu (1995), Bloch

(1996), and Yi (1997). Finally, proposals in our model can be both conditional or unconditional, while most papers on strategic bargaining only allow for conditional offers (the exception being Krishna and Serrano's (1996) analysis of unanimous bargaining games where unconditional offers are permitted). With conditional offers a rejection by even only one of the players receiving the offer blocks other players that have accepted the offer from exiting the game with the amount offered to them. Expanding the strategy set adds a new degree of realism to strategic bargaining, and allows for a Pareto efficient equilibrium outcome to arise.

The remainder of the paper is organized as follows: Section 2 presents the negotiation model, Section 3 establishes the uniqueness of the solution and the explicit formula for the coalitional bargaining value, Section 4 studies the economic properties of the solution, and Section 5 concludes. The Appendix contains the proofs of all theorems.

2. The Negotiation Model

Let $N = \{1, 2, 3\}$ represent three players each owning an indivisible tradeable resource or right (for simplicity, we also refer to the resource owned by player $i \in N$ as resource i). Players at every period of the game can buy or sell resources in exchange of a transfer of utility. Players that acquire resources may continue trading, and players that sell their resources leave the game. There are a total of five different ownership structures or *coalition structures (c.s.)*: the initial ownership structure $\{\{1\}, \{2\}, \{3\}\}$ where all resources are owned by different players; $\{\{i, j\}, \{k\}\}$ where one player, either i or j, owns both resources $\{i, j\}$ (this player is also referred to as the coalition $\{i, j\}$); and finally $\{N\}$ where one player owns all the resources.[4] Players are expected utility maximizers and have a common per period discount factor equal to $\delta \in (0, 1)$.

A set of parameters $\mathbf{v} = (v_i, V_k, V_{ij}, V)$ describes the flow of utility generated by the resources for all the possible coalition structures, and captures the presence of externalities. Accordingly, if the c.s. is $\{\{1\}, \{2\}, \{3\}\}$ the flow of utility to player i is $(1 - \delta) v_i$; if the c.s. is $\{\{i, j\}, \{k\}\}$ the flow of utility to coalition $\{i, j\}$ and k

[4] Alternatively, we can interpret the formation of a coalition as any kind of binding agreement (i.e., not necessarily an ownership agreement), in which the coalition act as an agent that is maximizing the aggregate utility of the coalition members. Although, apparently more general, both approaches are largely equivalent.

are, respectively, equal to $(1-\delta)\,V_{ij}$ and $(1-\delta)\,V_k$; and finally if the c.s. is $\{N\}$ the flow of utility to the grand coalition N is $(1-\delta)\,V$. Note that the specification above can capture any positive or negative externalities that the coalition $\{j,k\}$ creates for player i, whenever $v_i < V_i$ or $v_i > V_i$, respectively. The set of parameters $\mathbf{v}$ is also known as a *partition function form* (see Thrall and Lucas (1963) and Ray and Vohra (1999)), or simply a vector $\mathbf{v} \in R^{10}$. A *characteristic function form* corresponds to a special partition function form where $v_i = V_i$, and thus there are no externalities ($\mathbf{v} \in R^7$).

Without any loss of generality we consider only 0-normalized partition functions which corresponds to $v_i = 0$. Furthermore, all partition functions considered are weakly superadditive, which corresponds to $V_i + V_{jk} \leq V$ for all distinct i, j, and k and $V \geq 0$. Note that this assumption simply means that the grand coalition is (weakly) efficient.

We model negotiations as an infinite horizon non-cooperative game with complete information, utilizing the partition function as the basic underlying structure. The negotiation game evolves with players making offers (to acquire the resources of other players) followed by players that have received offers accepting or rejecting the offers, as in Rubinstein (1982). However, unlike most models on non-cooperative bargaining, we allow for a richer set of offers that includes both *conditional* and *unconditional* offers.

Specifically, the *strategy set* of i's offers includes the following types of offers: (1) Offers to only one player, such as an offer to buy player j at a price p_j. Player j's resource is exchanged for the offered amount conditional only on j's acceptance. (2) Joint offers to both players, such as an offer to buy both j and k at a price p_j and p_k, respectively. The joint offer must also specify the order in which the players sequentially respond to the offer, and one of the four types of conditions: (i) conditional on both player j's and k's acceptance decisions–that is an offer conditional on the formation of coalition N; (ii) conditional only on j's acceptance decision and unconditional on k's acceptance decision–or an offer conditional on the formation of coalition $\{i,j\}$; (iii) conditional only on k's acceptance decision and unconditional on j's acceptance decision–or an offer conditional on the formation of coalition $\{i,k\}$; or (iv) unconditional to acceptances. In particular, if the offer to player j is unconditional on k's acceptance decision then if j accepts the offer j's resource is transferred to i and j leaves the game with the amount offered, regardless of the response of player

k. If the offer to player j is conditional on k's acceptance decision, then j leaves the game with the amount offered in exchange for his resource if and only if both j and k accept the offer. Proposals can also be behavioral strategies, i.e. probability distributions over the set of offers.

The *coalition bargaining game* is the game with the following extensive form: At the beginning of each period one of the players belonging to the current coalition structure is randomly chosen, with equal probability, to be the proposer. If player i is the proposer he then chooses an offer from his strategy set, and players receiving the offer respond in the order specified, either accepting or rejecting the offer. An exchange of ownership of resources takes place according to the responses of the offer and the precise conditions attached to it (see previous paragraph). This defines a new coalition structure, and the game is repeated, after a lapse of one period of time, with a new proposer being randomly chosen as described.

Our notion of equilibrium is *stationary subgame perfect Nash equilibrium (SPE).* A strategy profile is SPE if it is a subgame perfect Nash equilibrium and the strategies are such that the choice at each stage of the game depends only on the current coalition structure and the current proposer, but neither on the history of the game nor on calendar time.

3. The Coalitional Bargaining Value: The Unique Equilibrium

The analysis of three-player coalitional bargaining games reduce to the well-known two-player bargaining game of Rubinstein (1982) and Ståhl (1972) after the formation of a pairwise coalition. Suppose that we are at the coalition structure $\{\{i\},\{j,k\}\}$, where player i and coalition $\{j,k\}$ are in a bilateral bargaining game. It is a well-known result (e.g., see Osborne and Rubinstein (1990) and Sutton (1986)) that the bilateral bargaining game has a unique stationary subgame perfect equilibrium, in which the two players with reservation values V_i and V_{jk} propose to form the grand coalition splitting by half the surplus $V - V_i - V_{jk}$. Thus in the subgame starting with a coalition structure $\{\{i\},\{j,k\}\}$, the expected equilibrium payoffs of player i

and coalition $\{j,k\}$ are, respectively, equal to

$$\Phi_i = \frac{1}{2}\left(V + V_i - V_{jk}\right) \text{ and } \Phi_{jk} = \frac{1}{2}\left(V - V_i + V_{jk}\right). \tag{1}$$

In this unique equilibrium whenever coalition $\{j,k\}$ or player i are chosen to propose (which happens with probability $\frac{1}{2}$) they propose to form the grand coalition offering, respectively, X_i and X_{jk} to acquire resources i and $\{j,k\}$ where

$$X_i = \delta\Phi_i + (1-\delta)\, V_i \text{ and } X_{ij} = \delta\Phi_{ij} + (1-\delta)\, V_{ij}. \tag{2}$$

It remains to determine what are the equilibrium strategies at the initial stage of the game when there are three players and the c.s. is $\{\{1\},\{2\},\{3\}\}$. Consider any SPE and let the expected equilibrium outcome of player i (before the choice of proposer) be equal to ϕ_i for all $i \in N$. Throughout the paper the stationarity assumption is used in a crucial way. At stage $\{\{1\},\{2\},\{3\}\}$, player i reasons that upon his rejection of an offer his payoff is either $\delta\phi_i$ or X_i, depending on whether the coalition structure in the next stage is, respectively, $\{\{1\},\{2\},\{3\}\}$ or $\{\{i\},\{j,k\}\}$.

Let us develop some intuition for what are the offers that are proposed in equilibrium. We show (see Lemma 1 in the Appendix) that the maximum expected utility that player i can achieve choosing any offer in his strategy set is

$$\max\left\{V - \delta\phi_j - \delta\phi_k, V - X_j - \delta\phi_k, V - \delta\phi_j - X_k\right\}. \tag{3}$$

Player i's best proposing strategy depends on what is the maximum value of expression (3), as we now argue.

Consider first the case where the maximum of (3) is $V - \delta\phi_j - \delta\phi_k$, which, naturally, is equivalent to $X_j \geq \delta\phi_j$ and $X_k \geq \delta\phi_k$. Then the best response strategy for proposer i is to offer $\delta\phi_j$ to j and $\delta\phi_k$ to k conditional on their joint acceptance (the order of response is not important) and it is a best response to players j and k to accept such offer. Note that it is important for player i to make the offer conditional on the acceptance of both j and k, because if the offer where, say unconditional on j's acceptance, then j would be better off rejecting the offer which yields him $X_j \geq \delta\phi_j$ whenever k accepts the offer.

In the second case, consider that the maximum of (3) is $V - X_j - \delta\phi_k$, which is equivalent to $X_j \leq \delta\phi_j$ and $X_j - \delta\phi_j \leq X_k - \delta\phi_k$.[5] Then the best response strategy for proposer i is to offer X_j to j and $\delta\phi_k$ to k conditional only on k's acceptance (the order of response is also not important) and it is a best response to players j and k to accept such offer. Note that now it is important for player i to make the offer unconditional on j's acceptance because otherwise j would reject the offer since $X_j \leq \delta\phi_j$. The strategy above is not the only best response strategy, though. It is also a best response to player i to offer X_j to j and $\delta\phi_k$ to k unconditional to acceptances, with k responding first followed by j. We point out that an unconditional offer with a reversed order of response may not be optimal for player i : after j's acceptance it is a best response to player k to reject the offer and get X_k rather than $\delta\phi_k$, whenever $X_k \geq \delta\phi_k$, which would lead to a lower payoff than $V - X_j - \delta\phi_k$ for player i.

We remark that the ability to use behavioral strategies is important, and we will see later on that there may not exist pure strategy equilibria (see Example 2). Observe that even though there may be several offers that are best response, any choice of strategy from the best-response set yields the same equilibrium outcome in all but one important case: Whenever $X_j - \delta\phi_j = X_k - \delta\phi_k < 0$ and player i is the proposer, the probability weights that player i places on each of his best response strategies determines the payoff of players j and k. This is so because if i offers X_j to j and $\delta\phi_k$ to k conditional on $\{i, k\}$ then j and k's payoff are respectively X_j and $\delta\phi_k$; and alternatively, if i offers $\delta\phi_j$ to j and X_k to k conditional on $\{i, j\}$ then j and k's payoff are respectively $\delta\phi_j$ and X_k. Since $X_j < \delta\phi_j$ player j's payoff is monotonically increasing in the probability that i puts on the latter offer (and the opposite holds for k).

Naturally, in order for ϕ_i to be the expected outcome of an stationary subgame perfect Nash equilibrium, it must be the case that the system of equations

$$\phi_i = \frac{1}{3}\left(\phi_i^i + \phi_i^j + \phi_i^k\right), \tag{4}$$

hold, where ϕ_i^j is the expected equilibrium outcome of player i given that player j has been chosen to be the proposer since all players can be chosen to propose with probability equal to $\frac{1}{3}$.

In Lemma 2 (see Appendix), using the system of equations (4), we derive explicit formulas of all possible SPE payoffs, showing that SPE payoffs are linear functions

[5] The case where the maximum is $V - \delta\phi_j - X_k$ is similar.

of the parameters of the game within certain convex conical regions. Roughly, these convex conical regions are given by inequality constraints, such as the ones considered in the previous paragraphs, and within each region proposers' best response strategies have similar conditionality requirements.

A key result of this paper is that, surprisingly, the uniqueness result of two-player bargaining games also extends to three-player coalitional bargaining games.

THEOREM 1: *There exists a unique SPE outcome* $\phi(\mathbf{v}, \delta)$ *for all three-player coalitional bargaining games* $(\mathbf{v}, \delta)$. *This outcome is Pareto efficient, is a continuous function of* $(\mathbf{v}, \delta)$, *and is a piecewise linear function of the game* $\mathbf{v}$, *where the linearity regions are convex cones.*

The uniqueness proof draws on concepts of convex geometry (see Ziegler (1994)) and is based on Lemmas 2 and 3. Lemma 3, by repeatedly applying the separation hyperplane theorem, shows that the conical regions are disjoint (more precisely they intersect only at common faces), which implies the uniqueness result.

The Pareto efficiency result is also surprising given that Chartterjee et al. (1993), Seidmann and Winter (1998), Okada (1996), and Ray and Vohra (1999) show that equilibria are inefficient except for very special types of games. The main reason why efficiency is attained in this model is because we allow for conditional and unconditional offers, while in the literature only conditional offers are permitted. The intuition is that players are better off proposing to form the grand coalition conditional on the formation of a pairwise coalition, rather than proposing to form an inefficient pairwise coalition. Thus, with conditional and unconditional offers, there are no delays in the formation of the grand coalition.

Of particular interest is the limit stationary subgame perfect equilibrium as the time interval between proposals becomes arbitrarily small, or the discount rate δ converges to one. The limit linearity regions and the limit equilibrium outcome have a particular simple and intuitive formula that can be easily obtained by taking the limit when $\delta \to 1$ of the equilibrium payoff expressions obtained in the proof of Theorem 1. The limit equilibrium outcome defines the *coalitional bargaining value (CBV)* of the game.

THEOREM 2: *(Coalitional bargaining value) The limit stationary subgame perfect Nash equilibrium outcome of all 0-normalized games* $\mathbf{v}$ *(i.e.,* $v_i = 0$*) is given by (where* $\overline{V}_{ij} = V_{ij} - V_k$ *for all distinct* $i, j,$ *and* k *in* N*):*

Case I : *If* $\overline{V}_{12} \leq \frac{V}{3}, \quad \overline{V}_{13} \leq \frac{V}{3},$ *and* $\overline{V}_{23} \leq \frac{V}{3}$ *then*

$$\phi_i = \frac{V}{3} \quad \text{for all } i;$$

Case $II(ij)$: *If* $\overline{V}_{ij} \geq \frac{V}{3}, \quad 2\overline{V}_{ik} + \overline{V}_{ij} \leq V,$ *and* $2\overline{V}_{jk} + \overline{V}_{ij} \leq V$ *then*

$$\phi_i = \phi_j = \frac{1}{4}\left(V + \overline{V}_{ij}\right), \quad \text{and } \phi_k = \frac{1}{2}\left(V - \overline{V}_{ij}\right);$$

Case $III(i)$: *If* $\overline{V}_{12} + \overline{V}_{13} + \overline{V}_{23} \leq V, \quad 2\overline{V}_{ik} + \overline{V}_{ij} \geq V,$ *and* $2\overline{V}_{ij} + \overline{V}_{ik} \geq V$ *then*

$$\phi_i = \frac{1}{2}\left(\overline{V}_{ik} + \overline{V}_{ij}\right), \ \phi_j = \frac{1}{2}\left(V - \overline{V}_{ik}\right), \quad \text{and } \phi_k = \frac{1}{2}\left(V - \overline{V}_{ij}\right);$$

Case IV : *If* $\overline{V}_{12} + \overline{V}_{13} + \overline{V}_{23} \geq V$ *then*

$$\phi_i = \frac{1}{6}\left(2V - 2\overline{V}_{jk} + \overline{V}_{ij} + \overline{V}_{ik}\right) \quad \text{for all } i.$$

Each of the cases above defines a convex conical region, and there are a total of eight regions when all permutations are included. Moreover, any given game belongs to the interior of only one of the eight regions.

Several remarks are warranted. As we pointed out before, the formation of coalitions may impose externalities on the non-members: in a partition function game coalition $\{i, j\}$ creates an externality worth V_k for player k. The results in this section suggests that it is convenient to measure the strength (or worth) of coalition $\{i, j\}$ in a partition function game by $\overline{V}_{ij} = V_{ij} - V_k$ because it captures not only how much value $\{i, j\}$ creates but also how much negative externality it imposes on k.[6]

Intuitively, region I above corresponds to the case where all pairwise coalitions

[6] Note that the expressions for Φ_i and Φ_{jk} in terms of $\overline{V}_{jk}$ are:

$$\Phi_i = \frac{1}{2}\left(V - \overline{V}_{jk}\right) \text{ and } \Phi_{jk} = \frac{1}{2}\left(V + \overline{V}_{jk}\right). \tag{5}$$

do not create much value (where the value of the coalition is measured by $\overline{V}_{ij}$). Region $II(ij)$ corresponds to the case where pairwise coalition $\{i, j\}$ is the only pairwise coalition that creates significant value. Region $III(i)$ corresponds to the case where the pairwise coalitions including i are the only pairwise coalitions that create significant value. Finally, region IV corresponds to the case where the aggregate value of all coalitions is greater than the value of the grand coalition.

Of course all results above also apply to characteristic function games (which correspond to games where $V_i = 0$, for all $i \in \{1, 2, 3\}$). All formulas for the characteristic function case are obtained by simply making $V_i = 0$ and $\overline{V}_{ij} = V_{ij}$ in all expressions.

Theorem 2 establishes a natural connection between the coalitional bargaining value and classical cooperative game theory concepts (developed for characteristic function games), such as the Nash Bargaining Solution, the Shapley value, and the nucleolus. More specifically, we will see that for games that satisfy $V_{ij} \leq \frac{V}{3}$ (region I) the CBV coincides with the Nash Bargaining Solution, for games that satisfy $V_{12} + V_{13} + V_{23} \leq V$ (regions I, II, or III) the CBV coincides with the nucleolus, and for games that satisfy $V_{12} + V_{13} + V_{23} \geq V$ (region IV) the CBV coincides with the Shapley value.

However, the advantage of the non-cooperative or strategic approach vis à vis the cooperative approach is that it enables us to understand precisely how players achieve their equilibrium value since this information is embedded in the strategies used by players. The limit equilibrium strategies used by players within each of the eight regions can be also easily obtained by taking the limit when $\delta \to 1$ of the expressions for the players' strategies obtained in the proof of Theorem 1.

THEOREM 3: *The limit SPE strategies for all 0-normalized games* $\mathbf{v}$ *are given by:*

If $\mathbf{v} \in I$ *then player* i *offers* ϕ_j *to* j *and* ϕ_k *to* k *conditional on* N;

If $\mathbf{v} \in II(12)$ *(the other two cases are symmetric) then* 1 *offers* ϕ_2 *to* 2 *and* Φ_3 *to* 3 *conditional on* $\{1, 2\}$ *(*2*'s offer is similar to* 1*'s and is also conditional on* $\{1, 2\}$*), and* 3 *offers* ϕ_1 *to* 1 *and* ϕ_2 *to* 2 *conditional on* N*;*

If $\mathbf{v} \in III(1)$ *(the other two cases are symmetric) then* 1 *offers* ϕ_2 *to* 2 *and* Φ_3 *to* 3 *conditional on* $\{1, 2\}$ *with probability* $\sigma_1(12) = \varphi\left(\frac{\overline{V}_{12} - \frac{V}{3}}{\overline{V}_{12} + \overline{V}_{13} - \frac{2V}{3}}\right)$*, and offers* Φ_2 *to* 2 *and* ϕ_3 *to* 3 *conditional on* $\{1, 3\}$ *with probability* $\sigma_1(13) = \varphi\left(\frac{\overline{V}_{13} - \frac{V}{3}}{\overline{V}_{12} + \overline{V}_{13} - \frac{2V}{3}}\right)$*, and*

$j = 2, 3$ offers ϕ_1 to 1 and Φ_k to k conditional on $\{1, j\}$, where the function $\varphi(x)$ is respectively, equal to 1, x, or 0 if $x \geq 1$, $x \in [0, 1]$, or $x \leq 0$;

If $\mathbf{v} \in IV$ then player i offers ϕ_j to j and Φ_k to k conditional on $\{i, j\}$ with probability $\sigma_i(ij)$, such that the overall probability that any of the three offers conditional on $\{i, j\}$ are chosen is equal to $\frac{1}{3}$ (i.e., $\frac{1}{3}(\sigma_i(ij) + \sigma_j(ij)) = \frac{1}{3}$).

The strategies used in each region have an intuitive interpretation, that serves to enhance our understanding of the associated cooperative solution concepts. In region I, the threat of forming a pairwise coalition is not credible and would only benefit the player left out because no pairwise coalitions create much value (accordingly, no offers made are conditional on the formation of a pairwise coalition or are unconditional to acceptances). In region $II(ij)$, players i and j condition their offers on the formation of coalition $\{i, j\}$, because the pairwise coalition $\{i, j\}$ is the only pairwise coalition that creates some significant value. In region $III(i)$, players j and k condition their offers on the formation of, respectively, $\{i, j\}$ or $\{i, k\}$ because only the pairwise coalitions including player i have some significant strength. Note that, as is intuitive, the weight that player i puts on the coalition $\{i, j\}$ is monotonically increasing in $\overline{V}_{ij}$, the value created by coalition $\{i, j\}$. Finally, in region IV all pairwise coalitions are equally likely to receive a conditional offer.

In the next section we further explore the economics of Theorems 2 and 3 and also provide several illustrative examples.

4. The Economics of Negotiations and the Coalitional Bargaining Value

4.1. *The Nash Bargaining Solution*

The Nash bargaining solution is a classical solution concept for n-person pure (or unanimous) bargaining games. In pure bargaining games the cooperation of all players is needed to achieve gains from trade, otherwise all players get their reservation value. An immediate corollary of Theorem 2, following from Case I, is that the coalitional bargaining value coincides with the Nash bargaining solution for three-person pure bargaining games. Moreover, the Nash bargaining solution is the appropriate solution concept for a more general class of multilateral bargaining problems where all three resources are 'more or less' essential.

EXAMPLE 1: *Multilateral bargaining games*: Consider the bargaining game $\mathbf{v}$, where $v_i = V_i = 0$, $V_{ij} \leq \frac{1}{3}$ for all pairs $\{i, j\}$, and $V = 1$, and players are very patient (δ close to one).

Note that this multilateral bargaining game is more general than a pure bargaining game, in which $v_i = V_i = 0$, $V_{ij} = 0$, and $V = 1$. According to Theorem 2 the bargaining value of the multilateral bargaining game is $\phi_i = \frac{1}{3}$, which is the Nash bargaining solution of the pure bargaining game. This generalizes the Nash bargaining solution as the appropriate solution concept for games where all coalitions $\{i, j\}$ can achieve a value V_{ij} less than or equal to $\frac{1}{3}$ of the grand coalition gains.

It is worth explaining the intuition of this result. The threat of any pair of players i and j to form coalition $\{i, j\}$ is not credible because the most the coalition $\{i, j\}$ can get, alienating player k, is $\Phi_{ij} = \frac{1}{2}(1 + V_{ij}) \leq \frac{2}{3}$, which is less than $\frac{2}{3}$, the amount they can get by agreeing to split the dollar equally. In other words, the ability of players to demand more than an equal split of the dollar by threatening to form a pairwise coalition is an outside option that is not credible (see Sutton (1986) and Osborne and Rubinstein (1990)). The coalitional bargaining value prediction has an interesting comparative statics implication: the expected outcome of players should be insensitive to changes in the values V_{ij} of pairwise coalitions, as long as the condition $V_{ij} \leq \frac{1}{3}$ is maintained.

4.2. *The Shapley Value and the Core*

We first provide a more general definition of the Shapley value that extends the concept to games in partition function. See Myerson (1979) for an alternative definition of the Shapley value for partition function games.[7]

[7] According to Myerson (1979) the Shapley value of player i in a three-player 0-normalized partition function is:

$$Sh_i(v) = \frac{1}{6}(2(V - V_{jk}) + 4V_i - 2V_j - 2V_k + V_{ij} + V_{ik}).$$

Definition 1: Let $v_S(\pi)$ be a game in partition function. Define the marginal contribution of player i to the coalition S, $i \notin S \subset N$, as $m(S,i) = v_{S\cup i}(\{S\cup i, N\backslash(S\cup i)\}) - v_S(\{S, N\backslash S\})$. Also, let $\theta \in \Pi$ denote any permutation of the players, and let $S(\theta,i)$ denote the coalition of players that come before player i in the ordering θ. The *Shapley value* of player i is defined as

$$Sh_i(\mathbf{v}) = \frac{1}{n!}\sum_{\theta\in\Pi} m(S(\theta,i),i),$$

the average marginal contribution of player i to his predecessors.

Naturally, the definition above coincides with the standard definition of the Shapley value for games in characteristic function. It can also be easily seen that the Shapley value for three-player partition function games is explicitly given by[8]

$$Sh_i(\mathbf{v}) = \frac{1}{6}(2V - 2(V_{jk} - V_i) + (V_{ij} - V_j) + (V_{ik} - V_k)), \tag{6}$$

which is also equivalent to

$$Sh_i(\mathbf{v}) = \Phi_i + \frac{1}{3}\left(V - \sum_{j=1}^{3}\Phi_j\right).$$

Another immediate corollary of Theorem 2, following from Case IV, is that the coalitional bargaining value coincides with the Shapley value whenever condition

$$(V_{12} - V_3) + (V_{13} - V_2) + (V_{23} - V_1) \geq V, \tag{7}$$

is satisfied (note that this condition is also equivalent to $\Phi_1 + \Phi_2 + \Phi_3 \leq V$).

In equilibrium, when condition (7) holds, a proposer i randomly chooses a player, say j, and offers him his Shapley value, conditional on forming the pairwise coalition $\{i,j\}$, and offers player k only $\Phi_k < Sh_k(\mathbf{v})$. Note that player i's payoff is greater than his Shapley value by $\frac{1}{3}\left(V - \sum_{l=1}^{3}\Phi_l\right) > 0$, and player k's payoff is smaller than

[8] In two of the six permutations player i is the last, and his marginal contribution is thus $V - V_{jk}$; in two of the permutations he is the first player, and his marginal contribution is thus V_i; in one permutation he comes second, just after player j, and his marginal contribution is thus $V_{ij} - V_j$; and in one permutation he comes second, just after player k, and his marginal contribution is thus $V_{ik} - V_k$.

his Shapley value by the same amount. Therefore, there is an advantage from being the proposer and a disadvantage from being excluded from a pairwise coalition, and the Shapley value arises as the equilibrium in situations where players are willing to form all pairwise coalitions. This property is consistent with Gul (1989) obtaining the Shapley value as the equilibrium outcome of his model, because in his framework any two pairs of players are equally likely to meet and form a coalition.

It is also worth pointing out the relationship between the coalition bargaining value and the core.[9] It is straightforward that the core of a three-player superadditive characteristic function game is non-empty if and only if

$$V_{12} + V_{13} + V_{23} \leq 2V. \tag{8}$$

Therefore, we conclude that the Shapley value is the coalitional bargaining value of all games with an empty core (because whenever the core is empty condition (7) holds).

Interestingly, we will see that even when the core is non-empty, the coalitional bargaining value may not belong to the core (see Example 2 and the discussion that follows). However, as we show in Proposition 4 below, this may only happen for games that satisfy condition $V \leq V_{12} + V_{13} + V_{23} \leq 2V$, and, whenever $V \leq V_{12} + V_{13} + V_{23}$, the coalitional bargaining value does belong to the core.

We finalize this section with the analysis of several well-known games where the Shapley value arises in equilibrium.

Example 2: *One-seller and two-buyer market game*: Consider the negotiation game $(\mathbf{v}, \delta)$ where $v_i = V_i = 0$, $V_{12} = v_H = 1$, $V_{13} = v_L$, $V_{23} = 0$, and $V = v_H = 1$, with $v_L < v_H = 1$. In this game player 1 is the seller, player 2 is the high valuation buyer, and player 3 is the low valuation buyer, and all player are very patient (δ close to 1).

In general we can readily tell when the bargaining value is the Shapley value by inspecting whether condition (7) is satisfied or not. By Theorem 2 we have that the bargaining value is the Shapley value

$$\phi_1 = \frac{v_H}{2} + \frac{v_L}{6}, \ \phi_2 = \frac{v_H - v_L}{2} + \frac{v_L}{6} = \frac{1}{2}v_H - \frac{1}{3}v_L, \text{ and } \phi_3 = \frac{v_L}{6},$$

because condition (7) is satisfied ($v_H + v_L = 1 + v_L > 1$).

[9] By definition a payoff vector ϕ_i belongs to the core of a characteristic function game if and only if $\sum_{i \in N} \phi_i = v_N$ and $\sum_{i \in S} \phi_i \geq v_S$ for all $S \subset N$.

This solution generalizes the solution of the one-seller two-buyer market game in Osborne and Rubinstein (1990) when players are allowed to use contracts and resell the resource. It is instructive to write down the equilibrium strategies of the game in the context of Osborne and Rubinstein's interpretation. Say that $\Delta = \frac{1}{6}v_L = \frac{1}{3}\left(V - \sum_{j=1}^{3} \Phi_j\right)$. The one-seller two-buyer game is an example of a game that has no equilibrium in pure strategy, and where the behavioral strategies are as follows: buyer 2 offers to buy the seller's resource for $\phi_1 - \Delta$ and pays buyer 3 $\phi_3 = \Delta$ to leave the market (with probability p_2), and offers to buy the seller's resource for ϕ_1 (with probability $1 - p_2$); buyer 2's payoff is $\phi_2 + \Delta$. Buyer 3 offers to buy the seller's resource for $\phi_1 - \Delta$ and resell it to buyer 2 for $\phi_1 + \Delta$ (with probability p_3), and offers to buy the seller's resource for ϕ_1 and resell it to buyer 2 for $\phi_1 + 2\Delta$ (with probability $1 - p_3$); buyer 3's payoff is 2Δ. The seller offers to sell his resource to buyer 3 for $\phi_1 + \Delta$, who then resell the resource to buyer 2 for ϕ_1 (with probability p_1), and offers to sell his resource to buyer 2 for $\phi_1 + \Delta$ (with probability $1 - p_1$); the seller's payoff is $\phi_1 + \Delta$.

Note that if the valuation of the buyers are the same $v_H = v_L = 1$ then the only point in the core of the market game is $(1, 0, 0)$, where the seller extracts all the surplus from the two buyers. Thus the coalitional bargaining value, which is equal to $\left(\frac{4}{6}, \frac{1}{6}, \frac{1}{6}\right)$, does not belong to the core.

Are the predictions of the coalitional bargaining value reasonable? Shouldn't we expect competition between the two buyers to drive the good's price to 1, as the core predicts? The main reason why the seller can't extract the entire surplus from the buyers is that both buyers have the option of forming a cartel to bid for the good, and then buy it at a very low price (0.5), rather than initiating a bidding war. The seller knows about that all to well, and rather than auctioning the good, the seller prefers to negotiate with one buyer an intermediate price (between 0.5 and 1), leaving the second buyer with nothing. Because all agreements are binding after a deal is sealed (i.e, either a buyers' cartel is formed or the good is sold) there is no way for the player left out to undue the deal enticing one of the players with a slightly better offer. In summary, we believe that the expected outcome of the one-seller and two-buyer market game should be very different than the core's prediction.

EXAMPLE 3: *Three-person majority voting game*: In this game $V_{ij} = V = 1$ and $v_i = V_i = 0$.

Note that this game satisfies condition (7) because $(V_{12} - V_3) + (V_{13} - V_2) + (V_{23} - V_1) = 3 > V = 1$ and thus the bargaining value coincides with the Shapley value, which is just an equal split of the dollar, and the core is empty.

EXAMPLE 4: *Three-person zero-sum game*: In this game $v_i = 0$, $V_{jk} = -V_i = c_i > 0$, and $V = 0$.

This game also satisfies condition (7) because $(V_{12} - V_3)+(V_{13} - V_2)+(V_{23} - V_1) = 2\,(c_1 + c_2 + c_3) > 0$, and thus the bargaining value also coincides with the Shapley value (and the core is empty) which is

$$\phi_1 = -c_1 + \frac{(c_1 + c_2 + c_3)}{3},\ \phi_2 = -c_3 + \frac{(c_1 + c_2 + c_3)}{3},\ \text{and } \phi_3 = -c_3 + \frac{(c_1 + c_2 + c_3)}{3}.$$

Interestingly, this solution coincides with the solution proposed by von Neumann and Morgenstern (1944) for three-person zero-sum games.

4.3. *The Nucleolus: Natural Coalitions and Pivotal Players*

We have seen that the Nash bargaining is the equilibrium outcome when players are not willing to form any pairwise coalition, and that the Shapley value is the equilibrium outcome when players are willing to form all pairwise coalitions. A novel element of our theory is that neither the Nash bargaining solution nor the Shapley value seems to be the right solution concept for a broad class of games: those satisfying the conditions of cases $II(ij)$ and $III(i)$ of Theorem 2.

Our next result shows that the outcome of the coalitional bargaining game in these two regions coincides with the nucleolus for characteristic function games. We recall that Schmeidler (1969) proved that the nucleolus always exists and is a unique point belonging to the core of the game, whenever the core is non-empty. Kohlberg (1971) then showed that the nucleolus is a piecewise linear function of the characteristic function of the game, and Brune (1983) computed the nucleolus with its regions of linearity for three-person games.[10] Comparing the formula for the nucleolus with the

[10] According to Brune (1983) the nucleolus for a three-person superadditive game satisfying $V_{12} \geq V_{13} \geq V_{23}$ is:
If $V_{12} \leq \frac{V}{3}$ then $\phi = (\frac{V}{3}, \frac{V}{3}, \frac{V}{3})$,
If $V_{12} \geq \frac{V}{3}$ and $V_{12} + 2V_{13} \leq V$ then $\phi = (\frac{V+V_{12}}{4}, \frac{V+V_{12}}{4}, \frac{V-V_{12}}{2})$,

formula for the coalitional bargaining value yields the following proposition.

PROPOSITION 4: *If* $\mathbf{v}$ *is a* 0*-normalized superadditive characteristic function game satisfying* $V_{12} + V_{13} + V_{23} \leq V$ *then the coalitional bargaining value coincides with the nucleolus, and belongs to the core. Otherwise, if* $V_{12} + V_{13} + V_{23} \geq V$ *holds then the coalitional bargaining value coincides with the Shapley value, is distinct from the nucleolus, and may not belong to the core.*

While the nucleolus is a concept that is mathematical very attractive and simple, economists have had difficulties in developing a motivation for it. Using the result that the nucleolus coincides with the CBV in the regions II and III (as well as I) we provide a new economic interpretation for the nucleolus using outside options.

First, for games that satisfy the conditions of case $II(ij)$ there exists a pair of players $\{i,j\}$ (*natural partners*) that are willing to form a pairwise coalition. According to Theorem 2, the outcome of negotiations when i and j are natural partners is $\phi_k = \Phi_k$, $\phi_i = \phi_j = \frac{\Phi_{ij}}{2}$, whenever case $II(ij)$ holds, which one can easily see is equivalent to

$$\Phi_k \leq \frac{V}{3}, \Phi_{ik} \leq \phi_i + \phi_k, \text{ and } \Phi_{jk} \leq \phi_j + \phi_k,$$

(these inequalities can be verified by substituting expressions (1) for Φ_i and Φ_{jk}). The intuition for the result now follows. Note first that the proposed solution is consistent with only the pairwise coalition $\{i,j\}$ being proposed to form: say that the coalition $\{i,k\}$ forms then the payoff for the coalition is Φ_{ik} and the payoff of the player left out is Φ_j. But since $\Phi_{ik} \leq \phi_i + \phi_k$ then the coalition $\{i,k\}$ is worse off (with respect to the proposed equilibrium). The payoffs of the players i and j, $\phi_i = \phi_j = \frac{\Phi_{jk}}{2}$, are also consistent with the fact that only the pairwise coalition $\{i,j\}$ may form: players i and j bargain over Φ_{ij} using as disagreement points their status quo values if they do not reach an agreement which are equal to $v_i = v_j = 0$.

Second, for games that satisfy the conditions of case $III(i)$ there is one *pivotal player* i that is included in all pairwise coalitions that are proposed, but the pairwise coalition between the non-pivotal players is never proposed. According to Theorem

If $V_{12} + 2V_{23} \leq V$ and $V_{12} + 2V_{13} \geq V$ then $\phi = (\frac{V_{12}+V_{13}}{2}, \frac{V-V_{13}}{2}, \frac{V-V_{12}}{2})$,
If $-V_{12} + 2(V_{13} + V_{23}) \geq V$ then $\phi = (\frac{V+V_{12}+V_{13}-2V_{23}}{3}, \frac{V+V_{12}+V_{23}-2V_{13}}{3}, \frac{V+V_{13}+V_{23}-2V_{12}}{3})$,
If $V_{12}+2V_{23} \geq V$ and $-V_{12}+2(V_{13}+V_{23}) \leq V$ then $\phi = (\frac{V+2V_{13}+V_{12}-2V_{23}}{4}, \frac{V+2V_{23}+V_{12}-2V_{13}}{4}, \frac{V-V_{12}}{2})$.

2, the outcome of negotiations when player i is pivotal is $\phi_i = V - \Phi_j - \Phi_k$, $\phi_j = \Phi_j$, $\phi_k = \Phi_k$, whenever case $III(i)$ holds, which one can easily see is equivalent to

$$\Phi_{jk} \leq \Phi_j + \Phi_k, \Phi_j \leq \frac{\Phi_{ij}}{2}, \text{ and } \Phi_k \leq \frac{\Phi_{ik}}{2}.$$

The intuition for this result is that players j and k cannot demand a higher payoff than Φ_j and Φ_k from player i by threatening to form the coalition $\{j,k\}$ because they would be worse off pursuing this strategy as $\Phi_{jk} \leq \Phi_j + \Phi_k$. Also, note that players j and k are not willing to accept any offer lower than Φ_j and Φ_k because they can guarantee this amount by credibly holding out. This is so because if j holds out then i would successfully bargain with k to form a coalition; k's gain are $\frac{\Phi_{ik}}{2} \geq \Phi_k$, and thus k does not want to hold out when j holds out.

It is worth comparing the predictions of the coalitional bargaining value, the Nash bargaining solution, and the Shapley value in situations where there are natural coalitions and pivotal players.

COROLLARY 5: *The Nash bargaining and the Shapley value are systematically biased with respect to the coalitional bargaining value in the following ways:*
(1) Whenever $\{i,j\}$ forms a natural coalition then $\phi_i \geq \frac{V}{3}$, $\phi_j \geq \frac{V}{3}$, and $\frac{V}{3} \geq \phi_k \geq Sh_k(\mathbf{v})$.
(2) Whenever player i is a pivotal player then $\frac{V}{3} \leq \phi_i \leq Sh_i(\mathbf{v})$, $\frac{V}{3} \geq \phi_j \geq Sh_j(\mathbf{v})$, and $\frac{V}{3} \geq \phi_k \geq Sh_k(\mathbf{v})$.

As we expected the players i and j when forming a natural coalition are able to strengthen their bargaining position and get more than their Nash bargaining value (the opposite happening with the player left out). Interestingly, the Shapley value underestimates the equilibrium outcome of player k. In the situation where i is a pivotal player then player i gets more than the Nash bargaining solution, but less then the Shapley value, and the opposite happens with players j and k.

A better understanding of negotiations can be grasped by analyzing more closely two examples, each one illustrating one of the two new situations. First, we provide an example of an oligopolistic industry where there are gains from merging, in which there are two natural merger partners. Then we provide an example of formation of labor unions where unions are better off bargaining separately with the firm rather than forming a larger union to collectively bargain for wages.

Example 5: *Oligopolistic industries and mergers and acquisitions*: Consider the game $v_i = V_i = 0$, $V = 1$, $V_{12} = v_H$, $V_{13} = v_{L_1}$, $V_{23} = v_{L_2}$ where $v_H \in \left[\frac{1}{3}, 1\right]$ and $v_{L_1} \leq v_{L_2} \leq \frac{1-v_H}{2} \leq v_H$.

In this example there are three firms competing in an industry where there are gains from consolidation. What are the prices at which firms merge? Are there any natural merger partners in this industry?

The bargaining value and strategies provide a direct answer to the questions above. One can easily verify that $\mathbf{v} \in II(12)$ and thus the bargaining value is

$$\phi_1 = \frac{1+v_H}{4},\ \phi_2 = \frac{1+v_H}{4},\ \text{and}\ \phi_3 = \frac{1-v_H}{2},$$

where $\phi_1 = \phi_2 \geq \phi_3$.

It is worth exploring several issues that are behind this solution. Note that the industry does not consolidate in a random fashion. If firms 1 and 3 merge their profitability increases by v_L. However, there are still gains from further consolidation with firm 2. What are the gains for each merging firm? Say that the initial merger between firms 1 and 3 is irreversible or divesting is too costly to be considered a viable option. Firm 2 and conglomerate $\{1,3\}$ will then split the merger gains in a Nash bargaining way, each getting, respectively, $\frac{1}{2}(1-v_L)$ and $\frac{1}{2}(1+v_L)$. Note that the value of the conglomerate $\{1,3\}$ is $\frac{1}{2}(1+v_L) \leq \frac{1}{4}(3-v_H) = \phi_1 + \phi_3$. Therefore, one can predict that firms 1 and 3 are not going to merge and, by the same reasoning, one can also rule out a merger between firms 2 and 3.

Consider now a merger between firms 1 and 2. The value of the conglomerate $\{1,2\}$ is equal to $\frac{1}{2}(1+v_H)$ and the value of firm 3 is $\frac{1}{2}(1-v_H)$ (see previous paragraph). How should the value of the conglomerate $\{1,2\}$ be split among firms 1 and 2? Firm 2 has an apparent stronger bargaining position than firm 1 because $v_{L_1} \leq v_{L_2}$ and thus it seems reasonable that firm 2 should receive a higher share of the value than firm 1. However, this intuitive idea is wrong: Firm 2 does not have any credible outside options other than to merge with firm 1, and thus the Nash bargaining solution is an equal split of the value of the conglomerate $\{1,2\}$.

Example 6: *Formation of labor unions*: In this game $v_i = V_i = 0$, $V_{12} = v$, $V_{13} = v$, $V_{23} \leq 1 - 2v$ where $v \in \left[\frac{1}{3}, 1\right]$, and $V = 1$. The firm is player 1, and players 2 and 3 are the two unions.

How much are the firm's profits and the employee wages? Are the workers better off forming only one union to collectively bargain for wages?

Since the conditions for player 1 to be pivotal holds, thus the bargaining value is equal to

$$\phi_1 = v, \ \phi_2 = \frac{1-v}{2}, \text{ and } \phi_3 = \frac{1-v}{2},$$

where $\phi_1 \geq \phi_2 = \phi_3$. Therefore, in equilibrium, the wages are equal to $\frac{1-v}{2}$ for workers in both unions, and the firm's profit is v.

Note that the unions are not willing to agree to a wage lower than $\frac{1-v}{2}$. Otherwise, the union could just wait until the firm signs a contract with the other union and bargain with the firm for a wage equal to half of the extra profits that the union could create, which results in a wage equal to $\frac{1}{2}(1-v)$. Interestingly, the threat of forming only one union to bargain for higher wages is not credible. The larger union can bargain for a total wage package equal to half of the surplus that it creates, which is equal to $\frac{1}{2}(1+V_{23}) \leq 1-v$. Therefore, collective bargaining results in a wage per worker lower than the amount the firm is willing to offer to employees in the first place. This stylized example illustrates that the theory of negotiations in this paper can bring new insights to collective bargaining and unionization models.

5. Conclusions

This paper introduces a new concept of value for coalition bargaining games–the coalitional bargaining value. The coalitional bargaining value is Pareto efficient, and is the unique stationary subgame perfect Nash equilibrium of a dynamic non-cooperative game where there are externalities, contracts can be renegotiated, and players are allowed to make conditional or unconditional offers while negotiating.

The theory developed in the paper provides a unified framework that selects an economically intuitive solution for all three-player coalitional bargaining games. Also, we propose a simple way to deal with externalities: add to the value of a pairwise coalition the amount of negative externalities (or subtract the amount of positive externalities) that it creates for the excluded player. Using this corrected measure of value, we show that the coalitional bargaining value can either be the Nash bargaining solution, in the case where the value of all pairwise coalition is less than a third of the grand coalition value; the Shapley value, in the case where the sum of the values created by all pairwise coalitions is greater than the grand coalition value; or the nucleolus, in the case where only the 'natural coalition' among two 'natural partners' creates significant value, and in the case where only the two pairwise coalitions including a 'pivotal player' create significant value.

Although we restricted our analysis to three-player games, the framework of this paper is suitable to generalizations to n-player coalitional bargaining games (see Gomes (2001)). However, it remains an open issue whether the uniqueness of stationary subgame perfect solutions hold for n-player games, and what is the connection with cooperative game theory concepts in more general settings.

Appendix

Lemma 1: *Any SPE of a coalitional bargaining game* $(\mathbf{v}, \delta)$ *satisfy the following conditions below:*
(1) If $X_j \geq \delta\phi_j$ *and* $X_k \geq \delta\phi_k$ *then* $\phi_i^i = V - \delta\phi_j - \delta\phi_k$, $\phi_j^i = \delta\phi_j$, *and* $\phi_k^i = \delta\phi_k$;
(2) If $X_j \leq \delta\phi_j$ *and* $X_j - \delta\phi_j < X_k - \delta\phi_k$ *then* $\phi_i^i = V - \delta\phi_k - X_j$, $\phi_j^i = X_j$, *and* $\phi_k^i = \delta\phi_k$;
(3) If $X_j - \delta\phi_j = X_k - \delta\phi_k \leq 0$ *then* $\phi_i^i = V - X_j - \delta\phi_k$, $\phi_j^i = \mu_i X_j + (1 - \mu_i)\delta\phi_j$ *and* $\phi_k^i = (1 - \mu_i) X_k + \mu_i \delta\phi_k$, *where* $\mu_i \in [0, 1]$ *is the probability that* i *puts on the offer* X_j *to* j *and* $\delta\phi_k$ *to* k *conditional only on* k*'s acceptance, and* $(1 - \mu_i)$ *is the probability that* i *puts on the offer* $\delta\phi_j$ *to* j *and* X_k *to* k *conditional only on* j*'s acceptance;*
(4) The system of equations hold:

$$\phi_i = \frac{1}{3}\left(\phi_i^i + \phi_i^j + \phi_i^k\right), \tag{9}$$

where ϕ_j^i *is the expected equilibrium outcome of player* j *given that player* i *has been chosen to be the proposer.*
Conversely, if there exists a set of numbers ϕ_i^j *and* ϕ_i *for* $i, j \in \{1, 2, 3\}$ *satisfying all conditions above, then there exists an SPE of the coalitional bargaining game* $(\mathbf{v}, \delta)$ *with an equilibrium outcome equal to* ϕ_i *and* ϕ_j^i.

Proof of Lemma 1: Suppose that we have an SPE of the game $(\mathbf{v}, \delta)$. The best acceptance strategy for players j and k for any proposed offer is as follows:

(i) If i offers p_j to player j then j's best response is to accept if and only if $p_j \geq \delta\phi_j$.

(ii) If i offers p_j to j and p_k to k conditional on their joint acceptance then j and k's best response is to accept if and only if $p_j \geq \delta\phi_j$ and $p_k \geq \delta\phi_k$, regardless of the order of response.

(iii) If i offers p_j to j unconditional on k's acceptance, and offer p_k to k conditional on j's acceptance then the best responses are as follows. Player j's best response, regardless of whether he is the first or last to respond, is to accept if and only if $p_j \geq \delta\phi_j$. Player k's best response, also independent of the order of response, is to accept if and only if $p_k \geq X_k$.

Note that the strategy above is a best response because if j accepts the offer and k rejects the offer then k's payoff is equal to X_k, and if j rejects the offer then k's acceptance decision is irrelevant for his payoff.

(iv) If i offers p_j to j unconditional on k's acceptance and offers p_k to k unconditional on j's acceptance then the best responses are as follows, where the order of response is relevant for the strategies that each player chooses. Say that j is the first player to respond followed by k. Player k's best response is to accept if and only if $p_k \geq X_k$ and player j has previously accepted the offer, or accept if $p_k \geq \delta\phi_k$ and player j has previously rejected the offer. Player j's best response is to accept if and only if $p_j \geq X_j$ and $p_k \geq \delta\phi_k$, or accept if $p_j \geq \delta\phi_j$ and $p_k < \delta\phi_k$.

This strategy is indeed a best response. If player j rejects he knows that player k's best response is to accept if $p_k \geq \delta\phi_k$, in which case j gets a payoff of X_j. Therefore, player j's best response is to accept any $p_j \geq X_j$, whenever $p_k \geq \delta\phi_k$. On the other hand, if $p_k < \delta\phi_k$, player j knows that the best response of player k is to reject any offer $p_k < \delta\phi_k$ if player j has previously rejected the offer, in which case j's payoff is equal to $\delta\phi_j$. Therefore, player j's best response is to accept if and only if $p_j \geq \delta\phi_j$, whenever $p_k < \delta\phi_k$.

Given that we already know how players respond to offers, let us determine the best proposal strategy. Obviously, if i proposes an offer that is unacceptable to all other players (or if he chooses not to propose), then his payoff is equal to $\delta\phi_i$.

(i) If i proposes an offer only to player j, then the highest payoff i can get is $X_{ij} - \delta\phi_j$. This is so because in the continuation game the equilibrium payoff of the coalition $\{i, j\}$ is X_{ij} and the minimum that player j accepts is $p_j = \delta\phi_j$.

(ii) If i proposes an offer to both players j and k conditional on their joint acceptance, then the highest payoff i can get is $V - \delta\phi_j - \delta\phi_k$. This is immediately true because the minimum that players j and k accept is $p_j = \delta\phi_j$ and $p_k = \delta\phi_k$, respectively.

(iii) If i proposes an offer both to players j and k conditional on j accepting but unconditional on k accepting, then the highest payoff i can get is $V - \delta\phi_j - X_k$. This is an immediate implication of the best response strategies of j and k, because players j and k accept a minimum offer of $p_j = \delta\phi_j$ and $p_k = X_k$, respectively.

(iv) If i proposes an offer to both players j and k unconditional on the acceptance of both, then the highest payoff i can achieve is

$$\begin{cases} \max\left\{V - \delta\phi_j - X_k, V - X_j - \delta\phi_k, X_{ij} - \delta\phi_j, X_{ik} - \delta\phi_k, \delta\phi_i\right\}, & \text{if } X_k \leq \delta\phi_k \text{ or } X_j \leq \delta\phi_j \\ \max\left\{V - X_j - X_k, X_{ij} - \delta\phi_j, X_{ik} - \delta\phi_k, \delta\phi_i\right\}, & \text{if } X_k \geq \delta\phi_k \text{ and } X_j \geq \delta\phi_j \end{cases}. \tag{10}$$

Let us prove this claim. Obviously, player i can achieve $X_{ij} - \delta\phi_j$, $X_{ik} - \delta\phi_k$ or $\delta\phi_i$ with an offer that only one or none of the players accepts. Consider then the best offer that i can make that is acceptable by both j and k. Assume that i chooses player j to be the first player to respond to the offer. We have seen before that j accepts the offer if and only if $p_j \geq X_j$ and $p_k \geq \delta\phi_k$, or $p_j \geq \delta\phi_j$ and $p_k < \delta\phi_k$, and that player k accepts the offer if and only if $p_k \geq X_k$, if player j has previously accepted the offer, or if player j has previously rejected the offer, if $p_k \geq \delta\phi_k$. Therefore, player i can buy players' j and k resources at a minimum cost equal to $\min\left\{X_j + \delta\phi_k, \delta\phi_j + X_k\right\}$ if $X_k \leq \delta\phi_k$, and equal to $X_j + X_k$ if $X_k \geq \delta\phi_k$, when player j is the first player to respond. But since player i can choose either j or k to be the first player to respond then he can buy both players' resources at a minimum cost equal to $\min\left\{X_j + \delta\phi_k, \delta\phi_j + X_k\right\}$ if $X_k \leq \delta\phi_k$ or $X_j \leq \delta\phi_j$, and equal to $X_j + X_k$ if $X_k \geq \delta\phi_k$ and $X_j \geq \delta\phi_j$, which proves the claim.

We have shown, so far, that player i's expected utility when chosen to propose, ϕ_{ii}, is equal to

$$\phi_i^i = \max\left\{\delta\phi_i, X_{ij} - \delta\phi_j, X_{ik} - \delta\phi_k, V - \delta\phi_j - \delta\phi_k, V - \delta\phi_j - X_k, V - X_j - \delta\phi_k\right\}.$$

But $X_{ij} - \delta\phi_j \leq V - \delta\phi_j - X_k$ (and also $X_{ik} - \delta\phi_k \leq V - \delta\phi_k - X_j$), and that the inequality holds strictly if $V_k + V_{ij} < V$ holds strictly (similarly we also have that $X_{ik} - \delta\phi_k \leq V - \delta\phi_k - X_j$). Note first that $X_{ij} - \delta\phi_j \leq V - \delta\phi_j - X_k$ is equivalent to $X_{ij} + X_k \leq V$, and

$$\begin{aligned} X_{ij} + X_k &= \delta X_{ij} + (1-\delta)V_{ij} + \delta X_k + (1-\delta)V_k \\ &= \delta\left(X_k + X_{ij}\right) + (1-\delta)\left(V_k + V_{ij}\right) \\ &= \delta V + (1-\delta)\left(V_k + V_{ij}\right), \end{aligned}$$

However, since $V_k + V_{ij} \leq V$ then $X_{ij} + X_k \leq V$, and if $V_k + V_{ij} < V$ then $X_{ij} - \delta\phi_j < V - \delta\phi_j - X_k$. Obviously, also $\delta\phi_i \leq V - \delta\phi_j - \delta\phi_k$, and that the inequality holds strictly if $V \geq 0$ holds strictly, because $\phi_i + \phi_j + \phi_k \leq V$.

We thus have that the highest expected utility that player i can achieve, conditional on being chosen to be the proposer, is equal to

$$\phi_i^i = \max\left\{V - \delta\phi_j - \delta\phi_k, V - \delta\phi_j - X_k, V - X_j - \delta\phi_k\right\}. \tag{11}$$

Using this result and the best acceptance strategies proposed we can easily prove that all conditions of the lemma are true, which completes the necessary part of the theorem.

We now prove the converse of the theorem. Suppose that we are given payoffs ϕ_i^j and ϕ_i for $i, j \in \{1, 2, 3\}$ satisfying all stated conditions. We claim that the stationary strategy profile σ of proposals and responses considered above is a stationary subgame perfect equilibrium. We use the one-stage deviation principle for infinite-horizon games to prove the claim. This proposition states that in any infinite-horizon game with observed actions that is continuous at infinity, a strategy profile σ is subgame perfect if and only if there is no player i and strategy σ_i' that agrees with σ_i except at a single stage t of the game and history h^t, such that σ_i' is a better response to σ_{-i} than σ_i conditional on history h^t being reached (see Fudenberg and Tirole (1991)).

Note first that the game is continuous at infinity: for each player i his utility function is such that, for any two histories h and h' such that the restrictions of the histories to the first t periods coincides, then the payoff of player i, $|u_i(h) - u_i(h')|$, converges to zero as t converge to infinity. It is immediately clear that the negotiation game is continuous at infinity because $|u_i(h) - u_i(h')| \leq M\left(\delta^{t+1} + \delta^{t+2} + \cdots\right) = \frac{M}{1-\delta}\delta^{t+1}$, where $M \geq V - \min\{0, V_j\} - \min\{0, V_k\}$.

The strategy profile σ_i is such that, by construction, no single deviation σ_i' at both the proposal and response stage can lead to a better response than σ_i. Therefore, by the one-stage deviation principle, the stationary strategy profile σ is a subgame perfect Nash equilibrium. Q.E.D.

Proof of Theorem 1: Suppose that ϕ_i is an SPE outcome. It must then be true that $X_i - \delta\phi_i$ belongs to at least one of the following cases, where the triple $(i,j,k) \in \Pi$ belongs to the set of permutations Π of the N players:

$$
\begin{aligned}
&I : X_1 - \delta\phi_1 \geq 0,\ X_2 - \delta\phi_2 \geq 0, \text{ and } X_3 - \delta\phi_3 \geq 0,\\
&II\,(i) : X_i - \delta\phi_i \leq 0,\ X_j - \delta\phi_j \geq 0, \text{ and } X_k - \delta\phi_k \geq 0,\\
&III_1\,(i,j,k) : X_i - \delta\phi_i < X_j - \delta\phi_j \leq 0, \text{ and } X_k - \delta\phi_k \geq 0,\\
&III_2\,(k) : X_i - \delta\phi_i = X_j - \delta\phi_j < 0, \text{ and } X_k - \delta\phi_k \geq 0,\\
&IV_1\,(i,j,k) : X_i - \delta\phi_i < X_j - \delta\phi_j < X_k - \delta\phi_k \leq 0,\\
&IV_2\,(i) : X_i - \delta\phi_i < X_j - \delta\phi_j = X_k - \delta\phi_k < 0,\\
&IV_3 : X_i - \delta\phi_i = X_j - \delta\phi_j = X_k - \delta\phi_k < 0,\\
&IV_4\,(k) : X_i - \delta\phi_i = X_j - \delta\phi_j < X_k - \delta\phi_k \leq 0.
\end{aligned}
$$

Define for each triple (i,j,k) the set of eight cases above as

$$\mathbb{Q}(i,j,k) = \{I, II(i), III_1(i,j,k), III_2(k), IV_1(i,j,k), IV_2(i), IV_3, IV_4(k)\}.$$

The set of all cases is given by $\mathbb{Q} = \underset{(i,j,k)\in\Pi}{\cup} \mathbb{Q}(i,j,k)$, when we consider all permutations of

N. Due to the symmetry of the problem we can concentrate on the analysis of the eight cases in $\mathbb{Q}(i,j,k)$.

The following lemma is the result of the separate analysis of each case.

LEMMA 2: *A payoff $\phi = (\phi_i)$ is an SPE equilibrium outcome if and only if there exists $Q \in \mathbb{Q}(i,j,k)$, such that the payoff ϕ is equal to*

$$\phi = \frac{V}{3} + \Phi_Q \cdot \omega \text{ for all games that satisfy the inequalities } \Omega_Q \cdot \omega \leq 0,$$

where Φ_Q and Ω_Q are matrices given explicitly below, and ω is the linear transformation of variables,

$$\omega_i = X_i - \frac{\delta V}{3} \text{ for } i \in N. \tag{12}$$

PROOF OF LEMMA 2: We first provide an outline of the steps involved in the proof that are repeatedly applied to all cases Q.

Necessity part: Assume that ϕ_i is an SPE outcome. Then there is a case $Q \in \mathbb{Q}(i,j,k)$ such that all inequalities in case Q hold. The conditions in Lemma 1 define explicit expressions for ϕ_j^i as a function of ϕ. Substituting these expressions for ϕ_j^i into the system of equations (9) and solving for the system yields a unique solution that can be expressed as $\phi = \frac{V}{3} + \Phi_Q \cdot \omega$. We then substitute this expression for the equilibrium outcome into the inequality restrictions that define case Q, resulting in a system of linear inequalities $\Omega_Q \cdot \omega \leq 0$.

Converse part: Suppose that the game $\mathbf{v}$ is such that $\Omega_Q \cdot \omega \leq 0$ and $\phi = \frac{V}{3} + \Phi_Q \cdot \omega$ for any case $Q \in \mathbb{Q}(i,j,k)$. Then, by the converse of Lemma 1, the payoff ϕ is an SPE outcome.

Consider now each of the cases $Q \in \mathbb{Q}(i,j,k)$.

I. $X_1 - \delta\phi_1 \geq 0$, $X_2 - \delta\phi_2 \geq 0$, and $X_3 - \delta\phi_3 \geq 0$.

The players' best response strategy yield the following system of equilibrium payoffs,

$$\begin{array}{ccc} \phi_1^1 = V - \delta\phi_2 - \delta\phi_3 & \phi_2^1 = \delta\phi_2 & \phi_3^1 = \delta\phi_3 \\ \phi_1^2 = \delta\phi_1 & \phi_2^2 = V - \delta\phi_1 - \delta\phi_3 & \phi_3^2 = \delta\phi_3 \\ \phi_1^3 = \delta\phi_1 & \phi_2^3 = \delta\phi_2 & \phi_3^3 = V - \delta\phi_1 - \delta\phi_2 \end{array}$$

where if all the inequalities are strict the unique best response strategy of all players is to choose the offer to all players conditional on N. The equilibrium payoffs satisfy the system of equations (9), whose unique solution is $\phi_i = \frac{V}{3}$ for all i, (define $\Phi_I = 0$). Moreover, the conditions that must be satisfied by the solution are $X_i - \delta\phi_i \geq 0$, which is equivalent to,

$$\omega_i \geq 0 \text{ for all } i.$$

$II(i)$. $X_i - \delta\phi_i \leq 0$, $X_j - \delta\phi_j \geq 0$, and $X_k - \delta\phi_k \geq 0$.

The players' best response strategy yield the following system of equilibrium payoffs,

$$\begin{array}{ccc} \phi_i^i = V - \delta\phi_k - \delta\phi_j & \phi_j^i = \delta\phi_j & \phi_k^i = \delta\phi_k \\ \phi_i^j = X_i & \phi_j^j = V - \delta\phi_k - X_i & \phi_k^j = \delta\phi_k \\ \phi_i^k = X_i & \phi_j^k = \delta\phi_j & \phi_k^k = V - \delta\phi_j - X_i \end{array}$$

where if all the inequalities are strict the unique best response strategy of players j and k is to choose the offer to all players conditional on $\{j,k\}$, and that of player i is to choose the offer to all players conditional on N. The equilibrium payoffs satisfy the system of equations (9), whose unique solution is

$$\begin{aligned} \phi_i &= \frac{2X_i + V(1-\delta)}{3-\delta}, \\ \phi_j &= \frac{V - X_i}{3-\delta}, \\ \phi_k &= \frac{V - X_i}{3-\delta}, \end{aligned}$$

which corresponds to $\phi = \frac{V}{3} + \Phi_Q \cdot \omega$ where

$$\Phi_Q = \frac{1}{(3-\delta)} \begin{bmatrix} 2 & 0 & 0 \\ -1 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}.$$

This solution is an equilibrium if the system of inequalities, $X_i - \delta\phi_i \leq 0$, $X_j - \delta\phi_j \geq 0$, and $X_k - \delta\phi_k \geq 0$, holds. This system of inequalities corresponds, after simplifications, to

$$\Omega_Q \cdot \omega = \begin{bmatrix} 1 & 0 & 0 \\ -\delta & -(3-\delta) & 0 \\ -\delta & 0 & -(3-\delta) \end{bmatrix} \begin{bmatrix} \omega_i \\ \omega_j \\ \omega_k \end{bmatrix} \leq 0.$$

$III_1(i,j,k)$. $X_i - \delta\phi_i < X_j - \delta\phi_j \leq 0$, and $X_k - \delta\phi_k \geq 0$.

The players' best response strategy yield the following system of equilibrium payoffs

$$\begin{array}{ccc} \phi_i^i = U - \delta x_k - X_j & \phi_j^i = X_j & \phi_k^i = \delta x_k \\ \phi_i^j = X_i & \phi_j^j = U - \delta x_k - X_i & \phi_k^j = \delta x_k \\ \phi_i^k = X_i & \phi_j^k = \delta x_j & \phi_k^k = U - \delta x_j - X_i \end{array}$$

where if all the inequalities are strict the unique best response strategy of players j and k is to choose the offer to all players conditional on $\{j,k\}$, and that of player i is to choose the offer to all players conditional on $\{i,k\}$. The equilibrium payoffs satisfy the system of equations (9), whose unique solution is

$$\begin{aligned} \phi_i &= \frac{(1-\delta)\left((3-\delta)U - 3Y_j\right) + (6-5\delta)X_i}{9+\delta^2-9\delta}, \\ \phi_j &= \frac{3(1-\delta)(U-X_i) + (3-2\delta)X_j}{9+\delta^2-9\delta}, \\ \phi_k &= \frac{(3-2\delta)U - (3-2\delta)X_i - X_j\delta}{9+\delta^2-9\delta}, \end{aligned}$$

which corresponds to $\phi = \frac{V}{3} + \Phi_Q \cdot \omega$ where

$$\Phi_Q = \frac{1}{\left(9+\delta^2-9\delta\right)} \begin{bmatrix} (6-5\delta) & -3(1-\delta) & 0 \\ -3(1-\delta) & (3-2\delta) & 0 \\ -(3-2\delta) & -\delta & 0 \end{bmatrix}.$$

This solution is an equilibrium if the system of inequalities, $X_i - \delta\phi_i < X_j - \delta\phi_j \le 0$, and $X_k - \delta\phi_k \ge 0$ holds. This system of inequalities corresponds, after some simplifications, to:

$$\begin{aligned} \left(9+\delta^2-9\delta\right)X_k + \left(3\delta-2\delta^2\right)X_i + X_j\delta^2 &\ge \left(3\delta-2\delta^2\right)U, \\ (3-\delta)X_j + X_i\delta &\le \delta U, \\ (9-6\delta)X_j - 9(1-\delta)X_i &> \delta^2 U, \end{aligned}$$

which is equivalent to

$$\Omega_Q \cdot \omega = \begin{bmatrix} -\left(3\delta-2\delta^2\right) & -\delta^2 & -\left(9+\delta^2-9\delta\right) \\ \delta & (3-\delta) & 0 \\ 3(1-\delta) & -(3-2\delta) & 0 \end{bmatrix} \begin{bmatrix} \omega_i \\ \omega_j \\ \omega_k \end{bmatrix} \le 0,$$

if we replace the strict inequality $(9-6\delta)X_j - 9(1-\delta)X_i > \delta^2 U$ by a weak inequality. Note that due to the upper hemi-continuity of the Nash equilibrium correspondence (see

Fudenberg and Tirole (1991)), if $\Omega_Q \cdot \omega \leq 0$ then $\phi = \frac{V}{3} + \Phi_Q \cdot \omega$ is also an SPE outcome.

$III_2\,(k)$. $X_i - \delta\phi_i = X_j - \delta\phi_j < 0$, and $X_k - \delta\phi_k \geq 0$.

The best response strategy of player k is to choose an offer to all players conditional on $\{i,k\}$ with probability $p \in [0,1]$ and to choose an offer to all players conditional on $\{j,k\}$ with probability $(1-p)$, and that of player j and i to choose, respectively, an offer to all players conditional on $\{j,k\}$ and $\{i,k\}$. This result in the following system of equilibrium payoffs:

$$\begin{array}{ccc} \phi_i^i = V - X_j - \delta\phi_k & \phi_j^i = X_j & \phi_k^i = \delta\phi_k \\ \phi_i^j = X_i & \phi_j^j = V - \delta\phi_k - X_i & \phi_k^j = \delta\phi_k \\ \phi_i^k = (1-p)\,X_i + p\delta\phi_i & \phi_j^k = pX_j + (1-p)\,\delta\phi_j & \phi_k^k = V - \delta\phi_i - X_j \end{array}$$

The equilibrium payoffs satisfy the system of equations (9), whose unique solution is

$$\begin{aligned} \phi_i &= \frac{(3-2\delta)\,X_i + (1-\delta)\,(2\delta V - 3X_j)}{\delta\,(6-5\delta)}, \\ \phi_j &= \frac{(3-2\delta)\,X_j + (1-\delta)\,(2\delta V - 3X_i)}{\delta\,(6-5\delta)}, \\ \phi_k &= \frac{(2-\delta)\,V - X_i - X_j}{6-5\delta}, \\ p &= \frac{(9\delta-9)\,X_i + (9-6\delta)\,X_j - \delta^2 V}{(3X_i + 3X_j - 2\delta V)\,\delta}, \end{aligned}$$

which corresponds to $\phi = \frac{V}{3} + \Phi_Q \cdot \omega$ where

$$\Phi_Q = \frac{1}{\delta\,(6-5\delta)} \begin{bmatrix} (3-2\delta) & -3\,(1-\delta) & 0 \\ -3\,(1-\delta) & (3-2\delta) & 0 \\ -\delta & -\delta & 0 \end{bmatrix}.$$

This solution is an equilibrium if the system of inequalities, $X_i - \delta\phi_i = X_j - \delta\phi_j < 0$, and $X_k - \delta\phi_k \geq 0$, holds (note that the condition $X_j - \delta\phi_j = X_i - \delta\phi_i$ is already satisfied), and in addition $p \in [0,1]$. This is equivalent to

$$\begin{aligned} \delta\left(\frac{(2-\delta)\,V - X_i - X_j}{6-5\delta}\right) &\leq X_k \\ \delta\left(\frac{(3-2\delta)\,X_j + (1-\delta)\,(2\delta V - 3X_i)}{\delta\,(6-5\delta)}\right) &> X_j \end{aligned}$$

and the inequalities arising from the restriction $p \in [0,1]$ are

$$\begin{aligned} \frac{(9\delta - 9) X_i + (9 - 6\delta) X_j - \delta^2 V}{(3X_j + 3X_i - 2\delta V)\,\delta} &\leq 1, \\ \frac{(9\delta - 9) X_i + (9 - 6\delta) X_j - \delta^2 V}{(3X_j + 3X_i - 2\delta V)\,\delta} &\geq 0. \end{aligned}$$

Note that the first two inequalities can be restated as

$$\begin{aligned} \delta\left((2-\delta) V - X_i - X_j\right) &\leq (6 - 5\delta) X_i, \\ 3X_j + 3X_i &< 2\delta V. \end{aligned}$$

But since $3X_j + 3X_i - 2\delta V < 0$ then the denominator of the conditions imposed on p is negative and the inequalities are equivalent to

$$\begin{aligned} (9 - 6\delta) X_i - 9(1-\delta) X_j &\leq \delta^2 V, \\ (9 - 6\delta) X_j - 9(1-\delta) X_i &\leq \delta^2 V. \end{aligned}$$

Therefore, the system of inequalities is equivalent to

$$\begin{aligned} (6 - 5\delta) X_k + \delta(X_i + X_j) &\geq \delta(2-\delta) V, \\ 3X_j + 3X_i &< 2\delta V, \\ (9 - 6\delta) X_i - 9(1-\delta) X_j &\leq \delta^2 V, \\ (9 - 6\delta) X_j - 9(1-\delta) X_i &\leq \delta^2 V. \end{aligned}$$

The upper hemi-continuity of the Nash equilibrium correspondence implies that the same equilibrium outcome given by the formula above also holds when the second inequality holds strictly, $3X_j + 3X_i \leq 2\delta V$. But note that if we add up both of the two last inequalities we obtain $\delta(3X_i + 3X_j) \leq 2\delta^2 V$, and therefore the second inequality can be dropped from the system of inequalities yielding,

$$\Omega_Q \cdot \omega = \begin{bmatrix} -\delta & -\delta & -(6-5\delta) \\ (3-2\delta) & -3(1-\delta) & 0 \\ -3(1-\delta) & (3-2\delta) & 0 \end{bmatrix} \begin{bmatrix} \omega_i \\ \omega_j \\ \omega_k \end{bmatrix} \leq 0.$$

$IV_1(i,j,k)$. $X_i - \delta\phi_i < X_j - \delta\phi_j < X_k - \delta\phi_k \leq 0.$

This case is similar to case $III_1(i,j,k)$ and has a similar best response strategy. Repeating the same reasoning we obtain that $\phi = \frac{V}{3} + \Phi_Q \cdot \omega$ where $\Phi_{IV_1(i,j,k)} = \Phi_{III_1(i,j,k)}$. This solution is the equilibrium if the system of inequalities, $X_i - \delta\phi_i < X_j - \delta\phi_j < X_k - \delta\phi_k \leq 0$, holds. This system of inequalities is equivalent to

$$\Omega_Q \cdot \omega \begin{bmatrix} 3(1-\delta) & -(3-2\delta) & 0 \\ -\delta^2 & -\left(12\delta - 2\delta^2 - 9\right) & -\left(9+\delta^2-9\delta\right) \\ \delta(3-2\delta) & \delta^2 & \left(9-9\delta+\delta^2\right) \end{bmatrix} \begin{bmatrix} \omega_i \\ \omega_j \\ \omega_k \end{bmatrix} \leq 0.$$

$IV_2(i)$. $X_i - \delta\phi_i \leq X_j - \delta\phi_j = X_k - \delta\phi_k < 0$.

The best response strategy of player i is to choose an offer to all players conditional on $\{i,k\}$ with probability $p \in [0,1]$ and to choose an offer to all players conditional on $\{i,j\}$ with probability $(1-p)$, and that of player j and k to choose an offer to all players conditional on $\{j,k\}$. This result in the following system of equilibrium payoffs:

$$\begin{array}{ccc} \phi_i^i = V - X_j - \delta\phi_k & \phi_j^i = pX_j + (1-p)\,\delta\phi_j & \phi_k^i = (1-p)\,X_k + p\delta\phi_k \\ \phi_i^j = X_i & \phi_j^j = V - X_i - \delta\phi_k & \phi_k^j = \delta\phi_k \\ \phi_i^k = X_i & \phi_j^k = \delta\phi_j & \phi_k^k = V - X_i - \delta\phi_j \end{array}$$

The system of equation (9) is

$$\begin{aligned} \phi_i &= \frac{1}{3}\left(V - X_j - \delta\phi_k + 2X_i\right), \\ \phi_j &= \frac{1}{3}\left(pX_j + (1-p)\,\delta\phi_j + V - X_i - \delta\phi_k + \delta\phi_j\right), \\ \phi_k &= \frac{1}{3}\left((1-p)\,X_k + p\delta\phi_k + \delta\phi_k + V - X_i - \delta\phi_j\right), \\ \delta\phi_j - X_j &= \delta\phi_k - X_k. \end{aligned}$$

Note that the first and last equation determines the value of ϕ_i, and $\phi_j = \delta^{-1}\left(\delta\phi_k + X_j - X_k\right)$ as a function of ϕ_k. It remains then only to solve for the values of ϕ_k and p using the second and third equations:

$$\begin{aligned} \delta^{-1}\left(\delta\phi_k + X_j - X_k\right) &= \frac{1}{3}\left(pX_j + (2-p)\left(\delta\phi_k + X_j - X_k\right) + V - X_i - \delta\phi_k\right), \\ \phi_k &= \frac{1}{3}\left((1-p)\,X_k + (1+p)\,\delta\phi_k + V - X_i - \left(\delta\phi_k + X_j - X_k\right)\right), \end{aligned}$$

collecting all terms in ϕ_k and p yields:

$$\begin{aligned} (3-\delta)\,\phi_k - X_k p + p\delta\phi_k + \frac{3}{\delta}\left(X_j - X_k\right) - V - 2X_j + 2X_k + X_i &= 0, \\ -3\phi_k + \delta p\phi_k + (1-p)\,X_k - X_j + V - X_i + X_k &= 0. \end{aligned}$$

Subtracting the first and second equations cancels all terms in p and gives us the solution for ϕ_k :

$$\phi_k = \frac{2\delta V - 2X_i\delta - (3-\delta)X_j + 3X_k}{(6-\delta)\delta}.$$

The last equation can be rewritten as $(\delta\phi_k - X_k)p - 3\phi_k + V + 2X_k - X_j - X_i = 0$. Since $\delta\phi_k - X_k > 0$ then there is a unique solution for p. We have then found a unique solution:

$$\begin{aligned}
\phi_i &= \frac{(2-\delta)V - X_j - X_k + 4X_i}{6-\delta}, \\
\phi_j &= \frac{2\delta V - 2X_i\delta + 3X_j - (3-\delta)X_k}{(6-\delta)\delta}, \\
\phi_k &= \frac{2\delta V - 2X_i\delta - (3-\delta)X_j + 3X_k}{(6-\delta)\delta}, \\
p &= \frac{\left(-9 - 2\delta^2 + 12\delta\right)X_k + \left(9 + \delta^2 - 9\delta\right)X_j + X_i\delta^2 - \delta^2 V}{\delta\left((3-\delta)X_k + (3-\delta)X_j - 2\delta V + 2X_i\delta\right)},
\end{aligned}$$

which corresponds to $\phi = \frac{V}{3} + \Phi_Q \cdot \omega$ where

$$\Phi_Q = \frac{1}{\delta(6-5\delta)}\begin{bmatrix} 4\delta & -\delta & -\delta \\ -2\delta & 3 & -(3-\delta) \\ -2\delta & -(3-\delta) & 3 \end{bmatrix}.$$

This is the equilibrium of the game if the system of inequalities $X_i - \delta\phi_i \leq X_j - \delta\phi_j = X_k - \delta\phi_k < 0$ holds (note that the condition $X_j - \delta\phi_j = X_k - \delta\phi_k$ is already satisfied) and $p \in [0,1]$, which is equivalent to:

$$\begin{aligned}
(3-2\delta)X_j + (7\delta - 6)X_i + (3-2\delta)X_k &\geq \delta^2 V, \\
(3-\delta)X_k + (3-\delta)X_j + 2X_i\delta &< 2\delta V, \\
\frac{\left(-9 - 2\delta^2 + 12\delta\right)X_k + \left(9 + \delta^2 - 9\delta\right)X_j + X_i\delta^2 - \delta^2 V}{\delta\left((3-\delta)X_k + (3-\delta)X_j - 2\delta V + 2X_i\delta\right)} &\leq 1, \\
\frac{\left(-9 - 2\delta^2 + 12\delta\right)X_k + \left(9 + \delta^2 - 9\delta\right)X_j + X_i\delta^2 - \delta^2 V}{\delta\left((3-\delta)X_k + (3-\delta)X_j - 2\delta V + 2X_i\delta\right)} &\geq 0,
\end{aligned}$$

But note that the second inequality implies that the denominators in the third and fourth inequalities are negative. Therefore the system of inequalities is equivalent to:

$$\begin{aligned}
(3-2\delta) X_j + (7\delta - 6) X_i + (3-2\delta) X_k &\geq \delta^2 V, \\
(3-\delta) X_k + (3-\delta) X_j + 2\delta X_i &< 2\delta V, \\
\left(12\delta - 2\delta^2 - 9\right) X_j + \left(9+\delta^2 - 9\delta\right) X_k + \delta^2 X_i &\leq \delta^2 V, \\
\left(12\delta - 2\delta^2 - 9\right) X_k + \left(9+\delta^2 - 9\delta\right) X_j + \delta^2 X_i &\leq \delta^2 V.
\end{aligned}$$

The upper hemi-continuity of the Nash equilibrium correspondence implies that the equilibrium outcome also holds when the second inequality holds strictly, $(3-\delta) X_k + (3-\delta) X_j + 2\delta X_i \leq 2\delta V$. But note that adding up the last two inequalities we get $\delta \left((3-\delta) X_k + (3-\delta) X_j + 2\delta X_i\right) \leq 2\delta^2 V$ and thus the second inequality becomes redundant. Finally, the system of inequalities is equivalent to

$$\Omega_Q \cdot \omega = \begin{bmatrix} -(7\delta-6) & -(3-2\delta) & -(3-2\delta) \\ \delta^2 & \left(12\delta - 2\delta^2 - 9\right) & \left(9+\delta^2-9\delta\right) \\ \delta^2 & \left(9+\delta^2-9\delta\right) & \left(12\delta - 2\delta^2 - 9\right) \end{bmatrix} \begin{bmatrix} \omega_i \\ \omega_j \\ \omega_k \end{bmatrix} \leq 0.$$

IV$_3$. $X_i - \delta\phi_i = X_j - \delta\phi_j = X_k - \delta\phi_k < 0$

The best response strategy of player i is to choose an offer to all players conditional on $\{i,k\}$ and $\{i,j\}$ with probabilities p_i and $(1-p_i)$, the strategy of player j is to choose an offer to all players conditional on $\{j,k\}$ and $\{i,j\}$ with probabilities p_j and $(1-p_j)$, and the strategy of player k is to choose an offer to all players conditional on $\{j,k\}$ and $\{i,k\}$ with probabilities p_k and $(1-p_k)$. This result in the following system of equilibrium payoffs:

$$\begin{array}{ccc}
\phi_i^i = V - X_j - \delta\phi_k & \phi_j^i = p_i X_j + (1-p_i)\,\delta\phi_j & \phi_k^i = (1-p_i) X_k + p_i \delta\phi_k \\
\phi_i^j = p_j X_i + (1-p_j)\,\delta\phi_i & \phi_j^j = V - X_i - \delta\phi_k & \phi_k^j = (1-p_j) X_k + p_j \delta\phi_k \\
\phi_i^k = p_k X_i + (1-p_k)\,\delta\phi_i & \phi_j^k = (1-p_k) X_j + p_k \delta\phi_j & \phi_k^k = V - X_i - \delta\phi_j
\end{array}$$

where p_i, p_j and p_j all belong to the interval $[0,1]$. The system of equation (9), implies that $\phi_i + \phi_j + \phi_k = V$. Imposing the condition $\delta\phi_i - X_i = \delta\phi_j - X_j = \delta\phi_k - X_k$ we immediately get that there is unique solution equal to

$$\begin{aligned}
\phi_i &= \frac{X_i}{\delta} + \frac{1}{3}\frac{\delta V - X_i - X_j - X_k}{\delta}, \\
\phi_j &= \frac{X_j}{\delta} + \frac{1}{3}\frac{\delta V - X_i - X_j - X_k}{\delta}, \\
\phi_k &= \frac{X_k}{\delta} + \frac{1}{3}\frac{\delta V - X_i - X_j - X_k}{\delta},
\end{aligned}$$

which corresponds to $\phi = \frac{V}{3} + \Phi_Q \cdot \omega$ where

$$\Phi_Q = \frac{1}{3\delta} \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & 0 & 2 \end{bmatrix}.$$

The restrictions $X_i - \delta\phi_i = X_j - \delta\phi_j = X_k - \delta\phi_k < 0$ imply that

$$\delta\left(\phi_i + \phi_j + \phi_k\right) = \delta V > X_i + X_j + X_k,$$

which is equivalent to

$$X_i + X_j + X_k < \delta V.$$

We are now interested in solving the system of equations for p_i, p_j, and p_k:

$$\begin{aligned}
&V - X_j - \delta\phi_k + p_j X_i + (1 - p_j)\,\delta\phi_i + p_k X_i + (1 - p_k)\,\delta\phi_i - 3\phi_i = 0, \\
&p_i X_j + (1 - p_i)\,\delta\phi_j + V - \delta\phi_i - X_k + (1 - p_k)\,X_j + p_k\delta\phi_j - 3\phi_j = 0, \\
&(1 - p_i)\,X_k + p_i\delta\phi_k + (1 - p_j)\,X_k + p_j\delta\phi_k + V - X_i - \delta\phi_j - 3\phi_k = 0.
\end{aligned}$$

After rearranging terms, the system is equivalent to:

$$\begin{aligned}
&(\delta\phi_i - X_i)\,(p_j + p_k) = V - X_j - \delta\phi_k + 2\delta\phi_i - 3\phi_i, \\
&\left(\delta\phi_j - X_j\right)(p_i - p_k) = -\delta\phi_i + \delta\phi_j + X_j + V - 3\phi_j - X_k, \\
&(\delta\phi_k - X_k)\,(p_i + p_j) = -2X_k - V + X_i + \delta\phi_j + 3\phi_k,
\end{aligned}$$

and substituting the expressions for ϕ_i, ϕ_j, ϕ_k results in:

$$\begin{aligned}
-\delta\,(\delta V - X_i - X_j - X_k)\,(p_j + p_k) &= 4\delta X_k - 5X_i\delta - 3X_j - 3X_k + 6X_i + 4X_j\delta - \delta^2 V, \\
\delta\,(\delta V - X_i - X_j - X_k)\,(p_i - p_k) &= 3\,(X_k + 2X_j\delta - X_i\delta - \delta X_k + X_i - 2X_j)\,, \\
\delta\,(\delta V - X_i - X_j - X_k)\,(p_i + p_j) &= -7\delta X_k + 2X_i\delta + 2X_j\delta + \delta^2 V + 6X_k - 3X_i - 3X_j.
\end{aligned}$$

We can immediately verify that any vector p_i, p_j and p_k is the unique solution of the system of linear equations:

$$\begin{aligned}
p_i &= p_k + \frac{(1-\delta)(3X_k - 6X_j + 3X_i)}{\delta(\delta V - X_i - X_j - X_k)}, \\
p_j &= \frac{\delta^2 V + (4X_k - 5X_i + 4X_j)(1-\delta) - (X_i + X_j + X_k)}{\delta(\delta V - X_i - X_j - X_k)} - p_k, \\
p_k &= p_k.
\end{aligned}$$

Note that p_i, p_j and p_j all belong to the interval $[0,1]$ (we must also have $X_i + X_j + X_k < \delta V$). This imposes the following six additional inequalities that must hold:

$$\begin{aligned}
p_k + \frac{(1-\delta)(3X_k - 6X_j + 3X_i)}{\delta(\delta V - X_i - X_j - X_k)} &\geq 0, \\
p_k + \frac{(1-\delta)(3X_k - 6X_j + 3X_i)}{\delta(\delta V - X_i - X_j - X_k)} &\leq 1, \\
\frac{\delta^2 V + (4X_k - 5X_i + 4X_j)(1-\delta) - (X_i + X_j + X_k)}{\delta(\delta V - X_i - X_j - X_k)} - p_k &\geq 0, \\
\frac{\delta^2 V + (4X_k - 5X_i + 4X_j)(1-\delta) - (X_i + X_j + X_k)}{\delta(\delta V - X_i - X_j - X_k)} - p_k &\leq 1, \\
p_k &\geq 0, \\
p_k &\leq 1.
\end{aligned}$$

We use the Fourier-Motzkin elimination method (see Dantzig (1963) and Ziegler (1994)) to eliminate the parameter p_k from the above system of inequalities. We first rewrite the system of inequalities as follows:

$$\begin{aligned}
p_k + \frac{(1-\delta)(3X_k - 6X_j + 3X_i)}{\delta(\delta V - X_i - X_j - X_k)} &\geq 0, \\
p_k - \frac{\delta^2 V + (4X_k - 5X_i + 4X_j)(1-\delta) - (X_i + X_j + X_k)}{\delta(\delta V - X_i - X_j - X_k)} &\geq -1, \\
p_k &\geq 0, \\
-p_k - \frac{(1-\delta)(3X_k - 6X_j + 3X_i)}{\delta(\delta - X_i - X_j - X_k)} &\geq -1, \\
-p_k + \frac{\delta^2 V + (4X_k - 5X_i + 4X_j)(1-\delta) - (X_i + X_j + X_k)}{\delta(\delta V - X_i - X_j - X_k)} &\geq 0, \\
-p_k &\geq -1,
\end{aligned}$$

where in the first three inequalities the coefficient of p_k is $+1$ and in the last three inequalities the coefficient of p_k is -1. By the Fourier-Motzkin elimination method we can eliminate the variable p_k by adding each of the first three inequalities to each of the last three inequalities.

Then the system is equivalent to

$$
\begin{gathered}
0 \geq -1, \\
\frac{(1-\delta)(3X_k - 6X_j + 3X_i)}{\delta(\delta V - X_i - X_j - X_k)} + \frac{\delta^2 V + (4X_k - 5X_i + 4X_j)(1-\delta) - (X_i + X_j + X_k)}{\delta(\delta V - X_i - X_j - X_k)} \geq 0, \\
\frac{(1-\delta)(3X_k - 6X_j + 3X_i)}{\delta(\delta V - X_i - X_j - X_k)} \geq -1, \\
-\frac{\delta^2 V + (4X_k - 5X_i + 4X_j)(1-\delta) - (X_i + X_j + X_k)}{\delta(\delta V - X_i - X_j - X_k)} - \frac{(1-\delta)(3X_k - 6X_j + 3X_i)}{\delta(\delta V - X_i - X_j - X_k)} \geq -2, \\
0 \geq -1, \\
-\frac{\delta^2 V + (4X_k - 5X_i + 4X_j)(1-\delta) - (X_i + X_j + X_k)}{\delta(\delta V - X_i - X_j - X_k)} \geq -2, \\
-\frac{(1-\delta)(3X_k - 6X_j + 3X_i)}{\delta(\delta V - X_i - X_j - X_k)} \geq -1, \\
\frac{\delta^2 V + (4X_k - 5X_i + 4X_j)(1-\delta) - (X_i + X_j + X_k)}{\delta(\delta V - X_i - X_j - X_k)} \geq 0, \\
0 \geq -1.
\end{gathered}
$$

Note that the first, fifth, and last inequality are always satisfied, and the remaining six inequalities can be simplified to:

$$
\begin{aligned}
(3-2\delta)(X_i + X_j + X_k) - 9(1-\delta) X_i &\leq \delta^2 V, \\
(3-2\delta)(X_i + X_j + X_k) - 9(1-\delta) X_j &\leq \delta^2 V, \\
(3-2\delta)(X_i + X_j + X_k) - 9(1-\delta) X_k &\leq \delta^2 V, \\
(4\delta-3)(X_i + X_j + X_k) + 9(1-\delta) X_i &\leq \delta^2 V, \\
(4\delta-3)(X_i + X_j + X_k) + 9(1-\delta) X_j &\leq \delta^2 V, \\
(4\delta-3)(X_i + X_j + X_k) + 9(1-\delta) X_k &\leq \delta^2 V.
\end{aligned}
$$

The upper hemi-continuity of the Nash equilibrium correspondence implies that the equilibrium outcome also holds when the inequality $X_i + X_j + X_k \leq \delta V$ holds strictly. Also, adding up the first three equations (or the last three) yields $X_i + X_j + X_k \leq \delta V$. Finally, the system of inequalities is equivalent to

$$\Omega_Q \cdot \omega = \begin{bmatrix} (7\delta - 6) & (3-2\delta) & (3-2\delta) \\ (3-2\delta) & (7\delta-6) & (3-2\delta) \\ (3-2\delta) & (3-2\delta) & (7\delta-6) \\ (6-5\delta) & (4\delta-3) & (4\delta-3) \\ (4\delta-3) & (6-5\delta) & (4\delta-3) \\ (4\delta-3) & (4\delta-3) & (6-5\delta) \end{bmatrix} \begin{bmatrix} \omega_i \\ \omega_j \\ \omega_k \end{bmatrix} \leq 0.$$

$IV_4(k)$. $X_i - \delta\phi_i = X_j - \delta\phi_j < X_k - \delta\phi_k \leq 0$.

This case is similar to case $III_2(k)$ and has a similar best response strategy. Repeating the same reasoning we obtain that $\phi = \frac{V}{3} + \Phi_Q \cdot \omega$ where $\Phi_{IV_4(k)} = \Phi_{III_2(k)}$. This solution is the equilibrium if the system of inequalities, $X_i - \delta\phi_i = X_j - \delta\phi_j < X_k - \delta\phi_k \leq 0$, holds. This system of inequalities is equivalent to

$$\Omega_Q \cdot \omega = \begin{bmatrix} -(4\delta-3) & -(4\delta-3) & -(6-5\delta) \\ \delta & \delta & (6-5\delta) \\ -3(1-\delta) & (3-2\delta) & 0 \\ (3-2\delta) & -3(1-\delta) & 0 \end{bmatrix} \begin{bmatrix} \omega_i \\ \omega_j \\ \omega_k \end{bmatrix} \leq 0.$$

The results we have just obtained from the analysis of all cases $Q \in \mathbb{Q}(i,j,k)$ complete the prove of Lemma 2.

Q.E.D. LEMMA 2

In order to establish the uniqueness of the equilibrium payoffs we must now show that for any given game $\mathbf{v}$ if $\Omega_Q \cdot \omega \leq 0$ and $\Omega_{Q'} \cdot \omega \leq 0$ where Q and Q' are any of the cases in $\mathbb{Q}$, and ω is the linear function of $\mathbf{v}$ given by (12) then $\Phi_Q \cdot \omega = \Phi_{Q'} \cdot \omega$.

We first characterize the set of games $\mathbf{v}$ such that $\Omega_Q \cdot \omega \leq 0$ and $\Omega_{Q'} \cdot \omega \leq 0$. We obtain this characterization in Lemma 3 using the key representation result from the theory of polytopes (see Ziegler (1994)): a polyhedral cone $H = H(Q) = \{\omega \in R^3 : \Omega_Q \cdot \omega \leq 0\}$ represented in terms of a system of inequalities (or intersection of a finite number of half-spaces or the H-representation of the cone) with lineality zero can be represented as $H = cone\,(ext(H))$, the convex hull of its extremal rays (the V-representation of the cone).[11]

[11]For completeness we recall some basic concepts: Given any finite set of points $V \subset R^3$, we denote its conical hull by $cone\,(V) = \{\sum_{i=1}^{n} \lambda_i v_i : \lambda_i \geq 0 \text{ and } v_i \in V\}$; an extremal ray of cone $H \subset R^3$ is any point $\omega \in H$, $\omega \neq 0$, such that there exists a vector $p \in R^3$ where p is a supporting hyperplane to the cone H, and $H \cap \{\phi \in R^3 : p \cdot \phi = 0\} = \lambda\omega : \lambda \geq 0$; a vector p defines a supporting

We now define the set of points that are the candidates to be the extremal rays of the polyhedral cones $H(Q)$. Define the set of points $\mathbb{V} = \underset{(i,j,k)\in\Pi}{\cup} \{a_i, b_i, c_{ijk}, d_{ijk}\} \subset R^3$ where,

$$\begin{aligned}
a_i &= e_i, \\
b_i &= -(3-\delta)\, e_i + \delta e_j + \delta e_k, \\
c_{ijk} &= -(3-2\delta)\, e_i - 3(1-\delta)\, e_j + \delta e_k, \\
d_{ijk} &= -(3-2\delta)\, e_i - 3(1-\delta)\, e_j + (4\delta - 3)\, e_k,
\end{aligned}$$

and where $e_i \in R^3$ is the ith unit vector. Associate each element $Q \in \mathbb{Q}$ to a subset of $\mathbb{V}$ (one-to-one correspondence) as follows:

$$\begin{aligned}
&I = \{a_1, a_2, a_3\}, && II(i) = \{b_i, a_j, a_k\}, \\
&III_1(i,j,k) = \{b_i, c_{ijk}, a_k\}, && III_2(k) = \{c_{ijk}, c_{jik}, a_k\}, \\
&IV_1(i,j,k) = \{d_{ijk}, c_{ijk}, b_i\}, && IV_2(i) = \{d_{ijk}, d_{ikj}, b_i\}, \\
&IV_3 = \{d_{ijk}, d_{ikj}, d_{jik}, d_{jki}, d_{kij}, d_{kji}\}, && IV_4(k) = \{c_{ijk}, c_{jik}, d_{ijk}, d_{jik}\}.
\end{aligned}$$

We then have the following characterization of the polyhedral cones $H(Q) = \left\{\omega \in R^3 : \Omega_Q \cdot \omega \leq 0\right\}$.

Lemma 3: *For all $Q \in \mathbb{Q}$, $H(Q) = cone(Q)$ and Q is the set of extremal rays of the cone $H(Q)$. Moreover, for any Q and Q' in $\mathbb{Q}$ with $Q' \neq Q$ then $H(Q) \cap H(Q') = cone(Q \cap Q')$.*

Proof of Lemma 3: We refer to $Q \in \mathbb{Q}$, interchangeably, as a subset of $\mathbb{V}$ using the one-to-one correspondence above. We use the following result in order to obtain the set of extremal rays of the cone $H = \{\omega : \Omega \cdot \omega \leq 0\}$: A vector $\phi \in H$ is an extremal ray of the cone H if and only if $\phi \in H$ and $\Omega_i \phi = 0$ and $\Omega_j \phi = 0$, for Ω_i and Ω_j two linearly independent row vectors of the matrix Ω.

First note that any two ν and ν' in $\mathbb{V}$ are linearly independent. This is true for all $\delta \in (0,1)$ because

$$-(3-\delta) < -(3-2\delta) < -3(1-\delta) < (4\delta - 3) < \delta < 1$$

for $\delta \in (0,1)$ and the definitions of $\nu \in \mathbb{V}$. Now, for all $Q \in \mathbb{Q}$ the matrix Ω_Q of the H-representation of the cones $H(Q)$ have rank equal to 3 (full rank). Therefore, $lineal(H(Q)) = \left\{\phi \in R^3 : \Omega_Q \cdot \phi = 0\right\} = \{0\}$ and thus all cones $H(Q)$ have lineality zero. Finally, one can easily verify, using the result stated in the previous paragraph, that Q is

hyperplane if for all $\phi \in H$ then $p \cdot \phi \leq 0$; the set of all extremal rays of a cone H is denoted $ext(H)$; the lineality space of a cone $H = \left\{\phi \in R^3 : \Omega \cdot \phi = 0\right\}$ is equal to the linear space $lineal(H) = \left\{\phi \in R^3 : \Omega \cdot \phi = 0\right\}$, and the lineality of a cone is the dimension of the lineality space.

the set of extremal rays of the cone $H(Q)$ for all $Q \in \mathbb{Q}$. Thus by the representation result for cones we have that $H(Q) = cone(Q)$.

We now show that for any Q and Q' in $\mathbb{Q}$ with $Q' \neq Q$ then $H(Q) \cap H(Q') = cone(Q \cap Q'))$.

Claim 1: *Suppose that for any two cones $H(Q) = cone(Q)$ and $H(Q') = cone(Q')$, there exists a separating hyperplane H such that if $\nu \in (Q \cup Q') \setminus (Q \cap Q')$ then $\nu \notin H$. Then $H(Q) \cap H(Q') = cone(Q \cap Q')$.*

Proof of Claim 1: Recall that any vector $p \in R^3$ can be associated with the hyperplane H, where $H = \{\omega : p.\omega = 0\}$. A hyperplane H is separating if and only if for all $\omega \in H(Q)$ and $\omega' \in H(Q')$ then $p.\omega \leq 0$ and $p.\omega' \geq 0$.

It is obvious that $cone(Q \cap Q') \subset H(Q) \cap H(Q')$. We need to prove that $cone(Q \cap Q') \supset H(Q) \cap H(Q')$. It is obvious that $H(Q) \cap H(Q')$ is a cone and that $H(Q) \cap H(Q') \subset cone(Q \cup Q')$. But the separating hyperplane H is such that $H(Q) \cap H(Q') \subset H$ and all $\nu \in (Q \cup Q') \setminus (Q \cap Q')$ are such that $\nu \notin H$, and thus $\nu \notin H(Q) \cap H(Q')$. Therefore, $H(Q) \cap H(Q') \subset cone(Q \cap Q')$. Q.E.D. claim 1

We now proceed showing that for each pair Q and Q' in $\mathbb{Q}$ with $Q' \neq Q$ there exists a separating hyperplane H, associated with a vector p, such that $\nu \notin Q \cap Q'$ implies that $\nu \notin H$.

1. First, consider the case with $Q = I$.

Consider the separating hyperplane associated with $p = e_i$. By Claim 1 it is straightforward that $H(I) \cap H(II(i)) = cone(a_j, a_k)$, $H(I) \cap H(III_1(i,j,k)) = H(I) \cap H(III_2(k)) = cone(a_k)$, $H(I) \cap H(IV_1(i,j,k)) = H(I) \cap H(IV_2(i)) = H(I) \cap H(IV_4(k)) = \{0\}$. Also, $H(I) \cap H(IV_3) = \{0\}$, because $p = e_i + e_j + e_k$ defines a separating hyperplane: $p \cdot \nu > 0$ for $\nu \in I$ and $p \cdot \nu < 0$ for all $\nu \in IV$.

2. Now consider the case with $Q = II(i)$.

The intersection $H(II(i)) \cap H(III_1(i,j,k)) = cone(b_i, a_k)$, because $p = \delta e_i + (3-\delta) e_j$ defines a separating hyperplane: for cone $H(II(i))$, $p \cdot b_i = 0$, $p \cdot a_k = 0$, $p \cdot a_j = 3 - \delta > 0$, and for cone $H(III_1(i,j,k))$, $p \cdot c_{ijk} = 9\delta - \delta^2 - 9 < 0$. Similarly, we have that $H(II(j)) \cap H(III_1(i,j,k)) = cone(a_k)$, and $H(II(k)) \cap H(III_1(i,j,k)) = \{0\}$ (separating hyperplane is $p = e_k$).

The intersection $H(II(i)) \cap H(III_2(k)) = cone(a_k)$ and $H(II(i)) \cap H(III_2(i)) = \{0\}$ because $p = e_i$ defines a separating hyperplane.

The intersection $H(II(i)) \cap H(IV_1(i,j,k)) = cone(b_i)$, because $p = \delta e_i + (3-\delta) e_j$ defines a separating hyperplane: for the cone $H(II(i))$ see the previous paragraph, and for the cone $H(IV_1(i,j,k))$ we have that $p \cdot b_i = 0$, $p \cdot d_{ijk} = 9\delta - \delta^2 - 9 < 0$, $p \cdot c_{jik} = 9\delta - \delta^2 - 9 < 0$. Similarly, we also have that $H(II(i)) \cap H(IV_1(j,i,k)) = H(II(i)) \cap H(IV_1(j,k,i)) = \{0\}$.

The intersection, $H(II(i)) \cap H(IV_2(i)) = cone(b_i)$, because $p = \left(3\delta - 2\delta^2\right) e_i + \delta^2 e_j + \left(9 + \delta^2 - 9\delta\right) e_k$ defines a separating hyperplane: for cone $H(II(i))$ we have $p \cdot b_i = 0$, $p \cdot a_j = \delta^2 > 0$, and $p \cdot a_k = 9 + \delta^2 - 9\delta > 0$, and for cone $H(IV_2(i))$ we have $p \cdot d_{ijk} = -3(1-\delta)\left(9 + \delta^2 - 9\delta\right) < 0$, and $p \cdot d_{ikj} = -3(1-\delta)(\delta - 3)^2 < 0$.

The intersection $H(II(i)) \cap H(IV_3) = \{0\}$ because $p = \left(3\delta - 2\delta^2\right) e_i + \delta^2 e_j + \left(9 + \delta^2 - 9\delta\right) e_k$ defines a separating hyperplane: for cone $H(II(i))$ see the previous paragraph, and for cone $H(IV_3)$, $p \cdot d_{ijk} = -3(1-\delta)\left(9 + \delta^2 - 9\delta\right) < 0$, $p \cdot d_{ikj} = -3(1-\delta)(3-\delta)^2 < 0$, $p \cdot d_{jki} = -3(1-\delta)\left(9 - \delta^2 - 3\delta\right) < 0$, $p \cdot d_{kji} = -3(1-\delta)\left(9 - \delta^2 - 6\delta\right) < 0$, $p \cdot d_{jik} = -27(1-\delta)^2 < 0$, $p \cdot d_{kij} = -9(1-\delta)(3-\delta) < 0$.

Finally, the intersection $H(II(i)) \cap H(IV_4(k)) = \{0\}$ because $p = \delta e_i + (3-\delta) e_j$ defines a separating hyperplane: for cone $H(II(i))$ see the first paragraph of item 2, and for cone $H(IV_4(k))$, $p \cdot c_{ijk} = 9\delta - \delta^2 - 9 < 0$, $p \cdot c_{jik} = 6\delta + \delta^2 - 9 < 0$, $p \cdot d_{ijk} = 9\delta - \delta^2 - 9 < 0$, and $p \cdot d_{jik} = 6\delta + \delta^2 - 9 < 0$.

3. Now consider the case with $Q = III_1(i,j,k)$.

The intersection $H(III_1(i,j,k)) \cap H(III_2(k)) = cone(c_{ijk}, a_k)$ because $p = -9(1-\delta) e_i + (9 - 6\delta) e_j$ defines a separating hyperplane: for cone $H(III_1(i,j,k))$, $p \cdot c_{ijk} = 0$, $p \cdot a_k = 0$, and $p \cdot b_i = -27\delta + 27 + 3\delta^2 > 0$, and for cone $H(III_2(k))$, $p \cdot c_{jik} = 3\delta(-6 + 5\delta) < 0$. Similarly, we also have that $H(III_1(i,j,k)) \cap H(III_2(i)) = H(III_1(i,j,k)) \cap H(III_2(j)) = \{0\}$.

The intersection $H(III_1(i,j,k)) \cap H(IV_1(i,j,k)) = cone(c_{ijk}, b_i)$ because $p = \left(3\delta - 2\delta^2\right) e_i + \delta^2 e_j + \left(9 + \delta^2 - 9\delta\right) e_k$ defines a separating hyperplane: for cone $H(III_1(i,j,k))$, $p \cdot b_i = 0$, $p \cdot c_{ijk} = 0$, $p \cdot a_k = 9 + \delta^2 - 9\delta > 0$ and for cone $H(IV_1(i,j,k))$, $p \cdot d_{ijk} = -3(1-\delta)\left(9 + \delta^2 - 9\delta\right) < 0$. Similarly, we also have that $H(III_1(i,j,k)) \cap H(IV_1(i,k,j)) = cone(b_i)$ and $H(III_1(i,j,k)) \cap H(IV_1(j,i,k)) = H(III_1(i,j,k)) \cap H(IV_1(j,k,i)) = \{0\}$.

The intersection $H(III_1(i,j,k)) \cap H(IV_2(i)) = cone(b_i)$ because $p = \left(3\delta - 2\delta^2\right) e_i + \delta^2 e_j + \left(9 + \delta^2 - 9\delta\right) e_k$ defines a separating hyperplane: see items 2 and 3 above. Similarly, $H(III_1(i,j,k)) \cap H(IV_2(j)) = H(III_1(i,j,k)) \cap H(IV_2(k)) = \{0\}$. We also have that $H(III_1(i,j,k)) \cap H(IV_3) = \{0\}$, because $p = \left(3\delta - 2\delta^2\right) e_i + \delta^2 e_j + \left(9 + \delta^2 - 9\delta\right) e_k$ also defines a separating hyperplane: see items 2 and 3 above.

The intersection $H(III_1(i,j,k)) \cap H(IV_4(k)) = cone(c_{ijk})$ because $p = 3(1-\delta) e_i + (2\delta - 3) e_j$ defines a separating hyperplane: for cone $H(III_1(i,j,k))$, $p \cdot c_{ijk} = 0$, $p \cdot a_k = 0$, and $p \cdot b_i = 9\delta - \delta^2 - 9 < 0$, and for cone $H(IV_4(k))$, $p \cdot c_{jik} = \delta(6 - 5\delta) > 0$, $p \cdot d_{ijk} = 0$, and $p \cdot d_{jik} = \delta(6 - 5\delta) > 0$. Similarly, we have that $H(III_1(i,j,k)) \cap IV_4(j) = H(III_1(i,j,k)) \cap H(IV_4(k)) = \{0\}$.

4. Now consider the case with $Q = III_2(k)$.

The intersection $H(III_2(k)) \cap H(IV_1(i,j,k)) = cone(c_{ijk})$, because $p = (3\delta - 2\delta^2) e_i + \delta^2 e_j + (9 + \delta^2 - 9\delta) e_k$ defines a separating hyperplane: for cone $H(III_2(k))$, $p \cdot c_{ijk} = 0$, $p \cdot c_{jik} = 3\delta^2(1 - \delta) > 0$, $p \cdot b_k = 9 + \delta^2 - 9\delta > 0$, and for cone $H(IV_1(i,j,k))$, see item 3 above. Similarly, $H(III_2(j)) \cap H(IV_1(i,j,k)) = H(III_2(i)) \cap H(IV_1(i,j,k)) = \{0\}$.

The intersection $H(III_2(k)) \cap H(IV_2(i)) = \{0\}$ and $H(III_2(k)) \cap H(IV_3) = \{0\}$ because $p = (3\delta - 2\delta^2) e_i + \delta^2 e_j + (9 + \delta^2 - 9\delta) e_k$ defines a separating hyperplane: see previous the paragraph for cone $H(III_2(k))$ and item 3 for cones $H(IV_2(i))$ and $H(IV_3)$.

The intersection $H(III_2(k)) \cap H(IV_4(k)) = cone(c_{ijk}, c_{jik})$ because $p = \delta e_i + \delta e_j + (6 - 5\delta) e_k$ defines a separating hyperplane: for cone $H(III_2(k))$, $p \cdot c_{ijk} = p \cdot c_{jik} = 0$, $p \cdot b_k = 6 - 5\delta > 0$, and for cone $H(IV_4(k))$, $p \cdot d_{ijk} = p \cdot d_{jik} = -3(1 - \delta)(6 - 5\delta) < 0$.

5. Now consider the case with $Q = IV_1(i,j,k)$.

The intersection $H(IV_1(i,j,k)) \cap H(IV_2(i)) = cone(b_i, d_{ijk})$ because $p = \delta^2 e_i + (12\delta - 2\delta^2 - 9) e_j + (9 + \delta^2 - 9\delta) e_k$ defines a separating hyperplane: for cone $H(IV_1(i,j,k))$, $p \cdot d_{ijk} = 0$, $p \cdot c_{ijk} = 3(1 - \delta)(9 + \delta^2 - 9\delta) > 0$, $p \cdot b_i = 0$, and for cone $H(IV_2(i))$, $p \cdot d_{ikj} = -3\delta(1 - \delta)(6 - \delta) < 0$.

The intersection $H(IV_1(i,j,k)) \cap H(IV_3) = cone(d_{ijk})$ because $p = (4\delta - 3) e_i + (4\delta - 3) e_j + (6 - 5\delta) e_k$ defines a separating hyperplane: for cone $H(IV_1(i,j,k))$, $p \cdot d_{ijk} = 0$, $p \cdot c_{ijk} = 3(\delta - 1)(5\delta - 6) > 0$, $p \cdot b_i = 3(\delta - 1)(\delta - 3) > 0$, and for cone $H(IV_3)$, $p \cdot d_{ikj} = 9\delta(\delta - 1) < 0$, $p \cdot d_{kji} = 18\delta(\delta - 1) < 0$, $p \cdot d_{jki} = 9\delta(\delta - 1) < 0$, $p \cdot d_{jik} = 0$, $p \cdot d_{kij} = 18\delta(\delta - 1) < 0$.

The intersection $H(IV_1(i,j,k)) \cap H(IV_4(k)) = cone(c_{ijk}, d_{ijk})$ because $p = 3(1 - \delta) e_i + (2\delta - 3) e_j$ defines a separating hyperplane: for cone $H(IV_1(i,j,k))$, $p \cdot d_{ijk} = 0$, $p \cdot c_{ijk} = 0$, $p \cdot b_i = 9\delta - 9 - \delta^2 < 0$, and for cone $H(IV_4(k))$, $p \cdot c_{jik} = p \cdot d_{jik} = \delta(6 - 5\delta) > 0$. Similarly, $H(IV_1(i,j,k)) \cap H(IV_4(j)) = H(IV_1(i,j,k)) \cap H(IV_4(i)) = \{0\}$.

6. Now consider the case with $Q = IV_2(i)$.

The intersection $H(IV_2(i)) \cap H(IV_3) = cone(d_{ijk}, d_{ikj})$, because $p = (7\delta - 6) e_i + (3 - 2\delta) e_j + (3 - 2\delta) e_k$ defines a separating hyperplane: for cone $H(IV_2(i))$, $p \cdot d_{ijk} = 0$, $p \cdot d_{ikj} = 0$, and $p \cdot b_i = 3(\delta - 1)(\delta - 6) > 0$, and for cone $H(IV_3)$, $p \cdot d_{kji} = 18\delta(\delta - 1) < 0$, $p \cdot \omega_{IV(j,ki)} = 18\delta(\delta - 1) < 0$, $p \cdot d_{jik} = 9\delta(\delta - 1) < 0$, $p \cdot d_{kij} = 9\delta(\delta - 1) < 0$.

The intersection $H\left(IV_2\left(i\right)\right) \cap H\left(IV_4\left(k\right)\right) = cone\left(d_{ijk}\right)$, because $p = \delta^2 e_i + \left(12\delta - 2\delta^2 - 9\right) e_j + \left(9 + \delta^2 - 9\delta\right) e_k$ defines a separating hyperplane: for cone $H\left(IV_2\left(i\right)\right)$ see item 5 above, and for cone $H\left(IV_4\left(k\right)\right)$, $p \cdot c_{ijk} = -3\left(\delta - 1\right)\left(9 + \delta^2 - 9\delta\right) > 0$, $p \cdot c_{jik} = 9\left(\delta - 1\right)\left(-3 + 2\delta\right) > 0$, $p \cdot d_{ijk} = 0$, $p \cdot d_{jik} = 3\delta\left(\delta - 1\right)\left(\delta - 3\right) > 0$. Similarly, $H\left(IV_2\left(i\right)\right) \cap H\left(IV_4\left(i\right)\right) = \{0\}$.

7. Finally, consider the case with $Q = IV_3$.

The intersection $H\left(IV_3\right) \cap H\left(IV_4\left(k\right)\right) = cone\left(d_{ijk}, d_{jik}\right)$, because $p = \left(4\delta - 3\right) e_i + \left(4\delta - 3\right) e_j + \left(6 - 5\delta\right) e_k$ defines a separating hyperplane: for cone $H\left(IV_3\right)$ see item 5 above, and for cone $H\left(IV_4\left(k\right)\right)$, $p \cdot c_{ijk} = 3\left(\delta - 1\right)\left(5\delta - 6\right) > 0$, $p \cdot c_{jik} = 3\left(\delta - 1\right)\left(5\delta - 6\right) > 0$, $p \cdot d_{ijk} = 0$, $p \cdot d_{jik} = 0$.

We have then proved for all possible pairs Q and Q' in $\mathbb{Q}$ with $Q' \neq Q$ that $H\left(Q\right) \cap H\left(Q'\right) = cone\left(Q \cap Q'\right)$. Q.E.D. LEMMA 3

We now finalize the proof of Theorem 1 showing that if $\omega \in H\left(Q\right) \cap H\left(Q'\right)$ then $\Phi_Q \cdot \omega = \Phi_{Q'} \cdot \omega$.

First, note that $\Phi_Q \cdot \nu = \Phi\left(\nu\right)$ where,

$$\begin{aligned}
\Phi\left(a_i\right) &= 0, \\
\Phi\left(b_i\right) &= -2e_i + e_j + e_k, \\
\Phi\left(c_{ijk}\right) &= -e_i + e_k, \\
\Phi\left(d_{ijk}\right) &= -e_i + e_k,
\end{aligned}$$

for all extremal rays $\nu \in Q$ and for all cases $Q \in \mathbb{Q} \subset \mathbb{V}$.

Now suppose that there exists $Q, Q' \in \mathbb{Q}$ with $Q \neq Q'$ such that $\omega \in H\left(Q\right) \cap H\left(Q'\right) = cone(Q \cap Q')$. By Lemma 3 then $\omega \in cone(Q \cap Q')$ and thus $\omega = \sum_{\nu \in Q \cap Q'} \alpha_\nu \nu$ where $\alpha_\nu \geq 0$. But the linearity of Φ_Q and $\Phi_{Q'}$ and the fact that $\Phi_Q \cdot \nu = \Phi\left(\nu\right)$ for the extremal points imply that $\Phi_Q \cdot \omega = \sum_{\nu \in Q \cap Q'} \alpha_\nu \Phi\left(\nu\right) = \Phi_{Q'} \cdot \omega$, for all $\omega \in H\left(Q\right) \cap H\left(Q'\right)$.

Q.E.D. THEOREM 1

PROOF OF THEOREM 2: We take the limit when $\delta \to 1$ of the expressions for Φ_Q and Ω_Q derived in the proof of Lemma 2 for all possible cases Q. Note that since $\omega_i \to X_i - \frac{V}{3}$ the results of cases I and $II(i)$ immediately follows. Also, note that all cases $III_1(i,j,k)$, $III_1(j,i,k)$, and $III_2(k)$ have the same limit coalitional bargaining value and that case $III\left(k\right)$ above is equivalent to $H\left(III_1(i,j,k)\right) \cup H\left(III_1(j,i,k)\right) \cup H\left(III_2(k)\right)$. Equivalently, case $III\left(k\right)$, which is associated with the polyhedral cone

$$H\left(III\left(k\right)\right) = \left\{\omega \in R^3 : \sum \omega \geq 0,\ \omega_i + 2\omega_j \leq 0,\ 2\omega_i + \omega_j \leq 0\right\},$$

satisfies $H(III(k)) = H(III_1(i,j,k)) \cup H(III_1(j,i,k)) \cup H(III_2(k))$, where

$$\begin{aligned} H(III_1(i,j,k)) &= \left\{\omega \in R^3 : \sum \omega \geq 0,\ \omega_i + 2\omega_j \leq 0,\ \omega_j \geq 0\right\}, \\ H(III_1(j,i,k)) &= \left\{\omega \in R^3 : \sum \omega \geq 0,\ 2\omega_i + \omega_j \leq 0,\ \omega_i \geq 0\right\}, \\ H(III_2(k)) &= \left\{\omega \in R^3 : \sum \omega \geq 0,\ \omega_i \leq 0,\ \omega_j \leq 0\right\}. \end{aligned}$$

We first show that $H(III_1(i,j,k)) \cup H(III_1(j,i,k)) \cup H(III_2(k)) \subset H(III(k))$. Suppose that $\omega \in H(III_1(i,j,k)) \cup H(III_1(j,i,k)) \cup H(III_2(k))$. If $\omega \in H(III_1(i,j,k))$ then $2\omega_i + 4\omega_j \leq 0$ and $-3\omega_j \leq 0$, which imply $2\omega_i + \omega_j \leq 0$ and thus $\omega \in H(III_2(k))$ (a similar argument holds for $III_1(j,i,k)$). Obviously, if $\omega \in H(III_2(k))$ then $\omega \in H(III(k))$. Now we show that $H(III(k)) \subset H(III_1(i,j,k)) \cup H(III_1(j,i,k)) \cup H(III_2(k))$. Suppose that $\omega \in H(III(k))$. Then we have either $\omega_j \leq 0$ or $\omega_j \geq 0$, and either $\omega_k \leq 0$ or $\omega_k \geq 0$. If either $\omega_j \geq 0$ or $\omega_k \geq 0$ holds then either ω belongs either to case $III_1(i,j,k)$ or to case $III_1(j,i,k)$. Otherwise, we must have both $\omega_j \leq 0$ and $\omega_k \leq 0$, which then imply that ω belongs to case $III_2(k)$.

Finally, all the different polyhedral cones of type IV collapse into the polyhedral cone $H(IV_3)$: $H(IV_1(i,j,k)) \cup H(IV_2(i)) \cup H(IV_4(k)) \subset H(IV_3)$. The unique limit case IV is simply determined by one linear inequality $\omega_1 + \omega_2 + \omega_3 \leq 0$, which is equivalent to $X_1 + X_2 + X_3 \leq V$. Q.E.D.

Proof of Theorem 3: Now take the limit when $\delta \to 1$ of the players' best response strategies derived in the proof of Lemma 2 for all possible cases Q. Cases I and $II(i)$ immediately follows.

Consider case $III(k)$ which by the results of Theorem 2 is equal to $III(k) = H(III_1(i,j,k)) \cup H(III_1(j,i,k)) \cup H(III_2(k))$. For all $\omega \in H(III(k))$ the strategies of players i and j are to make an offer to all players conditional on $\{i,k\}$ and $\{j,k\}$, respectively. If $\omega \in H(III_1(i,j,k))$ then the probability that player k choose an offer conditional on $\{j,k\}$ is equal to 1, and if $\omega \in H(III_1(i,j,k))$ then the probability that player k choose an offer conditional on $\{i,k\}$ is equal to 1. Also, whenever $\omega \in H(III_2(k))$ then the probability that player k choose an offer conditional on $\{i,k\}$ converges to,

$$\sigma_k(ik) = p = \lim_{\delta \to 1} \frac{(9\delta - 9)X_i + (9 - 6\delta)X_j - \delta^2 V}{(3X_i + 3X_j - 2\delta V)\delta} = \frac{X_j - \frac{V}{3}}{\left(X_j - \frac{V}{3}\right) + \left(X_i - \frac{V}{3}\right)},$$

and similarly for the probability that player k choose an offer conditional on $\{j,k\}$. Replacing the value of X_i as a function of $\mathbf{v}$ yields the desired result.

Finally, consider the case where $\omega \in H(IV_3)$. From the analysis of case IV_3 in the proof of Lemma 2 we have that the probability that the offer conditional on $\{i, j\}$ is chosen converges to

$$\lim_{\delta\to 1} \frac{1}{3}\left((1-p_i)+(1-p_j)\right) = \lim_{\delta\to 1} \frac{1}{3} \frac{-\delta^2 U + 4X_i\delta + 4X_j\delta - 5X_k\delta + 6X_k - 3X_j - 3X_i}{\delta\left(-\delta U + X_i + X_j + X_k\right)} = \frac{1}{3},$$

Proceeding similarly for the other pairs we conclude the proof. Q.E.D.

Proof of Proposition 4: Let $\phi(\mathbf{v})$ and $\eta(\mathbf{v})$ be respectively the coalitional bargaining value and the nucleolus of $\mathbf{v}$. If $\mathbf{v} \in I$ then $\phi(\mathbf{v}) = \eta(\mathbf{v})$. If $\mathbf{v} \in II(12)$, which corresponds to $V_{12} \geq \frac{V}{3}$, $2V_{13} + V_{12} \leq V$, and $2V_{23} + V_{12} \leq V$ (note that this implies that $V_{12} \geq V_{13}$ and $V_{12} \geq V_{23}$) then $\phi(\mathbf{v}) = (\frac{V+V_{12}}{4}, \frac{V+V_{12}}{4}, \frac{V-V_{12}}{2})$ by Theorem 2. But if $V_{13} \geq V_{23}$ and $\mathbf{v} \in II(12)$ then $2V_{23}+V_{12} \leq V$ holds and thus $\phi(\mathbf{v}) = \eta(\mathbf{v})$, and, by a similar argument, if $V_{23} \geq V_{13}$ then $\phi(\mathbf{v}) = \eta(\mathbf{v})$. Thus if $\mathbf{v} \in II$ then $\phi(\mathbf{v}) = \eta(\mathbf{v})$. Finally if $\mathbf{v} \in III(1)$, which corresponds to $2V_{13}+V_{12} \geq V$, $V_{12}+V_{13}+V_{23} \leq V$, and $2V_{12}+V_{13} \geq V$ (note that this also implies that $V_{12} \geq V_{13}$ and $V_{12} \geq V_{23}$) then $\phi(\mathbf{v}) = (\frac{V_{12}+V_{13}}{2}, \frac{V-V_{13}}{2}, \frac{V-V_{12}}{2})$ by Theorem 2. Suppose that $V_{13} \geq V_{23}$. Note that the first and second inequalities correspond to (A) $2V_{13} + V_{12} \geq V$ and (B) $V - V_{12} - V_{13} - V_{23} \geq 0$, and that adding both inequalities, $(A)+2*(B) \geq 0$, yields $V_{12}+2V_{23} \leq V$. Thus if $\mathbf{v} \in III(1)$ and $V_{13} \geq V_{23}$ then $V_{12} \geq V_{13} \geq V_{23}$ and $V_{12} + 2V_{23} \leq V$ and $V_{12} + 2V_{13} \geq V$ which implies that $\phi(\mathbf{v}) = \eta(\mathbf{v})$. A similar argument show that if $\mathbf{v} \in III(1)$ and $V_{23} \geq V_{13}$ then $\phi(\mathbf{v}) = \eta(\mathbf{v})$. Thus the coalitional bargaining value coincides with the nucleolus whenever $V_{12} + V_{13} + V_{23} \leq V$. In addition, if $V_{12} + V_{13} + V_{23} \leq V$ then the game has a non-empty core. Using Schmeidler's result that the nucleolus belong to the core, when the core is non-empty, yields that the coalitional bargaining value belong to the core if $V_{12} + V_{13} + V_{23} \leq V$.

If $\mathbf{v} \in IV$ then we have seen that the coalitional bargaining value coincides with the Shapley value which is equal to $Sh_i(\mathbf{v}) = \frac{1}{6}\left(2\left(V - V_{jk}\right) + V_{ij} + V_{ik}\right)$ and is distinct from the nucleolus. Example 2 shows that the CBV may not belong to the core. Q.E.D.

Proof of Corollary 5: (1) The Shapley value underestimates the equilibrium outcome of player k, because we have that $V \leq \sum_{l=1}^{3} \Phi_l$ and $Sh_k(\mathbf{v}) = \Phi_k + \frac{1}{3}\left(V - \sum_{l=1}^{3} \Phi_l\right) \leq \Phi_k = \phi_k$.

(2) Player i gets more than the Nash bargaining solution, but less then the Shapley value, and the opposite happens with players j and k. This is so because both players j and k get more than the Shapley value, $Sh_j(\mathbf{v}) = \Phi_j + \frac{1}{3}\left(V - \sum_{l=1}^{3} \Phi_l\right) \leq \Phi_j = \phi_j$ and, similarly, $\phi_k \geq Sh_k(\mathbf{v})$, which implies that $\phi_i \leq Sh_i(\mathbf{v})$, because $\sum_{l=1}^{3} Sh_l(\mathbf{v}) = \sum_{l=1}^{3} \phi_l = V$. Q.E.D.

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