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Dynamic Asset Allocation under Inflation

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Dynamic Asset Allocation under Inflation

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Abstract

This paper develops a simple framework for analyzing the asset allocation problem of a long-horizon investor when there is inflation and only nominal assets are available for trade. The investor's optimal investment strategy is given in simple closed form using the equivalent martingale method. The investor's hedge demands depend on both the investment horizon and the maturities of the bonds in which he invests. The optimal strategy can be decomposed into three components: first, a portfolio that mimics a hypothetical indexed bond with maturity equal to the investment horizon; secondly, the mean-variance tangency portfolio; thirdly, an additional investment in the hypothetical indexed bond to hedge against changes in the investment opportunity set. When short positions are precluded, the investor's optimal strategy consists of investments in cash, equity and a single nominal bond. When the model is calibrated to recent data on US interest rates and inflation, only high frequency movements in real interest rates are detected so that the optimal allocation between stock and bond is found to be relatively insensitive to the horizon. A longer calibration period reveals low frequency variation in real interest rates that induces more pronounced horizon effects. Reasons for the differences in the two calibration exercises are suggested.

An investor who has a long term horizon and can invest only in nominal bonds or stock faces a basic problem. Is it better to purchase a zero coupon bond corresponding to his horizon and bear the inflation risk, to follow a policy of rolling over short term bonds, or to adopt some quite different strategy? Despite the simplicity of this issue, there is still no well accepted framework for analyzing it because nominal long-term bonds have two important characteristics that cannot be represented adequately within the classical static, single-period, framework introduced by Markowitz (1959): first, the prices of bonds decline as interest rates rise so that, as Merton (1973) originally pointed out, long term bonds can provide a hedge against adverse shifts in the investor's future investment opportunity set; secondly, and somewhat weakening the hedging role of long term bonds, is the sensitivity of their prices to changes in expectations about future inflation. Therefore, a satisfactory counterpart to classical static portfolio theory that would enable us to address the problem faced by the hypothetical long term investor, must, at a minimum, yield simple closed form expressions for optimal portfolios for investors with a range of attitudes towards risk, and deal realistically with both the price and return characteristics of long term bonds and with inflation. In this paper we develop a simple model that satisfies these criteria. The investor's optimal portfolio is shown to be a sum of three components: first, a portfolio that mimics as closely as possible a hypothetical indexed bond with maturity equal to the investment horizon; secondly, the mean-variance tangency portfolio; thirdly, an additional investment in the hypothetical indexed bond mimicking portfolio to hedge against changes in the investment opportunity set.

The analysis of optimal portfolio strategies for long-lived investors starts with Latane and Tuttle (1967), Mossin (1968), Hakansson (1970), Merton (1969) and Samuelson (1969). These authors were primarily concerned with analyzing the investor's optimal allocation between stock and cash when returns are *i.i.d.* Merton (1971) was the first to consider the effect of a stochastic investment opportunity set, and to demonstrate that this creates a set of 'hedge' demands in addition to the standard myopic demand. However, further work on optimal strategies under stochastic investment opportunities languished until revived by empirical work demonstrating apparent asset return predictability in the 1990's. Brennan *et. al.* (1997) analyzed numerically the portfolio problem of a long lived investor who can invest in bonds, stock or cash, when there is stochastic variation in the interest rate, and the equity premium is predictable by the dividend yield. Kim and Omberg (1996), and Wachter (1999) have analyzed the optimal strategy of an investor when the interest rate is constant but the equity premium follows an Ornstein-Uhlenbeck process.[1] Sorensen (2000), Brennan and Xia (1999) and Omberg (1999) compute optimal dynamic strategies when the interest rate follows a Vasicek (1977) process[2] and risk premia are constant, while Liu (1999) is able to allow for both predictability in the Sharpe ratio and a stochastic interest rate. Barberis (2000), Kandel and Stambaugh (1999) and Xia (2000) have considered the implications of *uncertain* predictability of asset returns.

None of the above papers allows for stochastic inflation, takes account of borrowing and short sales constraints, or provides a rationale for the bond maturity choice. In this paper, we analyze the portfolio problem of a finite lived investor[3] who can invest in stock or nominal bonds, when the interest rate and the expected rate of inflation follow correlated Ornstein-Uhlenbeck processes and the risk premia are constant.[4]

The composition of the optimal bond portfolio is not determinate within the model when there are no constraints; only the optimal loadings on innovations in the estimated real interest rate, r, and expected rate of inflation, π, can be determined. Calibration of the model to data on US interest rates, stock returns, and inflation yields mixed results. When the calibration is made to monthly data on bond yields and inflation, strong mean reversion is found for the shadow real interest rate. This makes the optimal portfolio holdings relatively insensitive to the investment horizon beyond five years, and the gains from following a fully dynamic strategy relatively small except for high levels of risk aversion. When the calibration is made to annual interest rates and

[1] Wachter (1999) assumes that the innovation in the equity premium is perfectly correlated with the stock return.

[2] Brennan and Xia (1999) allow for a generalized Vasicek process.

[3] For completeness, we analyze both a utility of lifetime consumption model and a utility of final wealth model. These are essentially equivalent.

[4] While the assumption of constant risk premia may seem restrictive, Bossaerts and Hillion (1999) find no evidence of *out-of-sample* excess return predictability in fourteen countries using as potential predictors lagged excess returns, January dummies, bond and bill yields, dividend yields,etc.

inflation over a long period, much less mean reversion is found. With the lower mean reversion parameter, substantial horizon effects appear in the optimal portfolio strategies, and the gains from the optimal dynamic strategy become large. In both cases, as risk aversion increases, the investor holds less stock and the return on his bond portfolio tends to become less sensitive to innovations in the expected rate of inflation. In the limit, as risk aversion becomes infinite, the stock allocation goes to zero and the investor's dynamic strategy in bonds mimics as closely as possible the returns on an inflation indexed bond with maturity equal to the remaining investment horizon.

The foregoing results rely on the assumption that the investor is able to take unlimited short positions. When the investor is constrained to take only long positions, the optimal portfolio can be achieved with positions in only a single bond of the optimally chosen maturity, cash, and stock. The optimal bond maturity depends on both the investor's horizon and risk aversion. As in the unconstrained case, horizon effects are pronounced only for the annual data calibration. For both sets of calibrated parameters, the ratio of bonds to stock increases as the horizon increases, which is at odds with the popular view that long horizon investors should hold more stock. The bond-stock ratio also increases as risk aversion increases, which is consistent with the portfolio recommendations of popular financial advisors that have puzzled Canner *et al.* (1997). The maturity of the optimal bond decreases as risk aversion increases. For both sets of calibrated parameters it is never optimal to hold cash - a pure bond holding dominates a mixture of cash and bonds.

There are at least two possible explanations for the differences between the monthly and the annual calibrations. The first possibility is that the expected rate of inflation series estimated using monthly data on bond yields and price index changes is too smooth because non-price signals about the expected rate of inflation are ignored; this would cause the high frequency changes in the true market assessment of the expected rate of inflation, that are reflected in bond yields, to be impounded in our estimates of the real interest rate. The second possibility is that, whereas we use a one factor model to describe the dynamics of the real interest rate, these may be better represented by a two factor model in which one factor has high frequency[5] and the other low frequency; in this case the monthly calibration may be picking up the high frequency factor while the annual calibration may place more emphasis on the low frequency component.

In the paper that is closest to this one, Campbell and Viceira (1999)(CV) develop an approximately optimal portfolio strategy for an infinitely lived investor with recursive utility, in a discrete time setting in which the real interest rate and the expected rate of inflation follow similar stochastic processes. The major differences between their paper and this one are that: we develop closed form expressions for the investor's optimal policy whereas the CV formulation relies on linear approximation and numerical analysis; unlike CV, we are able to analyze the problem faced by a finite horizon investor, and are therefore able to give explicit consideration to the effect of the horizon on both the optimal bond-stock mix and the maturity of the optimal bond portfolio. Finally, in considering the problem when the investor is subject to short sales constraints, CV take the maturity of the bond as given, whereas we allow for the optimal choice of bond maturity and are able to consider the effect of risk aversion and horizon on the optimal maturity.

We present the basic model of stochastic real interest rates, inflation and expected stock returns in Section I. The optimal portfolio problem is derived in Section II. We calibrate the model to the U.S. postwar nominal interest rate, inflation and stock return data in Section III. Some representative calculations and discussions are offered in Section IV. We summarize the results and conclude the paper in Section V.

I. The Economy

In order to analyze the decision problem of an investor with a long run horizon we shall first describe his opportunity set. We assume that the investor can invest in a nominal instantaneous

[5]Perhaps because monetary policy has a short run impact on the real interest rate.

risk free asset, a stock and in nominal bonds with different maturities. The real returns on the nominal bonds are risky, both because the rate of inflation is stochastic and because the (shadow) real interest rate is stochastic.[6] We first describe the stochastic process for inflation and the price level, and then the (real) pricing kernel for the economy and the dynamics of the shadow real riskless rate. These assumptions are sufficient for us to derive the prices and stochastic processes for the stock and bonds in both real and nominal terms.

A. Inflation, the Pricing Kernel, and Security Prices

We assume that the price level Π follows a diffusion process:

$$\frac{d\Pi}{\Pi} = \pi dt + \sigma_\Pi dz_\Pi, \tag{1}$$

This implies that the rate of inflation is locally stochastic. The volatility of the inflation rate, σ_Π, is a constant, but the instantaneous expected rate of inflation, or proportional drift of the price level, π, follows an Ornstein-Uhlenbeck process:

$$d\pi = \alpha(\bar{\pi} - \pi)dt + \sigma_\pi dz_\pi, \tag{2}$$

We capture the investment opportunities of the investor in a parsimonious manner by describing the characteristics of the pricing kernel of the economy which determines the real and nominal expected returns on all securities. The (real) pricing kernel, M_t, is assumed to follow

[6] By the shadow real interest rate we mean the instantaneous real return that would prevail for an asset whose instantaneous real return were non-stochastic, if such an asset were to exist. In our model there is no instantaneous real riskless asset. However, as we shall show below, when the rate of inflation is spanned by the nominal returns on assets, as will be the case if the expected rate of inflation is unobservable and must be inferred from the price level realization, the model allows for the construction of a portfolio of nominal assets whose instantaneous real return is riskless.

the stochastic process:

$$\frac{dM}{M} = -rdt + \phi_S dz_S + \phi_r dz_r + \phi_\pi dz_\pi + \phi_u dz_u = -rdt + \phi' dz + \phi_u dz_u \tag{3}$$

where $\phi = [\phi_S, \phi_r, \phi_\pi]'$, $dz = [dz_S, dz_r, dz_\pi]'$, ϕ_i $(i = S, r, \pi, u)$ are constants that determine the prices of risk, λ_S, λ_r, λ_π and λ_u, that are associated with the four innovations, dz_S, dz_r, dz_π, and dz_u. These prices of risk are constant because the ϕ's are constant. dz_u, which is defined in equation (7) below, is proportional to the component of the inflation rate, $\frac{d\Pi}{\Pi}$, that is orthogonal to dz and therefore to the nominal returns on all assets: dz_u is the increment to the Brownian motion that drives the component of inflation that is not spanned by the asset returns and therefore cannot be hedged. We denote the drift of the pricing kernel by $-r$ because it is well known that, in an economy with a fully indexed riskless asset, the instantaneous (real) riskless interest rate is equal to the (negative of) the drift of the pricing kernel.

Following Vasicek (1977), we assume that r follows the Ornstein-Uhlenbeck process:

$$dr = \kappa(\bar{r} - r)dt + \sigma_r dz_r, \tag{4}$$

where $\bar{r}$, the long run mean, σ_r, the volatility, and κ, the mean reversion coefficient, are known constants. If an instantaneously riskless *real* asset existed, then its instantaneous real rate of return would r.

However, we assume that in this economy no instantaneously riskless real asset exists,[7] and that the investor is able to invest only in a single stock index and in nominal bonds with different maturities. The nominal stock price is assumed to follow a Geometric Brownian motion,

$$\frac{dS}{S} = \mu^N{}_S dt + \sigma_S dz_S, \tag{5}$$

[7] Even countries such as Canada, the U.S. and the U.K. which have inflation indexed bonds, have them for only a few (long) maturities.

and the nominal bond price process is similarly defined.

Since the Brownian increments in equation (3), dz and dz_u, are orthogonal, the pricing kernel relative can be rewritten as the product of two independent stochastic integrals:

$$\begin{aligned} M_s/M_t &= \exp\left\{\int_t^s \left(-r(u) - \frac{1}{2}\phi'\rho\phi\right) du + \int_t^s \phi' dz\right\} \exp\left\{\int_t^s \left(-\frac{1}{2}\phi_u^2\right) du + \int_t^s \phi_u dz_u\right\} \\ &\equiv \zeta_1(t,s)\zeta_2(t,s), \end{aligned} \quad (6)$$

where $\rho \equiv \begin{bmatrix} 1 & \rho_{Sr} & \rho_{S\pi} \\ \rho_{Sr} & 1 & \rho_{r\pi} \\ \rho_{S\pi} & \rho_{r\pi} & 1 \end{bmatrix}$ is the correlation matrix of dz_S, dz_r and dz_π. Note that $\zeta_1(t,s)$ and $\zeta_2(t,s)$ are orthogonal to each other.

In general, the realized rate of inflation given by equation (1) will not not perfectly correlated with the change in the expected rate of inflation given by equation (2). However, it can be shown that if the expected rate of inflation is not observable but must be inferred from observation of the price level itself,[8] then (the investor's assessment of) the change in the expected rate of inflation will be perfectly correlated with the realized rate of inflation: for reasons that will be apparent below we call this the 'complete markets case'. In the general case the innovation in the rate of inflation can be written as a linear function of the innovations dz_S, dz_r and dz_π and the projection residual, $\xi_u dz_u$:

$$\frac{d\Pi}{\Pi} = \pi dt + \sigma_\Pi dz_\Pi = \pi dt + \xi_S dz_S + \xi_r dz_r + \xi_\pi dz_\pi + \xi_u dz_u = \pi dt + \xi' dz + \xi_u dz_u \quad (7)$$

where $\xi \equiv [\xi_S, \xi_r, \xi_\pi]'$, and dz is defined above. In the case $\xi_S = \xi_r = \xi_u = 0$, $d\Pi/\Pi$ is perfectly correlated with $d\pi/\pi$ and the market is complete.

Equation (7) implies that the price level relative, Π_s/Π_t $(s > t)$, can also be written as the

[8] And not from any other information.

product of two independent stochastic integrals:

$$\begin{aligned} \Pi_s/\Pi_t &= \exp\left\{\int_t^s \left(\pi(u) - \frac{1}{2}\xi'\rho\xi\right) du + \int_t^s \xi' dz\right\} \exp\left\{\int_t^s \left(-\frac{1}{2}\xi_u^2\right) du + \int_t^s \xi_u dz_u\right\} \\ &\equiv \eta_1(t,s)\eta_2(t,s), \end{aligned} \tag{8}$$

where $\eta_1(t,s)$ and $\eta_2(t,s)$ are orthogonal so that $\sigma_\Pi^2 = \xi'\rho\xi + \xi_u^2$. $\eta_1(t,s)$ is the component of the price level change that can be hedged by investing in the available securities while $\eta_2(t,s)$ is the unhedgeable component.

The nominal and real prices of nominal bonds are determined by the pricing kernel as follows. Let $P(t,T)$ and $p(t,T)$ denote the nominal and real prices at time t of a nominal bond which matures at time T with a nominal payoff of \$1. Then, it follows from the definition of the pricing kernel that $p(t,T)$, the real price of the bond, satisfies:

$$p(t,T)M_t = \mathrm{E}_t\left[p(T,T)M_T\right] = \mathrm{E}_t\left[\frac{1}{\Pi_T}\frac{M_T}{M_t}\right], \tag{9}$$

since $p(T,T) = \frac{1}{\Pi_T}$. Similarly, since $P(t,T) = p(t,T)\Pi_t$, the nominal price of the bond is given by:

$$P(t,T) = \mathrm{E}_t\left[\frac{M_T/M_t}{\Pi_T/\Pi_t}\right] = \mathrm{E}_t\left[\frac{\zeta_1(t,T)}{\eta_1(t,T)}\right]\mathrm{E}_t\left[\frac{\zeta_2(t,T)}{\eta_2(t,T)}\right]. \tag{10}$$

Taking the expectation in equation (10), it is shown in Appendix A that $P(t,T)$, the nominal price at time t of a nominal bond that pays \$1 at time T, is an exponential affine function of r_t, the (shadow) real interest rate, and π_t, the investor's assessment of the current expected rate of inflation:

$$P(t,T) = \exp\left\{A(t,T) - B(t,T)r_t - C(t,T)\pi_t\right\} \tag{11}$$

where $A(t,T)$, $B(t,T)$ and $C(t,T)$ are time dependent constants, expressions for which are given in Appendix A.

Using Ito's Lemma and the expressions for $A(t,T)$, $B(t,T)$, and $C(t,T)$ given in Appendix A, the stochastic process for the bond price becomes:

$$\begin{aligned} \frac{dP}{P} &= [r + \pi - B\sigma_r\lambda_r - C\sigma_\pi\lambda_\pi - \xi_S\lambda_S - \xi_r\lambda_r - \xi_\pi\lambda_\pi - \xi_u\lambda_u]\, dt \\ &- B\sigma_r dz_r - C\sigma_\pi dz_\pi. \end{aligned} \tag{12}$$

where the λ's are the market prices of the risks associated with the stock return and innovations in the real interest rate, and expected and the unhedgeable component of unexpected inflation, expressions for which are given in equations (A16 - A19).

The instantaneous nominal riskfree interest rate, R_f, is obtained by taking the limit of the return on the nominal bond in equation (12) as $(t - T) \to 0$:

$$R_f = r + \pi - \xi_S\lambda_S - \xi_r\lambda_r - \xi_\pi\lambda_\pi - \xi_u\lambda_u \tag{13}$$

Note that $-\xi_S\lambda_S - \xi_r\lambda_r - \xi_\pi\lambda_\pi - \xi_u\lambda_u$ is the risk premium for the nominal instantaneous riskfree asset. The Fisher equation does not hold in this economy unless the market prices of stock return risk, real interest rate risk, expected and unexpected inflation risk, λ_S, λ_r, λ_π, and λ_u are all zero, so that the nominal risk free asset has a zero risk premium. Using the definition of the nominal risk free interest rate in equation (13), the nominal return on the nominal bond in equation (12) can also be written as:

$$\frac{dP}{P} = [R_f - B\sigma_r\lambda_r - C\sigma_\pi\lambda_\pi]\, dt - B\sigma_r dz_r - C\sigma_\pi dz_\pi, \tag{14}$$

which shows that the nominal risk premium on a bond depends only on its exposure to innovations in the real interest rate and expected rate of inflation.

For completeness, we calculate the price of a hypothetical real bond in this economy (though we do not assume that such a bond exists). This is the price at which such a bond would trade *if* it existed. Denote the (real) price at time t of a real bond that pays (real) \$1 at T by $p^*(t,T)$. Then, as shown in Appendix A,

$$p^*(t,T) = \mathrm{E}_t\left[M_T/M_t\right] = \exp\left\{\hat{A}(t,T) - B(t,T)r_t\right\} \tag{15}$$

where $\hat{A}(t,T)$ is given by equation (A22) in Appendix A.

B. Real Returns

We complete our description of the economy by deriving expressions for the real returns on the stock and the nominal bonds. Let $s \equiv S/\Pi$ denote the real stock price. Then, using Ito's Lemma, the real stock return is given by:

$$\frac{ds}{s} = \left[{\mu^N}_S - \pi + \xi'\rho\xi + \xi_u^2 - \sigma_S e_1'\rho\xi\right]dt + (\sigma_S - \xi_S)dz_S - \xi_r dz_r - \xi_\pi dz_\pi - \xi_u dz_u \tag{16}$$

where $e_1 = [1, 0, 0]'$. Similarly, the real return on a nominal bond is:

$$\begin{aligned}\frac{dp(t,T)}{p(t,T)} &= [r + \xi_S(\phi_S + \phi_r\rho_{Sr} + \phi_\pi\rho_{S\pi}) + (B\sigma_r + \xi_r)(\phi_S\rho_{Sr} + \phi_r + \phi_\pi\rho_{r\pi}) \\ &+ (C\sigma_\pi + \xi_\pi)(\phi_S\rho_{S\pi} + \phi_r\rho_{r\pi} + \phi_\pi) + \xi_u\phi_u]\,dt \\ &- \xi_S dz_S - (B\sigma_r + \xi_r)dz_r - (C\sigma_\pi + \xi_\pi)dz_\pi - \xi_u dz_u,\end{aligned} \tag{17}$$

where $B \equiv B(t,T)$ and $C \equiv C(t,T)$ are given by equations (A5) and (A6).

The current expected rate of inflation, π, does not affect the expected real return on the bonds given by the drift term in equation (17). However, the innovation in the expected rate of inflation, represented by the Brownian increment, $\sigma_\pi dz_\pi$, affects the instantaneous real return on the bond through the term $-C\sigma_\pi dz_\pi$ reflecting the effect of the change in the expected rate of inflation on the *nominal* price of the bond. The real return process of the nominal bond is also affected by the term, $-(\xi' dz' + \xi_u dz_u)$, which represents the direct reduction in the real value of the bond caused by the unexpected increase in the price level.

The following theorem, which summarizes the above discussion, relates the nominal and real returns on bonds and stock to the coefficients of the stochastic process of the pricing kernel, (3). We defer the proof of all theorems to Appendix B.

Theorem 1 *Define the unit market price of risk vector, λ, for dz_S, dz_r and dz_π:*

$$\lambda \equiv \begin{bmatrix} \lambda_S \\ \lambda_r \\ \lambda_\pi \end{bmatrix} \equiv \begin{bmatrix} (\xi_S + \xi_r \rho_{Sr} + \xi_\pi) - (\phi_S + \phi_r \rho_{Sr} + \phi_\pi) \\ (\xi_S \rho_{Sr} + \xi_r + \xi_\pi \rho_{r\pi}) - (\phi_S \rho_{Sr} + \phi_r + \phi_\pi \rho_{r\pi}) \\ (\xi_S \rho_{S\pi} + \xi_r \rho_{r\pi} + \xi_\pi) - (\phi_S \rho_{S\pi} + \phi_r \rho_{r\pi} + \phi_\pi) \end{bmatrix} = \rho(\xi - \phi), \tag{18}$$

where ρ, ξ, and ϕ are vectors defined earlier. Also define the market price of risk associated with dz_u as $\lambda_u \equiv \xi_u - \phi_u$.

Then, (1) The nominal risk premium on the stock, ${\mu^N}_S - R_f$, is given by

$${\mu^N}_S - R_f = \lambda_S \sigma_S, \tag{19}$$

and the real risk premium of the (nominal) stock is given by

$$\begin{aligned} \mu_S - r \;\; &\equiv \;\; (\xi_S - \sigma_S)(\phi_S + \phi_r \rho_{Sr} + \phi_\pi \rho_{S\pi}) + \xi_r(\phi_S \rho_{Sr} + \phi_r + \phi_\pi \rho_{r\pi}) \\ &+ \;\; \xi_\Pi(\phi_S \rho_{S\pi} + \phi_r \rho_{r\pi} + \phi_\pi \rho_{S\pi}) + \xi_u \phi_u. \end{aligned} \tag{20}$$

(2) The nominal risk premium on the nominal bond with maturity T, $\mu^N(t,T) - R_f$, is

$$\mu^N(t,T) - R_f = -B\sigma_r\lambda_r - C\sigma_\pi\lambda_\pi. \tag{21}$$

and the real risk premium of the nominal bond with maturity T is

$$\begin{aligned} \mu(t,T) - r \equiv & \; \xi_S(\phi_S + \phi_r\rho_{Sr} + \phi_\pi\rho_{S\pi}) + (B\sigma_r + \xi_r)(\phi_S\rho_{Sr} + \phi_r + \phi_\pi\rho_{r\pi}) \\ + & \; (C\sigma_\pi + \xi_\pi)(\phi_S\rho_{S\pi} + \phi_r\rho_{r\pi} + \phi_\pi) + \xi_u\phi_u, \end{aligned} \tag{22}$$

where $B \equiv B(t,T)$ and $C \equiv C(t,T)$.

Note that the real risk premium on the stock is equal to the covariance between the real return of the (nominal) stock, ds/s, and the real pricing kernel: $\mu_s - r = cov\left(\frac{ds}{s}, -\frac{dM}{M}\right)$. Similarly, for the real return of (nominal) bonds: $\mu_B(t,T) - r = cov\left(\frac{dp}{p}, -\frac{dM}{M}\right)$.

Theorem 1 implies a simple relation between the unit market price of risk vector, λ, and the expected excess nominal rates of return on the stock and any two nominal bonds. Let Λ be the the vector of nominal risk premiums for the stock and two nominal bonds with maturities T_1 and T_2, and define the factor loadings matrix of the three securities as:

$$\sigma \equiv \begin{bmatrix} \sigma_S & 0 & 0 \\ 0 & -B(t,T_1)\sigma_r & -C(t,T_1)\sigma_\pi \\ 0 & -B(t,T_2)\sigma_r & -C(t,T_2)\sigma_\pi \end{bmatrix}, \tag{23}$$

Then it follows directly from equations (19) and (21) that $\Lambda = \sigma\lambda$. Thus, the unit market price of risk vector is is related to the risk premium vector by:

$$\lambda = \sigma^{-1}\Lambda, \tag{24}$$

and the coefficients of the real pricing kernel are related to the risk premium vector by:

$$\phi = \xi - \rho^{-1}\lambda. \tag{25}$$

Note that ϕ_u cannot be determined from tradable asset prices because no nominal security returns load on dz_u, and all real returns have the same loading,ξ_u.

II. Optimal Portfolio Choice

Having described the securities that are available to the investor, we now turn to the issue of optimal portfolio strategies for long-lived investors. We shall consider two classical cases. In the first, the investor is assumed to be concerned with maximizing the expected utility of wealth on some fixed horizon date, T. This problem has the merit, both of simplicity, for it admits a closed form solution, and of clarifying the role of the horizon, for with this simple objective there is no ambiguity about the duration of the consumption stream that is being financed. The problem can be thought of as corresponding to that faced by an individual who has set aside predetermined savings for retirement and wishes to maximize the expected utility of wealth on his retirement date;[9] we are simplifying the full problem by ignoring the optimal investment and consumption plan during retirement. The second case that we consider is that of an investor who is concerned with maximizing the expected value of a time-additive utility function defined over lifetime consumption. This problem, which is only slightly more complicated, corresponds to the

[9]We are simplifying by ignoring labor income and the consumption-investment decision.

consumption-portfolio choice problem of an individual who is retired and faces a known date of death with no bequest motives.

There are four potential sources of uncertainty in the model economy that we have described: real interest rate risk represented by the innovations in r as shown in Equation (4); inflation risk due to unanticipated changes in the price level as shown in Equation (7); unanticipated changes in the expected rate of inflation as shown in Equation (2); and finally, the unanticipated stock return shown in Equation (5).

If $\phi_u = \xi_u = 0$ so that the change in the price level is an exact linear function of dz_S, dz_r, and dz_π, one dimension of risk faced by the investor is eliminated.[10] Then the market will be complete if there are at least four securities whose instantaneous variance-covariance matrix has rank three.[11] It may be verified that the variance-covariance matrix of real returns on cash, stock and any two finite maturity bonds with different maturities has rank three, so the market is complete whenever $\phi_u = \xi_u = 0$. It is well known that in this complete market setting, the investor's optimal portfolio problem may be solved using the martingale pricing approach of Cox and Huang (1989).

When the market is incomplete, ϕ_u cannot be derived from the observable security prices, but any specific value of ϕ_u is associated with a unique pricing kernel. Pagés (1987) shows that if a wealth (consumption) plan is feasible, i.e., financed by a trading strategy using the nominal securities, the expectation under any pricing kernel can serve the same purpose. Therefore, we can map the original dynamic portfolio choice problem into the static variational problem of Cox and Huang (1989) with only feasible wealth (consumption) plans under an arbitrary given value of ϕ_u.[12] We first solve the optimal terminal wealth and consumption allocation under the static budget constraint with a given ϕ_u, then derive the optimal strategy financing the optimal

[10] Formally, the *martingale multiplicity* of the investor's information structure is equal to two (Duffie and Huang (1985)). This condition will be satisfied if, for example, the only information that is available about expected inflation is derived from the historical inflation series.

[11] See Cox and Huang (1989) for a formal analysis

[12] See Pagés (1987) and Karatzas et. al. (1991) for a rigorous proof. In particular, see Pagés (1987) Proposition 2.15.

wealth and consumption plan, and verify that the optimal terminal wealth and consumption are financed by tradeable strategies. Therefore, the solution to the static variational problem is also the solution to the original portfolio choice problem.

A. State Contingent Wealth and Consumption

Suppose that $\xi_u \neq 0$ so that the market is incomplete since it is no longer possible to construct a portfolio whose return is riskless in real terms. The component of inflation that is orthogonal to asset returns and cannot be hedged ($\xi_u dz_u$) does not affect portfolio choice for a given level of wealth and price level. Moreover, as stated in the following Lemma which follows from the homogeneity of both the utility function and the inflation process, the optimal proportional allocation is independent of real wealth and the price level. Consequently, the market incompleteness caused by non-hedgeable inflation has no effect on the optimal proportional portfolio allocation.

Lemma 1 *For an investor with isoelastic utility, the optimal proportional portfolio allocation is independent of real wealth, $w_t \equiv \frac{W_t}{\Pi_t}$, and the price level, Π_t.*

The Lemma allows us to characterize the investor's problem in the incomplete market setting via the extended Cox-Huang method in incomplete market, which is first developed by Pagès (1987) and then by Karatzas et. al. (1991). In order to represent the market incompleteness caused by non-hedgeable inflation, fix a probability space $(\mathcal{O}, \mathcal{F}, \mathcal{P})$, where $\mathcal{O} = \mathcal{O}_1 \times \mathcal{O}_2$. The vector of standard Brownian motions $dz \equiv [dz_S, dz_r, dz_\pi]'$ is defined on $\mathcal{O}_1$, while $\xi_u dz_u$, the component of realized inflation that is orthogonal to all security returns and therefore non-hedgeable, is defined on $\mathcal{O}_2$. The standard filtration $\mathcal{F}_1$ is generated by $\{dz_s : 0 \leq s \leq t\}$ and $\mathbf{F}_2$ by dz_u, and $\mathbf{F} \equiv \mathbf{F}_1 \times \mathbf{F}_2$. It follows from the Lemma that an optimal trading strategy is a process x which specifies the proportion of wealth invested in each security and is adapted to $\mathbf{F}_1$.

Consider first the problem of an investor with an iso-elastic utility function who is concerned with maximizing the expected utility of real wealth at time T. Pagés (1987) and Karatzas et. al. (1991) show that the investor's optimal portfolio choice problem can be mapped into the following static variational problem:

$$\max_{W(\tau):t\le\tau\le T} E_t\left\{\frac{(W_T/\Pi_T)^{1-\gamma}}{1-\gamma}\right\}, \tag{26}$$

$$\text{subject to} \quad (1)\ \mathrm{E}_t\left[\frac{M_T}{M_t}(W_T/\Pi_T)\right] = W_t/\Pi_t \equiv w_t, \tag{27}$$

and (2) W is financed by a feasible trading strategy with initial investment W_0

where x is a (3×1) vector of proportional wealth allocations to the stock, and two nominal bonds, $1 - x'\mathbf{i}$ denotes the proportional wealth invested in (*nominal*) cash, and equation (27) is the static budget constraint. The following theorem solves this static variational problem.

Theorem 2 *Optimal Terminal Wealth Allocation*

For an investor who is concerned with maximizing the expected value of an isoelastic utility function defined over terminal wealth, in the incomplete markets setting in which $\xi_u \neq 0$*:*

(i) The optimal terminal real wealth allocation, $w_T^* \equiv W^*{}_T(M_T,\Pi_T)/\Pi_T$*, is*

$$w_T^* \;=\; w_t\eta_2^{-1}(t,T)\,(\zeta_1(t,T))^{-\frac{1}{\gamma}}\,F_1(t,T)^{-1}F_2(t,T)^{-1} \tag{28}$$

where $F_1(t,T)$ *and* $F_2(t,T)$ *are given by*

$$F_1(t,T) \;=\; \mathrm{E}_t\left[\zeta_1(t,T)^{1-\frac{1}{\gamma}}\right] \tag{29}$$

$$F_2(t,T) \;=\; \mathrm{E}_t\left[(\zeta_2(t,T)/\eta_2(t,T))\right] \tag{30}$$

(ii) The investor's expected utility at time t under the optimal policy, $J(W_t, r_t, \Pi_t, t)$, is separable in real wealth, W_t/Π_t, and can be written as:

$$J(W, r, \Pi, t) \equiv E_t \left\{ \frac{(W_T^*/\Pi_T)^{1-\gamma}}{1-\gamma} \right\} = \left(\frac{(W_t/\Pi_t)^{1-\gamma}}{1-\gamma} \right) \psi_1(r, t, T) \tag{31}$$

where W_T^* is the wealth at period T under the optimal policy, and

(iii) $\psi_1(r, t, T)$ represents the contribution to the investor's expected utility of the remaining investment opportunities up to the horizon:

$$\psi_1(r, t, T) = \exp^{(1-\gamma)[B(t,T)r_t + a_1(t,T)]}, \tag{32}$$

where

$$\begin{aligned} a_0(t, T) &= \frac{\phi' \rho \phi (T-t)}{2\gamma} + \left[\bar{r} - \frac{(1-\gamma)\sigma_r \phi' \rho e_2}{\gamma \kappa} \right] [(T-t) - B(t, T)] \\ &+ \frac{(1-\gamma)\sigma_r^2}{4\gamma\kappa^3} \left[2\kappa(T-t) - 3 + 4e^{\kappa(t-T)} - e^{2\kappa(t-T)} \right] \qquad (33) \\ a_1(t, T) &= a_0(t, T) + \left[\phi_u \xi_u - \frac{\gamma}{2} \xi_u^2 \right] (T-t), \qquad (34) \end{aligned}$$

$B(t, T)$ is defined in equation (A5), and T is the investor's horizon.

The indirect utility function, (31), can also be expressed in terms of the investor's real wealth, w, and $p^*(r; t, T)$, the real price of a hypothetical real discount bond with maturity T given in equation (15):[13]

$$J(w,r,t) = \frac{1}{1-\gamma}\left(\frac{w}{p^*(r;t,T)}\right)^{1-\gamma}\hat{\psi}(t,T), \tag{35}$$

where $\hat{\psi}(t,T) = \exp\left\{(1-\gamma)\left(a_0(t,T) + \hat{A}(t,T)\right)\right\}$.

The complete market problem ($\xi_u = 0$) is a special case of Theorem 2: $\xi_u = 0$ implies that $\eta_2(t,T) = F_2(t,T) = 1$. Then the optimal real wealth allocation and expected utility under the optimal policy are:

$$w^*{}_T(M_T,\Pi_T) = w_t\left(\zeta_1(t,T)\right)^{-\frac{1}{\gamma}} F_1(t,T)^{-1}, \tag{36}$$

$$J(r,W,\Pi,t) = \left(\frac{w_t^{1-\gamma}}{1-\gamma}\right)\exp^{(1-\gamma)[B(t,T)r_t + a_0(t,T)]}. \tag{37}$$

The effect of market incompleteness on the real terminal wealth allocation, w_T^*, in equation (28) is reflected in the terms $\eta_2^{-1}(t,T)$ and $F_2(t,T)$: the former represents the cost of the unhedgeable inflation realization on terminal wealth, while the latter corresponds to the expected reward for exposure to unhedgeable inflation risk. The effect of market incompleteness on investor welfare may be assessed by comparing the investor's certainty equivalent wealth[14] in the incomplete market to that in the complete market using the expressions for expected utility given by (37) and (31). The ratio of the certainty equivalents in the complete market to the certainty equivalent in the incomplete market, $\frac{CEW_C}{CEW_I}$, is equal to $\exp^{\left[\frac{\gamma}{2}\xi_u^2 - \phi_u\xi_u\right](T-t)}$. Thus, the investor is made better off by the unhedgeable inflation component dz_u only if $\gamma < \frac{2\phi_u}{\xi_u}$; in other words, only if he is not too risk averse given the risk premium and risk associated with the unhedgeable inflation.

[13] Sorensen (1999) obtains a similar result in a model without inflation.

[14] We define the certainty equivalent wealth as the sure amount at the horizon that the investor would exchange for $1 of current wealth and the investment opportunities up to the horizon.

A similar theorem is presented for the interim consumption case, in which an investor is concerned with maximizing the expected value of a time-additive iso-elastic utility function defined over lifetime consumption:

$$\max_{C(s):t\leq s\leq T} E_t \left\{ \int_t^T \frac{\left(C(s)/\Pi_s\right)^{1-\gamma}}{1-\gamma} ds \right\}, \tag{38}$$

$$\text{subject to} \qquad E_t \left[\int_t^T \frac{M_s}{M_t} \frac{C(s)}{\Pi_s} ds \right] = W_t/\Pi_t \equiv w_t, \tag{39}$$

$$\text{and} \qquad (2) \;\; C \text{ is financed by a feasible trading strategy with initial investment } W_0$$

where $C(\tau)$ is the investor's (nominal) consumption flow at time $\tau : \tau \in [t, T]$. The optimal consumption plan and indirect utility function are characterized in the following theorem:

Theorem 3 *Optimal Allocation of Lifetime Consumption*

For an investor who is concerned with maximizing the expected value of an isoelastic utility function defined over lifetime consumption in the incomplete markets setting in which $\xi_u \neq 0$:

(i)The real optimal consumption program, $c^*(s) \equiv C^*(s)/\Pi_s$, of the investor is:

$$c^*(s) = \eta_2^{-1}(t,s) \left(\zeta_1(t,s)\right)^{-\frac{1}{\gamma}} Q_1^{-1}(t,s) w_t \tag{40}$$

where w_t is real wealth at time t, $Q_1(t,T)$ is constant chosen to satisfy the budget constraint:

$$Q_1(t) \;\; = \;\; \int_t^T F_1(t,s)F_2(t,s)ds = \int_t^T q(t,s) \exp\left\{ \frac{1-\gamma}{\gamma}[B(t,s)r_t + a_1(t,s)] \right\} ds, \tag{41}$$

and $q(t,s) = \exp\left\{ \left[\frac{(3-\gamma)\xi_2^2}{2} - \frac{\phi_u \xi_u}{\gamma} \right] (s-t) \right\}$.

(ii) The indirect utility function $J(W_t, r_t, \Pi_t, t)$ is separable in W_t/Π_t and can be written as:

$$J(r, W, \Pi, t) \equiv E_t \left\{ \int_t^T \frac{(C^*(s)/\Pi_s)^{1-\gamma}}{1-\gamma} ds \right\} = \left(\frac{(W_t/\Pi_t)^{1-\gamma}}{1-\gamma} \right) \psi_2(r, t, T), \tag{42}$$

where $C^*(s)$ is the optimized consumption flow at time $s : t \leq s \leq T$, and

(iii) $\psi_2(r, t, T)$ represents the contribution to the investor's expected utility of the remaining investment opportunities up to the horizon:

$$\psi_2 = \left[\int_t^T q(t, s) \exp^{\frac{1-\gamma}{\gamma}[B(t,s)r_t + a_1(t,s)]} ds \right]^{\gamma-1} \left[\int_t^T q^{1-\gamma}(t, s) \exp^{\frac{1-\gamma}{\gamma}[B(t,s)r_t + a_1(t,s)]} ds \right]. \tag{43}$$

where $B(t, s)$ and $a_1(t, s)$ are given in equations (A5) and (34) with s replacing T.

The complete market problem is again a special case of Theorem 3 with $\xi_u = 0$: setting $\eta_2(t, s) = F_2(t, s) = 1$, the investor's optimal consumption and expected utility under the optimal policy are:

$$\begin{aligned} c^*(s) &= Q(t, T)^{-1} (\zeta_1(t, s))^{-\frac{1}{\gamma}} w_t && (44) \\ J(r, W, \Pi, t) &= \left(\frac{w_t^{1-\gamma}}{1-\gamma} \right) Q(t, T)^{\gamma} && (45) \\ Q(t, T) &= \left[\int_t^T \exp^{\frac{\gamma-1}{\gamma}[B(t,s)r_t + a_0(t,s)]} ds \right]^{\gamma}, && (46) \end{aligned}$$

where $Q(t, T)$ is derived from $Q_1(t, T)$ by setting $\xi_u = 0$ so that $q(t, s) = 1$ and $a_1(t, T) = a_0(t, T)$.

Comparing the optimal real consumption $c^*(s)$ for the complete market setting (44) with the optimal terminal wealth $w^*(T)$ for the complete market setting in equation (36), we see that they are of similar form: $w^*(T)$ is determined by the pricing kernel relative $\zeta_1(t, T)$ and initial real wealth, w_t, scaled by the constant $F_1(t,T)$; similarly, $c^*(s)$ is determined by the pricing kernel relative $\zeta_1(t,s)$ and initial real wealth, w_t, scaled by the sum of $F_1(t,s)$ for all $s \in [t,T]$. From equation (45), we also note that the expected lifetime utility for an investor with interim consumption is the sum of $F_1(t,s)$ $(s \in [t,T])$ raised to the power γ, while the expected utility of a terminal wealth investor is simply $F_1(t,T)$ raised to the power γ. Therefore, the problem with interim consumption could be interpreted as a summation of terminal wealth problems with the horizon s varying from t to T and the initial wealth allocated to each problem given by $\frac{F_1(t,s)}{\int_t^T F_1(t,s)ds}w_t$: first, the investor sets aside $\frac{F_1(t,s)}{\int_t^T F_1(t,s)ds}$ of w_t for each date s $(s \in [t,T])$, and then separately carries out each of the terminal-wealth optimization problems.

We can also interpret the interim consumption problem in the incomplete market setting as a two-stage problem in which the investor first allocates his initial wealth across time s $(s \in [t,T])$, $\frac{F_1(t,s)F_2(t,s)}{\int_t^T F_1(t,s)F_2(t,s)ds}w_t$, and then carries out the terminal-wealth optimization problem for each s. The expected utility for the interim consumption problem is still the sum of the expected utilities of the individual terminal-wealth problems. However, the result no longer simplifies as in the complete market case because of η_2,.

B. Unconstrained Optimal Portfolio Strategies

The optimal portfolio strategy is the one that replicates the optimal terminal wealth and consumption allocations by trade in tradable securities. The optimal strategies are of the same form in the complete and incomplete markets settings as established in the following theorems.

Theorem 4 *Optimal Portfolio Strategy for Terminal Wealth Problem*

The vector of optimal risky asset portfolio allocations for problem (26), (27), $x^* \equiv ({x_s}^*, {x_1}^*, {x_2}^*)'$ *can be written equivalently as:*

(i)

$$x^* = \frac{1}{\gamma}\begin{bmatrix} \frac{\xi_S - \phi_S}{\sigma_S} \\ D^{-1}\left[-C_2\left(\frac{\xi_r-\phi_r}{\sigma_r}\right) + B_2\left(\frac{\xi_\pi-\phi_\pi}{\sigma_\pi}\right)\right] \\ D^{-1}\left[C_1\left(\frac{\xi_r-\phi_r}{\sigma_r}\right) - B_1\left(\frac{\xi_\pi-\phi_\pi}{\sigma_\pi}\right)\right] \end{bmatrix} + \left(1-\frac{1}{\gamma}\right)\begin{bmatrix} \frac{\xi_S}{\sigma_S} \\ D^{-1}\left[-C_2\left(\frac{\xi_r-B(t,T)\sigma_r}{\sigma_r}\right) + B_2\left(\frac{\xi_\pi}{\sigma_\pi}\right)\right] \\ D^{-1}\left[C_1\left(\frac{\xi_r-B(t,T)\sigma_r}{\sigma_r}\right) - B_1\left(\frac{\xi_\pi}{\sigma_\pi}\right)\right] \end{bmatrix} \tag{47}$$

where $D = (B_1C_2 - C_1B_2)$*,* $B(t,T) = \frac{1-e^{\kappa(t-T)}}{\kappa}$*,* $B_i = B(t,T_i)$*,* $C_i = C(t,T_i)$*,* $(i = 1,2)$*,* T_i *is the maturity date of bond* i*,* $\xi = [\xi_S, \xi_r, \xi_\pi]'$*,* $\phi = [\phi_S, \phi_r, \phi_\pi]'$*.*

(ii)

$$x^* = \left[-\frac{J_W}{WJ_{WW}}\Omega^{-1}\Lambda - \frac{J_{Wr}}{WJ_{WW}}\left(\Omega^{-1}\sigma\rho e_2\sigma_r\right) - \frac{J_{W\Pi}}{WJ_{WW}}\left(\Omega^{-1}\sigma\rho\xi\Pi\right)\right], \tag{48}$$

where J *is the indirect utility function,* W *is the nominal wealth,* $\Lambda \equiv \sigma\lambda$ *is the vector of risk premia of stock and two nominal bonds, the variance covariance matrix of the nominal security returns is* $\Omega \equiv (\sigma\rho\sigma)$*,* σ *is the factor loading matrix of the nominal security returns, and* ρ *is the correlation matrix between Brownian motions* dz_S*,* dz_r*, and* dz_π*.*

(iii) The balance of the portfolio, $\mathbf{1} - \mathbf{x}'\mathbf{i}$*, is invested in the nominal riskless asset at the rate* R_f*.*

Theorem 4 establishes that there is a *feasible* trading strategy that finances the optimal wealth and consumption allocations derived in Theorems 2 and 3. Therefore, the optimal strategies given in Theorem 4 are indeed the solution to the original portfolio choice problem.

Equation (47) expresses the optimal risky portfolio allocation as the weighted sum of two portfolios. The first portfolio is proportional to the nominal mean-variance tangency portfolio.[15] The amount invested in the tangency portfolio is inversely related to the investor's relative risk aversion. The second portfolio is a "minimum risk" portfolio, since it is the portfolio that would be held by a highly risk averse investor, ($\gamma \to \infty$). In fact, it is the portfolio whose return most closely mimics the return on an indexed bond with a maturity equal to the remaining horizon: such a bond has a real interest rate sensitivity of $B(t, T)$ and no sensitivity to expected or unexpected inflation. It is intuitive that a highly risk averse investor will purchase an indexed bond if one is available and, if one is not available, then sythesize one as closely as possible by a dynamic strategy. The balance of the investor's portfolio, $\mathbf{1} - x^{*\prime}\mathbf{i}$, is invested in the (nominal) risk free asset at rate R_f.

Equation (48) expresses the optimal portfolio as the sum of three portfolios, in a form that is familiar from Merton (1973). The first portfolio is the *nominal* mean variance tangency portfolio. The second portfolio, $\Omega^{-1}\sigma\rho e_2\sigma_r$, is the hedge portfolio for the state variable r, and $\Omega^{-1}\sigma\rho\xi\Pi$ is the hedge portfolio for the stochastic price level Π. Therefore, the equation expresses the optimal portfolio as the weighted sum of an optimal myopic portfolio using *nominal* returns, a hedge portfolio associated with interest rate risk, and a hedge portfolio associated with price level risk.

The optimal strategy can also be thought of in terms of the exposures to the various sources of risk that the investor chooses, namely the portfolio loadings on the stock return and the innovations in r and π. These loadings for an investor with risk aversion γ and horizon T, $[x_S(\gamma, T), B_p(\gamma, T), C_p(\gamma, T)]$, are given in the following proposition.

Proposition 1 *Optimal Factor Loadings for Nominal Portfolio Returns*

[15] Where the tangency is from the nominal riskless rate R_f.

(i) For an investor with very large risk aversion ($\gamma \to \infty$) and horizon T,

$$x_S(\infty) = \xi_S/\sigma_S, \quad B_p(\infty, T) = \left(\xi_r - B(t,T)\sigma_r\right)/\sigma_r, \quad C_p(\infty, T) = \xi_\pi/\sigma_\pi \tag{49}$$

(ii) For an investor with risk aversion γ and horizon T,

$$\begin{aligned} x_S(\gamma) &= x_S(\infty) + \frac{\xi_S - \phi_S}{\gamma\sigma_S} - \frac{\xi_S}{\gamma\sigma_S}, & (50) \\ B_p(\gamma, T) &= B_p(\infty, T) + \frac{\xi_r - \phi_r}{\gamma\sigma_r} - \frac{\xi_r - B(t,T)\sigma_r}{\gamma\sigma_r}, & (51) \\ C_p(\gamma, T) &= C_p(\infty, T) + \frac{\xi_\pi - \phi_\pi}{\gamma\sigma_\pi} - \frac{\xi_\pi}{\gamma\sigma_\pi} & (52) \end{aligned}$$

(iii) Or, using the definitions of $x_S(\infty)$ etc. in equation (49),

$$\begin{aligned} x_S(\gamma) &= \frac{1}{\gamma}\frac{\xi_S - \phi_S}{\sigma_S} + \left(1 - \frac{1}{\gamma}\right) x_S(\infty), & (53) \\ B_p(\gamma, T) &= \frac{1}{\gamma}\frac{\xi_r - \phi_r}{\sigma_r} + \left(1 - \frac{1}{\gamma}\right) B_p(\infty, T), & (54) \\ C_p(\gamma, T) &= \frac{1}{\gamma}\frac{\xi_\pi - \phi_\pi}{\sigma_\pi} + \left(1 - \frac{1}{\gamma}\right) C_p(\infty, T) & (55) \end{aligned}$$

To understand the proposition, consider the price process of a hypothetical indexed bond with maturity equal to the investment horizon T:

$$\frac{dp^*}{p^*} = \left[r - B(t,T)\sigma_r\hat{\lambda}_r\right] dt - B(t,T)\sigma_r dz_r \tag{56}$$

where $\hat{\lambda}_r \equiv \text{Cov}\left(-\frac{dM}{M}, dz_r\right) = -\phi'\rho e_2$ is the real risk premium per unit risk of dz_r. The nominal return on this indexed bond follows from Ito's Lemma:

$$\begin{aligned}\frac{dP^*}{P^*} &= [r + \pi - B(t,T)\sigma_r\lambda_r]\,dt + \left(\frac{\xi_S}{\sigma_S}\right)\sigma_S dz_S \\ &+ \left(\frac{\xi_r}{\sigma_r} - B(t,T)\right)\sigma_r dz_r + \left(\frac{\xi_\pi}{\sigma_\pi}\right)\sigma_\pi dz_\pi + \xi_u dz_u \end{aligned} \tag{57}$$

Therefore, although the investor cannot perfectly synthesize an indexed bond using only nominal assets, part (i) of the proposition implies that a very risk averse investor will choose loadings on the innovations dz_S, dz_r, and dz_π that match those of the hypothetical indexed bond with maturity T, leaving himself exposed only to the unhedgeable component of inflation $\xi_u dz_u$. We call this portfolio that replicates an index bond up to the unhedgeable inflation risk, a pseudo index bond.

Part (ii) of the proposition implies that, as γ decreases, the investor adds a horizon independent amount $1/\gamma$ of the (nominal) mean-variance tangency portfolio. This can be easily seen from Part (ii) of Theorem 4, where the mean variance tangency portfolio *weight* is denoted by $\Omega^{-1}\Lambda \equiv (\sigma\rho\sigma')^{-1}\Lambda$. By the definition of $\Lambda \equiv \sigma\lambda = \sigma\rho(\xi - \phi)$, the tangency portfolio *weight* can be written as $(\sigma')^{-1}(\xi - \phi)$. If we were to characterize the mean variance tangency portfolio in terms of its factor loadings on the Brownian increments dz_S, dz_r and dz_π, then the loadings would be $\left[(\sigma')^{-1}(\xi - \phi)\right]'\sigma = (\xi - \phi)'$. However, in the proposition we characterize portfolios in terms of their factor loadings on the innovations, $\sigma_S dz_S$, $\sigma_r dz_r$ and $\sigma_\pi dz_\pi$; therefore, after scaling, the loadings of the tangency portfolio are $[(\xi_S - \phi_S)/\sigma_S, (\xi_r - \phi_r)/\sigma_r, (\xi_\pi - \phi_\pi)/\sigma_\pi]$ which correspond to the loadings of the second portfolio in Part (ii) of the Proposition. Finally, Part (ii) of the proposition shows that the investor shorts $1/\gamma$ of the pseudo index bond with maturity T to hedge against changes in the investment opportunity set. The portfolio separation result implied by Theorem 4 and the proposition is illustrated graphically in Figure 1. Part (iii) of the Proposition shows that the optimal factor loadings can also be written as a weighted average of the loadings of the mean variance tangency portfolio loadings and of the pseudo index bond of maturity T.

The optimal portfolio strategy for the interim consumption problem (38), (39) is determined by finding the portfolio strategy that yields the optimal consumption program $\hat{c}^*(s)$, $s \in [t, T]$, given by equation (40). Because the interim-consumption problem can be interpreted as a two-stage terminal wealth optimization problem, the optimal portfolio strategy is of a similar interpretation as well: it is a weighted average of the optimal strategies for terminal wealth problems with horizons at s, $s \in [t, T]$.

Theorem 5 *Optimal Portfolio Strategy With Interim Consumption*

The vector of optimal portfolio allocations for problem (38,39), $x^* \equiv ({x_s}^*, {x_1}^*, {x_2}^*)'$ is given by expression (47), with $B \equiv B(s,T)$ replaced by $\hat{B} \equiv \hat{B}(s,T)$, where:

$$\hat{B}(s,T) \equiv \int_s^T \frac{q(s,u)\exp\left\{\frac{1-\gamma}{\gamma}\left[B(s,u)r_s + a_1(s,u)\right]\right\}}{\int_s^T q(s,\nu)\exp\left\{\frac{1-\gamma}{\gamma}\left[B(s,\nu)r_s + a_1(s,\nu)\right]\right\}d\nu} B(s,u)du \tag{58}$$

is a weighted average of $B(s,u)$ ($u \in [s,T]$), and $q(s,u)$ and $a_1(s,u)$ are defined in Theorem 3.

C. Constrained Optimal Portfolio Strategies

Since many investors are constrained from taking short positions, it is important to consider constrained strategies also. Any portfolio strategy can be characterized in terms of the loadings of the (nominal) portfolio return on the innovations in the stock return, the real interest rate and inflation rate, x_S, $B_p \equiv -(x_1 B_1 + x_2 B_2)$ and $C_p \equiv -(x_1 C_1 + x_2 C_2)$, where x_i is the proportion of wealth invested in bond i, $i = 1, 2$. While Theorem 4 showed that the unconstrained optimal holdings generically involve two bonds, the following proposition establishes that the investor's constrained optimal allocation can be achieved by holding only a single bond in combination with cash and stock.

Proposition 2 *An investor who is constrained from borrowing or taking short positions in bonds, but who has available a continuum of possible bond maturities up to a maximum, τ_{max}, can achieve his constrained optimal portfolio allocation by investing in a single bond of an optimally chosen maturity, $\tau^* \leq \tau_{max}$, cash, and stock.*

Figure 2, which represents the feasible region of constrained portfolio factor loadings, shows how all feasible combinations of bond factor loadings can be achieved by investment in a combination of cash and a single bond of the appropriately chosen maturity. When the investor has a CRRA utility function, the indirect utility $J(r, W, \Pi, t)$ is homogeneous in real wealth so that $J(r, W, \Pi, t)$ can be expressed as $\frac{(W/\Pi)^{1-\gamma}}{1-\gamma}\psi(r, t)$. The simplified Bellman equation for the constrained portfolio problem can then be written as:

$$
\begin{aligned}
& \max_{\left\{\begin{array}{c} 0 \leq x_s, x_b \leq 1 \\ 0 \leq \tau \leq \tau_{max} \end{array}\right\}} \left\{ \frac{1}{2}\psi_{rr}\sigma_r^2 + \psi_r \left[(1-\gamma)\left(x_s\sigma_{rS} - x_b B(t,\tau)\sigma_r^2 - x_b C(t,\tau)\sigma_{r\pi} - \sigma_{r\Pi}\right)\right.\right. \\
+ \quad & \left. \kappa(\bar{r} - r)\right] + \psi\left[-\beta - (1-\gamma)^2\left(x_s\sigma_{S\Pi} - x_b B(t,\tau)\sigma_{r\Pi} - x_b C(t,\tau)\sigma_{\pi\Pi}\right)\right. \\
+ \quad & (1-\gamma)\left(r + x_s\sigma_s\lambda_s - x_b B(t,\tau)\sigma_r\lambda_r - x_b C(t,\tau)\sigma_\pi\lambda_\pi - \xi_S\lambda_S - \xi_r\lambda_r - \xi_\pi\lambda_\pi - \xi_u\lambda_u\right) \\
- \quad & \frac{1}{2}(1-\gamma)(\gamma-2)\sigma_\Pi^2 - \frac{1}{2}\gamma(1-\gamma)\left[x_s^2\sigma_s^2 - 2x_s x_b(B(t,\tau)\sigma_{rS} + C(t,\tau)\sigma_{S\Pi})\right. \\
+ \quad & \left.\left.\left. x_b^2(B(t,\tau)^2\sigma_r^2 + 2B(t,\tau)C(t,\tau)\sigma_{r\pi} + C(t,\tau)^2\sigma_\pi^2)\right]\right] + \psi_t = 0\right\}, \qquad (59)
\end{aligned}
$$

where x_s and x_b are the proportion of wealth invested in stock and bond respectively, and $\tau = T - t$ is the maturity of the bond chosen by the investor. Equation (59) is solved numerically to yield the constrained portfolio strategies that are reported in Section IV below.

D. Indexed Bonds

When there are no borrowing or short sales constraints and the market is complete ($\xi_u = 0$), indexed bonds are redundant assets and their introduction does not affect investor welfare. When the market is incomplete because inflation cannot be hedged with nominal assets ($\xi_u \neq 0$), the introduction of index bonds will complete the market. The derivation of the investor's optimal state-contingent wealth and consumption is then the same as that used in Theorems 2 and 3 when the market is incomplete, but allows for the fact that the investor can now vary his exposure to the previously unhedgeable price level innovation $\xi_u dz_u$. For example, the expected utility at time t of an investor concerned with the utility of wealth at time T when there are indexed bonds and $\xi_u \neq 0$, $J^{IB}(W, r, \Pi, t)$, is given by:

$$J^{IB}(W, r, \Pi, t) = \frac{(W_t/\Pi_t)^{1-\gamma}}{1-\gamma} \psi_1(r, t, T) \exp^{\frac{1-\gamma}{2\gamma}(\phi_u - \gamma\xi_u)^2(T-t)}. \tag{60}$$

Comparing expressions (31) and (60), we see that, for a given pricing kernel,[16] the introduction of index bonds increases investor welfare by the factor $\exp^{\frac{1}{2\gamma}(\phi_u - \gamma\xi_u)^2(T-t)} \geq 1$. Therefore, except for investors for whom $\gamma = \phi_u/\xi_u$, the introduction of indexed bonds increases welfare by permitting trade in the previously unhedgeable inflation component $\xi_u dz_u$.[17]

The introduction of indexed bonds will also generally increase investor welfare when the investor is subject short-selling or borrowing constraints. With index bonds, the investor's portfolio choice problem involves the nominal bond maturity, the indexed bond maturity, and the proportions of wealth invested in stock, nominal and indexed bonds.[18]

[16] It is important to remember that the introduction of a new security may change the prices of existing securities.

[17] Similarly, if π is updated only from the time series of the price level so that dz_Π and dz_π are perfectly correlated and ξ_u is zero, then the introduction of an extraneous information signal about expected inflation destroys perfect correlation and generically raises investor welfare (for a given pricing kernel) when index bonds are available.

[18] The constrained optimization problem can only be solved numerically and the availability of indexed bonds significantly increases the dimension of the problem. Therefore we do not consider indexed bonds in our calculations in Section IV. Campbell and Viceira (1999) offer a welfare analysis of indexed bonds for constrained investors but do not allow for maturity choice.

III. Model Calibration

The parameters of the model were estimated using a Kalman filter in which the unobserved state is described by r and π.[19] In order to reduce the number of parameters to be estimated, we assume that inflation risk is completely unhedgeable with the nominal assets so that $\xi_S = 0$, $\xi_r = 0$, $\xi_\pi = 0$ and $\xi_u = \sigma_\Pi$. The transition equation for the state variables, r_t and π_t, is the discrete time equivalent of equations (2) and (4):

$$\begin{pmatrix} r_t \\ \pi_t \end{pmatrix} = \begin{pmatrix} e^{-\kappa\Delta t} & 0 \\ 0 & e^{-\alpha\Delta t} \end{pmatrix} \begin{pmatrix} r_{t-\Delta t} \\ \pi_{t-\Delta t} \end{pmatrix} + \begin{pmatrix} \bar{r}\left[1 - e^{-\kappa\Delta t}\right] \\ \bar{\pi}\left[1 - e^{-\alpha\Delta t}\right] \end{pmatrix} + \begin{pmatrix} \eta_r(t) \\ \eta_\pi(t) \end{pmatrix}, \tag{61}$$

where the vector of innovations is:

$$\eta(t) \equiv \begin{pmatrix} \eta_r(t) \\ \eta_\pi(t) \end{pmatrix} = \begin{pmatrix} \sigma_r e^{-\kappa(t-\Delta t)} \int_{t-\Delta t}^{t} e^{-\kappa\tau} dz_r(\tau) \\ \sigma_\pi e^{-\alpha(t-\Delta t)} \int_{t-\Delta t}^{t} e^{-\alpha\tau} dz_\pi(\tau) \end{pmatrix}, \tag{62}$$

and the variance-covariance matrix of $\eta(t)$ is

$$Q = \begin{pmatrix} \frac{\sigma_r^2}{2\kappa}\left[1 - e^{-2\kappa\Delta t}\right] & \frac{\sigma_r\sigma_\pi\rho_{r\pi}}{\kappa+\alpha}\left[1 - e^{-(\kappa+\alpha)\Delta t}\right] \\ \frac{\sigma_r\sigma_\pi\rho_{r\pi}}{\kappa+\alpha}\left[1 - e^{-(\kappa+\alpha)\Delta t}\right] & \frac{\sigma_r^2}{2\kappa}\left[1 - e^{-2\kappa\Delta t}\right] \end{pmatrix}. \tag{63}$$

The observation equations consist first of yields at time t, $y_{\tau_j,t}$, on n bonds of maturity τ_j, $j = 1, \cdots, n$, which are related to the unobserved state variables, r,π, by

$$y_{\tau_j,t} \equiv -\frac{\ln P(t, t+\tau_j)}{\tau_j} = -\frac{A(t,\tau_j)}{\tau_j} + \frac{B(\tau_j)}{\tau_j} r_t + \frac{C(\tau_j)}{\tau_j}\pi_t + \epsilon_{\tau_j}(t), \tag{64}$$

where $\epsilon_{\tau_j,t}$ is the measurement error in the observed yields. The final observation equation is

[19] See Harvey (1989).

given by the relation between the realized inflation rate, $\frac{\Pi_t - \Pi_{t-1}}{\Pi_{t-1}}$, and the expected inflation rate:

$$\frac{\Pi_t - \Pi_{t-1}}{\Pi_{t-1}} = \pi \Delta t + \epsilon_\Pi(t). \tag{65}$$

The system was estimated using monthly data on eleven constant maturity U.S. treasury bond yields with maturities of 1, 3, 6 and 9 months, and 1, 2, 3, 4, 5, 7, and 10 years, and CPI inflation for the period January 1970 to December 1995.[20] To improve identification of the two factors, their long run means, $\bar{r}$ and $\bar{\pi}$, were imposed exogenously.[21] $\bar{r}$ was taken as 1.24%, the arithmetic sample mean real return of the three month Treasury Bill and $\bar{\pi}$ as 5.44%, the arithmetic sample mean inflation rate. The stock returns were taken as the returns on the value-weighted CRSP index for the same period. Table 1 reports the parameter estimates, along with their standard errors. The mean reversion coefficients imply half-lives for innovations in the real interest rate and expected inflation of 1.1 and 25.7 years respectively.[22] The market prices of both interest rate risk and inflation risk are negative and significant.[23] Since the loadings of bond returns on innovations in these variables are negative and grow with maturity, estimated bond risk premia are positive and increasing with maturity. The standard deviation of unexpected inflation, σ_Π, is about 133 basis points per year, which compares with 411 basis points for the unconditional standard deviation of inflation and of 136 basis points for σ_π, the standard deviation of innovations in expected inflation. The standard deviation of innovations in the real interest rate, σ_r, of 260 basis points seems high, but should be considered in conjunction with the strong mean reversion which implies that only about 53% of any innovation remains after one year. The correlation between innovations in the real interest rate and in inflation (expected and realized) is -0.06 which is consistent with the Mundell-Tobin model and with the empirical findings of Fama and Gibbons (1982) for the period 1953-77.

[20]The data on yields were kindly provided by David Backus. The CPI data are from the Bureau of Labor Statistics.

[21]See Dai and Singleton (1997) for discussion of identification problems in affine term structure models.

[22]These estimates are comparable to those of Campbell and Viceira (1999) for the period 1952-1996. The slow mean reversion in the estimated expected rate of inflation is consistent with Fama and Gibbons (1982, p 305) who report that 'the short term expected inflation rate is close to a random walk'.

[23]Note that by ignoring the standard errors of $\bar{\pi}$ and $\bar{r}$, the standard errors of all the parameters are understated.

The standard errors for the bond yield estimates include model error as well as sampling error. The standard errors of these 'estimation errors' are quite low, except for maturities up to one year which vary from 20 basis points for one year to 99 basis points for one month; for longer maturities the standard errors are in the range of zero to 12 basis points.

Figure 3 plots the time series of the estimated real interest rate, r, and expected rate of inflation, π. The estimated real interest rate varies between -4% and 7%. The series exhibits high short run variability and strong mean reversion. In contrast, the expected rate of inflation series exhibits much less mean reversion. It is possible that the high frequency variability in r is due to the model's attempts to fit variation in the yields of medium to long term bonds - this would account also for the relatively poor fit of the model at the short end of the term structure.

To assess the extent to which realized inflation may be hedged by a portfolio of nominal securities we estimate the discrete time equivalent of equation (7) which relates the inflation realization to the Brownian increments that drive the security returns: the discrete time equivalents of the Brownian increments dz_S, dz_r, dz_π and dz_Π were estimated from the monthly time series of filtered state variables r and π, of stock returns and of realized inflation rates, using the estimated parameters κ, α, σ_S, σ_r and σ_π:

$$\Delta z_S(t) = \frac{\left[R_S(t) - \left(r(t) + \pi(t) - \sigma_\Pi \lambda_\pi + \sigma_S \lambda_S\right)/12\right]}{\sigma_S/\sqrt{12}} \tag{66}$$

$$\Delta z_r(t) = \frac{r(t) - e^{-\kappa/12} r(t-1) - \left(1 - e^{-\kappa/12}\right)\bar{r}}{\sigma_r \sqrt{\left(1 - e^{-\kappa/6}\right)/(2\kappa)}} \tag{67}$$

$$\Delta z_\pi(t) = \frac{\pi(t) - e^{-\alpha/12} \pi(t-1) - \left(1 - e^{-\alpha/12}\right)\bar{\pi}}{\sigma_\pi \sqrt{\left(1 - e^{-\alpha/6}\right)/(2\alpha)}} \tag{68}$$

$$\Delta z_\Pi(t) = \frac{I - \pi(t)/12}{\sigma_\Pi/\sqrt{12}}. \tag{69}$$

Then the estimate of unexpected inflation, $\sigma_\Pi \Delta z_\Pi(t)$, was regressed on $\Delta z_S(t)$, $\Delta z_r(t)$, $\Delta z_\pi(t)$ and the results are reported in Table 2. The coefficients, which are estimates of ξ_S, ξ_π, ξ_r, are all insignificant, although the intercept is highly significant. This means that it is not possible to hedge the inflation risk using the nominal securities we consider, and that ξ_u is essentially equal to σ_Π. The result is consistent with our assumption that $\xi_u = \sigma_\Pi$. However, the inability to hedge against the inflation realization is not very costly. The certainty equivalent cost of unhedgeable inflation may be assessed by comparing the expressions for investor utility in the complete and incomplete markets, equations (31) and (37). Assuming that the reward for bearing inflation risk is zero ($\phi_u = 0$), the certainty equivalent cost imposed by unhedgeable inflation is $\exp^{\frac{\gamma \xi_u^2}{2}(T-t)}$: this amounts to only about 1% of wealth for a twenty year horizon investor with $\gamma = 5$ because ξ_u, the volatility of the unhedgeable inflation is only 0.0131.[24]

IV. Unconstrained and Constrained Dynamic Strategies

Optimal investment strategies were determined for the calibrated parameter values of the model under the assumption of non-hedgeable inflation: $\xi_S = \xi_r = \xi_\pi = 0$, $\xi_u = \sigma_\Pi$. Table 3 summarizes the optimal strategies and the certainty equivalent wealth[25] for different horizons and risk aversion parameters, γ.[26] Since $\alpha \neq \kappa$, any two bonds with different maturities are sufficient to span the (nominal or real) returns on all possible bond portfolios which are characterized by their loadings on the interest and (expected) inflation rates dr and $d\pi$; the amounts invested in bonds depend on which bonds are used to achieve the portfolio sensitivities. Therefore in the table we report (the negative of) the loadings of the nominal portfolio returns, $B_p \equiv -(x_1 B(t,T_1) + x_2 B(t,T_2))$, and $C_p \equiv -(x_1 C(t,T_1) + x_2 C(t,T_2))$; expressions for these loadings follow immediately from equation (B17). The optimal stock allocation and inflation

[24] It would of course be much more costly for the investor if he were not able to hedge against changes in *expected* inflation.

[25] The certainty equivalent wealth is that amount of wealth at the horizon that would leave the investor indifferent between receiving it for sure and having $1 today to invest in the stock and bonds.

[26] We analyze the optimal strategies only for the terminal wealth problem and not for the lifetime consumption problem since, as shown in Theorem 5, the optimal stock holding is the same for the two problems and the bond allocation for the second problem with horizon T is a weighted average of the allocations for the terminal wealth problem for horizons from 0 to T.

loadings are both independent of the horizon[27] and, while the interest rate loading is increasing in the horizon for $\gamma > 1$ and decreasing in the horizon for $\gamma < 1$,[28] this loading is relatively insensitive to the horizon for $T > 5$. Thus, in contrast to Xia (2000) who find strong horizon effects even at long horizons in models with excess return predictability, horizon effects here are limited to 5 years. Nevertheless, the hedge component of the optimal bond portfolio is significant. For example, when $\gamma = 3$ the myopic strategy has a loading of 2.29 on r, while the optimal strategy for a five-year investment horizon has a loading of 3.25, so the hedge demand as measured by the loading on r is about 42% of the myopic demand. When the horizon increases from five to twenty years, the hedge demand only increases by a further 2% of the myopic demand. The importance of the hedge demand increases rapidly as risk aversion increases;[29] for example when $\gamma = 5$ the optimal loading on r for a five year horizon is 182% of the loading for a myopic investor.

Figure 5 plots the optimal portfolio holdings of cash, stock, one-year and ten-year nominal bonds as a function of the investment horizon for an investor with $\gamma = 3$. The figure confirms the discussion in terms of factor loadings. The optimal bond allocation changes little beyond about 4 years, although there is a big difference between the myopic and the optimal allocation for a longer term investor: the myopic allocation in the one-year bond is 3.24, while the optimal allocation is 4.94 for $T = 5$, so that the hedge demand is 1.7, or 52% of the myopic allocation.

Although not shown here, the optimal portfolios are also quite sensitive to the risk aversion parameter, with the stock allocation and the loadings on both r and π decreasing as risk aversion increases. In the limit, as risk aversion becomes infinite, the stock allocation goes to zero and the investor's dynamic strategy in bonds synthesizes the returns on an inflation indexed bond with maturity equal to the remaining investment horizon. Note that since $C(t, T) > 0$ investment in a bond with positive maturity increases the investor's exposure to inflation risk. If ϕ_π were equal to zero so that there was no reward for bearing inflation risk, the investor would avoid all inflation

[27] See equation (B17).

[28] $\gamma = 1$ corresponds to log utility for which the portfolio is independent of the horizon.

[29] The hedge demand is zero for log utility ($\gamma = 1$) and negative for $\gamma < 1$.

risk by taking a short position in at least one of the bonds. For the parameter estimates in Table 1, all of the optimal portfolios in Table 3 involve at least one short bond or cash position.[30] Since many investors are precluded from taking short positions it is important to consider also the constrained optimal dynamic strategies.

Table 4 reports the constrained optimal strategies for the same parameter values as Table 3: the portfolio allocation is now shown as the proportion of wealth allocated to bonds and stock together with the optimal maturity of the bond. In this case, the investor tries to replicate the loadings on stock return, real interest rate and expected inflation as close as possible. Given the high reward associated with stock, the investor first tries to get the loading on stock return close enough to the unconstrained case. Then, he allocates the remainder to cash and one bond so that loadings on r and π are close to those in unconstrained case. The optimal maturity of bond is chosen in light of the tradeoff between optimal loadings on r versus that on π. For example, when $\gamma \to \infty$, the investor ideally would like to put all wealth in the bond with maturity equal to the investment horizon so that the loading on r exactly matches that without constraint. However, such a choice would make the loading on π deviate quite a lot from the ideal case. The comprise would force the investor to choose an optimal maturity smaller than the investment horizon. For these parameter values it is never optimal to hold cash - a pure bond holding dominates a mixture of cash and bonds. The constraints reduce investor welfare as measured by certainty equivalent wealth but the reduction is not great. For example, an investor with a risk aversion parameter of 3 and a horizon of 20 years who is subject to the constraints requires only about 13% additional wealth to compensate for the constraints. The portfolio allocation is relatively insensitive to the investment horizon: the stock-bond ratio is virtually independent of the horizon, and the maturity of the optimal bond shows little variation beyond year 5. However, the optimal portfolio is quite sensitive to the risk aversion parameter. As risk aversion increases, the ratio of bonds to stock increases at all horizons; moreover, the maturity of the optimal bond decreases as shown in Figure 7.

[30]This can be verified by checking whether ${B^*}_p, {C^*}_p$ lie in the region $\mathcal{S}$ where ${B^*}_p = \frac{B_p}{1-x_1}$ etc.

The efficiency gain from employing a dynamic strategy can be assessed by calculating the ratio of the certainty equivalent wealth under the optimal dynamic strategy to the certainty equivalent wealth under a myopic strategy in which the portfolio allocation depends only on the vector of risk premia and the covariance matrix.[31] When the investor does not face any short-sales or borrowing constraints, the efficiency gain ratio, EGR, is given by:

$$\begin{aligned} EGR &= \exp\left\{\frac{(1-\gamma)^2}{2\gamma} var\left(\int_t^T r(s)ds\right)\right\} \geq 1 \\ &= \exp\left\{\frac{(1-\gamma)^2}{\gamma}\left[\frac{\sigma_r^2}{4\kappa^3}\left(2\kappa(T-t) - 3 - e^{-2\kappa(T-t)} + 4e^{-\kappa(T-t)}\right)\right]\right\}. \end{aligned} \tag{70}$$

The efficiency gain depends only on the risk aversion parameter and variance of the cumulative real interest rate; for values of γ close to unity the gain is small; the gain is also small if either σ_r is small, or the mean reversion parameter κ is large. As we have seen, the estimated value of κ from the first set of data is very high. As a result, the estimates of EGR reported in the Table 3 are very small except for long horizons and strong risk aversion - for γ equal to 15 the efficiency gain over 20 years is around 21%. When the investor faces constraints, EGR is calculated numerically. Estimates of EGR reported in Table 4 are all close to one, reflecting the fact that the investor's optimal portfolio strategy is close to the myopic one when there are constraints.

It is possible that the estimate of the real interest rate mean reversion parameter, κ, is too high in light of the high frequency oscillations in the estimated real interest rate series that it implies, as seen in Figure 3. Figure 4 plots the annual 'realized' real interest rates for the period $1890-1985$.[32] It is clear that there is much less mean reversion even in this noisy series than that in the estimated series in Figure 3. Therefore we re-estimated the parameters of the stochastic processes for r and π with a Kalman filter using only annual data on the nominal interest rate and

[31] The expected utility under the myopic strategy can be simply calculated by inserting the myopic portfolio allocation in the process for real wealth and calculating $E\left[w^{1-\gamma}/\gamma\right]$.

[32] The 'realized' real interest rate for a year is the difference between the nominal interest rate and the realized rate of inflation. All data are from Shiller (1989).

inflation for the period $1890 - 1985$. The results are striking - the new value of κ is only 0.105 in contrast to 0.631, while the two estimates of σ_r are virtually identical: the filtered estimates of r track the realized values of r quite well.

Tables 5 and 6 report the unconstrained and constrained portfolio strategies for the new value of κ, holding the other parameters unchanged. The unconstrained strategies now exhibit strong horizon effects in B_p even at long horizons:[33] when γ is less than unity B_p decreases with the horizon, while the reverse is true for γ greater than unity. The strong horizon effect is also evident in Figure 6, which plots the optimal portfolio holdings when the investor has $\gamma = 3$ and can invest in cash, stock, one year and ten year bonds. Most significantly, the efficiency gain over the myopic strategy is now substantial: when the horizon is 20 years, the gain is 119% for γ equal 5 and 252% for γ equal 7 in the unconstrained case, and the respective gain is 54% and 121% in the constrained case.

Horizon effects are now also evident for the constrained strategies shown in Table 6. First, in contrast to the popular view that long-horizon investors should hold more stock than short horizon investors,[34] the optimal stock holding decreases with the horizon; for example, for $\gamma = 3$, the optimal holding of stock is 66% for a one-month horizon and only 40% for a 20 year horizon. Similarly, the stock-bond ratio also decreases with the horizon beyond year 1. These results are obviously sensitive to the assumption of the model that the equity risk premium is constant. In contrast to the previous example, a myopic investor with large enough risk aversion parameter may want to hold cash, and in such circumstances the constraints are not binding. However, when the investment horizon is longer than one year, a mix of stock and bond (no cash) dominates portfolios with cash. The second horizon effect, which is shown in Figure 8, is that the maturity of the optimally chosen bond increases with the horizon - the effect is much more pronounced with the reduced degree of mean reversion in r which generally leads the long term investor to hold a much longer maturity bond: the maturity of the optimally chosen bond for a 20 year

[33] There are no horizon effects in C_p and x_s.

[34] 'Stocks should constitute the overwhelming proportion of all long-term financial portfolios.' Siegel (1998, p 283).

investor with $\gamma = 3$ rises from 7 years when κ is 0.63 to 13.5 years when κ is 0.11.

V. Conclusion

In this paper we have derived the optimal dynamic strategies for an investor with power utility in an economy with stochastic inflation and real interest rates, and a constant equity premium, when there exists no riskless security. Closed form expressions were obtained for the optimal portfolio when the investor is free to take unconstrained portfolio positions, and it was shown that the optimal portfolio position can be achieved by investments in stock, cash, and two nominal bonds. In this setting the optimal allocation to stock and the optimal portfolio loading on the innovation in inflation are independent of the horizon, while the optimal loading on the innovation in the shadow real interest rate is increasing in the horizon for investors more risk averse than the log. The efficiency gain of the optimal dynamic strategy over the myopic strategy was shown to be a function of both the investor's risk aversion and the variance of the cumulative real interest rate over the investor's horizon: the gain is small for risk aversion close to the log, and when either the variance of innovations in the real interest rate is small or the mean reversion in the real interest rate is large. When the investor is constrained from holding short positions or borrowing, the optimal portfolio was shown to be achievable with an investment in stock, cash, and a single bond with an optimally chosen maturity.

The model was calibrated to monthly data on U.S. Treasury bond yields and inflation for the period $1970 - 1995$. The resulting parameter estimates implied an unreasonably high degree of mean reversion in the real interest rate and yielded very small estimates of the efficiency gain of the dynamic strategy, and only limited horizon effects in optimal portfolios. A striking characteristic of the optimal constrained portfolios is that, not only does the allocation to bonds increase with risk aversion, but the maturity of the optimal bond decreases as risk aversion increases.

When the real interest rate mean reversion parameter is calibrated to a long history of annual data it falls from 0.63 to 0.11. With this parameterization both horizon effects and the efficiency gains of the optimal dynamic strategy become large. Thus, the importance of dynamic considerations in optimal asset allocation depends critically on the stochastic characteristics of the investment opportunity set. Further work is required to assess more precisely the dynamics of the real interest rate in the U.S.

Appendix

A. Bond Prices

1 Price of Nominal Bond

The price of a nominal bond, $P(t,T)$ is given by:

$$
\begin{aligned}
P(t,T) &= \mathrm{E}_t\left[\frac{M_T/M_t}{\Pi_T/\Pi_t}\right] = \mathrm{E}_t\left[\exp\left\{\ln\left(M_T/M_t\right) - \ln\left(\Pi_T/\Pi_t\right)\right\}\right] \\
&= \exp\left\{\mathrm{E}_t\left[\ln\left(M_T/M_t\right) - \ln\left(\Pi_T/\Pi_t\right)\right] + \frac{1}{2}Var_t\left[\ln\left(M_T/M_t\right) - \ln\left(\Pi_T/\Pi_t\right)\right]\right\} \quad \text{(A1)}
\end{aligned}
$$

Let

$$
\begin{aligned}
\phi_1^2 \equiv \phi'\rho\phi &= \phi_S^2 + \phi_r^2 + \phi_\pi^2 + 2\rho_{Sr}\phi_S\phi_r + 2\rho_{S\pi}\phi_S\phi_\pi + 2\rho_{r\pi}\phi_r\phi_\pi, && \text{(A2)} \\
\xi_1^2 \equiv \xi'\rho\xi &= \xi_S^2 + \xi_r^2 + \xi_\pi^2 + 2\rho_{Sr}\xi_S\xi_r + 2\rho_{S\pi}\xi_S\xi_\pi + 2\rho_{r\pi}\xi_r\xi_\pi, && \text{(A3)} \\
V_M &= \phi_1^2 + \phi_u^2, \qquad V_\Pi = \xi_1^2 + \xi_u^2 && \text{(A4)}
\end{aligned}
$$

where $\phi = [\phi_S, \phi_r, \phi_\pi]'$, $\xi = [\xi_S, \xi_r, \xi_\pi]'$, and

$$
\begin{aligned}
B(t,T) &= \frac{1}{\kappa}\left(1 - e^{\kappa(t-T)}\right), && \text{(A5)} \\
C(t,T) &= \frac{1}{\alpha}\left(1 - e^{\alpha(t-T)}\right), && \text{(A6)}
\end{aligned}
$$

We then have

$$
\ln\left(M_T/M_t\right) = \int_t^T \left(-r(s) - \frac{1}{2}V_M\right)ds + \int_t^T \phi' dz + \int_t^T \phi_u dz_u \qquad \text{(A7)}
$$

and

$$\ln\left(\Pi_T/\Pi_t\right) = \int_t^T \left(\pi(s) - \frac{1}{2}\sigma_\Pi^2\right) ds + \int_t^T \xi' dz + \xi_u \int_t^T dz_u \tag{A8}$$

Using equation (A7) and (A8),

$$\begin{aligned}
E_t\left[\ln\left(M_T/M_t\right)\right] &= -\bar{r}(T-t) + (\bar{r} - r_t)B(t,T) - \frac{1}{2}V_M(T-t) && \text{(A9)}\\
Var_t\left[\ln\left(M_T/M_t\right)\right] &= -\frac{{\sigma_r}^2}{2\kappa^3}\left[2\kappa\left(B(t,T) - (T-t)\right) + \kappa^2 B^2(t,T)\right] + V_M(T-t) && \\
&- \frac{2\sigma_r}{\kappa}\left(\phi_S\rho_{Sr} + \phi_r + \phi_\pi\rho_{r\pi}\right)\left[T - t - B(t,T)\right]; && \text{(A10)}
\end{aligned}$$

and

$$\begin{aligned}
E_t\left[\ln\left(\Pi_T/\Pi_t\right)\right] &= \bar{\pi}(T-t) - (\bar{\pi} - \pi_t)C(t,T) - \frac{1}{2}(\xi_1^2 + \xi_u^2)(T-t) && \text{(A11)}\\
Var_t\left[\ln\left(\Pi_T/\Pi_t\right)\right] &= -\frac{{\sigma_\pi}^2}{2\alpha^3}\left[2\alpha\left(C(t,T) - (T-t)\right) + \alpha^2 C^2(t,T)\right] + (\xi_1^2 + \xi_u^2)(T-t) && \\
&+ \frac{2\sigma_\pi}{\alpha}\left(\xi_S\rho_{S\pi} + \xi_r\rho_{r\pi} + \xi_\pi\right)\left[T - t - C(t,T)\right]. && \text{(A12)}
\end{aligned}$$

In addition, we have

$$\begin{aligned}
CV &\equiv cov\left[\ln\left(\frac{M_T}{M_t}\right), \ln\left(\frac{\Pi_T}{\Pi_t}\right)\right] \\
&= -\frac{\sigma_r\sigma_\pi\rho_{r\pi}}{\alpha\kappa}\left[(T-t) - B(t,T), -C(t,T) + \frac{1 - e^{(\alpha+\kappa)(t-T)}}{\alpha+\kappa}\right] \\
&- \frac{\sigma_r}{\kappa}\left(\xi_S\rho_{Sr} + \xi_r + \xi_\pi\rho_{r\pi}\right)\left[(T-t) - B(t,T)\right] + \phi_u\xi_u(T-t) \\
&+ \frac{\sigma_\pi}{\alpha}\left(\phi_S\rho_{S\pi} + \phi_r\rho_{r\pi} + \phi_\pi\right)\left[(T-t) - C(t,T)\right] + \phi'\rho\xi(T-t). \qquad \text{(A13)}
\end{aligned}$$

Therefore,

$$P(t,T) = \exp\left\{A(t,T) - B(t,T)r_t - C(t,T)\pi_t\right\}, \tag{A14}$$

where

$$
\begin{aligned}
A(t,T) &= [B(t,T)-(T-t)]\,\bar{r}^* + [C(t,T)-(T-t)]\,\bar{\pi}^* \\
&- \frac{\sigma_r^2}{4\kappa^3}\left[2\kappa(B(t,T)-(T-t))+\kappa^2 B^2(t,T)\right] \\
&- \frac{\sigma_\pi^2}{4\alpha^3}\left[2\alpha(C(t,T)-(T-t))+\alpha^2 C^2(t,T)\right] \\
&+ \frac{\sigma_r\sigma_\pi\rho_{r\pi}}{\kappa\alpha}\left[(T-t)-C(t,T)-B(t,T)+\frac{1-e^{(\alpha+\kappa)(t-T)}}{\alpha+\kappa}\right] \\
&+ (\xi_S\lambda_S+\xi_r\lambda_r+\xi_\pi\lambda_\pi+\xi_u\lambda_u)(T-t). \qquad \text{(A15)}
\end{aligned}
$$

In the above equation, $\bar{r}^* = \bar{r} - \lambda_r\frac{\sigma_r}{\kappa}$, and $\bar{\pi}^* = \bar{\pi} - \lambda_\pi\frac{\sigma_\pi}{\alpha}$ where

$$
\begin{aligned}
\lambda_S &\equiv (\xi_S+\xi_r\rho_{Sr}+\xi_\pi\rho_{S\pi})-(\phi_S+\phi_r\rho_{Sr}+\phi_\pi\rho_{S\pi}), && \text{(A16)} \\
\lambda_r &\equiv (\xi_S\rho_{Sr}+\xi_r+\xi_\pi\rho_{r\pi})-(\phi_S\rho_{Sr}+\phi_r+\phi_\pi\rho_{r\pi}), && \text{(A17)} \\
\lambda_\pi &\equiv (\xi_S\rho_{S\pi}+\xi_r\rho_{r\pi}+\xi_3)-(\phi_S\rho_{S\pi}+\phi_r\rho_{r\pi}+\phi_\pi), && \text{(A18)} \\
\lambda_u &\equiv \xi_u-\phi_u. && \text{(A19)}
\end{aligned}
$$

$\bar{r}^*$ can be interpreted as the long run mean of the real interest rate under the risk neutral measure and $\bar{\pi}^*$ the long run mean of the expected inflation rate.

2 Price of Real Bond

The real price of a real bond, $p^*(t,T)$, is given by:

$$
p^*(t,T) = \mathrm{E}_t\left[\frac{M_T}{M_t}\right] = \mathrm{E}_t\left[\exp\left\{\ln\left(M_T/M_t\right)\right\}\right] \qquad \text{(A20)}
$$

Substituting from (A9) and (A10) in equation (A20),

$$p^*(t,T) = \exp\left\{\hat{A}(t,T) - B(t,T)r_t\right\}, \tag{A21}$$

where

$$\hat{A}(t,T) = [B(t,T) - (T-t)]\,\bar{r}^{**} - \frac{{\sigma_r}^3}{4\kappa^3}\left[2\kappa\left(B(t,T) - (T-t)\right) + \kappa^2 B^2(t,T)\right]; \tag{A22}$$

and

$$\bar{r}^{**} \;\; = \;\; \bar{r} + \frac{\sigma_r}{\kappa}\left(\phi_S\rho_{Sr} + \phi_r + \phi_\pi\rho_{r\pi}\right) = \bar{r}^* + \frac{\sigma_{r\Pi}}{\kappa}. \tag{A23}$$

B. Proof of Theorems

1 Proof of Theorem 1

(1) The definition of the (real) pricing kernel (3) implies that for the stock the quantity, Ms, follows a martingale. Using Ito's Lemma and equations (3) and (16), we have

$$\begin{aligned} d(Ms) &= sdM + Mds + dMds \\ &= Ms[-r + \mu_S^N - \pi + \xi'\rho\xi + \xi_u^2 - \sigma_S e_1'\rho\xi + \sigma_S e_1'\rho\phi - \phi'\rho\xi - \phi_u\xi_u]dt \\ &+ Ms\left[(\sigma_S - \xi_S)dz_S - \xi_r dz_r - \xi_\pi dz_\pi - \xi_u dz_u\right]. \end{aligned} \tag{B1}$$

Since Ms is a martingale, the drift term in equation (B1) is zero. Thus,

$$\mu_S^N = r + \pi - \xi_S \lambda_S - \xi_r \lambda_r - \xi_\pi \lambda_\pi - \xi_u \lambda_u + \sigma_S \lambda_S \tag{B2}$$
$$= R_f + \sigma_S \lambda_S, \tag{B3}$$

where the first equality follows from the definition of λ's in equation (18) and the second equality follows from the definition of R_f.

(2) The expressions for the nominal and real expected rates of return on a zero-coupon bond with maturity T follow from Ito's Lemma and the drift terms in equation (12).

2 Proof of Theorem 2

Use the fact that η_1, ζ_1 are orthogonal to η_2, ζ_2, and note that $\Pi_T = \Pi_t \eta_1(t,T)\eta_2(t,T)$, we can write the Lagrange for *feasible* wealth process as:

$$\begin{aligned}\mathcal{L} &= \mathrm{E}_t\left[\frac{(W_T/\eta_1(t,T))^{1-\gamma}}{1-\gamma}\right]\mathrm{E}_t\left[\frac{1}{\eta_2(t,T)^{1-\gamma}}\right]\frac{1}{\Pi_t^{1-\gamma}} \\ &- \delta\left\{\mathrm{E}_t\left[\zeta_1(t,T)\frac{W_T}{\eta_1(t,T)}\right]\mathrm{E}_t\left[\zeta_2(t,T)\frac{1}{\eta_2(t,T)}\right]\frac{1}{\Pi_t} - \frac{W_t}{\Pi_t}\right\}.\end{aligned} \tag{B4}$$

The investor can choose different nominal wealth across different state in $\mathcal{O}_1$ by trading the nominal securities, but can only stay with the same level of nominal wealth for different state in $\mathcal{O}_2$. Therefore, the investor's objective function defined in (B4) has two extra constant terms $\mathrm{E}_t\left[\frac{1}{\eta_2(t,T)^{1-\gamma}}\right]$ and $\mathrm{E}_t\left[\zeta_2(t,T)\frac{1}{\eta_2(t,T)}\right]$ as compared to the Lagrange equations in the complete market case. Derive the first order conditions:

$$\left(\frac{W_T}{\eta_1(t,T)\Pi_t}\right)^{-\gamma}\mathrm{E}_t\left[\frac{1}{\eta_2(t,T)^{1-\gamma}}\right] = \delta\zeta_1(t,T)\mathrm{E}_t\left[\zeta_2(t,T)\frac{1}{\eta_2(t,T)}\right], \tag{B5}$$
$$\mathrm{E}_t\left[\zeta_1(t,T)\frac{W_T}{\eta_1(t,T)}\right]\mathrm{E}_t\left[\zeta_2(t,T)\frac{1}{\eta_2(t,T)}\right] = W_t. \tag{B6}$$

Equation (B5) can be rewritten as

$$W_T = (\hat{\delta})^{-\frac{1}{\gamma}}(\zeta_1(t,T))^{-\frac{1}{\gamma}}\eta_1(t,T)\Pi_t. \tag{B7}$$

Substitute equation (B7) into (B6) and solve for $\hat{\delta}$, then eliminate $\hat{\delta}$ from (B7) by substituting the value of $\hat{\delta}$, we get that

$$w_T^* \equiv W_T^*/\Pi_T = \eta_2^{-1}(t,T) w_t \zeta_1(t,T)^{-\frac{1}{\gamma}} F_1(t,T)^{-1} F_2(t,T)^{-1} \tag{B8}$$

where $F_1(t,T) = \mathrm{E}_t\left[\zeta_1(t,T)^{1-\frac{1}{\gamma}}\right]$ and $F_2(t,T) = \mathrm{E}_t\left[\zeta_2(t,T)/\eta_2(t,T)\right] = \exp^{(\xi_u^2 - \xi_u \phi_u)(T-t)}$.

Substituting w_T^* given in equation (B8) into the investor's J function (31), we get

$$J = \frac{(w_t)^{1-\gamma}}{1-\gamma} F_1(t,T)^{\gamma} F_2(t,T)^{\gamma-1} F_3(t,T), \tag{B9}$$

where $F_3(t,T) = \mathrm{E}_t\left[\eta_2(t,T)^{\gamma-1}\right] = \exp^{\frac{(1-\gamma)(2-\gamma)}{2}\xi_u^2(T-t)}$. The expression for $\psi_1(t,T)$ is derived by substituting in $F_1(t,T)$, $F_2(t,T)$ and $F_3(t,T)$ explicitly.

3 Proof of Theorem 3

Using similar techniques as in the previous proof we get the first order conditions:

$$\left(\frac{C^*(s)}{\eta_1(t,s)\Pi_t}\right)^{-\gamma} - \hat{\delta}\zeta_1(t,s) = 0 \tag{B10}$$

$$\int_t^T \mathrm{E}_t\left[\frac{\zeta_2(t,s)}{\eta_2(t,s)}\right] \mathrm{E}_t\left[\frac{\zeta_1(t,s)}{\eta_1(t,s)} C(s)\right] ds - W_t = 0 \tag{B11}$$

Eliminating δ between the first order conditions yields

$$c^*(s) \;=\; \eta_2^{-1}(t,s)\left(\zeta_1(t,s)\right)^{-\frac{1}{\gamma}} Q_1^{-1}(t,T)w_t \tag{B12}$$

where

$$\begin{aligned} Q_1(t,T) &= \mathrm{E}_t\left[\int_t^T \left(\zeta_1(t,s)\right)^{1-\frac{1}{\gamma}} \frac{\zeta_2(t,s)}{\eta_2(t,s)} ds\right] = \int_t^T F_1(t,s)F_2(t,s)ds \\ &= \int_t^T q(t,s)\exp\left\{\frac{1-\gamma}{\gamma}[B(t,s)r_t + a_1(t,s)]\right\} ds, \end{aligned} \tag{B13}$$

and $q(t,s) = \exp\left\{\left[\frac{(3-\gamma)\xi_2^2}{2} - \frac{\phi_u \xi_u}{\gamma}\right](s-t)\right\}$. $B(t,s)$ and $a_1(t,s)$ are given in equations (A5) and (34) with s replacing T.

The expression for the indirect utility function is obtained by substituting for $C^*(s)$ in (42):

$$\begin{aligned} J(W,r,\Pi,t) &= \mathrm{E}_t\left\{\int_t^T \frac{w_t^{1-\gamma}}{1-\gamma} Q_1^{\gamma-1}(t,T)\zeta_1(t,s)^{1-\frac{1}{\gamma}}\eta_2^{\gamma-1}(t,s)ds\right\} \\ &= \frac{w_t^{1-\gamma}}{1-\gamma} Q_1^{\gamma-1}(t,T)\left\{\int_t^T E_t\left[\zeta_1(t,s)^{1-\frac{1}{\gamma}}\right] E_t\left[\eta_2^{\gamma-1}(t,s)\right] ds\right\} \\ &= \frac{w_t^{1-\gamma}}{1-\gamma} Q_1^{\gamma-1}(t,T)\int_t^T F_1(t,s)F_3(t,s)ds \\ &= \frac{w_t^{1-\gamma}}{1-\gamma} Q_1^{\gamma-1}(t,T)Q_2(t,T) \equiv \frac{w_t^{1-\gamma}}{1-\gamma}\psi_2(r,t,T), \end{aligned} \tag{B14}$$

where

$$Q_2(t,T) \;=\; \int_t^T q^{1-\gamma}(t,s)\exp\left\{\frac{1-\gamma}{\gamma}[B(t,s)r_t + a_1(t,s)]\right\} ds. \tag{B15}$$

The expression for ψ_2 is given by substituting $Q_1(t,T)$ and $Q_2(t,T)$ into its definition.

4 Proof of Theorem 4

(i) Define $G_s \equiv G[M_s, r, s] \equiv E_s \left[\frac{W^*{}_T}{\Pi_T} \frac{M_T}{M_s} \right]$ as the (real) value at time s of the optimally chosen terminal payoff $W^*{}_T$. Then, using the definition of $W^*{}_T$, equation (28),

$$G_s = \frac{W_t}{\Pi_t} \frac{F_1(s,T)}{F_1(t,T)} \frac{F_2(s,T)}{F_2(t,T)} \frac{\zeta_1(t,s)^{-\frac{1}{\gamma}}}{\eta_2(t,s)}, \tag{B16}$$

where $F_1(t,T)$ and $F_2(t,T)$ are given in equations (29) - (30). It follows from equation (A7) that the (log) instantaneous real return on optimally invested wealth is:

$$d\ln G_s = g_1(r, T-s)dt - \frac{\phi_S}{\gamma} dz_s - \left[\frac{\phi_r}{\gamma} + \left(1 - \frac{1}{\gamma}\right) B(s,T)\sigma_r \right] dz_r - \frac{\phi_\pi}{\gamma} dz_\pi - \xi_u dz_u \tag{B17}$$

where the drift term $g_1(r,s)$ is a function of current real interest rate r_s and the remaining investment horizon $T - s$.

Now consider the log real return on portfolio x^*:

$$\begin{aligned} d\ln w &= g_2(r, T-s)dt + (x_1\sigma_s - \xi_S)\, dz_s - [(x_2 B_1 + x_3 B_2)\sigma_r + \xi_r]\, dz_r \\ &- [(x_2 C_1 + x_3 C_2)\sigma_\pi + \xi_\pi]\, dz_\pi - \xi_u dz_u \end{aligned} \tag{B18}$$

Since strategy x^* yields the terminal payoff $w^*{}_T$, it follows that the coefficients in equations (B17) and (B18) are identical. Equating coefficients yields (47).

(ii) The nominal wealth process for any given portfolio strategy x is given by:

$$\frac{dW}{W} = (R_f + x'\Lambda)\, dt + x'\sigma dz, \tag{B19}$$

and the Bellman equation is

$$
\begin{aligned}
\max_x \quad & \left\{ \frac{1}{2} W^2 \left(x'\sigma\rho\sigma' x \right) J_{WW} + W x'\sigma\rho e_2 \sigma_r J_{Wr} + W\Pi x'\sigma\rho\xi J_{W\Pi} + \Pi\xi'\rho e_2\sigma_r J_{r\Pi} \right. \\
+ \quad & \left. \frac{1}{2}\Pi^2 \xi'\rho\xi J_{\Pi\Pi} + \frac{1}{2}\sigma_r^2 J_{rr} + \kappa(\bar{r} - r)J_r + W(R_f + x'\Lambda)J_W + \Pi\pi J_\Pi + J_t \right\}. \quad \text{(B20)}
\end{aligned}
$$

The first order condition is then

$$
x^* = (\sigma\rho\sigma)^{-1} \left[-\frac{J_W}{W J_{WW}} \Lambda - \frac{J_{Wr}}{W J_{WW}} \left(\sigma\rho e_2 \sigma_r \right) - \frac{J_{W\Pi}}{W J_{WW}} \left(\sigma\rho\xi\Pi \right) \right]. \quad \text{(B21)}
$$

We can directly use Theorems 2 to get $J = \frac{(W/\Pi)^{1-\gamma}}{1-\gamma} \exp^{(1-\gamma)B(t,T)r_t + a(t,T)}$. Calculating J_W, J_{WW}, $J_{W\Pi}$ and J_{Wr} and substituting them into equation (B21), we have that

$$
\begin{aligned}
x^* &= \frac{1}{\gamma} \left(\sigma\rho\sigma' \right)^{-1} \left[\Lambda + (1-\gamma)B(t,T)\sigma_r \sigma\rho e_2 - (1-\gamma)\sigma\rho\xi \right] \\
&= \frac{1}{\gamma} \left(\sigma' \right)^{-1} \left[\gamma\xi - \phi + (1-\gamma)B(t,T)\sigma_r e_2 \right] \qquad \text{(B22)}
\end{aligned}
$$

where the second equation is derived by using $\Lambda = \sigma\lambda$ and $\left(\sigma\rho\sigma'\right)^{-1} = \left(\sigma'\right)^{-1} \rho^{-1}\sigma^{-1}$. Substituting $(\sigma)^{-1}$ in, we can verify that x^* in (48) is equal to the expression in (47).

Alternatively, we conjecture that $J(W, r, \Pi, t) = \frac{(W/\Pi)^{1-\gamma}}{1-\gamma} \exp^{(1-\gamma)[c(t,T)r_t + d(t,T)]}$ where $c(t,T)$ and $d(t,T)$ will be determined by method of undetermined coefficient, then the Bellman equation (B20) and the first order condition are simplified into:

$$
\begin{aligned}
0 &= \left(1 - \kappa c(t,T) + \partial c/\partial t\right) r + \partial d/\partial t - \xi'\lambda + \kappa\bar{r}c(t,T) + \frac{1}{2}(1-\gamma)\sigma_r^2 c^2(t,T) \\
&- \frac{\gamma - 2}{2}\xi'\rho\xi - (1-\gamma)\sigma_r c(t,T)\xi'\rho e_2 + \frac{1}{2}\gamma(x^*)'\sigma\rho\sigma(x^*), \qquad \text{(B23)} \\
x^* &= \frac{1}{\gamma} \left(\sigma'\right)^{-1} \left[\gamma\xi - \phi + (1-\gamma)c(t,T)\sigma_r e_2 \right]. \qquad \text{(B24)}
\end{aligned}
$$

Therefore,

$$\begin{aligned}
\partial c/\partial t &= \kappa c(t,T) - 1 && \text{(B25)}\\
\partial d/\partial t &= \xi'\lambda - \kappa\bar{r}c(t,T) - \frac{1}{2}(1-\gamma)\sigma_r^2 c^2(t,T) + \frac{\gamma-2}{2}\xi'\rho\xi \\
&+ (1-\gamma)\sigma_r c(t,T)\xi'\rho e_2 - \frac{1}{2}\gamma(x^*)'\sigma\rho\sigma(x^*) && \text{(B26)}
\end{aligned}$$

with terminal conditions $c(T,T) = 0$ and $d(T,T) = 0$. Solving this system of ordinary differential equations gives us $c(t,T) = B(t,T)$ and $d(t,T) = a(t,T)$. This result then provides a double check on the validity of previous theorems.

5 Proof of Theorem 5

Define $G_s \equiv G(M_s, r, s) \equiv \mathrm{E}_s\left[\int_s^T (M_u/M_s)\, C^*(u)/\Pi_u du\right]$. Then, using (40) and (B13) to substitute for $C^*(u)$,

$$\begin{aligned}
G_s &= w_t \frac{\zeta_1^{-\frac{1}{\gamma}}(t,s)}{\eta_2(t,s)} Q_1(t,T)^{-1}\mathrm{E}_s\left[\int_s^T \frac{(M_u/M_s)^{1-\frac{1}{\gamma}}}{\Pi_{u,2}}du\right] \\
&= w_t \frac{\zeta_1^{-\frac{1}{\gamma}}(t,s)}{\eta_2(t,s)} Q_1(t,T)^{-1} Q_1(s,T) && \text{(B27)}
\end{aligned}$$

where $Q_1(s,T) \equiv \mathrm{E}_s\left[\int_s^T F_1(s,u)F_2(s,u)du\right] = \int_s^T q(s,u)\exp\left\{\frac{1-\gamma}{\gamma}\left[B(s,u)r_s + a_1(s,u)\right]\right\} du.$

Therefore, using Ito's Lemma and definitions of $B(s,t)$ and $a_1(s,t)$

$$d\ln G_s = g_3(r,s)dt - \frac{1}{\gamma}\phi_S dz_S - \left[\frac{\phi_r}{\gamma} + \left(1-\frac{1}{\gamma}\right)\hat{B}\sigma_r\right]dz_r - \frac{\phi_\pi}{\gamma}dz_\Pi - \xi_u dz_u. \qquad \text{(B28)}$$

The remaining proof is then identical to that in Theorem 4.

6 Proof of Proposition 2

Suppose without loss of generality that $\kappa > \alpha$. Then, allowing bond maturity, τ, to vary, a bond's return loading on the real interest rate, B, is an increasing convex function of its loading on inflation, C. Then the set of achievable factor loading combinations is defined by $\mathcal{S} = \left\{(B, C) | \frac{C_{\tau max}}{B_{\tau max}} B \geq C \geq \frac{1-(1-\kappa B)^{\frac{\alpha}{\kappa}}}{\alpha}\right\}$. Any point in this set, (B_p, C_p) can be achieved by a convex combination of cash $(0, 0)$ and the loadings of a single bond with maturity $\tau^* \leq \tau_{max}$ such that $\frac{C_{\tau^*}}{B_{\tau^*}} = \frac{C_p}{B_p}$, where the weight on the bond is $\frac{B_p}{B_{\tau^*}}$.

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Parameter	Estimate		Standard Error
Stock Return Process: $\frac{dS}{S} = (R_f + \lambda_S \sigma_S)dt + \sigma_S dz_S$			
σ_S	0.158		
λ_S	0.343		0.057
Real Interest Rate: $dr = \kappa(\bar{r} - r)dt + \sigma_r dz_r$			
$\bar{r}$	0.012	$(0.017)^*$	0.002
κ	0.631	$(0.105)^*$	0.003
σ_r	0.026	$(0.026)^*$	0.004
λ_r	-0.209		0.077
Expected Inflation: $d\pi = \alpha(\bar{\pi} - \pi)dt + \sigma_\pi dz_\pi$			
$\bar{\pi}$	0.054		0.023
α	0.027		0.009
σ_π	0.014		0.005
λ_π	-0.105		0.005
Realized Inflation: $d\Pi/\Pi = \pi dt + \sigma_\Pi dz_\Pi$			
σ_Π	0.013		0.005
Parameters of the Pricing Kernel Process: Φ			
ϕ_S	-0.333		
ϕ_r	0.170		
ϕ_π	0.120		
ρ_{Sr}	-0.129		
$\rho_{S\pi}$	-0.024		
$\rho_{r\pi}$	-0.061		0.002

Standard Errors of Bond Yield Errors by Bond Maturities			
1 month	0.0099	3 month	0.0066
6 month	0.0043	9 month	0.0030
1 year	0.0020	2 year	0.0004
3 year	0.0003	4 year	0.0002
5 year	$<$ 0.0001	7 year	0.0005
10 year	0.0012		

Table 1:

Estimates of Model Parameters

Maximum Likelihood parameter estimates for the joint process of real interest rate, expected rate of inflation and stock returns estimated by Kalman Filter using monthly yields of eleven U.S. constant maturity treasury bonds, CPI data and CRSP value-weighted stock returns for the period from January 1970 to December 1995.

$*$ Parameter estimates using annual nominal interest rate and CPI data from 1890 to 1985.

Coefficient	Estimate	t-statistics	R^2	Adj. R^2	No. Obs.
	Regression: $\frac{\Delta\Pi}{\Pi}^u = b_S \Delta z_S + b_r \Delta z_r + b_\pi \Delta z_\pi + \epsilon_1$				
b_S	-0.0010	-1.21	-0.02	-0.03	311
b_r	0.0006	0.76			
b_π	0.0006	0.84			
$\hat{\sigma}_\epsilon$	0.0131	24.82*			
	Regression: $\frac{\Delta\Pi}{\Pi}^u = b_0 + b_S \Delta z_S + b_r \Delta z_r + b_\pi \Delta z_\pi + \epsilon_2$				
b_S	-0.0010	-1.26	0.02	0.002	311
b_r	0.0005	0.65			
b_π	0.0006	0.79			
b_0	-0.0023	-3.11*			
$\hat{\sigma}_\epsilon$	0.0129	24.78*			

Table 2:

Projection of Unexpected Inflation on State Variable Innovations

The time series of $\frac{\Delta\Pi}{\Pi}^u \equiv \sigma_\Pi \Delta z_\Pi(t)$, $dz_\S$, dz_r and dz_π are constructed using equations (66) to (69). The parameters are estimated by OLS with and without an intercept. Note that $\hat{\sigma}^2_\epsilon = s^2 = \frac{\hat{\epsilon}'\hat{\epsilon}}{N-K}$ where N is the number of observations and K is the number of regressors. The standard deviation of σ_ϵ is $\sqrt{\frac{\sigma^2_\epsilon}{2(N-K)}}$ which is derived from $var(s^2) = \frac{2\sigma^4_\epsilon}{N-K}$ via the delta method.

*significant at 1% level.

Horizon		Risk Aversion Parameter, γ						
		0.8	1.5	3.0	5.0	7.0	10.0	15.0
1 month								
	CE	1.01	1.01	1.00	1.00	1.00	1.00	1.00
	EGR	1.00	1.00	1.00	1.00	1.00	1.00	1.00
	x_S	2.51	1.34	0.67	0.40	0.29	0.20	0.13
	B_p	-8.37	-4.50	-2.29	-1.41	-1.03	-0.74	-0.52
	C_p	-9.64	-5.14	-2.57	-1.54	-1.10	-0.77	-0.51
1 year								
	CE	1.13	1.08	1.05	1.04	1.04	1.03	1.03
	EGR	1.00	1.00	1.00	1.00	1.00	1.00	1.00
	x_S	2.51	1.34	0.67	0.40	0.29	0.20	0.13
	B_p	-8.21	-4.72	-2.73	-1.94	-1.59	-1.34	-1.14
	C_p	-9.64	-5.14	-2.57	-1.54	-1.10	-0.77	-0.51
5 years								
	CE	1.77	1.43	1.27	1.21	1.18	1.16	1.14
	EGR	1.00	1.00	1.00	1.01	1.01	1.02	1.03
	x_S	2.51	1.34	0.67	0.40	0.29	0.20	0.13
	B_p	-8.01	-4.98	-3.25	-2.56	-2.26	-2.04	-1.86
	C_p	-9.64	-5.14	-2.57	-1.54	-1.10	-0.77	-0.51
10 years								
	CE	3.05	2.00	1.58	1.43	1.37	1.32	1.29
	EGR	1.00	1.00	1.01	1.02	1.03	1.05	1.09
	x_S	2.51	1.34	0.67	0.40	0.29	0.20	0.13
	B_p	-8.00	-5.00	-3.29	-2.61	-2.32	-2.10	-1.92
	C_p	-9.64	-5.14	-2.57	-1.54	-1.10	-0.77	-0.51
20 years								
	CE	9.02	3.92	2.43	2.01	1.84	1.72	1.63
	EGR	1.00	1.00	1.02	1.05	1.08	1.13	1.21
	x_S	2.51	1.34	0.67	0.40	0.29	0.20	0.13
	B_p	-8.00	-5.00	-3.29	-2.61	-2.32	-2.10	-1.93
	C_p	-9.64	-5.14	-2.57	-1.54	-1.10	-0.77	-0.51

Table 3:

Unconstrained Optimal Portfolio Strategy
for Investor with Terminal Wealth Objective
($\kappa = 0.631$)

This table reports the unconstrained optimal strategy for an investor with different values of risk aversion parameter, γ, and investment horizon, T. B_p (C_p) is the sensitivity of the optimal portfolio to innovations in r (π); x_S is the proportional portfolio allocation to the stock. CE is the certainty equivalent wealth at the horizon. The calculation is based on the parameter estimates in Table 1 for $\sigma = 3\%$. EGR is the ratio of the CE under the optimal strategy to the CE under a myopic strategy: it is calculated from equation (70). κ is the mean reversion coefficient for the shadow real interest rate process.

Horizon		Risk Aversion Parameter, γ						
		0.8	1.5	3.0	5.0	7.0	10.0	15.0
1 month								
	CE	1.01	1.01	1.00	1.00	1.00	1.00	1.00
	EGR	1.00	1.00	1.00	1.00	1.00	1.00	1.00
	x_S	1.00	1.00	0.64	0.40	0.29	0.20	0.13
	x_B	0.00	0.00	0.36	0.60	0.71	0.80	0.87
	τ	n.a.	n.a.	7.49	3.08	1.86	1.09	0.62
1 year								
	CE	1.07	1.06	1.05	1.04	1.04	1.03	1.03
	EGR	1.00	1.00	1.00	1.00	1.00	1.00	1.00
	x_S	1.00	1.00	0.63	0.40	0.29	0.20	0.14
	x_B	0.00	0.00	0.37	0.60	0.71	0.80	0.86
	τ	n.a.	n.a.	7.24	3.34	2.35	1.71	1.27
5 years								
	CE	1.36	1.31	1.23	1.18	1.16	1.14	1.12
	EGR	1.00	1.00	1.00	1.00	1.00	1.00	1.01
	x_S	1.00	1.00	0.62	0.39	0.29	0.20	0.14
	x_B	0.00	0.00	0.38	0.61	0.71	0.80	0.86
	τ	n.a.	n.a.	7.00	3.54	2.71	2.18	1.81
10 years								
	CE	1.81	1.68	1.47	1.37	1.31	1.27	1.22
	EGR	1.00	1.01	1.00	1.00	1.00	1.01	1.03
	x_S	1.00	1.00	0.62	0.39	0.28	0.20	0.14
	x_B	0.00	0.00	0.38	0.61	0.72	0.80	0.86
	τ	n.a.	n.a.	7.00	3.54	2.73	2.21	1.85
20 years								
	CE	3.22	2.77	2.16	1.82	1.65	1.49	1.31
	EGR	1.00	1.00	1.00	1.00	1.00	1.01	1.02
	x_S	1.00	1.00	0.63	0.40	0.29	0.21	0.14
	x_B	0.00	0.00	0.37	0.60	0.71	0.79	0.86
	τ	n.a.	n.a.	7.00	3.55	2.74	2.22	1.85

Table 4:

Constrained Optimal Portfolio Strategy
for Investor with Terminal Wealth Objective
($\kappa = 0.631$)

This table reports the constrained optimal strategy for an investor with different values of the risk aversion parameter, γ, and investment horizon, T. x_S and x_B are the proportional allocations to the stock and the optimal bond. τ is the maturity of the optimally chosen bond. CE is the certainty equivalent wealth at the horizon. The calculation is based on the parameter estimates in Table 1 for $r_0 = 3\%$. EGR is the ratio of the CE under the optimal strategy to the CE under a myopic strategy: it is calculated numerically. κ is the mean reversion coefficient for the shadow real interest rate process.

Horizon		Risk Aversion Parameter, γ						
		0.8	1.5	3.0	5.0	7.0	10.0	15.0
1 month								
	CE	1.01	1.01	1.00	1.00	1.00	1.00	1.00
	EGR	1.00	1.00	1.00	1.00	1.00	1.00	1.00
	x_S	2.52	1.34	0.67	0.40	0.29	0.20	0.13
	B_p	-8.37	-4.50	-2.29	-1.41	-1.03	-0.75	-0.52
	C_p	-9.63	-5.14	-2.57	-1.54	-1.10	-0.77	-0.51
1 year								
	CE	1.14	1.09	1.06	1.05	1.04	1.04	1.04
	EGR	1.00	1.00	1.00	1.00	1.00	1.00	1.00
	x_S	2.52	1.34	0.67	0.40	0.29	0.20	0.13
	B_p	-8.15	-4.79	-2.87	-2.10	-1.77	-1.53	-1.33
	C_p	-9.63	-5.14	-2.57	-1.54	-1.10	-0.77	-0.51
5 years								
	CE	1.85	1.51	1.35	1.28	1.26	1.24	1.22
	EGR	1.00	1.00	1.01	1.03	1.05	1.08	1.13
	x_S	2.52	1.34	0.67	0.40	0.29	0.20	0.13
	B_p	-7.42	-5.77	-4.83	-4.45	-4.29	-4.17	-4.08
	C_p	-9.63	-5.14	-2.57	-1.54	-1.10	-0.77	-0.51
10 years								
	CE	3.31	2.27	1.83	1.68	1.61	1.57	1.53
	EGR	1.00	1.01	1.08	1.19	1.33	1.56	2.05
	x_S	2.52	1.34	0.67	0.40	0.29	0.20	0.13
	B_p	-6.84	-6.54	-6.36	-6.29	-6.27	-6.24	-6.23
	C_p	-9.63	-5.14	-2.57	-1.54	-1.10	-0.77	-0.51
20 years								
	CE	10.19	5.02	3.35	2.84	2.64	2.49	2.37
	EGR	1.01	1.04	1.39	2.19	3.52	7.24	24.41
	x_S	2.52	1.34	0.67	0.40	0.29	0.20	0.13
	B_p	-6.30	-7.26	-7.81	-8.03	-8.12	-8.19	-8.25
	C_p	-9.63	-5.14	-2.57	-1.54	-1.10	-0.77	-0.51

Table 5:

Unconstrained Optimal Portfolio Strategy
for Investor with Terminal Wealth Objective
($\kappa = 0.105$)

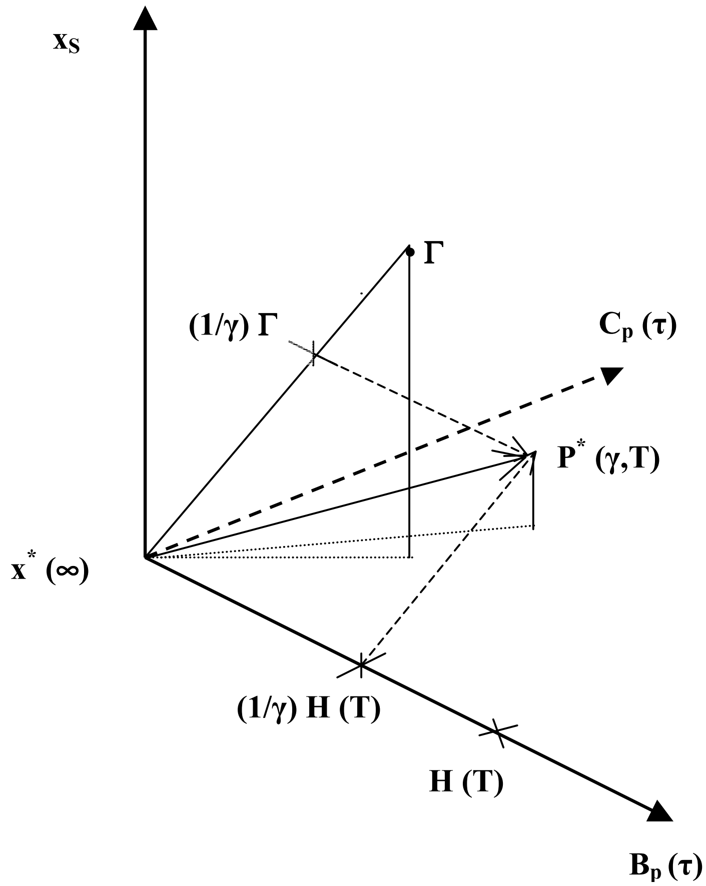
This table reports the unconstrained optimal strategy for an investor with different values of risk aversion parameter, γ, and investment horizon, T. B_p (C_p) is the sensitivity of the optimal portfolio to innovations in r (π); x_S is the proportional portfolio allocation to the stock. CE is the certainty equivalent wealth at the horizon. The calculation is based on the parameter estimates in Table 1 for $\underline{r} = 3\%$. EGR is the ratio of the CE under the optimal strategy to the CE under a myopic strategy: it is calculated from equation (70). κ is the mean reversion coefficient for the shadow real interest rate process.

Horizon		Risk Aversion Parameter, γ						
		0.8	1.5	3.0	5.0	7.0	10.0	15.0
1 month								
	CE	1.01	1.01	1.01	1.00	1.00	1.00	1.00
	EGR	1.00	1.00	1.00	1.00	1.00	1.00	1.00
	x_S	1.00	1.00	0.66	0.40	0.29	0.20	0.13
	x_B	0.00	0.00	0.34	0.44	0.31	0.22	0.15
	τ	n.a.	n.a.	9.71	3.72	3.72	3.72	3.72
1 year								
	CE	1.08	1.07	1.06	1.05	1.04	1.04	1.04
	EGR	1.00	1.00	1.01	1.00	1.00	1.00	1.00
	x_S	1.00	1.00	0.64	0.40	0.29	0.20	0.14
	x_B	0.00	0.00	0.36	0.60	0.71	0.80	0.86
	τ	n.a.	n.a.	10.81	3.69	2.40	1.72	1.30
5 years								
	CE	1.48	1.42	1.33	1.27	1.24	1.21	1.18
	EGR	1.04	1.04	1.01	1.01	1.04	1.06	1.10
	x_S	1.00	1.00	0.55	0.38	0.28	0.20	0.13
	x_B	0.00	0.00	0.45	0.62	0.72	0.80	0.87
	τ	n.a.	n.a.	12.60	7.15	5.61	4.72	4.21
10 years								
	CE	2.14	1.97	1.73	1.56	1.46	1.27	1.28
	EGR	1.06	1.10	1.03	1.11	1.20	1.36	1.72
	x_S	1.00	0.66	0.47	0.32	0.23	0.16	0.09
	x_B	0.00	0.34	0.53	0.68	0.77	0.84	0.91
	τ	n.a.	27.77	13.20	9.07	7.72	6.94	6.58
20 years								
	CE	4.39	3.61	2.63	2.00	1.66	1.37	n.a.
	EGR	1.11	1.24	1.12	1.54	2.21	3.97	n.a.
	x_S	1.00	0.63	0.40	0.24	0.16	0.09	n.a.
	x_B	0.00	0.37	0.60	0.76	0.84	0.91	n.a.
	τ	n.a.	25.98	13.53	10.26	9.15	8.50	n.a.

Table 6:

Constrained Optimal Portfolio Strategy
for Investor with Terminal Wealth Objective
($\kappa = 0.105$)

This table reports the constrained optimal strategy for an investor with different values of the risk aversion parameter, γ, and investment horizon, T. x_S and x_B are the proportional allocations to the stock and the optimal bond. τ is the maturity of the optimally chosen bond. CE is the certainty equivalent wealth at the horizon. The calculation is based on the parameter estimates in Table 1 for $\underline{r} = 3\%$. EGR is the ratio of the CE under the optimal strategy to the CE under a myopic strategy: it is calculated numerically. κ is the mean reversion coefficient for the shadow real interest rate process.



Decomposition of Unconstrained Optimal Portfolio Loadings

The optimal portfolio factor loadings, $P^*(\gamma,T)$, is decomposed into its constituents, $x^*(\infty)$, Γ and $H(T)$. $x^*(\infty)$ is the vector of portfolio factor loadings of a highly risk averse investor, which most closely mimics those of an index bond with maturity T. $\Gamma=[(\xi_S - \phi_S)/\sigma_S\ ,\ (\xi_r - \phi_r)/\sigma_r\ ,\ (\xi_\pi - \phi_\pi)/\sigma_\pi]$ is the vector of factor loadings of the mean variance tangency portfolio and $H(T)$ is the hedge portfolio for an investor with horizon T, which equals $x^*(\infty)$. $P^*(\gamma,T) = x^*(\infty) + (1/\gamma)\ \Gamma - (1/\gamma)\ H$ is the optimal portfolio for an investor with risk aversion and horizon T.

Figure 1

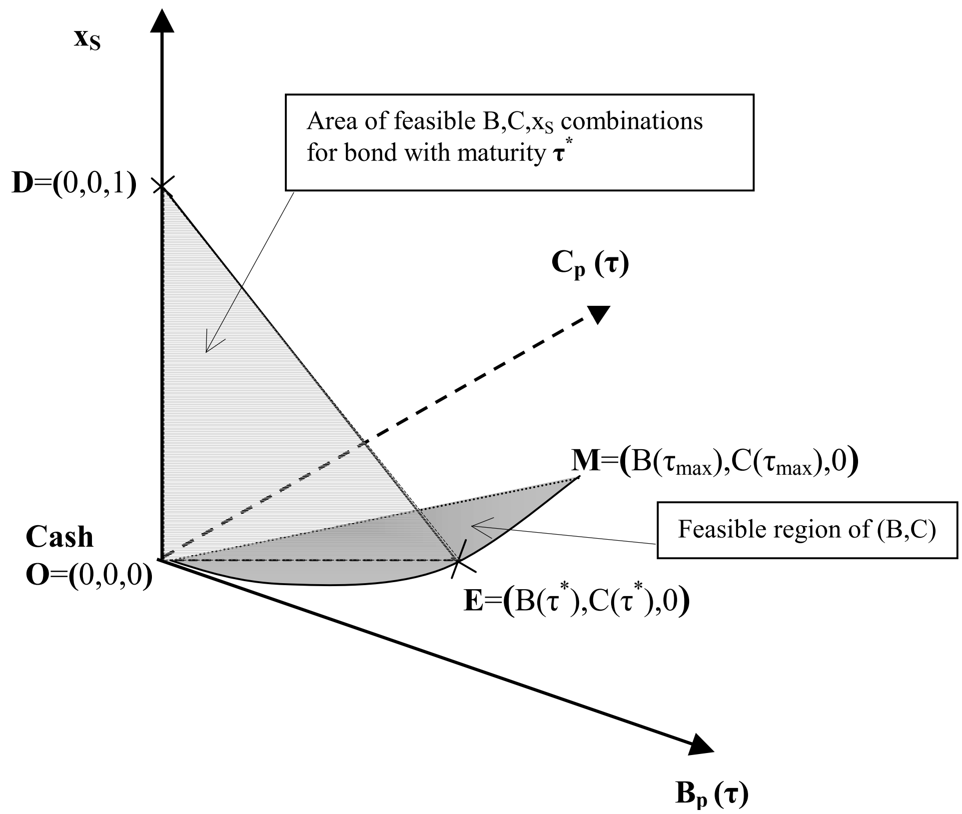
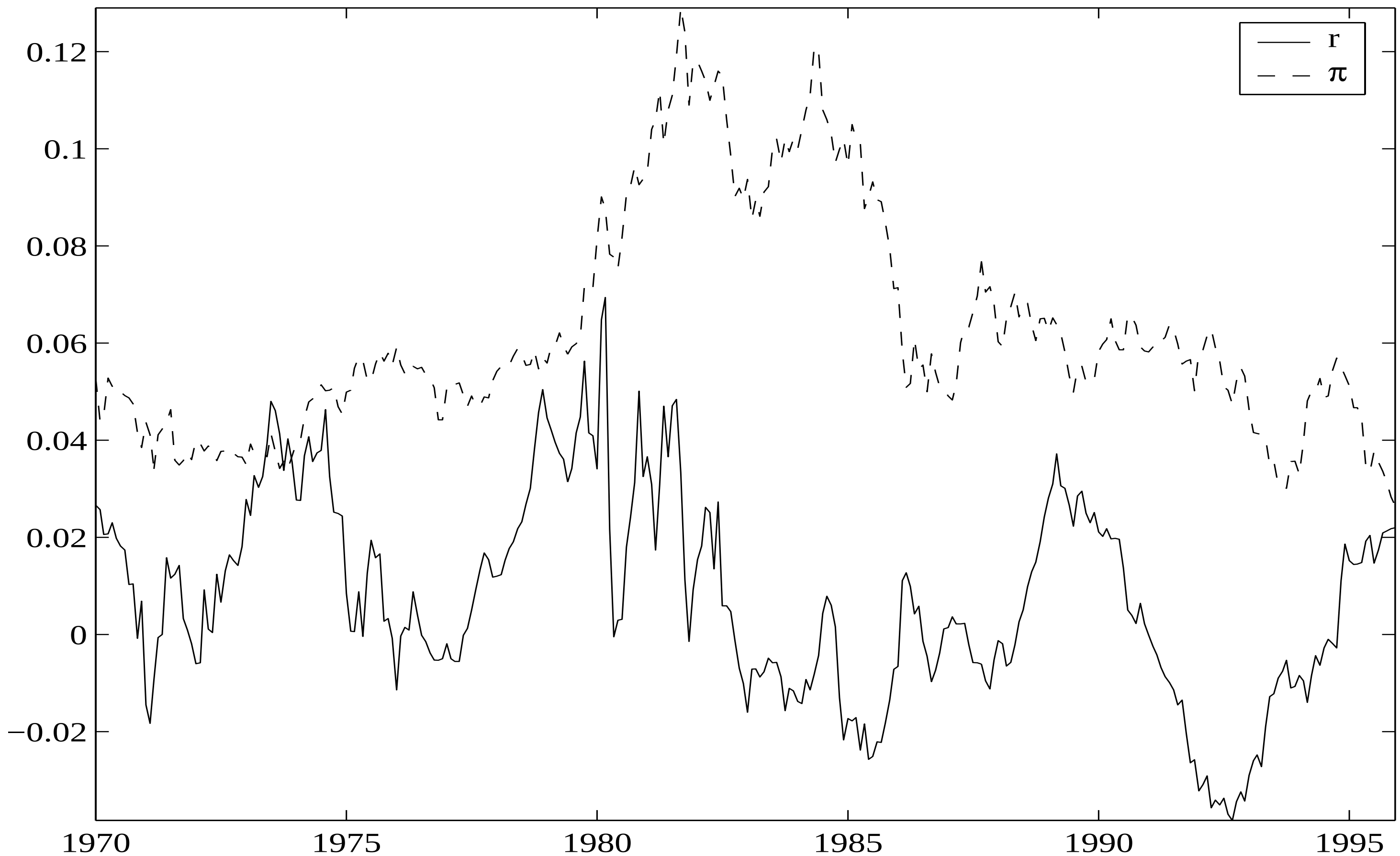


Illustration of the Feasible Region for a Portfolio Strategy with Short Sales Constraint

OEM is the locus of **(**B(τ),C(τ)**)** combinations as τ is varied from zero to τ_{max}. M corresponds to **(**B(τ_{max}),C(τ_{max})**).** The area OEM is the feasible region of (B,C) combinations that are attainable with cash and bonds. Given a selected bond maturity τ^*, the area DOE is the feasible region of (B,C,x_S) combinations.

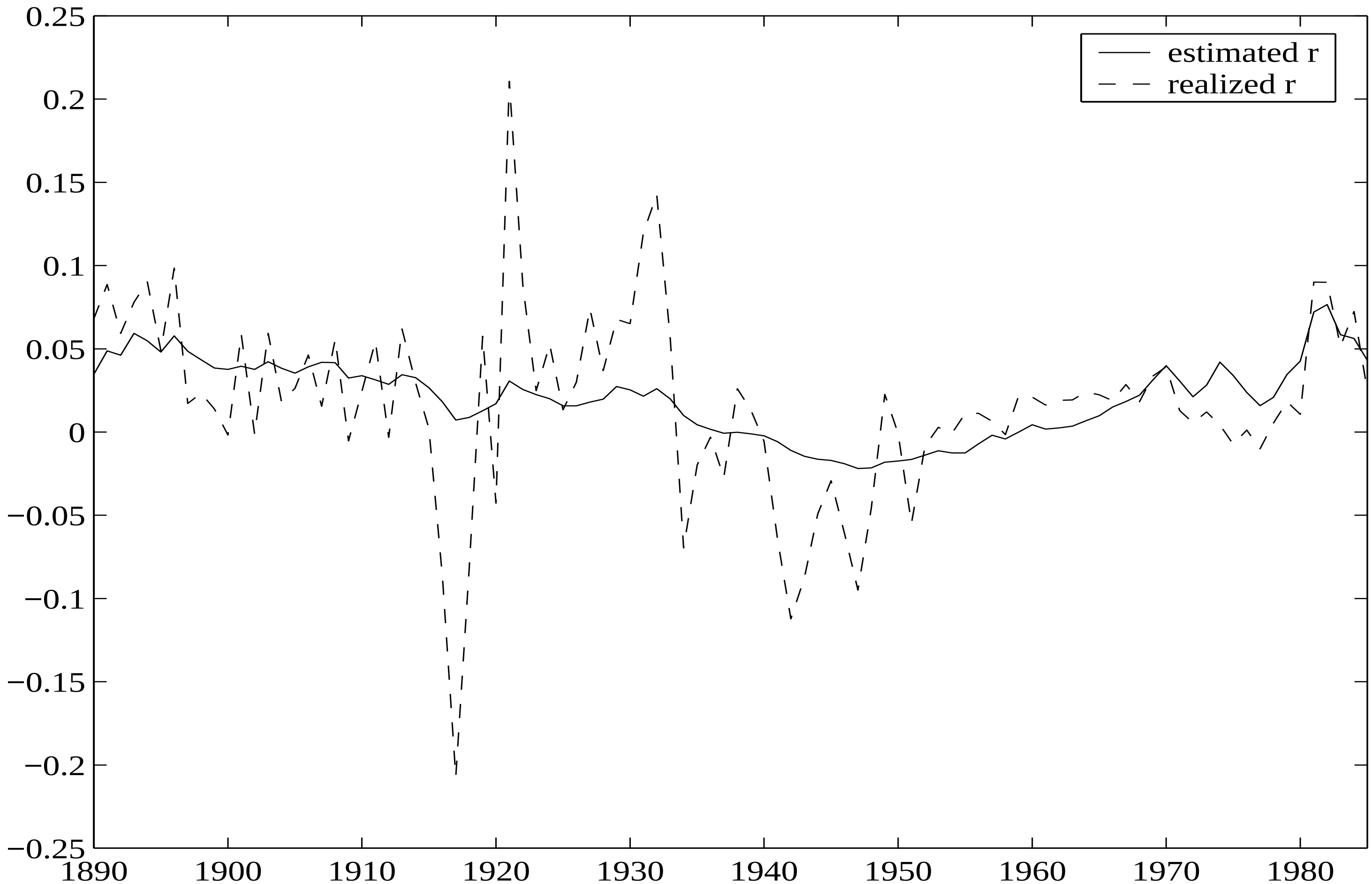
Figure 2



Estimated State Variables, r and π

This figures shows the estimated state variables, r, the real interest rate, and π, the expected rate of inflation, derived from the Kalman filter. The sample period is January 1970 to December 1995. The series are estimated using observations on monthly eleven U.S. treasury constant maturity bond yields and U.S. CPI data.

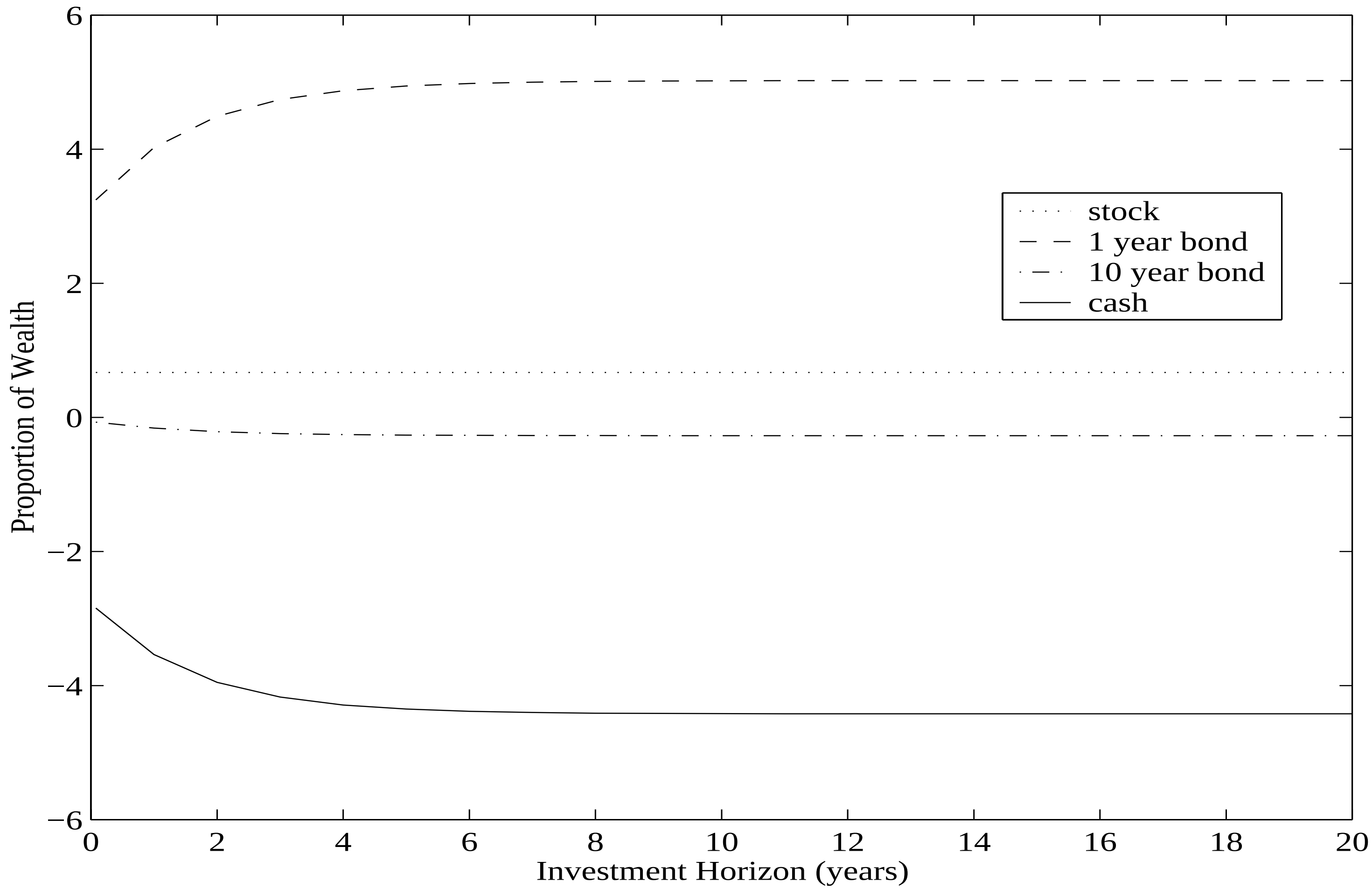
Figure 3



Estimated State Variables, r and its Realized Counterpart

This figures shows the estimated state variables, r, the real interest rate derived from the Kalman filter, and the realized real interest rate derived by subtracting actual inflation rate from the nominal interest rate. The sample period is 1890 to 1985. The series are estimated using observations on annual U.S. nominal interest rate and U.S. CPI data.

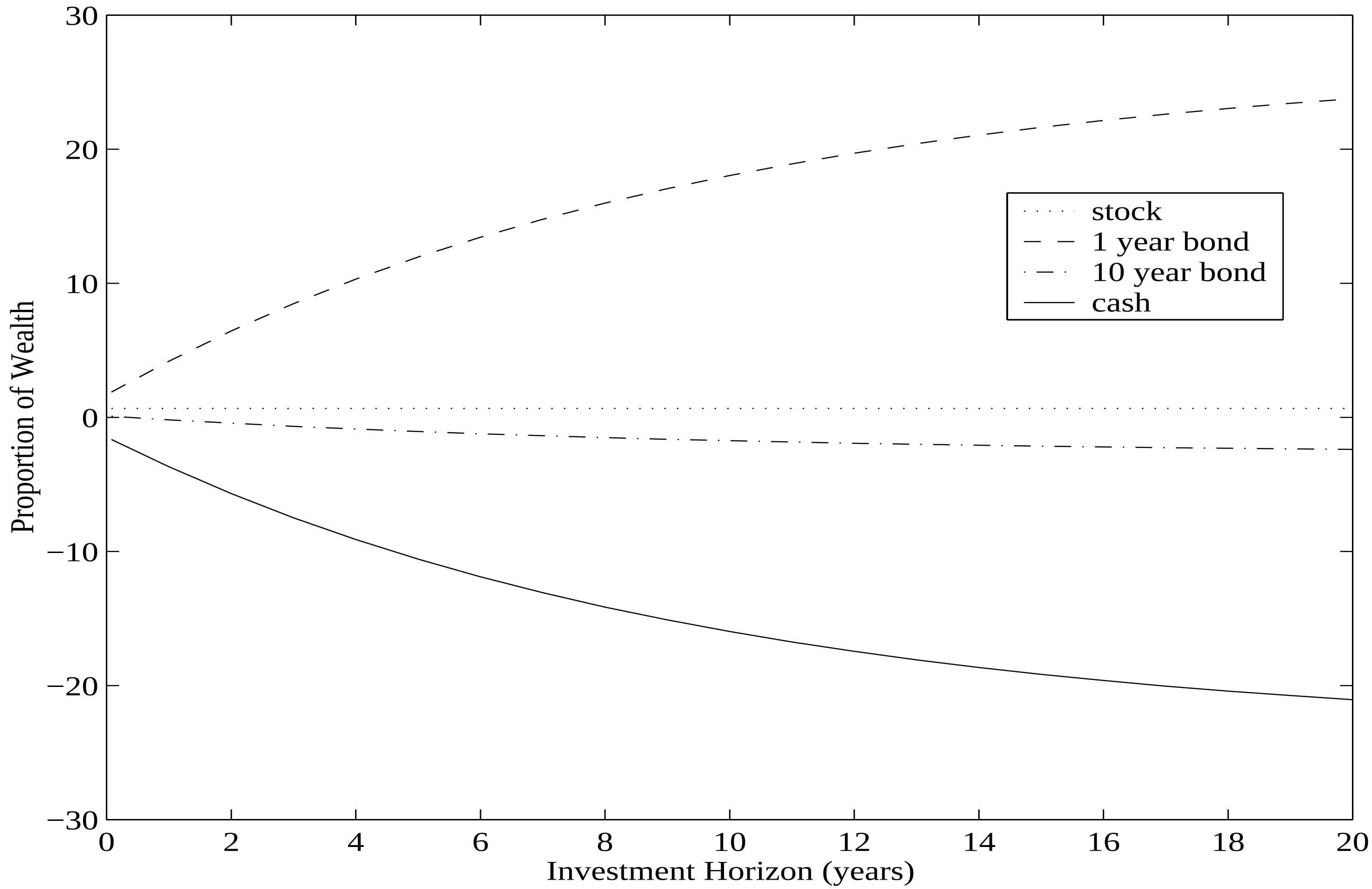
Figure 4



Unconstrained Optimal Asset Allocation for Different Horizons
($\gamma = 3$, $\kappa = 0.631$ and $r_0 = 3\%$)

The unconstrained optimal proportion of wealth invested in stock, 1 year bond, 10 year bond and cash. The optimal allocations are derived using the parameter estimates in Table 1 for $r_0 = 3\%$.

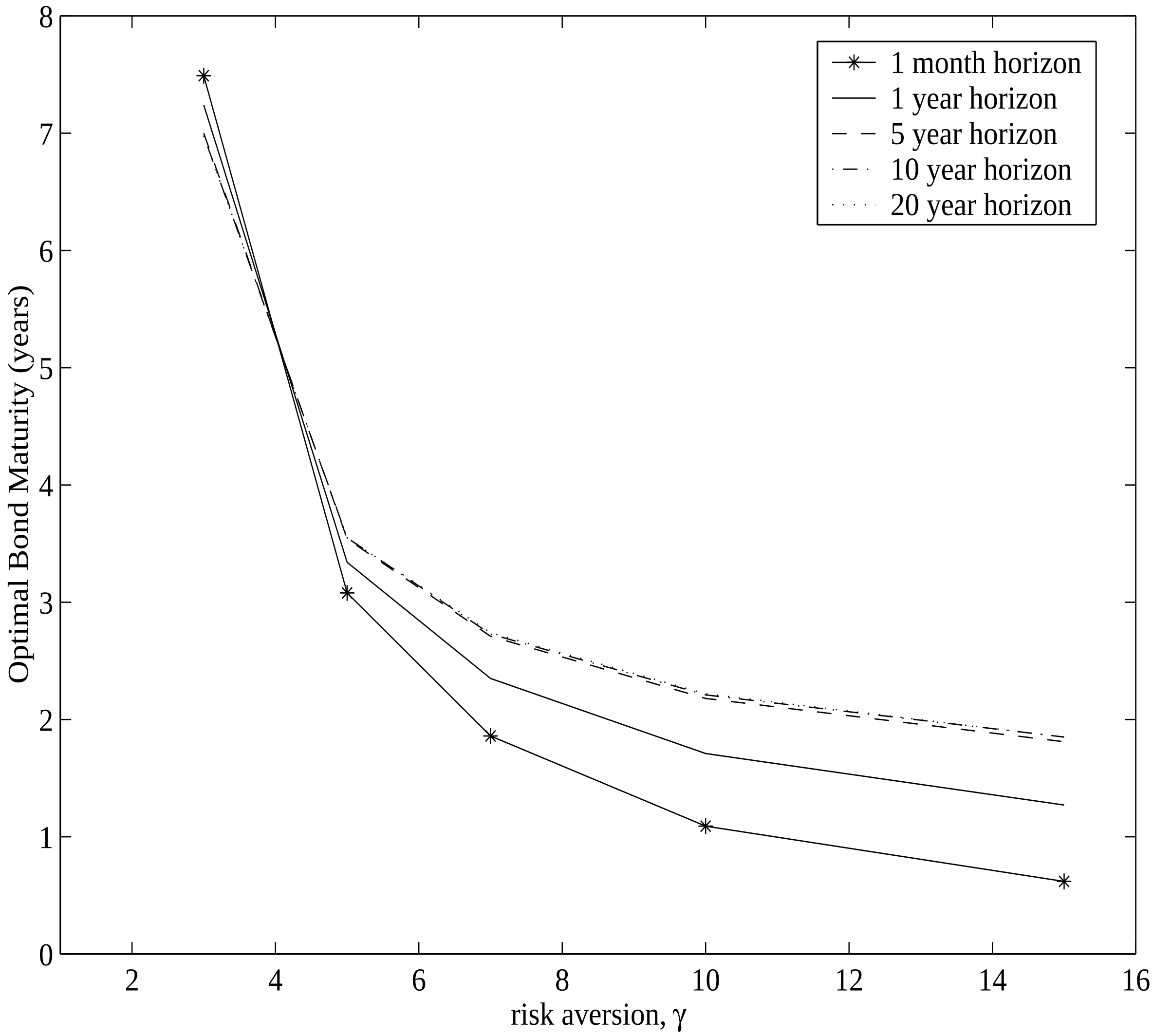
Figure 5



Unconstrained Optimal Asset Allocation for Different Horizons
($\gamma = 10$, $\kappa = 0.105$ and $r_0 = 3\%$)

This figures plots the unconstrained optimal proportion of wealth invested in stock, 1 year bond, 10 year bond and cash. The optimal allocations are derived using the parameter estimates in Table 1.

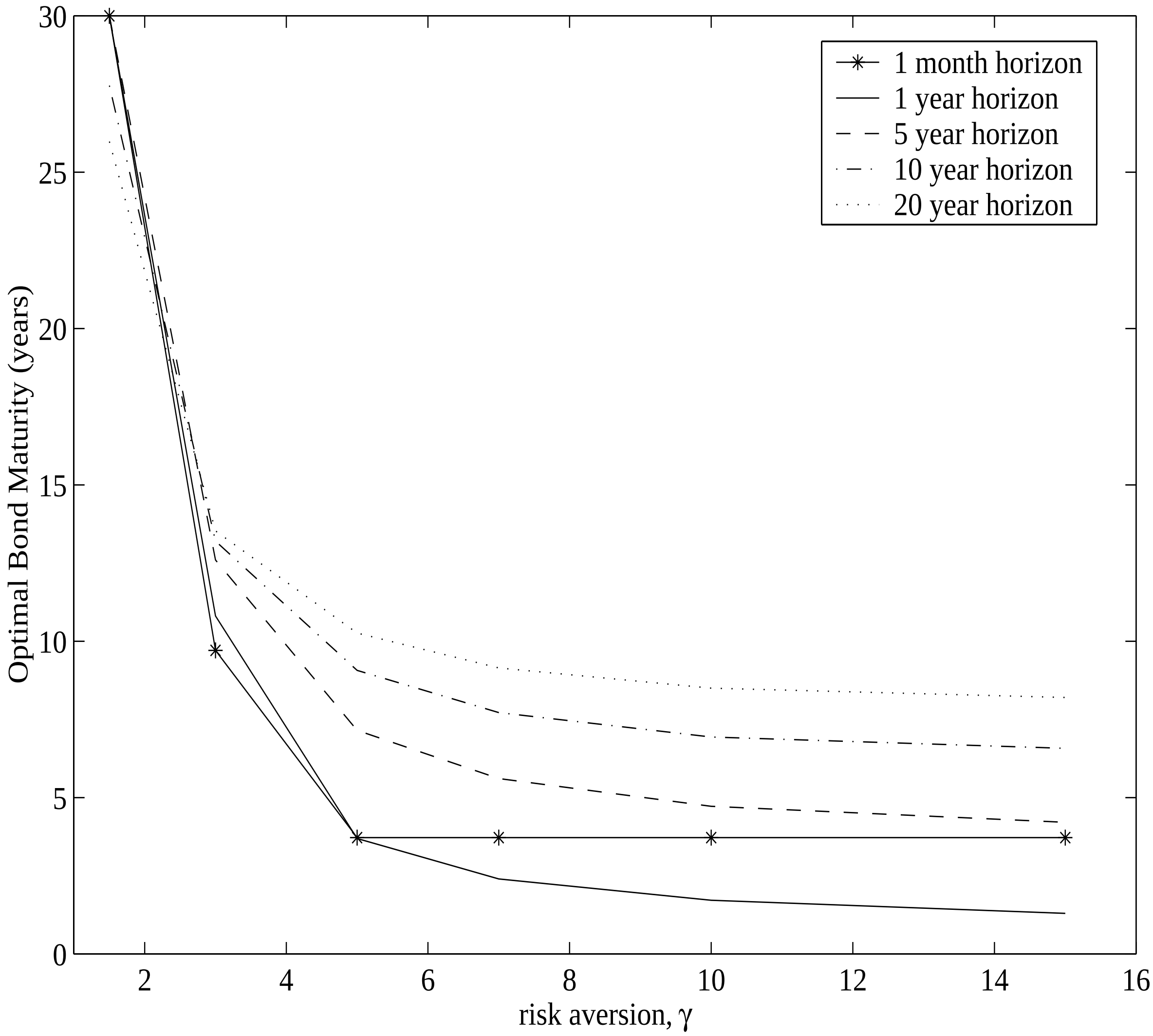
Figure 6



Optimal Bond Maturity and Risk Aversion for Different Horizons
($r_0 = 3\%$, $\kappa = 0.631$)

This figure plots the optimal bond maturity chosen by an investor who faces borrowing and short sales constraints for different values of the risk aversion parameter, γ, and investment horizon.

Figure 7



Optimal Bond Maturity and Risk Aversion for Different Horizons
($r_0 = 3\%$, $\kappa = 0.105$)

This figure plots the optimal bond maturity chosen by an investor who faces borrowing and short sales constraints for different values of the risk aversion parameter, γ, and investment horizon.

Figure 8