# Risk Aversion and Optimal Portfolio Policies in Partial and General Equilibrium Economies*

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**Abstract**

In this article, we examine analytically the optimal consumption and portfolio policies in an economy with incomplete financial markets where agents have power utility over intermediate consumption and bequest, and face portfolio constraints and a stochastic investment opportunity set. The source of changes in the investment opportunity set could be a stochastic instantaneous interest rate, stochastic volatility, and/or a stochastic risk premium. We find analytically the conditions under which investment in the risky asset can increase with risk aversion. We then nest this portfolio problem in a general equilibrium setting (for a production economy and also for an exchange economy) with multiple agents who differ in their degree of risk aversion. We derive the optimal portfolio policies when the evolution of the investment opportunity set is determined endogenously and also characterize explicitly the interest rate, stock price and risk premium in general equilibrium. The exact local comparative statics and approximate but analytical expressions for the optimal policies are obtained by developing a method based on perturbation analysis to expand around the solution for an investor with log utility.

*JEL classification*: G12, G11, D52, C63.
*Key words*: Asset allocation, stochastic investment opportunities, incomplete markets, borrowing constraints, asymptotic analysis.

## Contents

# 1 Introduction

In this article, our objective is to examine analytically the optimal portfolio rules in economies where the investment opportunity set is stochastic, agents have power utility over intermediate consumption, and financial markets are incomplete. We study the optimal portfolio policies in a partial-equilibrium single-agent setting and also in a general equilibrium economy. For both the partial- and general-equilibrium analysis, we consider the case without portfolio constraints and also the case with portfolio constraints. We rely on perturbation analysis, which allows us to obtain in closed-form the exact local comparative statics and the approximate portfolio policies.

Our contribution to the literature on portfolio selection is on three fronts. One, in a partial-equilibrium economy we obtain explicit expressions for the portfolio rules when there is an arbitrary vector process driving the stochastic investment opportunities and there are constraints on portfolio positions. In our model, we allow for intermediate consumption without requiring financial markets to be complete. This analysis nests the optimal portfolio policies for the cases where the source of changes in the investment environment is (i) stochastic volatility; (ii) stochastic interest rates, (iii) stochastic mean, and/or (iv) stochastic risk premium.

Two, we embed the above analysis in a general equilibrium setting with heterogenous agents, where the evolution of the investment opportunity set is now determined endogenously rather than being specified exogenously. We first analyze portfolio rules in a constant-returns-to-scale production economy with multiple agents who differ in their degree of risk aversion, and face borrowing constraints. In this economy, the interest rate process is determined endogenously. Following this, we analyze portfolio decisions in an exchange economy where, in addition to an endogenous interest rate, also the volatility and expected return on the risky asset are endogenous.

Our last contribution is methodological. Using perturbation or asymptotic analysis, we provide a general framework for characterizing explicitly optimal portfolio policies and equilibrium with incomplete financial markets. The basic idea of asymptotic methods is to formulate a general problem, find a particular case that has a known solution, and use this as a starting point for computing the solution to nearby problems.[1] In the context of portfolio problems, the insight

[1] A description of the application of this method to problems in economics, along with a discussion of the non-local properties of the approximate solution, can be found in Judd (1996) and Judd (1998, Chapters 13–15).

is that the solution for the investor with log utility provides a convenient starting point for the expansion.

We now discuss the existing literature on dynamic portfolio choice, and its relation to our analysis. In this discussion, we focus on the work studying intertemporal hedging demands arising from a stochastic investment opportunity set.[2] This literature can be divided into four strands:
(i) Single agent, without portfolio constraints (partial equilibrium);
(ii) Single agent, with portfolio constraints (partial equilibrium);
(iii) Multiagent, without portfolio constraints (general equilibrium); and
(iv) Multiagent, with portfolio constraints (general equilibrium);

**Single agent analysis without portfolio constraints**

Merton (1969, 1971) shows that in an environment where investment opportunities vary over time, investors optimizing over a single period will choose portfolios that are different from investors optimizing over multiple periods. This is because, compared to the static portfolio, the intertemporal optimal portfolio will be one that not only is instantaneously mean-variance efficient but also one that provides the best hedge against future shifts in the investment opportunity set.[3] While a general characterization of the optimal consumption and portfolio policies in an environment with a stochastic investment opportunity set is provided in the papers by Merton, these papers do not indicate how one can obtain explicit solutions. In the dynamic programming formulation of Merton, obtaining an explicit solution requires one to solve a nonlinear partial differential equation. Typically, closed-form solutions are not available; moreover, in many cases it is not even clear how to specify the boundary conditions for the differential equation.

In order to obtain explicit solutions to the intertemporal portfolio problem, research building on the work of Merton has had to make restrictive assumptions about investor preferences and financial markets: in particular, whether investors care about intermediate consumption or only about terminal wealth; and, whether financial markets are complete or incomplete. Based on

[2]For a more detailed description of the literature, along with an account of its historical development, we refer the reader to Karatzas and Shreve (1998).

[3]The static and dynamic portfolios will coincide only under specific conditions for the utility function (unit elasticity of intertemporal substitution) or asset returns (zero correlation between changes in the investment opportunity and asset returns). Results in the empirical literature suggest that it is unlikely that either condition is true; a discussion of this literature is in Campbell and Viceira (1999).

the modeling choices along these two dimensions, one can classify the existing partial equilibrium literature on portfolio choice as follows.

Kim and Omberg (1996) and Liu (1998) ignore intermediate consumption and assume that an investor wishes to maximize only the expected utility from terminal wealth. For this special case, Liu identifies the portfolio rules for the case of stochastic volatility, stochastic interest rates, and predictability in expected returns, while Kim and Omberg do this for the case where the risk premium follows a particular stochastic process. Numerical solutions for the optimal portfolio are provided by Barberis (1999) for the case of predictable expected returns, and by Brennan, Schwartz and Lagnado (1997) for the case of predictable expected returns and stochastic risk premium.[4] A limitation of these models is that the assumption of zero intermediate consumption may have strong implications for the demand for risky assets for purposes of intertemporal hedging; in contrast, our analysis will allow for intermediate consumption. Moreover, we do not need to rely on numerical analysis.

A second stream of the partial equilibrium literature has dealt with the problem of intermediate consumption by assuming that financial markets are complete, and hence the stochastic changes in investment opportunities are fully spanned by traded securities. With this assumption, one can then use the insight of Cox and Huang (1989) that the optimal consumption policy and portfolio rules can be determined in two distinct steps: first, consumption can be identified by solving a static optimization problem, and then the optimal portfolio rules can be obtained by solving a linear differential equation. This approach has been adopted by Liu (1998) and Wachter (1998).[5] In contrast to these papers, we will analyze economies where financial markets are not complete.

A third approach, developed by Campbell (1993), has been to allow for intermediate consumption *and* incomplete financial markets, but to make appropriate approximations in order to overcome the non-linearity of the problem. In this approach, for models set in discrete time one needs to log-linearize the budget equation and first order condition, and for models set in continuous time one needs to log-linearize the Hamilton-Jacobi-Bellman equation. Using the approximation

[4]See Brandt (1999) for a very different approach to this problem, where instead of specifying the stochastic processes for the state variables, one estimates the first-order conditions directly from the data.

[5]In Wachter (1998), the assumption of complete financial markets is made by specifying that there is perfect correlation between returns on the risky asset and the process driving variation in expected returns. A more general description of this approach is contained in Schroder and Skiadas (1998) and Fisher and Gilles (1999).

for discrete-time models, Campbell and Viceira (1998, 1999) examine the effect of a stochastic short-term interest rate and a stochastic risk premium, respectively.[6] In a continuous time setting, Chacko and Viceira (1999) study the implications of stochastic volatility.[7] Our work is closest in spirit, and complementary to, these papers. Relative to these papers, our contribution is to consider a more general specification for the changes in the investment opportunity set, including the case where there is more than one factor driving changes in investment opportunities,[8] to allow for constraints on portfolio positions even when the portfolio weights are time-varying, to nest this specification in general equilibrium, and to show how this can be extended to economies with multiple agents who differ in risk aversion. A detailed comparison of the method we use and the log-linear approximation approach developed in Campbell (1993) is provided in Section 3.3.

**Single agent analysis with portfolio constraints**

Another direction in which Merton's work on portfolio selection has been extended is to consider the effect of shortselling and borrowing constraints on the optimal portfolio. Some of the papers in this area include Cvitanic and Karatzas (1992), Grossman and Vila (1992), He and Pages (1993), He and Pearson (1991a, b), Karatzas, Lehoczky, Shreve and Xu (1991), Shreve and Xu (1992), Tepla (1998), Vila and Zariphopoulou (1994) and Zariphopoulou (1989). These papers typically focus on economies with a single agent; moreover, it is very difficult to obtain an explicit characterization of the optimal portfolio policies when the investment opportunity set is time-varying and markets are incomplete. Our model extends this strand of the literature by providing explicit analytic expressions for the optimal portfolio policies in the presence of constraints on portfolio weights for both single-agent and multiagent general-equilibrium economies.

[6] Campbell and Koo (1997) and Fisher and Gilles (1999) study the range of parameter values over which the approximation works well.

[7] Viceira (1999) uses the same kind of approximations to explore the effect of stochastic labor income on portfolio choice.

[8] Numerical analysis of such problem is considered in Campbell, Chan and Viceira (1999), and the effect of transactions costs in this setting is considered in Balduzzi and Lynch (1999). See Brennan, Schwartz and Lagnado (1997) for a discussion of the computational problems entailed in solving such problems.

**Multiagent economies without portfolio constraints**

Unconstrained multiagent economies have been studied in Merton (1973), Constantinides (1982) and Karatzas, Lehoczky and Shreve (1990). The focus of these papers is on the equilibrium pricing of assets (CAPM) and existence of equilibrium, rather than on providing an explicit characterization of portfolio decisions and the equilibrium riskless rate. The analysis of equilibrium behavior and interest rates in a multiagent economy where agents differ in risk aversion has been examined in Dumas (1989) for a production economy, and Wang (1996) for an exchange economy. Both these papers assume that financial markets are complete; moreover, Dumas has to resort to numerical methods to study the model, while Wang can solve the model in closed form only for particular values of the risk aversion parameter. We extend these models to allow for incomplete markets and borrowing constraints, and we do not need to rely on numerical methods to analyze the problem.

**Multiagent economies with portfolio constraints**

General equilibrium economies *with* portfolio constraints have been studied in a number of papers. In addition to Cuoco (1997), and Detemple and Murthy (1997), where the focus is on equilibrium pricing rather than the characterization of optimal policies, Sellin and Werner (1993) examine the effect of capital market segmentation in a a two-country international economy with each country populated by investors having logarithmic utility.[9] Saito (1996) studies a production economy where some agents face restrictions on the markets in which they can participate. Again, he studies economies in which all agents have logarithmic utility functions, and even for this case only a numerical solution is provided. In contrast to these papers, in our model agents can differ in their degree of risk aversion and we can analyze the model without having to rely on numerical methods. Compared to Basak and Cuoco (1998), who evaluate the equity risk premium in an exchange economy with two agents with different risk aversion and different access to capital markets, our focus is on portfolio decisions, and our formulation allows for a more general specification of constraints than the ones considered by Basak and Cuoco. Moreover, Basak and Cuoco provide an explicit characterization only for the case where both agents have log utility.

[9] Cuoco (1997) analyzes the effect of very general portfolio constraints that include restrictions on short selling and borrowing and also market incompleteness arising from stochastic endowment income that cannot be hedged perfectly.

On the methodological front, identifying the equilibrium in multiagent economies with incomplete financial markets is a difficult problem and to date the literature does not have an explicit characterization in terms of exogenous variables.[10] Cuoco and He (1994a,b) show that with incomplete markets one can still construct a representative agent, but in this case the weights assigned to individual agents in this aggregation evolve stochastically. However, their characterization of equilibrium is in terms of endogenous variables and is not very useful as a solution method. Until such a method is developed, our approach can be viewed as a convenient way of analyzing economies where financial markets are incomplete and agents differ in their degree of risk aversion.

We would like to alert the reader to two caveats. While the method we propose can be applied in a wide variety of settings, it may not apply in all settings, and in the course of our discussion we will describe how to identify economies where this method works well and others where it is less likely to fare as well. Also, we need to emphasize that while the method allows for exact comparative statics results around the case of log utility, it provides only approximations to the value function and portfolio rules. In some cases it will be possible to demonstrate that the approximation is good, and again, we will do this in what follows; however, general results on the convergence of the approximate solution to the true solution are currently not available.[11]

The rest of the paper is arranged as follows. In Section 2, we explain our approach for analyzing portfolio decisions in the context of a general vector process driving investment opportunities. In this analysis, we allow for constraints on portfolio positions. In Section 3, we apply our method to examine portfolio choice in a partial equilibrium setting. In Section 4, we examine the optimal portfolios in a constant-returns-to-scale, general equilibrium, production economy, where the interest rate process is determined endogenously. In Section 5, we study an exchange economy where, in addition to the endogenous interest rate, the volatility and expected return of the stock are also determined endogenously; moreover, financial markets are incomplete. We conclude in Section 6.

In order to make it easy to identify the main results, in each section the optimal portfolio and consumption policies, along with the value function of the log investor, are highlighted in propositions. The proofs for all propositions are collected in the appendix.

[10]For example, Telmer (1993) and Heaton and Lucas (1996) use numerical methods to solve for the equilibrium in such economies.

[11]See Judd (1996) for a discussion of issues related to convergence.

# 2 Optimal portfolio policies: A general approach

In this section, we undertake an asymptotic analysis of a partial equilibrium model of optimal consumption and portfolio selection with a stochastic investment opportunity set, when the agent derives utility from intermediate consumption and bequest, and faces constraints on her portfolio position. We show that one can obtain an explicit asymptotic expression for the solution of the intertemporal consumption-portfolio problem, as long as the solution of the analogous problem for the agent with logarithmic preferences is known in closed form.

The section is structured as follows. We start by describing an economy with an arbitrary stochastic vector process for the state variables that drives changes in the investment opportunity set. Then, we derive the optimal consumption and portfolio rules in the absence of constraints. We conclude by considering the effect of constraints on portfolio positions.

## 2.1 The economy

### Preferences

The utility function of the agent is time-separable and is given by

$$B \cdot \mathrm{E}_0 \left[ \int_0^T e^{-\rho t} \frac{1}{\gamma} \left( C_t^\gamma - 1 \right) \ dt \right] + (1 - B) \cdot e^{-\rho T} \mathrm{E}_0 \left[ \frac{1}{\gamma} \left( W_T^\gamma - 1 \right) \right],$$

where $\rho$ is the constant subjective time discount rate, $C_t$ is the flow of consumption, and $\gamma$ is the parameter dictating the agent's degree of risk aversion. The preference parameter $B$ controls the relative weight of intermediate consumption and the end-of-period wealth (bequest) in the agent's utility function. With this specification, the agent's relative risk aversion is given by $1 - \gamma$. For agents with unit risk aversion ($\gamma = 0$), utility is given by the logarithmic function:

$$B \cdot \mathrm{E}_0 \left[ \int_0^T e^{-\rho t} \log C_t \ dt \right] + (1 - B) \cdot e^{-\rho T} \mathrm{E}_0 \left[ \log W_T \right].$$

**Financial Assets**

The agent can allocate her wealth among two assets: a short-term riskless asset (bond) with rate of return $r_t$, and a stock (paying zero dividend). The price of the stock, $P_t$, evolves according to

$$\frac{dP_t}{P_t} = \mu_{Pt}\, dt + \sigma_{Pt}\, dZ_{Pt}, \tag{1}$$

where $\mu_{Pt}$ is the instantaneous expected return and $\sigma_{Pt}$ is the volatility. Our convention is to denote stochastic variables with a subscript "$t$"; thus, in the above specification, the riskless rate, $r_t$, the expected return on the stock, $\mu_{Pt}$, and the volatility of stock returns, $\sigma_{Pt}$, are permitted to be stochastic.

**The Investment Opportunity Set**

The investment opportunity set is described by the vector of state variables, $\mathbf{X}_t$. The state vector is assumed to change over time according to

$$d\mathbf{X}_t = \mu_{\mathbf{X}} (\mathbf{X}_t)\ dt + \sigma'_{\mathbf{X}} (\mathbf{X}_t) \cdot d\mathbf{Z}_{\mathbf{X}t}, \tag{2}$$

where

$$\operatorname{cov}\left(\frac{dP_t}{P_t}, d\mathbf{X}_t\right) = \begin{bmatrix} \sigma_P^2 (\mathbf{X}_t) & \sigma'_{P\mathbf{X}} (\mathbf{X}_t) \\ \sigma_{P\mathbf{X}} (\mathbf{X}_t) & \Sigma_{\mathbf{X}} (\mathbf{X}_t) \end{bmatrix} dt.$$

With the above specification, the riskless rate and the expected rate of return and volatility of the risky asset may depend on the state vector:

$$r_t = r (\mathbf{X}_t)\,, \qquad \mu_{Pt} = \mu (\mathbf{X}_t)\,, \qquad \sigma_{Pt} = \sigma (\mathbf{X}_t)\,,$$

implying that the instantaneous Sharpe ratio is also stochastic:

$$\phi_t = \phi (\mathbf{X}_t) \equiv (\mu_{Pt} - r_t) / \sigma_{Pt}.$$

At this stage, our objective is to see what we can say about portfolio decisions without having to specify exactly either the elements of the vector $\mathbf{X}_t$ or the stochastic process for these elements. In Section 3, we will study portfolio decisions for particular specifications of $\mathbf{X}_t$, and in the general equilibrium economies of Sections 4 and 5, the evolution of $\mathbf{X}_t$ will be determined endogenously.

## 2.2 Optimal policies in the absence of portfolio constraints

In the above economy, denoting by $\theta_t$ the proportion of the agent's wealth invested in the risky asset, the wealth of the agent evolves according to

$$dW_t = \left[\left(r_t + \theta_t \phi_t \sigma_{P_t}\right) W_t - C_t\right] \, dt + \theta_t \sigma_{P_t} W_t \, dZ_{P_t}. \tag{3}$$

The value function $J\left(W, \mathbf{X}, t\right)$ of the optimal control problem is defined by

$$J\left(W_t, \mathbf{X}_t, t\right) = \sup_{\{C_s, \theta_s\}} B \cdot \mathrm{E}_t \left[\int_t^T e^{-\rho(s-t)} \frac{1}{\gamma} \left(C_s^\gamma - 1\right) ds\right] + (1-B) \cdot e^{-\rho(T-t)} \mathrm{E}_t \left[\frac{1}{\gamma}\left(W_T^\gamma - 1\right)\right], \tag{4}$$

subject to equations (1), (2), and (3). Defining the consumption-wealth ratio $c \equiv \frac{C}{W}$, the function $J\left(W, \mathbf{X}, t\right)$, satisfies the Hamilton-Jacobi-Bellman equation

$$0 = \max_{c,\theta} \left\{ \begin{array}{c} \frac{B}{\gamma}\left((Wc)^\gamma - 1\right) + J_t - \rho J + \left(r + \theta\phi\sigma_P - c\right) J_W W + \frac{1}{2}\theta^2 W^2 J_{WW} \sigma_P^2 \\ + \mu_{\mathbf{X}}' \cdot J_{\mathbf{X}} + \frac{1}{2}\sigma_{\mathbf{X}}' \cdot J_{\mathbf{XX}} \cdot \sigma_{\mathbf{X}} + \theta W \sigma_{P\mathbf{X}}' \cdot J_{W\mathbf{X}} \end{array} \right\}. \tag{5}$$

Given the homogeneity of the utility function, the solution to this equation has the following functional form:

$$J\left(W, \mathbf{X}, t\right) = \frac{A(t)}{\gamma}\left(\left(e^{g(\mathbf{X},t)} W\right)^\gamma - 1\right), \tag{6}$$

where

$$A(t) = \left(1 - B\frac{1+\rho}{\rho}\right) e^{-\rho(T-t)} + \frac{B}{\rho}.$$

The exact solution for the optimal consumption policy and portfolio weight can be obtained from the first-order condition implied by the Hamilton-Jacobi-Bellman equation:

$$c(\mathbf{X}, t) = \left(\frac{1}{B} A(t) \; e^{\gamma \, g(\mathbf{X},t)}\right)^{1/(\gamma-1)}, \tag{7}$$

$$\begin{aligned} \theta(\mathbf{X}, t) &= -\frac{J_W}{W J_{WW}} \frac{\phi(\mathbf{X})}{\sigma_P(\mathbf{X})} - \frac{J_W}{W J_{WW}} \frac{J_{W\mathbf{X}}}{J_W} \frac{\sigma_{P\mathbf{X}}'(\mathbf{X})}{\sigma_P^2(\mathbf{X})} \\ &= \frac{1}{1-\gamma} \frac{\phi(\mathbf{X})}{\sigma_P(\mathbf{X})} + \frac{\gamma}{1-\gamma} \frac{\sigma_{P\mathbf{X}}'(\mathbf{X})}{\sigma_P^2(\mathbf{X})} \frac{\partial g(\mathbf{X}, t)}{\partial \mathbf{X}}, \end{aligned} \tag{8}$$

where the second line is obtained by using (6).

In general, the unknown function $g(\mathbf{X},t)$ cannot be computed in closed form. Our approach is to obtain an asymptotic approximation to $g(\mathbf{X},t)$, where the expansion is with respect to the risk aversion parameter $\gamma$. That is, we look for $g(\mathbf{X},t)$ as a power series in $\gamma$:

$$g(\mathbf{X},t) = g_0(\mathbf{X},t) + \gamma g_1(\mathbf{X},t) + O\left(\gamma^2\right), \tag{9}$$

where $g_0(\mathbf{X},t)$ is obtained from the value function of an agent with logarithmic utility ($\gamma = 0$):

$$J(W,\mathbf{X},t) = A(t)\Big(\log(W(t)) + g_0(\mathbf{X},t)\Big). \tag{10}$$

Note that the first-order asymptotic expansions are sufficient to obtain *exact* local comparative statics results for the dependence of the optimal policies on the risk aversion parameter. The asymptotic expansions will also approximate the optimal consumption and portfolio policies reasonably well when the risk aversion parameter $\gamma$ is sufficiently close to zero. Even for the case where $\gamma$ is not close to zero, we will see that the optimal policies are well-approximated by the asymptotic solution.[12]

We now derive the asymptotic expansions for the consumption-portfolio problem (by substituting (9) into (8)) and explain how one can obtain the function $g_0(\mathbf{X},t)$. Following this, we examine the comparative statics properties of the optimal policies.

**Proposition 2.1** *The first-order asymptotic expansions for the optimal consumption and portfolio choice are*

$$c(\mathbf{X},t) = \frac{B}{A(t)} - \gamma\frac{B}{A(t)}\Big(g_0(\mathbf{X},t) + \log(A(t))\Big) + O\left(\gamma^2\right), \tag{11}$$

$$\theta(\mathbf{X},t) = \frac{1}{1-\gamma}\frac{\phi(\mathbf{X})}{\sigma_P(\mathbf{X})} + \frac{\gamma}{1-\gamma}\frac{\sigma_{P\mathbf{X}}'(\mathbf{X})}{\sigma_P^2(\mathbf{X})}\frac{\partial g_0(\mathbf{X},t)}{\partial \mathbf{X}} + O\left(\gamma^2\right). \tag{12}$$

*An asymptotically equivalent expression for the portfolio choice is given by*

$$\theta(\mathbf{X},t) = \frac{\phi(\mathbf{X})}{\sigma_P(\mathbf{X})} + \gamma\left(\frac{\phi(\mathbf{X})}{\sigma_P(\mathbf{X})} + \frac{\sigma_{P\mathbf{X}}'(\mathbf{X})}{\sigma_P^2(\mathbf{X})}\frac{\partial g_0(\mathbf{X},t)}{\partial \mathbf{X}}\right) + O\left(\gamma^2\right). \tag{13}$$

The two expressions for the portfolio weight, (12) and (13), are equally easy to manipulate. However, the role of the risk aversion coefficient is more apparent in (13), while (12) provides a

[12] One can further improve the approximation by considering higher-order terms of the expansion.

better non-local approximation to the portfolio policy (see Section 4.4). We will use both expressions in the rest of the paper.

Comparing the asymptotic weight in (12) to the exact one in (8), we see that the only difference is that under the standard approach one needs to identify the unknown function $g(\mathbf{X}, t)$, while in our approach one needs to identify only $g_0(\mathbf{X}, t)$, the value function for the log investor. Typically, it is much easier to solve for the value function of the log investor. The intuition for this is well-known:[13] the substitution effect and the income effect arising from a change in the investment opportunity set are of exactly the same magnitude and opposite sign for an investor with log utility. Consequently, this investor has zero demand for hedging future changes in the investment opportunity set, and so her portfolio coincides with the myopic portfolio. Similarly, log-utility investors do not adjust their consumption-wealth ratio for changes in the investment opportunity set, and so it is easy to identify this ratio as a deterministic function of time, $A(t)$.

The function $g_0(\mathbf{X}, t)$ can be obtained by substituting the optimal consumption and portfolio policies in (11) and (12), with $\gamma = 0$, into the value function for the log investor.

**Proposition 2.2** *The function $g_0(\mathbf{X}, t)$ is given by*

$$\begin{aligned} g_0(\mathbf{X}, t) =& B \log(B) \frac{1 - e^{-\rho(T-t)}}{\rho A(t)} - B A^{-1}(t) \int_t^T e^{-\rho(s-t)} \log(A(s))\, ds \\ &+ A^{-1}(t) \mathrm{E}_t \left[ \int_t^T \left( A(t) - \frac{B}{\rho} \left( 1 - e^{-\rho(s-t)} \right) \right) \left( -\frac{B}{A(s)} + r(\mathbf{X}_s) + \frac{\phi(\mathbf{X}_s)^2}{2} \right) ds \right]. \quad (14) \end{aligned}$$

As along as this function is known in closed form, one can obtain explicit first-order asymptotic expressions for the optimal consumption and portfolio policies. For example, the class of affine processes will yield close-form solutions. In Section 3, we compute this function for particular exogenous specifications of the state process, $\mathbf{X}_t$, and in Sections 4 and 5 we repeat this exercise for the case where the state process is determined endogenously in equilibrium.

Analyzing the consumption-portfolio rules given in Proposition 2.1, we see that:

- The zero-order components of these expansions correspond to the well-known solution for the

[13] Early results on the properties of the log utility function are in Leland (1968) and Mossin (1968). These results were developed further in Hakansson (1971) and Merton (1971).

case where the agent has a logarithmic utility function ($\gamma = 0$): the optimal consumption-wealth ratio, $c \equiv C/W$, is given by the deterministic function $A^{-1}(t)$, and the optimal portfolio policy is myopic and independent of changes in the investment opportunity set.

- The first-order terms capture the effect of risk aversion when the coefficient of relative risk aversion deviates from one ($\gamma$ deviates from zero). In particular, one can interpret the expression for the optimal portfolio in (12) as

$$\theta(\mathbf{X},t) = \underbrace{\frac{1}{1-\gamma}\frac{\phi(\mathbf{X})}{\sigma_P}}_{\substack{\text{myopic}\\ \text{demand}}} + \underbrace{\frac{\gamma}{1-\gamma}\left(\frac{1}{\sigma_P^2}\sigma'_{P\mathbf{X}} \cdot \frac{\partial g_0(\mathbf{X},t)}{\partial \mathbf{X}}\right)}_{\substack{\text{hedging}\\ \text{demand}}} + O\left(\gamma^2\right),$$

  where the first bracketed term represents the portfolio weights under constant investment opportunity set, the myopic demand, and the second term characterizes the demand arising from the desire to hedge against changes in the investment opportunity set.

The equation above allows one to obtain the familiar comparative static results: the hedging demand is asymptotically proportional to the risk aversion parameter and vanishes as $\gamma$ approaches zero. The hedging demand is also proportional to the scalar product of the vector of "betas" of the state variables with respect to the risky asset, $\sigma_P^{-2}\sigma'_{P\mathbf{X}}$, and the "delta" of the function $g_0(\mathbf{X},t)$ with respect to the state vector, $\partial g_0(\mathbf{X},t)/\partial \mathbf{X}$. Finally, the equation shows that the hedging demand is zero when the shocks to the state variables are uncorrelated with the returns on the stock ($\sigma_{P\mathbf{X}} = 0$).

The asymptotic expansions (11) and (12) approximate the optimal consumption and portfolio policies when the risk aversion parameter $\gamma$ is sufficiently close to zero. They also provide exact local comparative statics results for the dependence of the optimal policies on the risk aversion parameter:

$$\left.\frac{\partial c(\mathbf{X},t)}{\partial \gamma}\right|_{\gamma=0} = -\frac{1}{A(t)}\left(g_0(\mathbf{X},t) + \log(A(t))\right),$$
$$\left.\frac{\partial \theta(\mathbf{X},t)}{\partial \gamma}\right|_{\gamma=0} = \frac{\phi(\mathbf{X})}{\sigma_P(\mathbf{X})} + \frac{1}{\sigma_P^2(\mathbf{X})}\sigma'_{P\mathbf{X}}(\mathbf{X}) \cdot \frac{\partial g_0(\mathbf{X},t)}{\partial \mathbf{X}}. \quad (15)$$

Equation (15) indicates that the optimal position in the risky asset can either increase or decrease with the risk aversion coefficient, depending on the magnitude of the second term in equation (15), which is the sensitivity of the hedging demand with respect to the parameter $\gamma$.

Motivated by papers such as Canner, Mankiw and Weil (1997), the comparative static in equation (15) has been the focus of several recent papers on portfolio choice. Because our approach allows us to express $g_0(\mathbf{X},t)$ explicitly in terms of the primitive parameters, we will be able to determine the sign of this partial derivative in a setting where there is intermediate consumption and financial markets are incomplete.

### Infinite-Horizon Economies

Infinite-horizon economies are a special case of the general formulation of the previous section. Because of the importance of such problems, we formulate the corresponding results below as separate propositions. The solution to the infinite-horizon problem can be obtained from the general formulation by setting $B = 1$ and taking a limit of $T \to \infty$. Taking these limits in Propositions 2.1 and 2.2 gives the following results.

**Proposition 2.3** *The first-order asymptotic expansions for the optimal consumption and portfolio choice are*

$$c(\mathbf{X}) = \rho - \gamma\rho\left(g_0(\mathbf{X}) - \log(\rho)\right) + O\left(\gamma^2\right), \tag{16}$$

$$\theta(\mathbf{X}) = \frac{1}{1-\gamma}\frac{\phi(\mathbf{X})}{\sigma_P(\mathbf{X})} + \frac{\gamma}{1-\gamma}\frac{\sigma'_{P\mathbf{X}}(\mathbf{X})}{\sigma_P^2(\mathbf{X})}\frac{\partial g_0(\mathbf{X})}{\partial \mathbf{X}} + O\left(\gamma^2\right). \tag{17}$$

*An asymptotically equivalent expression for the portfolio choice is given by*

$$\theta(\mathbf{X}) = \frac{\phi(\mathbf{X})}{\sigma_P(\mathbf{X})} + \gamma\left(\frac{\phi(\mathbf{X})}{\sigma_P(\mathbf{X})} + \frac{\sigma'_{P\mathbf{X}}(\mathbf{X})}{\sigma_P^2(\mathbf{X})}\frac{\partial g_0(\mathbf{X})}{\partial \mathbf{X}}\right) + O\left(\gamma^2\right). \tag{18}$$

**Proposition 2.4** *The function $g_0(\mathbf{X})$ is given by*

$$g_0(\mathbf{X}) = \log(\rho) - 1 + \mathrm{E}\left[\int_0^\infty e^{-\rho t}\left(r(\mathbf{X}_t) + \frac{\phi(\mathbf{X}_t)^2}{2}\right)dt \,\middle|\, \mathbf{X}_0 = \mathbf{X}\right]. \tag{19}$$

## 2.3 Optimal policies in the presence of portfolio constraints

Up to this point, it had been assumed that the agent's consumption-portfolio choice was unconstrained. We now extend the analysis to allow for constraints on the portfolio weights. To simplify the exposition, we analyze only the infinite-horizon problem explicitly. It should be clear from our presentation how the solution of the finite-horizon problem in Propositions 2.1 and 2.2 must be modified to account for constraints.

We consider constraints of the form that restrict the portfolio weight on the risky asset to lie between a lower and an upper bound:

$$\underline{\theta}(\mathbf{X}) \;\leq\; \theta(\mathbf{X}) \;\leq\; \overline{\theta}(\mathbf{X}),$$

where these bounds, are allowed to depend on the state of the economy.[14]

The value function of the agent's constrained optimization problem now satisfies

$$0 = \max_{c,\theta\in\left[\underline{\theta}(\mathbf{X}),\overline{\theta}(\mathbf{X})\right]} \left\{ \begin{array}{c} \frac{1}{\gamma}\left((Wc)^{\gamma}-1\right)-\rho J+\left(r+\theta\phi\sigma_P-c\right)J_W W+\frac{1}{2}\theta^2 W^2 J_{WW}\sigma_P^2 \\ +\mu_{\mathbf{X}}'\cdot J_{\mathbf{X}}+\frac{1}{2}\sigma_{\mathbf{X}}'\cdot J_{XX}\cdot\sigma_{\mathbf{X}}+\theta W\sigma_{P\mathbf{X}}'\cdot J_{W\mathbf{X}} \end{array} \right\}.$$

**Proposition 2.5** *In the presence of constraints, the optimal portfolio choice is given by*

$$\theta(\mathbf{X}) = \begin{cases} \widetilde{\theta}(\mathbf{X}), & \underline{\theta}(\mathbf{X}) \leq \widetilde{\theta}(\mathbf{X}) \leq \overline{\theta}(\mathbf{X}), \\ \underline{\theta}(\mathbf{X}), & \widetilde{\theta}(\mathbf{X}) < \underline{\theta}(\mathbf{X}), \\ \overline{\theta}(\mathbf{X}), & \widetilde{\theta}(\mathbf{X}) > \overline{\theta}(\mathbf{X}), \end{cases}$$

*where*

$$\widetilde{\theta} \equiv \frac{\phi(\mathbf{X})}{\sigma_P(\mathbf{X})} + \gamma\left(\frac{\phi(\mathbf{X})}{\sigma_P(\mathbf{X})} + \frac{1}{\sigma_P^2(\mathbf{X})}\sigma_{P\mathbf{X}}'(\mathbf{X})\cdot\frac{\partial g_0^{\mathsf{c}}(\mathbf{X})}{\partial\mathbf{X}}\right) + O\left(\gamma^2\right). \tag{20}$$

*and the optimal consumption policy is given by*

$$c(\mathbf{X}) = \rho - \gamma\rho\left(g_0^{\mathsf{c}}(\mathbf{X}) - \log(\rho)\right) + O\left(\gamma^2\right).$$

The function $g_0^{\mathsf{c}}(\mathbf{X})$ in (20), where the superscript "c" indicates the presence of constraints, is the counterpart of the function $g_0(\mathbf{X})$ in (17): it defines the value function of the log-utility

[14] For a more general specification of constraints on portfolio positions, see Cuoco (1997).

maximizer subject to the same portfolio constraints and the same investment opportunity set as the investor with (non-log) power utility function.

**Proposition 2.6** *The value function of the log investor in the presence of constraints is*

$$\begin{aligned} g_0^{\mathrm{c}}(\mathbf{X}) &= \log(\rho) - 1 \qquad (21) \\ &+ \mathrm{E}\left[ \int_0^\infty e^{-\rho t} \left( r(\mathbf{X}_t) + \theta_0(\mathbf{X}_t)\,\sigma_P(\mathbf{X}_t)\,\phi(\mathbf{X}_t) - \frac{1}{2}\theta_0^2(\mathbf{X}_t)\,\sigma_P^2(\mathbf{X}_t) \right) dt \,\middle|\, \mathbf{X}_0 = \mathbf{X} \right], \end{aligned}$$

*where*

$$\theta_0(\mathbf{X}_t) = \begin{cases} \phi(\mathbf{X}_t)/\sigma_P(\mathbf{X}_t), & \underline{\theta}(\mathbf{X}_t) \le \phi(\mathbf{X}_t)/\sigma_P(\mathbf{X}_t) \le \overline{\theta}(\mathbf{X}_t), \\ \underline{\theta}(\mathbf{X}_t), & \phi(\mathbf{X}_t)/\sigma_P(\mathbf{X}_t) < \underline{\theta}(\mathbf{X}_t), \\ \overline{\theta}(\mathbf{X}_t), & \phi(\mathbf{X}_t)/\sigma_P(\mathbf{X}_t) > \overline{\theta}(\mathbf{X}_t). \end{cases} \qquad (22)$$

As in the unconstrained case, an explicit asymptotic expression for the optimal consumption and portfolio policies is available as long as the solution of the analogous problem for the agent with the logarithmic utility function is known in closed form. We will provide some examples of this in the sections that follow.

An important qualitative implication of Proposition 2.6 is that the effect of constraints is separable across time. When portfolio constraints are not binding today, the effect of constraints imposed over future non-overlapping time intervals is additive. This follows from the integral representation of the function $g_0^{\mathrm{c}}$ in (21).

The results of this section can be used as building blocks in the analysis of fairly complicated models. In particular, they allow one to obtain asymptotic expressions for the prices of assets in equilibrium economies that otherwise can only be studied numerically. Successful application of our results is possible as long as it is possible to obtain explicit solutions for agents with logarithmic utility functions. In that case, the asymptotic demand functions, equations (11) and (12) for the finite-horizon case or (16) and (17) for the infinite-horizon case, are known in closed form and for equilibrium models the asset prices can be determined from the market clearing conditions. In the next section, we use our approach to examine portfolio decisions when the process for the investment opportunity set is given exogenously, and in the two sections that follow we analyze equilibrium models with heterogeneous agents, portfolio constraints and incomplete markets.

# 3 Portfolio decisions in partial equilibrium economies

The objective of this section is to illustrate the application of the method developed above to the consumption-portfolio choice problem for a particular specification for the state process, $\mathbf{X}_t$. We consider the case where the short rate *and* the risk premium on the stock are stochastic. The results for the case where only the interest rate is stochastic or only the risk premium is stochastic can be obtained as special cases of this multivariate specification.[15] We study both the case where portfolio positions are unconstrained and the case with constraints. Following this, we compare our approach to the log-linearization approach developed in Campbell (1993); this comparison is undertaken by considering a setting where the volatility of stock returns is stochastic.

## 3.1 Portfolio choice with a stochastic interest rate and predictable returns

We assume that the agent has an infinite horizon: $B = 1$, $T = \infty$ and lives in an economy where the interest rate and/or the risk premium follow a particular stochastic process, while volatility is assumed to be constant.

In particular, we assume that

$$\mathbf{X}_t = \begin{pmatrix} X_{1t} \\ X_{2t} \end{pmatrix},$$

where $X_1$ is the process driving interest rates and $X_2$ is the process driving the risk premium:

$$r_t = r(X_{1t}) \equiv \alpha_r + X_{1t}, \tag{23}$$

$$\phi_t = \phi(X_{2t}) \equiv \alpha_\phi + X_{2t}. \tag{24}$$

The processes we specify for $X_1$ and $X_2$ are

$$dX_{jt} = -\lambda_{X_j} X_{jt}\, dt + \sigma_{X_j}\, dZ_{X_j t}, \quad j = \{1, 2\}$$

implying that

$$X_{jt} = X_{j0}\, e^{-\lambda_{X_j} t} + \int_0^t e^{-\lambda_{X_j}(t-s)} \sigma_{X_j}\, dZ_{X_j s} \quad j = \{1, 2\}, \tag{25}$$

[15] The behavior of portfolio policies when expected returns on the risky asset are predictable can be obtained from studying the case where the risk premium is stochastic but the interest rate and volatility are constant.

with the covariance structure

$$\operatorname{cov}\left(dP_t, dX_{1t}, dX_{2t}\right) = \begin{pmatrix} \sigma_P^2 & \sigma_{PX_1} & \sigma_{PX_2} \\ \sigma_{PX_1} & \sigma_{X_1}^2 & \sigma_{X_1X_2} \\ \sigma_{PX_2} & \sigma_{X_1X_2} & \sigma_{X_2}^2 \end{pmatrix} dt.$$

Given this specification for the evolution of the investment opportunity set, the value function of the log-utility maximizer can be obtained in closed form by evaluating the integral in (19), after substituting from equations (23) and (24) the expressions for $r_t$ and $\phi_t$.

**Proposition 3.1** *The function $g_0\left(X_1, X_2\right)$ is:*

$$g_0\left(X_1, X_2\right) = a_0 + a_1 X_1 + a_2 X_2 + \frac{1}{2} a_{22} X_2^2, \tag{26}$$

*with*

$$a_0 = -1 + \frac{\alpha_r + \alpha_\phi^2/2}{\rho} + \frac{\sigma_{X_2}^2}{2\rho\left(2\lambda_{X_2} + \rho\right)} + \log(\rho)$$

$$a_1 = \frac{1}{\left(\lambda_{X_1} + \rho\right)}, \qquad a_2 = \frac{\alpha_\phi}{\left(\lambda_{X_2} + \rho\right)}, \qquad a_{22} = \frac{1}{\left(2\lambda_{X_2} + \rho\right)}.$$

We can now obtain an explicit expression for the portfolio policy.

**Proposition 3.2** *The optimal portfolio composition is given by*

$$\theta\left(\mathbf{X}\right) = \frac{1}{1-\gamma}\frac{\phi\left(X_2\right)}{\sigma_P} + \frac{\gamma}{1-\gamma}\beta_{r,P}\frac{1}{\lambda_{X_1} + \rho} + \frac{\gamma}{1-\gamma}\beta_{\phi,P}\left(\frac{\alpha_\phi}{\lambda_{X_2} + \rho} + \frac{\phi\left(X_2\right) - \alpha_\phi}{2\lambda_{X_2} + \rho}\right) + O\left(\gamma^2\right), \tag{27}$$

*where*

$$\beta_{r,P} \equiv \frac{\sigma_{PX_1}}{\sigma_P^2}, \qquad \beta_{\phi,P} \equiv \frac{\sigma_{PX_2}}{\sigma_P^2},$$

*are the "betas" of the processes for the interest rate and the Sharpe ratio with respect to the stock-return process.*

From (27), one sees that the demand for the risky asset can be *increasing* in risk aversion (decreasing in $\gamma$) as long as one of the "betas" is sufficiently large in absolute value and negative.

In particular, the hedging demand is proportional to the "betas" of the state variables. The hedging demand induced by uncertainty about future interest rates equals $\gamma(1-\gamma)^{-1}\beta_{r,P}/\left(\lambda_{X_1}+\rho\right)$, while that induced by uncertainty about future values of the Sharpe ratio, $\phi_t$, equals

$$\frac{\gamma}{1-\gamma}\beta_{\phi,P}\left[\frac{\alpha_\phi}{\lambda_{X_2}+\rho}+\frac{\phi(X_2)-\alpha_\phi}{2\lambda_{X_2}+\rho}\right].$$

When the "betas" are sufficiently large in absolute value and negative, the increase in hedging demand,

$$\frac{\gamma}{1-\gamma}\left[\beta_{r,P}\frac{1}{\lambda_{X_1}+\rho}+\beta_{\phi,P}\left(\frac{\alpha_\phi}{\lambda_{X_2}+\rho}+\frac{\phi(X_2)-\alpha_\phi}{2\lambda_{X_2}+\rho}\right)\right],$$

offsets the myopic reduction in holdings of the risky asset,

$$\frac{\phi(X_2)}{\sigma_P}\left|\frac{\gamma}{1-\gamma}\right|,$$

due to an increase in risk aversion and leads to a net increase in holdings of the risky asset. For example, when $\beta_{r,P}=0$ and the value of the Sharpe ratio equals its long-run mean, i.e., when $X_{2t}=0$, a sufficient condition for the position in the risky asset to increase with the degree of risk aversion is that $\beta_{\phi,P}<-\left(\lambda_{X_2}+\rho\right)/\sigma_P$.

Also, observe that the strength of hedging demand is independent of the current level of the interest rate, while it is a linear function of the Sharpe ratio.

### 3.2 Portfolio choice with position constraints

In this section we extend the results of Section 3.1 to incorporate constraints on the portfolio position. To simplify the exposition, we consider only a single constraint

$$\theta\left(\mathbf{X}\right)\leq 1+L.$$

One can think of this constraint as a restriction on borrowing: the agent is prohibited from borrowing more than a fraction $L$ of her wealth.

The optimal portfolio strategy in the presence of portfolio constraints is given by Proposition 2.6. To understand the effect of portfolio constraints, we compare the portfolio policy of Proposition 2.6

to the one in Proposition 2.4. In particular, we study the differences between functions $\partial g_0^{\mathsf{c}}(\mathbf{X})/\partial \mathbf{X}$ and $\partial g_0(\mathbf{X})/\partial \mathbf{X}$.

Given the results in equations (19) and (21),

$$g_0^{\mathsf{c}}(\mathbf{X}) - g_0(\mathbf{X}) = -\frac{1}{2}\mathrm{E}\left[\int_0^\infty e^{-\rho t}\left(\phi(\mathbf{X}_t) - \overline{\phi}\right)^2 1_{[\phi(\mathbf{X}_t)\geq\overline{\phi}]}dt \right| \mathbf{X}_0 = \mathbf{X}\Bigg],$$

and

$$\frac{\partial g_0^{\mathsf{c}}(\mathbf{X})}{\partial \mathbf{X}} - \frac{\partial g_0(\mathbf{X})}{\partial \mathbf{X}} =$$

$$-\mathrm{E}\left[\int_0^\infty e^{-\rho t}\left(\phi(\mathbf{X}_t) - \overline{\phi}\right) 1_{[\phi(\mathbf{X}_t)\geq\overline{\phi}]}\left(\left(\frac{\partial\phi(\mathbf{X}_t)}{\partial \mathbf{X}_t}\right)' \cdot \frac{\partial \mathbf{X}_t}{\partial \mathbf{X}_0}\right) dt \right| \mathbf{X}_0 = \mathbf{X}\Bigg],$$

where $\partial \mathbf{X}_t/\partial \mathbf{X}_0$ denotes the derivative of the process $\mathbf{X}_t$ with respect to the initial conditions and $\overline{\phi}$ is defined by

$$\overline{\phi} = (1+L)\sigma_P.$$

Functions $g_0^{\mathsf{c}}(\mathbf{X})$ and $g_0(\mathbf{X})$ are defined through the value functions of the logarithmic agent. The portfolio constraint is binding for the logarithmic agent when $\phi(\mathbf{X}_t) \geq \overline{\phi}$. Otherwise, her portfolio position is exactly the same as in the unconstrained case.

According to the definition of the function $\phi(\mathbf{X})$ and the process $\mathbf{X}_t$ in 25,

$$\frac{\partial\phi}{\partial X_1} = 0, \qquad \frac{\partial\phi}{\partial X_2} = 1, \qquad \frac{\partial X_{2t}}{\partial X_{20}} = e^{-\lambda_{X_2} t}.$$

Thus,

$$\left(\frac{\partial\phi(\mathbf{X}_t)}{\partial \mathbf{X}_t}\right)' \cdot \frac{\partial \mathbf{X}_t}{\partial \mathbf{X}_0} \equiv e^{-\lambda_{X_2} t},$$

and

$$\frac{\partial g_0^{\mathsf{c}}(\mathbf{X})}{\partial \mathbf{X}} - \frac{\partial g_0(\mathbf{X})}{\partial \mathbf{X}} = -\int_0^\infty e^{-(\lambda_{X_2}+\rho)t} H(\mathbf{X},t)\, dt \leq 0,$$

where

$$H(\mathbf{X},t) \equiv \mathrm{E}\left[\left(\phi(\mathbf{X}_t) - \overline{\phi}\right) 1_{[\phi(\mathbf{X}_t)\geq\overline{\phi}]}\right| \mathbf{X}_0 = \mathbf{X}\Bigg].$$

The function $H(\mathbf{X},t)$ is a conditional expectation of a monotonically increasing and convex function of $\phi(\mathbf{X}_t)$. Therefore, it is increased if the process $\phi(\mathbf{X}_t)$ is increased or if the latter is changed by superposition of a mean-preserving spread (this follows from the well-known results on first-order and second-order stochastic dominance).

The following proposition summarizes comparative statics results for the difference $\partial g_0^{\mathrm{c}}(\mathbf{X})/\partial\mathbf{X} - \partial g_0(\mathbf{X})/\partial\mathbf{X}$ and provides an explicit expression for this difference.

**Proposition 3.3** *The difference $\partial g_0^{\mathrm{c}}(\mathbf{X})/\partial\mathbf{X} - \partial g_0(\mathbf{X})/\partial\mathbf{X}$ is negative and is (i) decreasing in $\phi_0$; (ii) decreasing in $\alpha_\phi$; (iii) decreasing in $\sigma_{X_2}$; (iv) increasing in $\lambda_{X_2}$; (v) increasing in $\rho$; (vi) increasing in $L$; and, (vii) given explicitly by*

$$\begin{aligned}
\frac{\partial g_0^{\mathrm{c}}(\mathbf{X})}{\partial\mathbf{X}} - \frac{\partial g_0(\mathbf{X})}{\partial\mathbf{X}} &= -\int_0^\infty e^{-(\lambda_{X_2}+\rho)t} H(\mathbf{X},t)\,dt, \\
H(\mathbf{X},t) &= \left(m(t)-\overline{\phi}\right)\frac{1+\operatorname{erf}\left(\frac{m(t)-\overline{\phi}}{\sqrt{2}s(t)}\right)}{2} + \frac{1}{\sqrt{2\pi}}s(t)\,e^{-\frac{(m(t)-\overline{\phi})^2}{2s(t)^2}}, \qquad (28) \\
m(t) &= \alpha_\phi + (\phi_0-\alpha_\phi)\,e^{-\lambda_{X_2}t}, \\
s(t) &= \frac{\sigma_{X_2}}{\sqrt{2\lambda_{X_2}}}\left(1-e^{-2\lambda_{X_2}t}\right)^{1/2}.
\end{aligned}$$

When the agent is not myopic (has a non-logarithmic utility function), her optimal portfolio holdings are affected by position constraints even when these constraints are not binding at the current time. This is because such constraints change the future investment opportunity sets.

Proposition 3.3 demonstrates that position constraints can either increase or decrease the hedging demand of the agent, depending on the sign of $\gamma$ and the correlation between the state variable $X_{2t}$ and stock returns. The impact of the constraint on the hedging demand is proportional to $\gamma\cdot\sigma_{PX_2}$; in particular, the demand is increased if $\gamma\cdot\sigma_{PX_2}<0$. The proposition also shows that, intuitively, the impact of the constraint is stronger for smaller values of $L$ (tighter constraints) and when the constraint is closer to be binding, i.e., the magnitude of the impact is increasing in $\phi_0$. The magnitude of the constraint-induced hedging demand is increasing in $\alpha_\phi$ and $\sigma_{X_2}$, and decreasing in $\lambda_{X_2}$ and $\rho$.

### 3.3 Comparison with the log-linearization technique

In this section, we discuss the similarities and differences between our methodology and the log-linearization technique developed in Campbell (1993) that has been used in Campbell and Viceira (1998, 1999), Chacko and Viceira (1999), and Viceira (1999). For making this comparison clear, we consider the specific problem of optimal portfolio choice in the presence of stochastic volatility, which is the same problem studied in Chacko and Viceira (1999).

**Portfolio choice with stochastic volatility of returns**

The investment opportunity set in this economy is stochastic and is characterized by a single state variable,

$$\mathbf{X}_t = X_t,$$

where $X_t$ is the stochastic process driving the volatility of returns. As in Chacko and Viceira (1999), we set[16]

$$\sigma_P(X_t) = \frac{1}{\sqrt{X_t}}$$

and assume that the state variable $X_t$ is driven by the stochastic process

$$dX_t = -\lambda_X \left(X_t - \overline{X}\right) dt + \sigma_X \sqrt{X_t}\, dZ_{Xt}, \tag{29}$$

where

$$0 < 2\lambda_X \overline{X} - \sigma_X^2,$$

$$\operatorname{cov}(dP_t, dX_t) = \begin{pmatrix} X_t^{-1} & \sigma_{PX} \\ \sigma_{PX} & \sigma_X^2 X_t \end{pmatrix} dt.$$

Reasons for modeling the inverse of volatility, $\sigma_P(X_t) = \frac{1}{\sqrt{X_t}}$, and assuming the particular specification of $X_t$ in (29), will become clear below.

Given this specification for the evolution of the investment opportunity set, the value function of the log-utility maximizer can be obtained in closed form by evaluating the integral in (19).

[16]We choose this specification to facilitate comparison with Chacko and Viceira (1999); one could just as well have modeled volatility, rather than its inverse, as a standard square-root process.

**Proposition 3.4** *The value function of the agent with log utility is given by*

$$\frac{1}{\rho}\left(\log W + g_0\left(\mathbf{X}\right)\right),$$

*where the function* $g_0\left(X\right)$ *is given by*

$$g_0\left(X\right) = a_0 + a_1 X, \tag{30}$$

*with*

$$a_0 = \frac{\lambda_X\left(\mu - r\right)^2 \overline{X}}{2\rho\left(\lambda_X + \rho\right)} + \left(\frac{r}{\rho} - 1 + \log\left(\rho\right)\right), \quad a_1 = \frac{\left(\mu - r\right)^2}{2\left(\lambda_X + \rho\right)}.$$

We can now use (30) in (17) to obtain an explicit expression for the portfolio policy.

**Proposition 3.5** *The optimal portfolio weight is given by*

$$\theta\left(X\right) = \underbrace{\frac{1}{1-\gamma}\left(\mu - r\right)X}_{\text{myopic demand}} + \underbrace{\frac{\gamma}{1-\gamma}\left(\mu - r\right)^2 \frac{\sigma_{PX}}{2(\lambda_X + \rho)}X}_{\text{hedging demand}} + O\left(\gamma^2\right).$$

The various comparative statics results for the optimal portfolio weight based on Proposition 3.5 are similar to the ones derived in Proposition 2 of Chacko and Viceira (1999). For instance, the optimal position in the risky asset is a linear function of the state variable. The magnitude of the hedging demand,

$$\frac{\gamma}{1-\gamma}\left(\mu - r\right)^2 \frac{\sigma_{PX}}{2(\lambda_X + \rho)}X,$$

is proportional to the covariance between the state variable and stock returns and depends on the strength of mean-reversion in the state variable and the subjective discount rate. When the state variable is strongly mean-reverting or when the subjective discount rate is high, demand for hedging is small. In the former case, there is not much reason to hedge changes in the state variable, since it is expected to revert back to its long-run mean soon; in the latter case, agents discount future consumption sufficiently that they are unwilling to hedge changes in investment opportunity set. Also, the ratio of the hedging demand to the myopic demand component is independent of the current level of volatility.

The optimal position in the risky asset can be either increasing or decreasing in the risk aversion coefficient. Specifically,

$$\left.\frac{\partial\theta(X)}{\partial\gamma}\right|_{\gamma=0} = X\left(\mu - r + (\mu - r)^2\,\frac{\sigma_{PX}}{2(\lambda_X+\rho)}\right).$$

Thus, the optimal position in the stock can increase in individual risk aversion $(1-\gamma)$ only when the process for volatility is sufficiently negatively correlated with expected returns, i.e., when

$$\sigma_{PX} < -\frac{2(\lambda_X+\rho)}{\mu - r}.$$

Empirically, changes in conditional volatility are often found to be negatively correlated with stock returns (such behavior is often referred to as the leverage effect). Therefore, since the state variable $X$ is the inverse of conditional volatility, $\sigma_{PX}$ tends to be positive. As a result, both the the total and the hedging component of demand for the risky asset decrease with risk aversion. In fact, hedging demand is negative for agents with relative risk aversion higher than one $(\gamma < 0)$.

**The the log-linearization technique and its relation to our approach**

The Hamilton-Jacobi-Bellman equation for the portfolio choice problem with stochastic volatility results in the nonlinear second-order ordinary differential equation analogous to (5) for the unknown function $g(X)$, which is defined in (6):

$$\begin{aligned} 0 \;=\;& \gamma(\gamma-1)\,\sigma_X^2 X g''(X) + 2\gamma\left[\lambda_X(\gamma-1)\left(X-\overline{X}\right) + \gamma(\mu-r)\,\sigma_{PX}X\right]g'(X) \\ &+ \gamma^2\left(\sigma_X^2(\gamma-1)+\gamma\sigma_{PX}^2\right)\left(g'(X)\right)^2 + 2(\gamma-1)^2 c(X) + \gamma(\mu-r)^2 X + 2(\gamma-1)(\rho-\gamma r), \end{aligned} \tag{31}$$

where $c(X)$ is the optimal consumption-wealth ratio given by

$$c(X) = \rho^{\frac{1}{1-\gamma}} e^{\frac{\gamma}{\gamma-1}g(X)}.$$

Application of the log-linearization technique in continuous time requires approximating $c(X)$ around the long-run mean of the log-consumption-wealth ratio. Specifically, let

$$\widetilde{c}(X) \equiv \log(c(X)) = \frac{1}{1-\gamma}\log(\rho) + \frac{\gamma}{\gamma-1}g(X),$$

$$\kappa \equiv \lim_{t\to\infty} \mathrm{E}_0\left[\widetilde{c}(X)\right].$$

Then, considering a first-order expansion

$$\begin{aligned} c(X) &= \exp(\widetilde{c}(X)) \approx \exp(\kappa)(1+\widetilde{c}(X)-\kappa) \\ &= \exp(\kappa)\left(1+\frac{1}{1-\gamma}\log(\rho)+\frac{\gamma}{\gamma-1}g(X)-\kappa\right). \end{aligned} \tag{32}$$

Replacing the consumption-wealth ratio in equation (31) with this approximate expression leads to a somewhat simpler equation, in which the only nonlinear term is proportional to $(g'(X))^2$. In general, such an equation does not have a closed-from solution. However, given the particular parameterization of the process for stochastic volatility in (29), this equation has an explicit solution of the form

$$g(X) = A_0 + A_1 X,$$

where the unknown coefficients $A_0$ and $A_1$ satisfy

$$\begin{aligned} 0 &= (\mu-r)^2 + 2\left(e^{\kappa}(\gamma-1)+\lambda_X(\gamma-1)+\gamma(\mu-r)\sigma_{PX}\right)A_1 \\ &\quad + \gamma\left(\sigma_X^2(1-\gamma)+\gamma\sigma_{PX}^2\right)A_1^2, \end{aligned} \tag{33}$$

$$0 = \gamma\lambda_X\overline{X}A_1 + e^{\kappa}\left(1+\log(\rho)-\kappa+\gamma(\kappa-1)\right)+(r\gamma-\rho)-\gamma e^{\kappa}A_0. \tag{34}$$

Given a value for $\kappa$, equations (33) and (34) lead to two pairs of solutions for $(A_0, A_1)$. Chacko and Viceira (1999) show that one must select the pair $(A_0, A_1)$ corresponding to the negative root of the discriminant in (33). Finally, the value of the endogenous variable $\kappa$ must be determined. By definition,

$$\kappa = \lim_{t\to\infty} \mathrm{E}_0\left[\frac{1}{1-\gamma}\log(\rho)+\frac{\gamma}{\gamma-1}\left(A_0+A_1X_t\right)\right] = \frac{1}{1-\gamma}\log(\rho)+\frac{\gamma}{\gamma-1}\left(A_0+A_1\overline{X}\right),$$

because $\lim_{t\to\infty}\mathrm{E}_0[X_t] = \overline{X}$. This yields a nonlinear algebraic equation for $\kappa$:

$$\kappa = \frac{1}{1-\gamma}\log(\rho)+\frac{\gamma}{\gamma-1}\left(A_0(\kappa)+A_1(\kappa)\overline{X}\right),$$

in which we express explicitly the dependence of the solution of (33) and (34) on $\kappa$. This equation must be solved numerically.

As one can see, the log-linearization procedure does not lead to an explicit solution. Instead, in this context it reduces the original problem to a single nonlinear algebraic equation which must be solved numerically. When the investment opportunity set is driven by a vector of state variables one has to solve a system of such equations, which is a nontrivial computational problem. Moreover, generally speaking, one cannot be sure that such a system (or even a single equation as in this section) has a unique solution.

The first step of the log-linearization procedure resembles the asymptotic expansions of this paper, as it replaces the nonlinear term of the Hamilton-Jacobi-Bellman equation with a linear first-order expansion (32). However, unlike our explicit expansions, this one involves an endogenous variable $\kappa$, defined as the unconditional mean of the *optimal* (to be determined) log-consumption-wealth ratio. This is the first major difference between the two approaches: we make an explicit assumption about the "model" — that the preferences of the agent are close to logarithmic. This assumption allows one to build the expansions around the log-case solution, in powers of the risk aversion parameter. At the same time, our assumption guarantees that the consumption-wealth ratio is close to being constant and thus justifies the log-linearization. The original log-linearization technique does not make such explicit assumptions about the exogenous parameters of the model; instead, it imposes a "higher-level" assumption — directly on the endogenous variable, the optimal consumption-wealth ratio. As a result, one does not know *a priori* how the "point of expansion" is related to the exogenous model parameters and is forced to solve for an endogenous parameter $\kappa$.

Another difference between the two procedures can be seen from the comparison of the log-linearized equation (31) with our characterization of the function $g_0(X)$ in (19), which is equivalent to a linear Kolmogorov backwards equation. The equation obtained by log-linearization is still nonlinear, as it involves the square of the derivative of $g(X)$, and it can be solved only because of the special form assumed for the inverse of volatility in (29). This is an important limitation of the log-linearization technique — in general equilibrium economies the investment opportunity set is determined endogenously and one cannot expect the resulting nonlinear equation to lead to a closed-form solution. Thus, the log-linearization method is inherently partial-equilibrium in nature. In contrast, our asymptotic technique can be successfully applied to a variety of general equilibrium models, as illustrated in Sections 4 and 5.

Other differences between the two methods include their ability to handle finite-horizon problems (log-linearization relies on stationarity) and constrained portfolio optimization problems with a stochastic investment opportunity set (in which case the log-linearized Hamilton-Jacobi-Bellman equation is likely to be intractable).

We conclude this section by comparing numerically our asymptotic solution and the solution obtained using the log-linearization method with the "exact" numerical solution of the problem (using finite-difference methods) for a set of parameter values used in Chacko and Viceira (1999).[17] Our results are presented in Figure 1. The figure compares the accuracy of the asymptotic portfolio policy and the policy obtained by log-linearization. The first panel plots the percentage error of the approximate solutions: (approximate policy − true policy) / (true policy), where the "true" policy is based on the finite-difference solution of the original problem. The figure shows that the solution based on the log-linearization is more accurate than the asymptotic solution — the solution from log-linearization is practically indistinguishable from the numerical solution. However, the relative error of the asymptotic solution is small — less than 0.02 percent. The second panel of the figure compares the ratio of the hedging demand to the myopic demand, as given by the approximate policies and the numerical solution. The absolute magnitude of the hedging demand is approximately two percent of the myopic demand; the asymptotic solution yields approximately four percent for this ratio.

Thus, the advantage of the log-linearization method over our approach is that the latter has the potential to deliver better quality of approximation, at least when compared to the first-order asymptotic expansions. Formally, this can be expected, because the log-linearization preserves all but one of the original terms in the Hamilton-Jacobi-Bellman equation. A more fundamental reason is that log-linearization should be accurate whenever the consumption-wealth ratio is approximately constant, which can be true even when values of the risk aversion parameter are not close to zero.[18] Thus, one can view the log-linearization technique as being positioned somewhere between our method and solving the original problem exactly, both in terms of effectiveness in producing an explicit solution and the accuracy of the approximation.

[17] The parameter values are: $r = 0.015$, $\mu - r = 0.0799$, $\lambda_X = 0.3413$, $\overline{X} = 27.7088$, $\sigma_X = 0.6512$, $\sigma_{PX} = 0.5355\sigma_X$, $\rho = 0.06$ and $\gamma = -1$.

[18] When the investment opportunity set is almost constant, the log-linearized solution would be accurate for large values of $\gamma$, while our asymptotic solution may not be as accurate.

# 4 Portfolio choice in a general equilibrium production economy

In the previous section, the stochastic process for the state variables driving investment opportunities was specified exogenously. In this section, we examine a general equilibrium model in which the instantaneously riskless rate is endogenous, while the mean and volatility of stock returns is constant; in the next section, we consider a model where all three are stochastic.

The economy analyzed in this section is an extension of Dumas (1989), where the model studied is of a production economy with two agents who differ in risk aversion. The features of this economy are described below, with the extension relative to Dumas being the introduction of a constraint on borrowing.[19] Also, in contrast to the analysis in Dumas, which is based on numerical methods, we will examine the model using analytic methods. In addition to the aesthetic appeal of closed-form results, this allows us to get additional insights that are not transparent from a numerical analysis of the problem.

This section is structured as follows. We first describe the model, then analyze the unconstrained version of the model, and then examine the effect of borrowing constraints on equilibrium. We conclude by comparing the asymptotic solution with the "exact" (but numerical) solution developed by Dumas (1989). In the course of comparing the asymptotic solution to the solution in Dumas, we also discuss the non-local properties of the asymptotic solution.

## 4.1 The production economy

Following Dumas (1989), the economy is populated with two types of investors, with infinite horizons, the same subjective discount rate, but different risk aversion parameters. The first type of agent maximizes

$$\mathrm{E}_0\left[\int_0^\infty e^{-\rho t}\log\left(C_t^*\right)dt\right],$$

[19] We also study the case where agents have time-nonseparable recursive utility functions of the type described in Duffie and Epstein (1992), Epstein and Zin (1989), Kreps and Porteus (1978) and Weil (1989). The details of the analysis with recursive utility are not included in this draft.

while the second type of investor, who may be less or more risk averse than the log-utility investor ($\gamma > 0$ or $\gamma < 0$) maximizes

$$\mathrm{E}_0 \left[ \int_0^\infty e^{-\rho t} \frac{1}{\gamma} (C_t^\gamma - 1) dt \right].$$

We will adopt the convention of using "$*$" to indicate all quantities associated with the agent of the first type (who has log utility).

There is a single constant-returns-to-scale production technology available to agents in this economy, which leads to the following dynamics for the aggregate capital stock:

$$dS_t = (\alpha S_t - C_t - C_t^*) \; dt + \sigma S_t \, dZ_t,$$

There are two assets available for trading in the economy. The first asset is the stock on the production technology, generating the cumulative return process

$$dR_t = \alpha R_t \, dt + \sigma R_t dZ_t.$$

The number of shares of stock available in the economy is equal to the aggregate capital stock. The second asset is a short-term risk-free bond, available in zero net supply, which pays the interest rate $r_t$ that will be determined in equilibrium.

## 4.2 Equilibrium in the economy without constraints

The equilibrium in this economy is defined by the aggregate capital stock process, $S_t$, the interest rate process $r_t$, the portfolio policies $\{\theta_t^*, \theta_t\}$ and the consumption processes $\{C_t^*, C_t\}$, such that (1) given the price processes for financial assets, the consumption and portfolio choices are optimal for the agents, (2) the markets for the stock and the bond clear.

### Investment opportunity sets

Let $W_t^*$ and $W_t$ denote the individual wealth processes for the two types of agents. Following Dumas (1989), we define the state variable $\omega = W/\left(W^* + W\right)$ that summarizes the cross-sectional distribution of wealth in the economy. Since $\omega$ is the only state variable in the economy, the correspondence with the general formulation in Section 2 is that $\mathbf{X} = \{\omega\}$.[20] In equilibrium, $\omega$ characterizes completely the investment opportunity set faced by the agents: $r_t = r(\omega_t)$. The evolution of $\omega_t$ is given by

$$d\omega_t = \mu_{\omega t}\, dt + \sigma_{\omega t}\, dZ_t, \tag{35}$$

where the drift and diffusion coefficients are functions of the state: $\mu_{\omega t} = \mu_\omega(\omega_t)$, $\sigma_{\omega t} = \sigma_\omega(\omega_t)$.

If $\gamma$ were equal to zero, then both types of agents would have logarithmic preferences. As a result, they would hold the same portfolios and their wealth would be perfectly correlated; in this case, the cross-sectional distribution of wealth in the economy would not change over time and $\omega_t$ would be constant.

**Individual consumption-portfolio choice**

Based on the general asymptotic expression for the optimal consumption-portfolio choice in equations (16) and (18), we can characterize individual behavior in this economy. The optimal consumption and portfolio positions of the two types of investors are given by:

$$c^*(\omega) = \rho, \tag{36}$$

$$c(\omega) = \rho - \gamma\rho\left(g_0(\omega) - \log(\rho)\right) + O\left(\gamma^2\right), \tag{37}$$

$$\theta^*(\omega) = \frac{\alpha - r(\omega)}{\sigma^2}, \tag{38}$$

$$\theta(\omega) = \frac{\alpha - r(\omega)}{\sigma^2} + \gamma\left(\frac{\alpha - r(\omega)}{\sigma^2} + \frac{1}{\sigma^2}\sigma_{P\omega}\frac{\partial g_0(\omega)}{\partial\omega}\right) + O\left(\gamma^2\right). \tag{39}$$

To get a complete characterization of the above policies in terms of exogenous variables, we need to determine $g_0(\omega)$ and $r(\omega)$.

As discussed in Section 2, the function $g_0(\omega)$ characterizes the value function of the logarithmic agent in the economy. Since the returns on financial assets are determined endogenously in equilibrium, this function now depends implicitly on the parameter $\gamma$. In particular, it can be represented asymptotically as

$$g_0(\omega;\gamma) = g_{0,0} + \gamma g_{0,1}(\omega) + O\left(\gamma^2\right). \tag{40}$$

[20] In the unconstrained economy, because there is only one source of shocks, $\omega$ is perfectly correlated with the aggregate capital stock and markets are complete.

The leading term, $g_{0,0}$, defines the value function of the agent with log utility in the economy populated *entirely* with the logarithmic agents, which would be the case if $\gamma$ was equal to zero. Since the investment opportunity set in such an economy is constant, $g_{0,0}$ is a constant as well. According to (19), it is given by

$$g_{0,0} = \log(\rho) + \frac{\alpha - \sigma^2/2 - \rho}{\rho}. \tag{41}$$

The higher-order terms in (40) capture the indirect effect of $\gamma$ on the welfare of the logarithmic agents through its effect on equilibrium prices. We can safely ignore these higher-order terms when evaluating (38) and (39), because they do not affect the first-order asymptotic expansions of the individual consumption-portfolio strategies, which are:

$$\begin{aligned} c(\omega) &= \rho - \gamma\left(\alpha - \sigma^2/2 - \rho\right) + O\left(\gamma^2\right) \\ \theta(\omega) &= (1+\gamma)\frac{\alpha - r(\omega)}{\sigma^2} + O\left(\gamma^2\right). \end{aligned} \tag{42}$$

**Market clearing**

In this economy, it is sufficient to impose explicitly only the market clearing condition for one of the two markets, since the other market will clear automatically. We will use the market clearing condition in the stock market, which is

$$\theta\omega + \theta^*(1-\omega) = 1. \tag{43}$$

The short-term interest rate, $r_t = r(\omega_t)$, is the only price-process determined endogenously in equilibrium. It is derived by substituting the individual portfolio choices (38) and (42) into the market-clearing condition (43), which then allows us to provide the following asymptotic characterization of the competitive equilibrium.

**Proposition 4.1** *In the production economy described in this section,*
*(i) The equilibrium interest rate is given by*

$$r(\omega) = \alpha - \sigma^2 + \gamma\sigma^2\omega + O\left(\gamma^2\right); \tag{44}$$

*(ii) The optimal consumption and portfolio positions of the two types of investors are given by*

$$\begin{aligned} c^*(\omega) &= \rho, \\ c(\omega) &= \rho - \gamma\left(\alpha - \sigma^2/2 - \rho\right) + O\left(\gamma^2\right), \\ \theta^*(\omega) &= 1 - \gamma\omega + O\left(\gamma^2\right), \qquad (45) \\ \theta(\omega) &= 1 + \gamma(1-\omega) + O\left(\gamma^2\right); \qquad (46) \end{aligned}$$

*(iii) The cross-sectional wealth distribution evolves according to*

$$d\omega_t = \gamma\omega_t\left(1-\omega_t\right)\left(\alpha - \sigma^2/2 - \rho\right)\,dt + \gamma\omega_t\left(1-\omega_t\right)\sigma\,dZ_t + O\left(\gamma^2\right); \qquad (47)$$

*(iv) The aggregate capital stock evolves according to*

$$\frac{dS_t}{S_t} = \left(\alpha - \rho + \gamma\left(\alpha - \sigma^2/2 - \rho\right)\omega_t\right)\,dt + \sigma\,dZ_t + O\left(\gamma^2\right).$$

The analysis of the equilibrium relies on the fact that the consumption and portfolio policies can be characterized explicitly. In particular, our characterization of these policies can be viewed as an outcome of the following two-step procedure:

1. Solve for the equilibrium in the homogeneous-agent economy, populated *only* by logarithmic agents, and determine the value function for these agents, (given by $g_{0,0}$);

2. Use $g_{0,0}$ in place of $g_0$ in equations (38) and (39) to evaluate the equilibrium consumption and portfolio policies.

Once again, just as in the analysis of partial-equilibrium economies in Section 3, the ability to derive closed-form expressions for the consumption and portfolio policies depends on the ability to characterize explicitly the value function of logarithmic investors in an economy where *all* investors have logarithmic preferences. This test allows one to identify the models that can be analyzed using our approach.

**Properties of the equilibrium and comparison with Dumas (1989)**

We now discuss our asymptotic results, compare them with the numerical and analytical results in Dumas (1989), and also describe the additional insights available from the asymptotic analysis.

Studying the portfolio policies in equations (45) and (46), we see that the size of each investor's relative position in the stock market depends on her degree of relative risk aversion. The less risk averse agent always borrows at the risk-free rate to invest in the stock, which is Proposition 15 in Dumas (1989). Observe also that that the equilibrium portfolio holdings are asymptotically myopic, since the standard deviation of the state variable $\omega_t$ is of order $O(\gamma)$ in equilibrium, implying that the hedging demand is of order $O\left(\gamma^2\right)$.

In addition to these insights, the asymptotic analysis also allows us to interpret the portfolio weights of the two investors as the ratio of the individual's risk tolerance to the average risk tolerance. Risk tolerance for an individual is the inverse of the agent's risk aversion. Thus, the risk tolerance of the log investor is unity, while that for the non-log investor is $1/(1-\gamma)$. The wealth-weighted average of the individual risk tolerances is:

$$\omega \cdot \frac{1}{(1-\gamma)} + (1-\omega) \cdot 1.$$

Taking the first-order expansion of the ratio of risk tolerance for each individual to the average risk tolerance gives us the expressions for the portfolio weights in equations (45) and (46).

The open interest in the bond market (as a fraction of the aggregate wealth) is given by

$$OI_t \equiv \frac{1}{2}\Big(|1-\theta_t|\,\omega + |1-\theta_t^*|\,(1-\omega)\Big) = \ \gamma\,\omega_t\,(1-\omega_t)\,,$$

i.e., one type of investors lends this amount and another type borrows. Thus, the open interest achieves its maximum when $\omega = 1/2$, i.e., when the aggregate wealth is evenly distributed among the agents.

According to (44), the short-term interest rate in this production economy is asymptotically proportional to the state variable. This is consistent with the qualitative result in Dumas (1989, Proposition 17): $r(\omega)$ is increasing when $\gamma > 0$ and decreasing otherwise.

Dumas (1989, Proposition 17 and equation (12)) states that the interest rate in a heterogeneous-agent economy is *bounded* by the interest rates in the corresponding homogeneous economies, each populated by one of the two types. Another interpretation of the asymptotic expression for $r(\omega)$, which relies on the explicit solution and sharpens the corresponding result in Dumas, is the following: the interest rate is the wealth-weighted *average* of interest rates in two economies, each populated exclusively by logarithmic and non-logarithmic agents, respectively.[21] Also, as in the case of the portfolio weights, the interest rate can be interpreted in terms of average risk tolerance:

$$r(\omega) = \alpha - [\text{average risk aversion}] \cdot \sigma^2,$$

where average risk aversion is defined as the inverse of average risk tolerance. This, of course, corresponds to the expression for the interest rate in a representative-agent economy; the only difference is that in our case the average risk aversion is endogenous: it changes with $\omega$.

The volatility of the interest rate is proportional to the volatility of the cross-sectional wealth distribution, $\omega$, because of its linear dependence on the state variable. According to (47), the instantaneous standard deviation of the state variable equals $\gamma\omega_t\,(1-\omega_t)$. Thus, the standard deviation of the interest rate is of order $O\left(\gamma^2\right)$ and is a symmetric quadratic function of the cross-sectional wealth distribution $\omega$, achieving its highest value when the wealth in the economy is evenly distributed among the agents:

$$\sigma_{rt} = \gamma^2\sigma^3\omega_t\,(1-\omega_t)\,.$$

This observation, illustrated in Figure 2, is consistent with the numerical results in Dumas (1989) and his Figures 5 and 6. Also, in equilibrium the standard deviation of the interest rate is proportional to the open interest in the bond market. As Dumas notes, the process for the interest rate that emerges from equilibrium is quite different from that assumed in partial-equilibrium settings.

Finally, according to (47), the cross-sectional wealth distribution is non-stationary: over time, the less risk averse agent tends to accumulate wealth at a higher rate and eventually dominates the economy. This is consistent with the results in Dumas (1989, Section 5.1). According to a set of criteria presented there, the wealth distribution is *always* non-stationary when the risk aversion parameter of the non-logarithmic agent is sufficiently close to zero.

Dumas (1989, Section 5.1) also demonstrates that under a certain set of conditions the wealth distribution is stationary. Our results indicate that this stationarity cannot be captured without including higher-order terms in the asymptotic expansions.

[21] See Detemple and Murthy (1997) for a similar result in a different setting.

### 4.3 Equilibrium and optimal portfolio policies with constraints

We now study the economy described above, but with a constraint limiting the ability of agents to use leverage. Specifically, the positions of the two agents are constrained to satisfy

$$\begin{aligned} 1-\theta &\geq -L\gamma, \\ 1-\theta^* &\geq -L\gamma, \end{aligned}$$

where $L \geq 0$ is a constant that limits the proportion of wealth that investors can borrow. In equilibrium it is the less risk-averse agent who is the borrower; hence, it is this agent's constraint that ends up being binding in equilibrium.

One can consider two distinct cases: Case 1, where the non-log agent is less risk averse than the log agent; and, Case 2, where the non-log agent is more risk averse than the log agent. We analyze Case 1 below; the analysis for Case 2 is similar, and the details are not included.

Under Case 1, the non-log agent is less risk averse than log agent, which implies that the risk aversion of the non-log agent is less than one, i.e., $\gamma > 0$. As long as $L \geq 1$, the solution of the unconstrained problem, as given in Proposition 4.1, is feasible and therefore describes the equilibrium in the constrained economy. Therefore, we focus on the situation where $L < 1$.

To compute the equilibrium prices and allocations, one has to repeat the steps of Section 4.2 taking into account that the individual portfolio demands are now given by Proposition 2.5, instead of Proposition 2.3. Doing this, we find that the equilibrium of the constrained economy can be naturally described in terms of two regions of the state space. In the first region,

$$\omega > 1 - L,$$

the leverage constraint is not binding and the solution in this region (indicated henceforth by superscript "u") is identical to the one in the unconstrained case, as given in the Proposition 4.1.

Note that even if the borrowing constraint does not bind today, it is possible that it might be binding in the future; it turns out, however, that in equilibrium prices respond sufficiently so that the effect of the constraint binding on a future date is of higher order and does not affect the first-order asymptotic solution.

In the second region,

$$\omega \leq 1 - L,$$

the leverage constraint is binding today and has a direct effect on individual portfolio choices and equilibrium asset prices (we indicate the solution in this region by superscript "c").

Our results are summarized in the following proposition.

**Proposition 4.2** *In the region $\omega > 1-L$, the equilibrium in the constrained economy coincides with the equilibrium of the unconstrained economy described in Proposition 4.1. In the region $\omega \leq 1-L$, the borrowing constraint is binding and:*
*(i) The equilibrium interest rate is given by*

$$r^{\mathsf{c}}(\omega) = \alpha - \sigma^2 + \gamma\sigma^2 L \frac{\omega}{1-\omega} + O\left(\gamma^2\right);$$

*(ii) The optimal consumption and portfolio positions of the two types of investors are given by*

$$\begin{aligned} c^{*\mathsf{c}}(\omega) &= \rho, \\ c^{\mathsf{c}}(\omega) &= \rho - \gamma\left(\alpha - \sigma^2/2 - \rho\right) + O\left(\gamma^2\right), \\ \theta^{*\mathsf{c}}(\omega) &= 1 - \gamma \frac{L\omega}{1-\omega} + O(\gamma^2), \\ \theta^{\mathsf{c}}(\omega) &= 1 + \gamma L + O(\gamma^2); \end{aligned}$$

*(iii) The cross-sectional wealth distribution evolves according to*

$$d\omega_t = \gamma\omega_t\left(1-\omega_t\right)\left(\alpha - \sigma^2/2 - \rho\right)\, dt + \gamma\omega_t L\sigma dZ_t + O\left(\gamma^2\right).$$

*(iv) The aggregate capital stock evolves according to*

$$\frac{dS_t}{S_t} = \left(\alpha - \rho + \gamma\left(\alpha - \sigma^2/2 - \rho\right)\omega_t\right)\, dt + \sigma\, dZ_t + O\left(\gamma^2\right).$$

**Comparison with the unconstrained economy**

We now evaluate the impact of the leverage constraint on the behavior of endogenous variables in the economy. First, consider the evolution of the cross-sectional distribution of wealth $\omega_t$. While the drift of this state variable is not affected by the constraint, its volatility is. In particular,

$$\sigma_\omega^{\mathsf{c}}(\omega) = \sigma_\omega^{\mathsf{u}}(\omega)\,\frac{L}{1-\omega} + O\left(\gamma^2\right) \lesssim \sigma_\omega^{\mathsf{u}}(\omega)\,.$$

Thus, the the conditional volatility of the state variable is lower than in the unconstrained economy. This result can be understood in terms of the extreme case: $L = 0$. In this case, leverage is not allowed. Therefore, both agents invest only in the risky asset and their wealth is perfectly correlated. As a result, the volatility of the state variable, $\omega = W/(W + W^*)$, is equal to zero.

Next, consider the risk-free rate in the constrained economy. It is lower than in the corresponding unconstrained economy:

$$r^{\mathsf{c}}(\omega) = r^{\mathsf{u}}(\omega) - \gamma\sigma^2\,\frac{\omega\left((1-L)-\omega\right)}{1-\omega} + O\left(\gamma^2\right) \lesssim r^{\mathsf{u}}(\omega)\,.$$

Due to the presence of the constraint, the demand for borrowing by the non-log agent, as a function of the spot interest rate, is reduced, while the supply by the log agent remains the same. Thus, the interest rate is reduced.

The effect of the leverage constraint on the conditional volatility of the interest rate is more complicated. The conditional volatility is

$$\sigma_{rt}^{\mathsf{c}} = \sigma_r^{\mathsf{c}}(\omega) = \gamma^2\sigma^3\,\frac{L^2\omega}{(1-\omega)^2} + O\left(\gamma^3\right),$$

which can be higher or lower than in the unconstrained case. Specifically, the volatility is higher in the region $\omega \in \left[1 - L^{2/3}, 1 - L\right]$ and lower in the region $\omega \in \left[0, 1 - L^{2/3}\right]$. In addition, the volatility experiences a jump as a function of the state variable at $\omega = 1 - L$:

$$\sigma_r^{\mathsf{c}}(1-L) > \sigma_r^{\mathsf{u}}(1-L)\,.$$

The behavior of the interest rate and its volatility is illustrated in Figure 2.

Finally, the conditional volatility of the spot interest rate is an increasing function of $L$. Thus, if the leverage constraint were to be unexpectedly relaxed through a small increase in $L$, the short-term interest rate would become more volatile.

## 4.4 Comparison of asymptotic approximation with exact solution

In this section, we compare our asymptotic solution to the exact solution in Dumas (1989) for the unconstrained problem, which is obtained by solving the Hamilton-Jacobi-Bellman equation characterizing the problem of the non-log investor. Given that this is a non-linear partial differential equation, the "exact" solution can only be obtained using numerical methods.[22]

The Bellman equation for the non-log investor can be obtained by using the standard dynamic programming approach and substituting the optimal consumption and portfolio policies using the first order conditions for these variables. Because the details of this derivation are given in Dumas (1989, equation (24)), we state only the final equation, with the observation that that the function $I(\omega)$ in Dumas corresponds to $(1/\rho)\exp(\gamma g(\omega))$ in our model:[23]

$$\begin{aligned} 0 &= -\rho/\gamma + \frac{(1-\gamma)}{\gamma}\left(\frac{e^{\gamma\, g(\omega)}}{\rho}\right)^{1/(\gamma-1)} + \left[\alpha - \sigma^2 h(\omega) - \frac{1}{2}(\gamma-1)\left(\sigma\mu(\omega)h(\omega)\right)^2\right] \\ &\quad + g'(\omega)\,\omega(1-\omega)\left[\rho - \omega\sigma^2\left[\left(\mu(\omega)-1\right)h(\omega)\right]^2 - \left(\frac{e^{\gamma\, g(\omega)}}{\rho}\right)^{1/(\gamma-1)}\right] \\ &\quad + \frac{1}{2}\left[\gamma\, g'(\omega)^2 + g''(\omega)\right]\left[\omega(1-\omega)\sigma\left(\mu(\omega)-1\right)h(\omega)\right], \end{aligned} \tag{48}$$

with boundary conditions at $\omega = 0$ and $\omega = 1$:

$$\begin{aligned} g(0) &= \frac{1}{\gamma}\log\left[\rho\left(\frac{1-\gamma}{\rho-\gamma\left(\alpha-\sigma^2+\frac{\sigma^2}{2(1-\gamma)}\right)}\right)^{1-\gamma}\right], \\ g(1) &= \frac{1}{\gamma}\log\left[\rho\left(\frac{1-\gamma}{\rho-\gamma\left(\alpha-\frac{(1-\gamma)\,\sigma^2}{2}\right)}\right)^{1-\gamma}\right], \end{aligned}$$

where

$$\mu(\omega) \equiv \frac{1-\omega(1-\omega)\gamma\, g'(\omega)}{1-\gamma-\omega(1-\omega)\gamma\, g'(\omega)}, \qquad h(\omega) \equiv \frac{1}{\omega\mu(\omega)+1-\omega}.$$

[22] Bernard Dumas kindly provided to us the solution to this differential equation for the parameter values that he considered in his paper. His solution was obtained using a finite-difference scheme to solve the differential equation. We then wrote computer code also based on the finite-difference method, and verified it by comparing it to the solution provided; the difference between our numerical values and the ones provided by Dumas was never more than $10^{-5}$, and is typically of the order $10^{-7}$.

[23] The definition of $\mu(\omega)$ corresponds to that in Dumas (1989); it should not be confused with the drift of $\omega$, $\mu_\omega$.

To study the properties of the asymptotic solution for levels of risk aversion different from unity ($\gamma$ different from zero), we compare the portfolio policies and the interest rate obtained from the exact solution of the Bellman equation in (48) to that obtained using our approach. For the non-local analysis of the asymptotic solution, we derive the portfolio results based on equation (17) rather than equation (18). While these two equations give portfolio weights that are asymptotically equivalent, the first expression performs much better when $\gamma$ is significantly different from 0. The portfolio weights based on (17) can be interpreted as the ones obtained by substituting the asymptotic solution for the value function into the exact expression for the portfolio weights. The expressions for the optimal portfolio weights—based on (17), and asymptotically equivalent to the ones in (45) and (46)—are:

$$\begin{aligned}\theta^*(\omega) &= 1 - \frac{1-\gamma}{1-\gamma\,(1-\omega)} + O\left(\gamma^2\right), \\ \theta(\omega) &= 1 + \frac{1}{1-\gamma\,(1-\omega)} + O\left(\gamma^2\right).\end{aligned}$$

Using the market-clearing condition in (43) with these weights gives the following expression for the interest rate, which again is asymptotically equivalent to the interest rate in (44) :

$$r(\omega) = \alpha - \sigma^2 + \sigma^2 \frac{\gamma\,\omega}{1-\gamma\,(1-\omega)} + O\left(\gamma^2\right).$$

In Figure 3, we plot the portfolio weight for the non-log investor based on the exact solution (solid line) and that based on the asymptotic solution of order 1 (circles) for relative risk aversion of 0.5, 2 and 3. In all three plots, we see that the portfolio weights from the exact and the asymptotic solution are virtually the same. In Figure 4, we repeat this experiment for the interest rate. Again, we see that the interest rate obtained from from the exact solution (solid line) and the asymptotic solution of order 1 (circles) are very close to each other.

These results indicate that the non-local properties of the asymptotic approximation can be quite good for economic quantities such as consumption-portfolio policies and prices.[24]

[24]In general, the non-local properties of the asymptotic solution can be further improved in at least two ways. First, one can include higher-order terms, along with methods to accelerate the rate of convergence of a series; one such method is given in Press, Teukolsky, Vetterling and Flannery (1992, p. 166). A second approach is to use a Padé approximation, the details of which are in Judd (1996, p. 516).

# 5 Portfolio choice in a general equilibrium exchange economy

In the production economy described in the previous section, the assumption of constant-returns-to-scale implies that the expected return on the risky stock and the volatility of the return is constant, with only the return on the riskless asset being stochastic. In this section, we consider an exchange economy where the equilibrium interest rate process and also the expected return on the risky asset and the return volatility are endogenous. A second difference with the economy studied in the previous section is that now financial markets are incomplete.

## 5.1 The exchange economy

The preferences of agents in this economy and the structure of securities markets is identical to those in Section 4.1. The stock is a claim on the aggregate endowment, which is given by

$$de_t = X_t e_t \, dt + \sigma_e e_t \, dZ_{et}, \tag{49}$$

with

$$dX_t = \underbrace{-\lambda_X \left(X_t - \overline{X}\right)}_{\mu_X(X_t)} dt + \sigma_X \, dZ_{Xt}, \tag{50}$$

$$[dZ_{et}, dZ_{Xt}] = \rho_{eX} \, dt, \tag{51}$$

where $\sigma_e$, $\lambda_X$, $\sigma_X$ and $\rho_{eX}$ are constant parameters. The process for the state variable $X_t$ dictates the instantaneous expected growth rate of the aggregate endowment process.[25]

## 5.2 Equilibrium in the exchange economy

The equilibrium in this economy is defined by the stock price process, $P_t$, the interest rate process $r_t$, the portfolio policies $\{\theta_t^*, \theta_t\}$ and the consumption processes $\{C_t^*, C_t\}$, such that (1) given the price processes for financial assets, the consumption and portfolio choices are optimal for the agents, (2) the goods market and the markets for the stock and the bond clear. Our analysis of the equilibrium follows the same logic as in Section 4.2.

[25] With a constant expected growth rate, this model reduces to the complete-market economy studied in Wang (1996).

**Investment opportunity sets**

The state of the economy is now given by

$$\mathbf{X}_t = \begin{pmatrix} X_t \\ \omega_t \end{pmatrix},$$

where, as before, $\omega_t = W_t / (W_t^* + W_t)$. The first component of the state vector, $X_t$, is defined by equation (50). The evolution of $\omega_t$ is given by

$$d\omega_t = \mu_{\omega t}\, dt + \sigma_{\omega X,t}\, dZ_{Xt} + \sigma_{\omega e,t}\, dZ_{et}, \tag{52}$$

where the drift and the diffusion coefficients are functions of the state: $\mu_{\omega t} = \mu_\omega(\mathbf{X}_t)$, $\sigma_{\omega X,t} = \sigma_{\omega X}(\mathbf{X}_t)$, and $\sigma_{\omega e,t} = \sigma_{\omega e}(\mathbf{X}_t)$. Expanding the coefficients in (52) in powers of $\gamma$ gives:

$$\begin{aligned}
\mu_\omega(\mathbf{X}) &= \mu_{\omega,0}(\mathbf{X}) + \gamma\mu_{\omega,1}(\mathbf{X}) + O(\gamma^2), \\
\sigma_{\omega X}(\mathbf{X}) &= \sigma_{\omega X,0}(\mathbf{X}) + \gamma\sigma_{\omega X,1}(\mathbf{X}) + O(\gamma^2), \\
\sigma_{\omega e}(\mathbf{X}) &= \sigma_{\omega e,0}(\mathbf{X}) + \gamma\sigma_{\omega e,1}(\mathbf{X}) + O(\gamma^2),
\end{aligned}$$

where the functions $\mu_{\omega,0}(\mathbf{X})$, $\mu_{\omega,1}(\mathbf{X})$, $\sigma_{\omega X,0}(\mathbf{X})$, $\sigma_{\omega X,1}(\mathbf{X})$, $\sigma_{\omega e,0}(\mathbf{X})$ and $\sigma_{\omega e,1}(\mathbf{X})$ are determined endogenously in equilibrium. When $\gamma = 0$, both types of agents have logarithmic preferences. As a result, the cross-sectional distribution of wealth in the economy does not change over time and $\omega_t$ is constant, which implies that $\mu_{\omega,0}(\mathbf{X}) = 0$; $\sigma_{\omega X,0}(\mathbf{X}) = 0$; $\sigma_{\omega e,0}(\mathbf{X}) = 0$.

**Individual consumption-portfolio choice**

Based on the general asymptotic expression for the optimal consumption-portfolio choice (16) and (18), the optimal consumption and portfolio positions of the two types of investors are given by:

$$c^*(X,\omega) = \rho, \tag{53}$$

$$c(X,\omega) = \rho - \gamma\rho\,(g_0(X,\omega) - \log\rho) + O(\gamma^2), \tag{54}$$

$$\theta^*(X,\omega) = \frac{\mu_R(X,\omega) - r(X,\omega)}{\sigma_P^2(X,\omega)}, \tag{55}$$

$$\theta(X,\omega) = (1+\gamma)\,\frac{\mu_R(X,\omega) - r(X,\omega)}{\sigma_P^2(X,\omega)} \tag{56}$$

$$+\gamma\left(\frac{1}{\sigma_P^2(X,\omega)}\left[\begin{pmatrix} \sigma_{PX}(X,\omega) \\ \sigma_{P\omega}(X,\omega)\end{pmatrix}' \cdot \begin{pmatrix} \frac{\partial g_0(X,\omega)}{\partial X} \\ \frac{\partial g_0(X,\omega)}{\partial \omega}\end{pmatrix}\right]\right) + O(\gamma^2), \tag{57}$$

where $\mu_R(X,\omega)$ is the expected cumulative rate of return on the stock (taking into account both capital gains and the dividend process), $\sigma_P(X,\omega)$ is the diffusion coefficient of stock returns and $\sigma_{PX}(X,\omega)$, $\sigma_{P\omega}(X,\omega)$ are the covariances of stock returns with changes in the state variables $X$ and $\omega$. Asymptotically, the function $g_0(X,\omega)$ can be expanded as

$$g_0\left(X,\omega;\gamma\right) = g_{0,0}\left(X\right) + \gamma g_{0,1}\left(X,\omega\right) + O\left(\gamma^2\right),$$

where $g_{0,0}$ corresponds to the value function of the logarithmic investor in a homogeneous-agent economy in which *all* investors have logarithmic preferences. The precise form of $g_{0,0}\left(X\right)$ is given in the following proposition.

**Proposition 5.1** *The function $g_{0,0}$ is*

$$g_{0,0}\left(X_0\right) = a + bX_0 + \log\rho, \tag{58}$$

*where the constants $a$ and $b$ are:*

$$a = -\frac{\sigma_e^2}{2\rho} + \frac{\lambda_X \overline{X}}{\rho\left(\rho+\lambda_X\right)}, \quad b = \frac{1}{\rho+\lambda_X}.$$

### Market clearing

In equilibrium, the market-clearing condition in the stock market that determines the equilibrium price processes is

$$1 = \omega\theta + \left(1-\omega\right)\theta^*.$$

Using the market clearing condition, and the expression for $g_{00}$ in (58), one obtains the following characterization of the equilibrium in this economy.

**Proposition 5.2** *In the exchange economy of Section 5.1,*

*(i) The equilibrium stock price process is given by*

$$\begin{aligned} P_t &= p\left(X_t,\omega_t\right)e_t, \\ p\left(X,\omega\right) &\equiv \frac{1}{\rho}\left[1+\gamma\omega\left(a+bX\right)\right] + O\left(\gamma^2\right), \\ \frac{dP_t + e_t\,dt}{P_t} &= \left[X_t + \rho - \gamma\omega_t\left(X_t - \frac{\sigma_e^2}{2} - \frac{\rho_{eX}\sigma_e\sigma_X}{\rho+\lambda_X}\right)\right]dt \\ &\quad + \sigma_e dZ_{et} + \gamma b\sigma_X\omega_t\,dZ_{Xt} + O\left(\gamma^2\right); \end{aligned} \tag{59}$$

*(ii) The equilibrium interest rate is given by*

$$r_t = X_t - \sigma_e^2 + \rho - \gamma\omega_t \left( X_t - \frac{3}{2}\sigma_e^2 \right) + O\left(\gamma^2\right);$$

*(iii) The optimal consumption and portfolio positions of the two types of investors are given by*

$$\begin{aligned} c^* (X,\omega) &= \rho, \\ c(X,\omega) &= (\rho - \gamma\rho (a + bX - \log\rho)) + O\left(\gamma^2\right), \\ \theta^*(X,\omega) &= 1 - \gamma\omega \left(1 + \frac{\rho_{eX}\sigma_X}{\sigma_e(\rho+\lambda_X)}\right) + O\left(\gamma^2\right), \\ \theta(X,\omega) &= 1 + \gamma(1-\omega)\left(1 + \frac{\rho_{eX}\sigma_X}{\sigma_e(\rho+\lambda_X)}\right), \end{aligned}$$

*where $a$ and $b$ are defined in Proposition 5.1.*

*(iv) The cross-sectional wealth distribution evolves according to*

$$d\omega_t = \gamma\omega_t (1-\omega_t)(a + bX_t)\, dt + \gamma\omega_t(1-\omega_t)\sigma_e \left(1 + \frac{\rho_{eX}\sigma_X}{\sigma_e} b\right) dZ_{et} + O\left(\gamma^2\right).$$

**Properties of optimal portfolios**

There is an important difference between the equilibrium portfolio policies in this exchange economy and the portfolio policies in the production economy of Section 4. While in the model of Section 4 the less risk averse agent always borrows to invest in the stock market, here it is possible for the less risk averse agent to be a lender. This would be the case if

$$\frac{\rho_{eX}\sigma_X}{\sigma_e(\rho+\lambda_X)} < -1. \tag{60}$$

These results are driven by the hedging demand of the non-logarithmic agent, who hedges the changes in the risk-free rate (the Sharpe ratio in this economy is asymptotically constant). In equilibrium, this demand can be sufficiently large to cause the more risk averse agent to borrow from the less risk averse. Hedging demand has an impact on the equilibrium price processes as well. In particular, it affects the expected rate of return on the stock in (59).

Also, as the risk aversion of the non-logarithmic agent increases, her position in the stock can increase, as long as (60) is satisfied. This result is a general-equilibrium counterpart of the partial-equilibrium results of Section 3. It holds because the non-logarithmic agent hedges the changes in the risk-free rate and her hedging demand can be more sensitive to the risk aversion coefficient than her myopic demand.

As in the case of the production economy considered in the previous section, the instantaneously riskless rate, $r_t$, is the wealth-weighted average of the interest rates in two economies, each populated exclusively by logarithmic and non-logarithmic agents, respectively. This is now true also for the cumulative return on the stock.

## 6 Conclusion

In this article, we have provided an asymptotic analysis of the optimal consumption and portfolio decisions of an investor who has preferences over intermediate consumption and faces an economic environment with stochastic investment opportunities and incomplete financial markets. Our results include comparative statics results for optimal portfolios, and analytic expressions for the asymptotic value function and decision rules of an individual investor.

Following the analysis of the portfolio policy of a single agent, we have shown how the portfolio-choice problem in the presence of a stochastic investment opportunity set can be embedded in a general equilibrium exchange or production economy, with multiple investors who differ in their degree of risk aversion, when they face constraints on their portfolio positions, and financial markets are incomplete.

Our analysis relies on asymptotic expansions. This allows us to obtain local comparative static results that are exact. When the preferences of agents being studied are close to those of a log-utility maximizer, by construction the consumption and portfolio policies are close to the exact policies. However, even when the utility function of the agent under consideration is not close to logarithmic, the asymptotic solution can yield very good approximations for the decision rules. An example of this is provided in the context of a general equilibrium production economy with two agents who differ in their degree of risk aversion.

The general model developed here is for an economy with a single risky asset. The extension to multiple risky assets is straightforward. Also, we have assumed that agents have time-additive power utility rather than the more general recursive preferences described in Duffie and Epstein

(1992), Epstein and Zin (1989), Kreps and Porteus (1978), and Weil (1989). Given that log utility is a special case also of the Kreps-Porteus specification of recursive utility, it is possible to extend the asymptotic method to the case of recursive preferences. Chan and Kogan (2000) demonstrate how the method developed here can also be applied to an economy where agents exhibit habit-persistence.

One limitation of the analysis we have presented is that it applies only to those situations where there exists a closed-form solution for an investor with logarithmic utility. However, even when an explicit solution does not exist for the log investor, one may apply asymptotic analysis, but with the perturbation now being around a parameter different from that governing risk aversion. Also, in this paper we have not discussed results on evaluating the approximation error. However, there exist a number of methods to evaluate the quality of the approximate solution (for instance, see Den Haan and Marcet, 1994, and Judd, 1996 and 1998) which can also be applied to the portfolio problems considered here.

# A Proofs for all propositions

### Proof of Proposition 2.1

The result follows by substituting (9) into (8). First- and higher-order terms in the expansion of $g(\mathbf{X})$ do not affect the first-order asymptotic expansion of the optimal consumption-portfolio policy. The equivalent asymptotic expression (13) is obtained by expanding (12) in powers of $\gamma$ and eliminating terms of order two and higher.

### Proof of Proposition 2.2

Using the definition of the value function of the log-utility maximizer, with optimal consumption $c(\mathbf{X}_t, t) = B/A(t)$ substituted in,

$$J(W_t, \mathbf{X}_t, t) = B\mathrm{E}_t \left[ \int_t^T e^{-\rho(s-t)} \log\left(\frac{W_s}{A(s)}\right) ds \right] + (1-B)e^{-\rho(T-t)}\mathrm{E}_t \left[\log(W_T)\right], \qquad \text{(A1)}$$

where the wealth process $W_t$ evolves according to

$$\frac{dW_t}{W_t} = \left( -\frac{B}{A(t)} + r(\mathbf{X}_t) + \theta(\mathbf{X}_t, t)\sigma_P(\mathbf{X}_t)\phi(\mathbf{X}_t) \right) dt + \theta(\mathbf{X}_t, t)\sigma_P(\mathbf{X}_t)\, dZ_{Pt}.$$

Thus,

$$\log(W_s) = \log(W_t) + \int_t^s -\frac{B}{A(u)} + r(\mathbf{X}_u) + \frac{\phi(\mathbf{X}_u)^2}{2}\, du + \int_t^s \phi(\mathbf{X}_u)\, dZ_{Pu},$$

where we have used the expression for the optimal portfolio policy: $\theta(\mathbf{X}_t, t) = \phi(\mathbf{X}_t)/\sigma_P(\mathbf{X}_t)$. Substituting this into (A1) yields

$$J(W_t, \mathbf{X}_t, t) =$$

$$B\mathrm{E}_t \left[ \int_t^T e^{-\rho(s-t)} \left( \log(B) - \log(A(s)) + \left( \log(W_t) + \int_t^s -\frac{B}{A(u)} + r(\mathbf{X}_u) + \frac{\phi(\mathbf{X}_u)^2}{2}\, du \right) \right) ds \right]$$

$$+ (1-B)e^{-\rho(T-t)}\mathrm{E}_t \left[ \left( \log(W_t) + \int_t^T -\frac{B}{A(u)} + r(\mathbf{X}_u) + \frac{\phi(\mathbf{X}_u)^2}{2}\, du \right) \right].$$

Integration by parts completes the proof of the proposition.

**Proof of Proposition 2.3 and Proposition 2.4**

The results follow by setting $B = 1$ and taking a limit of $T \to \infty$ for the corresponding expressions in Propositions 2.1 and 2.2, while noting that in the limit, $A(t) = \rho^{-1}$.

**Proof of Proposition 2.5**

As in Proposition 2.3, this result follows by replacing the function $g(\mathbf{X})$ in the expression for the optimal consumption-portfolio policy with its asymptotic expansion. Only the leading term in the expansion must be retained, which corresponds to the solution of the log-utility maximizer's problem.

**Proof of Proposition 2.6**

The wealth process of the log-investor evolves according

$$\frac{dW_t}{W_t} = \Big(-\rho + r(\mathbf{X}_t) + \theta_0(\mathbf{X}_t)\sigma_P(\mathbf{X}_t)\phi(\mathbf{X}_t)\Big)\, dt + \theta_0(\mathbf{X}_t)\sigma_P(\mathbf{X}_t)\, dZ_{Pt},$$

where $\theta_0(\mathbf{X}_t)$ is the optimal portfolio policy of the log-utility maximizer, given by (22). Repeating the steps of the proof of Proposition 2.4, we obtain the desired result.

**Proof of Proposition 3.1**

The proof is similar to that for Proposition 3.4 below.

**Proof of Proposition 3.2**

The result follows by using (26) in (17).

**Proof of Proposition 3.3**

By first- and second-order stochastic dominance, function $H(\mathbf{X}, t)$ increases in a particular parameter if such a parameter either increases the process $\phi(t)$ or adds a mean-preserving spread to it.

Since the process $\phi(\mathbf{X}_t)$ satisfies

$$d\phi(\mathbf{X}_t) = -\lambda_{X_2}(\phi(\mathbf{X}_t) - \alpha_\phi)\, dt + \sigma_{X_2}\, dZ_{X_2 t}, \tag{A2}$$

it can be characterized as

$$\phi(t) = \alpha_\phi + (\phi_0 - \alpha_\phi)\, e^{-\lambda_{X_2} t} + \int_0^t \sigma_{X_2} e^{-\lambda_{X_2}(t-s)}\, dZ_{X_2 s}.$$

This explicit expression for $\phi(t)$ leads to the following results:

(i) $\partial\phi(t)/\partial\phi_0 = \exp(-\lambda_{X_2} t) > 0$, therefore $\phi(t)$ increases in $\phi_0$;

(ii) $\partial\phi(t)/\partial\alpha_\phi = (1 - \exp(-\lambda_{X_2} t)) > 0$, therefore $\phi(t)$ increases in $\alpha_\phi$;

(iii) The conditional distribution of $\phi(t)$ is Gaussian with mean $m(t)$ and variance $s^2(t)$. The mean $m(t)$ is independent of $\sigma_{X_2}$, while $s^2(t)$ is increasing in $\sigma_{X_2}$. Therefore, increasing $\sigma_{X_2}$ adds a mean-preserving spread to the distribution of $\phi(t)$;

(iv) The mean $m(t)$ is independent of $\lambda_{X_2}$, while $s^2(t)$ is decreasing in $\lambda_{X_2}$. Therefore, reducing $\lambda_{X_2}$ adds a mean-preserving spread to the distribution of $\phi(t)$ and increases $H(\mathbf{X}, t)$. Thus, $H(\mathbf{X}, t)$ is decreasing in $\lambda_{X_2}$, and so is the product $\exp[-(\lambda_{X_2} + \rho)t]\, H(\mathbf{X}, t)$.

(v) $H(\mathbf{X}, t)$ is independent of $\rho$, while $\exp[-(\lambda_{X_2} + \rho)t]$ is decreasing in it;

(vi) The function $\left(\phi(\mathbf{X}_t) - \overline{\phi}\right) 1_{[\phi(\mathbf{X}_t) \geq \overline{\phi}]}$ is decreasing in $\overline{\phi}$ and therefore decreasing in $L$, and so is $H(\mathbf{X}, t)$;

(vii) By Fubini theorem,

$$H(\mathbf{X}, t) = \mathrm{E}\left[\int_0^\infty \left(\phi(\mathbf{X}_t) - \overline{\phi}\right) 1_{[\phi(\mathbf{X}_t) \geq \overline{\phi}]} \middle| \mathbf{X}_0 = \mathbf{X}\right].$$

Because the conditional distribution of $\phi(t)$ is Gaussian with mean $m(t)$ and variance $s^2(t)$, the expectation can be computed explicitly, yielding (28).

**Proof of Proposition 3.4**

The Sharpe ratio of stock returns equals

$$\phi(X_t) = (\mu - r)\sqrt{X_t}.$$

Thus, according to (19),

$$g_0(X) = \rho \mathrm{E}\left[\int_0^\infty e^{-\rho t}\left(\log(\rho) + \int_0^t \left(r + \frac{(\mu - r)^2 X_t}{2} - \rho\right) ds\right) dt \middle| X_0 = X\right].$$

Since the conditional mean of the process $X_t$ equals

$$\mathrm{E}\left[X_t \middle| X_0 = X\right] = \overline{X} + \left(X - \overline{X}\right) e^{-\lambda_X t},$$

we find

$$\begin{aligned} g_0(X) &= \int_0^\infty e^{-\rho t}\left(\log(\rho) + \int_0^t \left(r + \frac{(\mu - r)^2 \left(\overline{X} + \left(X - \overline{X}\right) e^{-\lambda_X t}\right)}{2} - \rho\right) ds\right) dt \\ &= a_0 + a_1 X, \end{aligned}$$

where $a_0$ and $a_1$ are given in Proposition 3.4.

**Proof of Proposition 3.5**

Substituting (30) in (17) gives the portfolio policy in the proposition.

**Proof of Proposition 4.1**

Substituting the individual portfolio choices (38) and (39) into the market-clearing condition (43), gives $r(\omega)$. Substituting the equilibrium interest rate and equation (41) into the expressions for the optimal consumption and portfolio policies in equations (36) to (39) gives the results in the proposition.

**Proof of Proposition 4.2**

If the constraint is not binding, then the optimal policies are the same as in Proposition 4.1. If the constraint is binding, then the solution for the optimal policies can be obtained by applying Proposition 2.5.

**Proof of Proposition 5.1**

The value function of the log-utility representative agent equals

$$\frac{1}{\rho}\left(\log W_0 + g_{0,0}\left(X_0\right)\right) = \mathrm{E}_0\left[\int_0^{\infty} e^{-\rho t}\log\left(e_t\right)\, dt\right].$$

A straightforward calculation shows that the expectation on the right-hand side equals $\log e_0 + a + bX_0$. At the same time, $W_0$ is the aggregate wealth in the economy, which is equal to the price of the stock, $\rho^{-1}e_0$. Thus, $g_{0,0}\left(X_0\right) = a + bX_0 + \log\rho$.

**Proof of Proposition 5.2**

Because the aggregate financial wealth in the economy equals the total value of the stock market, it follows that

$$W = \omega P,$$
$$W^* = \left(1-\omega\right) P.$$

Using these relations along with (53) and (54), we relate the stock price to the aggregate endowment and to the cross-sectional distribution of wealth:

$$P_t = p\left(X_t, \omega_t\right) e_t,$$
$$p\left(X, \omega\right) \equiv \frac{1}{\rho}\left[1 + \gamma\omega\left(g_{0,0}\left(X\right) - \log\rho\right)\right] + O\left(\gamma^2\right).$$

By Itô's lemma,

$$\frac{dP_t}{P_t} = X_t\, dt + \sigma_e\, dZ_{et} + \frac{p_\omega}{p} d\omega_t + \frac{p_X}{p} dX_t$$
$$+ \frac{1}{2}\frac{p_{\omega\omega}}{p}\left[d\omega_t, d\omega_t\right] + \frac{1}{2}\frac{p_{XX}}{p}\left[dX_t, dX_t\right] + \frac{p_{X\omega}}{p}\left[dX_t, d\omega_t\right]$$
$$+ \frac{p_X}{p\cdot e}\left[dX_t, de_t\right] + \frac{p_\omega}{p\cdot e}\left[d\omega_t, de_t\right].$$

Given the dynamics of the state variable $\omega_t$ in (52), stock returns satisfy

$$\frac{dP_t}{P_t} = \left[X_t + \gamma\rho b\left(\mu_X\left(X_t\right) + \rho_{eX}\sigma_e\sigma_X\right)\omega_t\right] dt + \sigma_e dZ_{et} + \gamma\rho b\sigma_X\omega_t dZ_{Xt}$$
$$= \left[X_t + \gamma b\left(\mu_X\left(X_t\right) + \rho_{eX}\sigma_e\sigma_X\right)\omega_t\right] dt + \sigma_e dZ_{et} + \gamma b\sigma_X\omega_t\, dZ_{Xt} + O\left(\gamma^2\right). \qquad \text{(A3)}$$

The cumulative expected return equals

$$
\begin{aligned}
\frac{dP_t + e_t\,dt}{P_t} &= \Big[X_t + \gamma b\left(\mu_X\left(X_t\right) + \rho_{eX}\sigma_e\sigma_X\right)\omega_t + \rho\left[1 - \gamma\omega_t\left(a + bX_t\right)\right]\Big]\,dt \\
&\quad + \sigma_e dZ_{et} + \gamma b\sigma_X\omega_t\,dZ_{Xt} \\
&= \left[X_t + \rho - \gamma\omega_t\left(X_t - \frac{\sigma_e^2}{2} - \frac{\rho_{eX}\sigma_e\sigma_X}{\rho + \lambda_X}\right)\right]dt + \sigma_e dZ_{et} + \gamma b\sigma_X\omega_t\,dZ_{Xt} + O\left(\gamma^2\right).
\end{aligned}
$$

The dynamics of the state variable $\omega_t$ follows from the individual-wealth dynamics:

$$
d\omega_t = \omega_t\left(1 - \omega_t\right)\left[\left(\left(\theta_t - \theta_t^*\right)\left(\mu_{Pt} - \sigma_{Pt}^2 - r_t\right) - c_t + c_t^*\right)\,dt + \left(\theta_t - \theta_t^*\right)\sigma_{Pt}\,dZ_{Pt}\right].
$$

Using (A3), we find that

$$
d\omega_t = \gamma\omega_t\left(1 - \omega_t\right)\left(a + bX_t\right)\,dt + \gamma\omega_t\left(1 - \omega_t\right)\sigma_e\left(1 + \frac{\rho_{eX}\sigma_X}{\sigma_e}b\right)\,dZ_{et} + O\left(\gamma^2\right).
$$

To determine the equilibrium interest rate, we impose the market-clearing condition in the stock market (which implies clearing of the bond market). This condition leads to

$$
\begin{aligned}
r_t &= \mu_{Pt} - \sigma_{Pt}^2 + \gamma\omega_t\left(\sigma_{Pt}^2 + \rho_{eX}\sigma_X\sigma_{Pt}b\right) \\
&= \left[X_t + \rho - \gamma\omega_t\left(X_t - \frac{\sigma_e^2}{2} - \frac{\rho_{eX}\sigma_e\sigma_X}{\rho + \lambda_X}\right)\right] \\
&\quad - \left(\sigma_e^2 + 2\gamma b\rho_{eX}\sigma_e\sigma_X\omega_t\right) + \gamma\omega_t\left(\sigma_{Pt}^2 + \rho_{eX}\sigma_X\sigma_{Pt}b\right) + O\left(\gamma^2\right) \\
&= X_t - \sigma_e^2 + \rho - \gamma\omega_t\left(X_t - \frac{3}{2}\sigma_e^2\right) + O\left(\gamma^2\right),
\end{aligned}
$$

which is the expression in the proposition.

# B Determining higher-order terms of the asymptotic solution for the production economy

As in (9), we wish to identify the unknown function

$$g_0(\omega) = \sum_{n=0}^{N} \gamma^n g_{0,n}(\omega) + O(\gamma^{N+1}), \tag{B1}$$

so that it solves the Hamilton-Jacobi-Bellman equation for the non-log investor ($\gamma \neq 0$).

In order to identify (B1), we start by determining the first term of the expansion, $g_{0,0}(\omega)$. To do this, we consider the zero-order power series expansion of (B1)

$$g(\omega) = g_{0,0}(\omega) + O(\gamma), \tag{B2}$$

substitute this into the (48), and expand the resulting equation around the point $\gamma = 0$ upto zero order.[26] Solving this expansion for the unknown function gives the expression in (41). Observe that this solution is independent of $\omega$, and thus, constant.[27]

To obtain a solution accurate to the first order, we substitute the first-order expansion of equation (B1),

$$g_0(\omega) = g_{0,0}(\omega) + \gamma\, g_{0,1}(\omega) + O(\gamma^2), \tag{B3}$$

into (48), and expand this around the point $\gamma = 0$ to the first order. Into this expansion, we substitute $g_{0,0}(\omega)$ that was identified in the previous step, which then leaves us with a single equation that we can solve for $g_{0,1}(\omega)$:

$$g_{0,1}(\omega) = \frac{4\,(\alpha - \rho)^2 - 4\,(\alpha - 2\,\rho)\,\sigma^2 + \sigma^4}{8\,\rho^2}.$$

Substituting this and the solution for $g_{0,0}$ into (B3) gives the asymptotic solution of order one. Observe that this, too, is independent of $\omega$. The solution for the second and higher orders, however, does depend on $\omega$.

Having obtained the solution for order $n-1$, the solution upto the $n^{th}$-order can be obtained following the same procedure:

[26] Computer algebra software such as Mathematica makes this a significantly less unpleasant exercise. One can also use the results in Proposition 2.4 to find $g_{0,0}$.

[27] A quick and easy way to check for errors at each step of the expansion is to examine if the solution satisfies the the boundary conditions, expanded to the same order.

1. Start with the $n$-order expansion of (B1);
2. Substitute this into the Bellman equation (48), and expand this in $\gamma$ upto the $n^{th}$ order;
3. Substitute the previously identified expressions for $g_{0,0}(\omega), \ldots, g_{0,n-1}(\omega)$, and solve the equation one gets for the unknown function $g_{0,n}(\omega)$;
4. Construct the asymptotic solution by substituting into (B1) the expressions for $g_{0,0}(\omega), \ldots, g_{0,n}(\omega)$.

**Figure 1: Comparison with the log-linearization method**

The figure compares the accuracy of the asymptotic portfolio policy and the policy obtained by log-linearization. The first panel plots the proportional error of the approximate solutions: (approximate policy − true policy) / (true policy), where the "true" policy is based on the finite-difference solution of the original problem. The second panel compares the ratio of the hedging demand to the myopic demand as given by the approximate policies and the numerical solution. The solid line corresponds to the numerical solution, the dashed-line represents the asymptotic solution and circles represent the log-linearization-based solution. The figure shows that the solution based on log-linearization is more accurate than the asymptotic solution — it is practically indistinguishable from the numerical solution. The relative error of the asymptotic solution is less than 0.02. The absolute magnitude of the hedging demand is approximately two percent of the myopic demand; the asymptotic solution yields approximately four percent for this ratio. The parameter values for this figure are taken from Chacko and Viceira (1999): $r = 0.015$, $\mu - r = 0.0799$, $\lambda_X = 0.3413$, $\overline{X} = 27.7088$, $\sigma_X = 0.6512$, $\sigma_{PX} = 0.5355\sigma_X$, $\rho = 0.06$ and $\gamma = -1$.

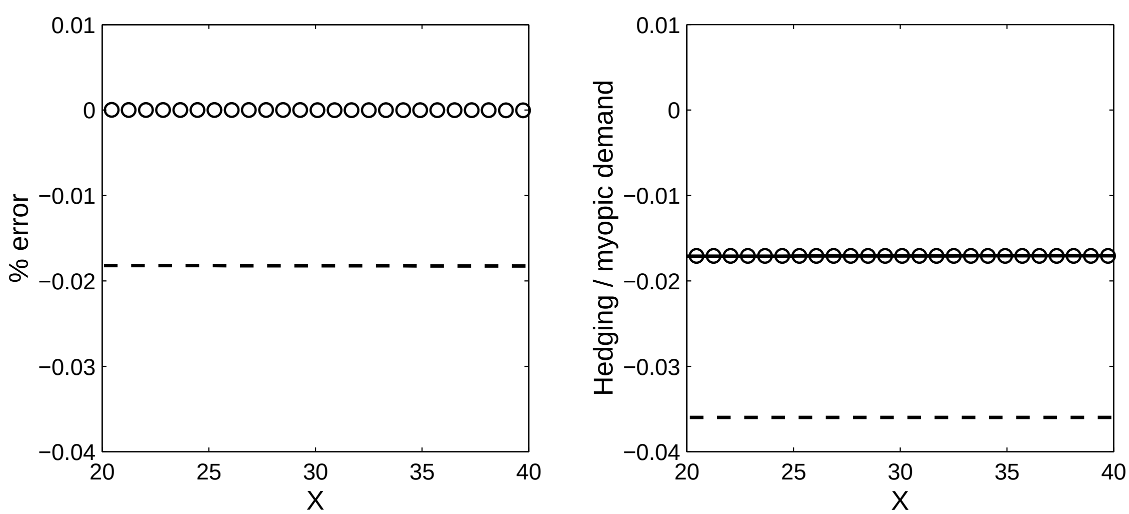


**Figure 2: The interest rate and its volatility**

The first panel plots, as a function of the state variable, the risk-free interest rate (scaled so that it is independent of the model parameters), while the second panel plots the instantaneous standard deviation of the interest rate. The solid line corresponds to the unconstrained economy, the dashed line corresponds to the economy with borrowing constraints. The borrowing constraint is set to $L = 1/3$.

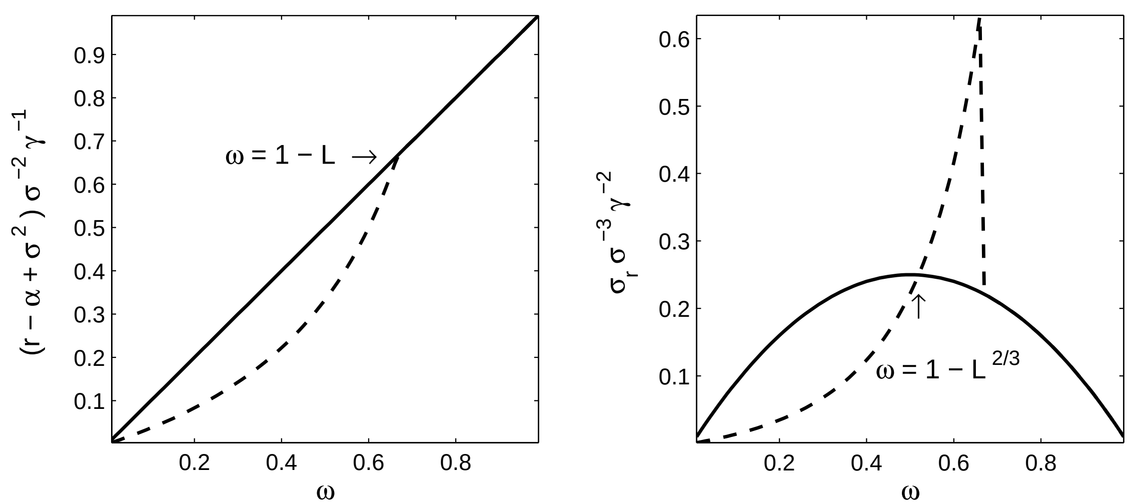


**Figure 3: Asymptotic and exact portfolio weight**

The figure shows the portfolio weight, $\theta(\omega)$, based on the exact solution (solid line), and that obtained from the asymptotic solution of the first order (circles) for different degrees of relative risk aversion (RRA). In the first panel, RRA $= 0.5$, in the second it is 2 and in the third panel it is 3. From the figure we see that the asymptotic portfolio weight from the first-order approximation is quite close to the optimal portfolio weight. The other parameter values are chosen to match those in Dumas (1989): $\rho = 0.106$, $\alpha = 0.11$, $\sigma = 0.10$.

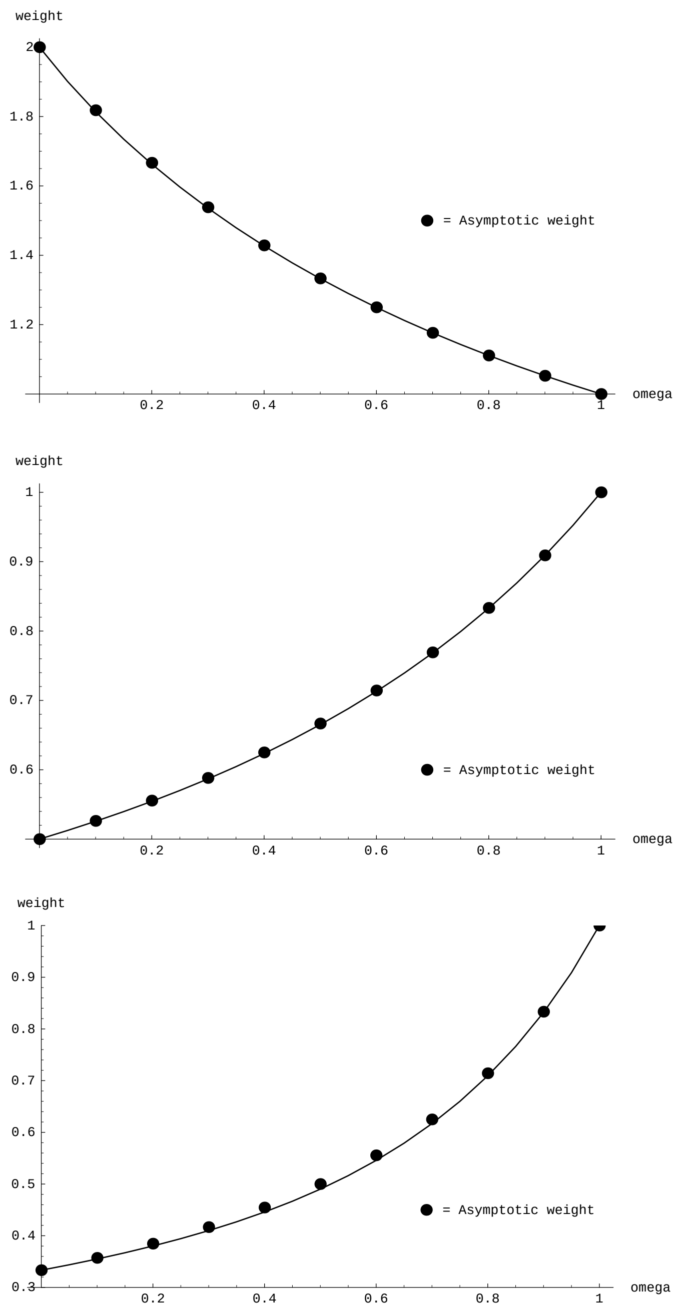
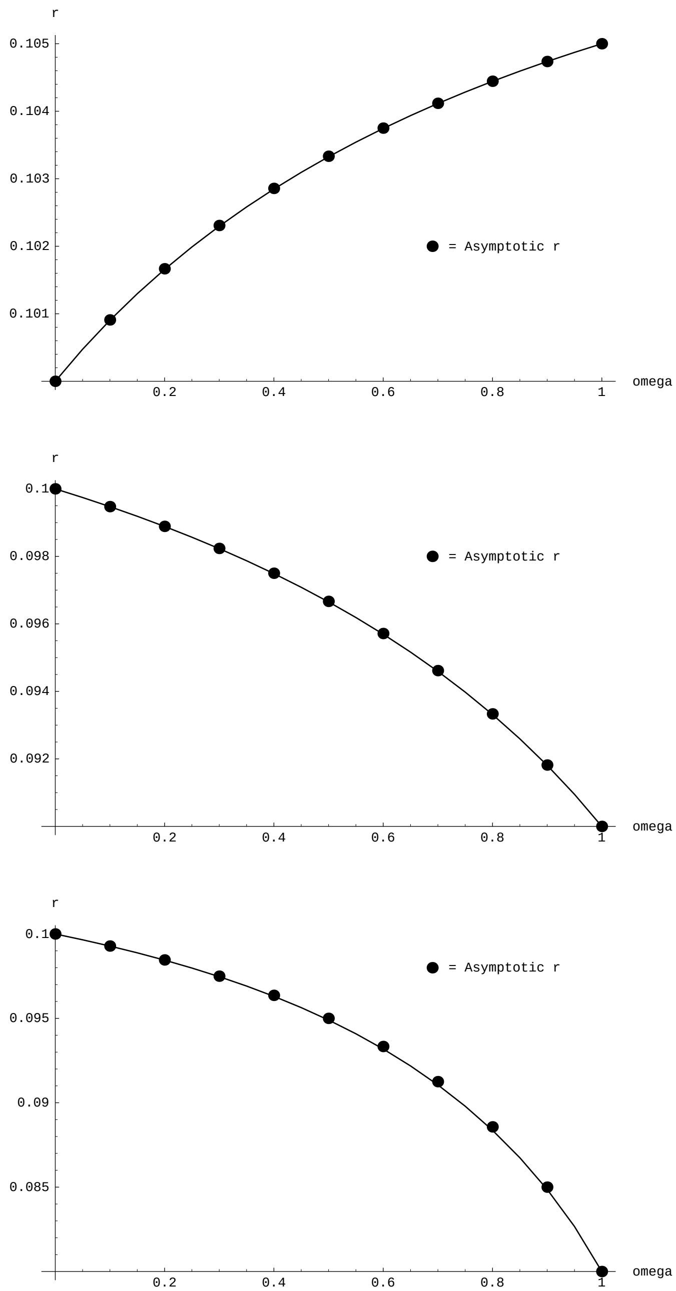


**Figure 4: Asymptotic and exact interest rate**

The figure shows the interest rate, $r(\omega)$, based on the exact solution (solid line), and that obtained from the asymptotic solution for the first order (circles) for different levels of relative risk aversion. In the first panel, RRA $= 0.5$, in the second it is 2 and in the third panel it is 3. From the figure we see that the asymptotic interest rate from the first-order approximation is quite close to the true value of the interest rate. The other parameter values are chosen to match those in Dumas (1989): $\rho = 0.106$, $\alpha = 0.11$, $\sigma = 0.10$.



# References

Balduzzi, Pierluigi and Anthony Lynch, 1997, Transactions costs and predictability: Some utility cost calculations, *Journal of Financial Economics* 52, 47-78.

Barberis, Nicholas C., 1999, Investing for the long run when returns are predictable, forthcoming, *Journal of Finance.*

Basak, Suleyman and Domenico Cuoco, An equilibrium model with restricted stock market participation, *Review of Financial Studies* 11.2, 309-341.

Brandt, Michael, 1999, Estimating portfolio and consumption choice: A conditional Euler equations approach, forthcoming *Journal of Finance.*

Brennan, Michael J., Eduardo S. Schwartz, and Ronald Lagnado, 1997, Strategic asset allocation, *Journal of Economic Dynamics and Control* 21, 1377-1403.

Campbell, John Y., 1993, Intertemporal asset pricing without consumption data, *American Economic Review* 83, 487-512.

Campbell, John Y., Yeung Lewis Chan, and Luis M. Viceira, 1999, A multifactor model of strategic asset allocation, Working paper, Harvard University.

Campbell, John Y., J. Cocco, F. Gomes, Pascal Maenhout and Luis M. Viceira, 1998, Stock market mean reversion and the optimal equity allocation of a long-lived investor, Working paper, Harvard University.

Campbell, John Y. and Hyeng Keun Koo, 1997, A comparison of numerical and analytical approximate solutions to an intertemporal consumption choice problem, *Journal of Economic Dynamics and Control* 21, 273-295.

Campbell, John Y. and Luis M. Viceira, 1998, Who should buy long-term bonds, NBER Working paper No. 6801.

Campbell, John Y. and Luis M. Viceira, 1999, Consumption and portfolio decisions when expected returns are time varying, *Quarterly Journal of Economics* 114, 433-495.

Canner, Niko, Gregory Mankiw, and David N. Weil, 1997, An asset allocation puzzle, *American Economic Review* 87.1, 181-191.

Chacko, George and Luis M. Viceira, 1999, Dynamic consumption and portfolio choice with stochastic volatility in incomplete markets, Working paper, Harvard University.

Chan, Yeung Lewis and Leonid Kogan, 2000, Heterogeneous preferences, catching up with the Joneses and the dynamics of asset prices, Working paper, U. of Pennsylvania.

Constantinides, George M., 1982, Intertemporal asset pricing with heterogenous consumers and without demand aggregation, *Journal of Business* 55, 253-267.

Cox, John C. and Chi-fu Huang, 1989, Optional consumption and portfolio policies when assets prices follow a diffusion process, *Journal of Economic Theory* 49, 33-83.

Cuoco, Domenico, 1997, Optimal consumption and equilibrium prices with portfolio constraints and stochastic income, *Journal of Economic Theory* 72, 33-73.

Cuoco, Domenico, and Hua He, 1994a, Dynamic aggregation and computation of equilibria in finite-dimensional economies with incomplete financial markets, Working paper, University of Pennsylvania.

Cuoco, Domenico and Hua He, 1994b, Dynamic equilibrium in infinite-dimensional economies with incomplete financial markets, Working paper, University of Pennsylvania.

Cvitanic, J. and Ioannis Karatzas, 1992, Convex duality in constrained portfolio optimization, *Annals of Applied Probability* 2, 767-818.

Den Haan, Wouter J. and Albert Marcet, 1994, Accuracy in simulations, *The Review of Economic Studies* 61.1, 3-17.

Detemple, Jerome and Shashidhar Murthy, 1997, Equilibrium asset prices and no-arbitrage with portfolio constraints, *Review of Financial Studies* 10.4, 1133-1174.

Duffie, D. and L. G. Epstein, 1992, Stochastic Differential Utility, *Econometrica*, 60, 353-394.

Dumas, B., 1989, Two-person dynamic equilibrium in the capital market, *Review of Financial Studies*, 2, 157-188.

Epstein, L. G., and S. Zin, 1989, Substitution, risk aversion and the temporal behavior of consumption and asset returns: A theoretical framework, *Econometrica,* 57, 937-969.

Fisher, Mark and Christian Gilles, 1999, Consumption and asset prices with recursive preferences, Working paper, Board of Governors of the Federal Reserve System (Atlanta).

Grossman, S. and J.-L. Vila, 1992, Optimal dynamic trading with leverage constraints, *Journal of Financial and Quantitative Analysis* 27, 151-168.

Hakansson, Nils H., 1971, On optimal myopic portfolio policies with and without serial correlation of yields, *Journal of Business* 44, 324-334.

He, Hua and H. F. Pages, 1993, Labor income, borrowing constraints, and equilibrium asset prices; A duality approach, *Economic Theory* 3, 663-696.

He, Hua and N. D. Pearson, 1991a, Consumption and portfolio policies with incomplete markets and short-sale constraints: The finite-dimensional case, *Mathematical Finance* 1, 1-10.

He, Hua and N. D. Pearson, 1991b, Consumption and portfolio policies with incomplete markets and short-sale constraints: The infinite-dimensional case, *Journal of Economic Theory* 54. 259-304.

Heaton, John and Deborah Lucas, 1996, Evaluating the effects of incomplete markets on risk sharing and asset pricing, *Journal of Political Economy* 104(3), 443-487.

Judd, Kenneth L., 1996, Approximation, perturbation, and projection methods for economic growth models, in H. Amman, D. Kendrick, and J. Rust, eds., *Handbook of Computational Economics*, Elsevier, Amsterdam.

Judd, Kenneth L., 1998, *Numerical Methods in Economics*, MIT Press, Cambridge, MA.

Judd, Kenneth L. and Sy-Ming Guu, 1993, Perturbation solution methods for economic growth models, in Hal R. Varian ed. *Economic and Financial Modeling with Mathematica*, Springer-Verlag (TELOS), New York.

Karatzas, Ioannis, J. P. Lehoczky, and Steven E. Shreve, 1990, Existence and uniqueness of multi-agent equilibrium in a stochastic, dynamic consumption/investment model, *Math. Operations Research* 15, 80-128.

Karatzas, Ioannis, J. P. Lehoczky, Steven E. Shreve and G. L. Xu, 1991, Martingale and duality methods for utility maximization in an incomplete market, *SIAM Journal of Control and Optimization* 29, 702-730.

Karatzas, Ioannis and Steven E. Shreve, 1998, *Methods of Mathematica Finance*, Springer-Verlag, New York.

Kim, Tong Suk and Edward Omberg, 1996, Dynamic nonmyopic portfolio behavior, *Review of Financial Studies* 9, 141-161.

Kreps, David and E. Porteus, 1978, Temporal resolution of uncertainty and dynamic choice theory," *Econometrica*, 46, 185-200.

Leland, Hayne (1968), Dynamic portfolio theory, Ph.D. thesis, Harvard University.

Liu, Jun, 1998, Portfolio selection in stochastic environments, Working paper, Stanford University.

Merton, Robert C., 1969, Lifetime portfolio selection under uncertainty: The continuous time case, *Review of Economics and Statistics* 51, 247-257.

Merton, Robert C., 1971, Optimum consumption and portfolio rules in a continuous time model, *Journal of Economic Theory* 3, 373-413.

Merton, Robert C., 1973, An intertemporal capital asset pricing model, *Econometrica* 41, 867-888.

Merton, Robert C., 1990, *Continuous Time Finance*, Basil Blackwell, Cambridge, MA.

Mossin, Jan, 1968, Optimal multiperiod portfolio policies, *Journal of Business* 41, 215-229.

Press, William H., Saul A. Teukolsky, William T. Vetterling, Brian P. Flannery, 1992, *Numerical Recipes in C: The Art of Scientific Computing*, second ed., Cambridge University Press, New York, NY.

Saito, Makoto, 1996, Limited participation and asset pricing, Working paper, University of British Columbia.

Sellin, P. and Ingrid Werner, 1993, International barriers in general equilibrium, *Journal of International Economics* 34, 2107-2138.

Schroder, Mark and Costis Skiadas, 1998, Optimal consumption and portfolio selection with Stochastic Differential Utility, Working paper, Northwestern University.

Telmer, Chris, 1993, Asset-pricing puzzles and incomplete markets, *Journal of Finance* 48.5, 1803-1832.

Teplá, Lucie, 1998, Optimal portfolio policies with borrowing and shortsale constraints, Working paper, INSEAD.

Viceira, Luis M.,1999, Optimal portfolio choice for long-horizon investors with nontradable labor income," NBER Working paper no. 7409.

Vila, J.-L. and Zariphopoulou, T., 1994, Optimal consumption and portfolio choice with borrowing constraints, Working paper, MIT Sloan.

Wachter, Jessica, 1998, Portfolio and consumption decisions under mean-reverting returns: An exact solution for complete markets, Working paper, Harvard University.

Wang, Jiang, 1996, The term structure of interest rates in a pure exchange economy with heterogeneous investors , *Journal of Financial Economics* 41.1,75-110.

Weil, Philippe, 1989, The equity-premium puzzle and the risk-free rate puzzle, *Journal of Monetary Economics* 24, 401-421.

Xu, G.L. and Steven E. Shreve, 1992, A duality method for optimal consumption and investment under short-selling prohibition. I. General market coefficients, *Annals of Applied Probability* 2, 87-112.

Xu, G.L. and Steven E. Shreve, 1992, A duality method for optimal consumption and investment under short-selling prohibition. II. Constant market coefficients, *Annals of Applied Probability* 2, 314-328.

Zariphopoulou, T., 1989, Consumption-investment models with constraints, *SIAM Journal of Control and Optimization* 32, 59-85.