STATISTICAL TESTS OF CONTINGENT CLAIMS ASSET-PRICING MODELS: A NEW METHODOLOGY

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A new methodology for statistically testing contingent claims asset-pricing models based on asymptotic statistical theory is proposed. It is introduced in the context of the Black-Scholes option pricing model, for which some promising estimation, inference, and simulation results are also presented. The proposed methodology is then extended to arbitrary contingent claims by first considering the estimation problem for general Ito-processes and then deriving the asymptotic distribution of a general contingent claim which depends upon such an Ito-process.

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1. Introduction.

Since Black and Scholes (1973) and Merton (1973, 1976) introduced their now famous option-pricing models, their methodology has been applied to the pricing of a variety of other assets whose payoffs are contingent upon the value of some other underlying or "fundamental" asset. By assuming that the fundamental asset price process is of the Itô-type and that trading takes place continuously in time, the price of a contingent claim can often be derived by using the hedging and no-arbitrage arguments of Black-Scholes and Merton. Since the deduced pricing formulas are almost always functions of unknown parameters of the fundamental asset price processes, any empirical application of contingent-claims analysis must first consider the statistical estimation of fundamental asset price parameters. In addition, since parameter-estimates are ultimately employed in the pricing formulas in place of the true but unknown parameters, the sampling variation of parameter-estimates will of course induce sampling variation in the estimated contingent-claims prices about their true values. The practical value of contingent claims analysis then depends critically on how parameter-estimation errors affect the accuracy of the resulting contingent-claims price estimator. Furthermore, some measure of the induced estimation error is required if the model is to be empirically tested. Indeed, although a number of papers have studied the discrepancies between estimated and observed prices for particular contingent claims, to date there have been few direct statistical tests of these models.[1] In a spirit similar to Gibbons' (1982) examination of the capital-asset pricing model, this paper proposes a new framework in which tests of contingent-claims asset pricing models may be performed and in which the accuracy of contingent-claims price estimates may be quantified statistically.

This new approach seems particularly fruitful for several reasons. Although it is introduced in the context of the Black-Scholes call-option pricing model, later sections show that the suggested methodology can be applied to any contingent claim for which the associated fundamental asset price parameters may be estimated. Few additional assumptions beyond those common to all contingent claims models are required in order to apply the proposed methods. In addition, the results derived in this paper are computationally quite simple to implement. Furthermore, such a framework is well-suited to the standard tools of statistical inference, estimation, and forecasting. In fact, since the distribution of the contingent-claims estimator is derived in closed form, all the usual hypothesis testing and forecasting techniques may be applied to contingent claims analysis. This is achieved through the use of large-sample or asymptotic statistical theory which, essentially, consists of applying laws of large numbers and central limit theorems to otherwise intractable estimation and inference problems. By appealing to large-sample arguments, it is possible to derive explicitly the limiting distribution of highly nonlinear functions (such as the Black-Scholes formula) of fundamental parameter estimates. In fact, it will be shown that even if a closed-form solution does not exist for the contingent claim's price, the limiting distribution may still be calculated using numerical methods.

Of course, exact small-sample properties are always preferable to their large-sample counterparts when available. However, due to the nonlinear nature of most contingent claims price estimators, their exact distributions usually do not exist in closed form. The use of asymptotic distributions as approximations is a natural alternative. Moreover, because financial econometricians actually have at their disposal relatively "large samples," applications of asymptotic approximations may yield quite accurate results. In order to demonstrate the practical value of this new methodology and also to clarify the particular econometric issues at hand, Section 2 derives the large-sample properties of the Black-Scholes (BS) call-option price estimator. The derived asymptotic statistics are then calculated using data for options written on three specific stocks and some simple hypothesis tests are performed. To explore the accuracy of the proposed estimators, some simulation evidence is presented in Section 3. In Section 4 the methodology is developed in its most general form, and we conclude in Section 5.

2. Estimation and Inference for the BS Call Option Pricing Model.

Let $S(t)$ denote the price of a stock at time t and let $F(S, E, r, \tau, \sigma^2)$ be the price of a corresponding call-option at time t with exercise price E and time-to-maturity τ, where r is the interest rate on a default-free pure discount bond with time-to-maturity τ and σ^2 is the variance rate of the underlying stock price process $S(t)$. Under the assumptions of the BS model, F is determined by the well-known formula:

$$F = S\,\Phi(d_1) - Ee^{-r\tau}\Phi(d_2) \tag{1a}$$

$$d_1 = \frac{1}{\sigma\sqrt{\tau}}\left[\ln\left(\frac{S}{E}\right) + \left(r+\frac{1}{2}\sigma^2\right)\tau\right] \tag{1b}$$

$$d_2 = d_1 - \sigma\sqrt{\tau} \tag{1c}$$

where Φ is the standard normal cumulative distribution function. Although the stock price, exercise price, time-to-maturity, and interest rate are in principle observable without error, the variance σ^2 of the underlying stock is unknown. Recent studies by Whaley (1982) and Bhattacharya (1983) have used variances implied by call option prices in performing various tests of the BS model. Indeed, Latane and Rendleman's (1976) empirical results seem to indicate that these implied volatilities may be better forecasts of future volatility than estimates derived from historical data. However, because such an approach assumes that the BS model obtains, actual tests of the model itself are difficult to construct in such a framework. In contrast to this approach, the following analysis takes as its starting point the assumption that the stock price process S(t) is the usual lognormal diffusion process given by:

$$\frac{dS}{S} = \mu dt + \sigma dw \qquad (2)$$

The BS model is not assumed to obtain, but instead forms the null hypothesis which is to be tested. Since an estimate $\hat{\sigma}^2$ of σ^2 may be obtained by using historical data, evaluating F at $\hat{\sigma}^2$ yields an estimate of the corresponding option price. Although the resulting option estimator is clearly not unbiased, it is consistent if the variance estimator is consistent. Consistency is a particularly desirable property since by definition a consistent estimator approaches the true value with probability one as the sample size grows. This is distinct from an unbiased estimator which, although is correct on average, may fluctuate considerably about its true value even in very large samples.[2]

Given a consistent estimator of the option price, a direct statistical test of the BS model can be constructed by comparing this estimate with the actual market price. Since the estimated price is subject to sampling variation, a measure of its "spread" is needed in order to perform a meaningful comparison. More formally, a test of whether or not the estimated option price differs significantly from the actual market price requires the calculation of the standard error about the estimated option price and the estimator's sampling distribution. In this section, the asymptotic distribution of the option price estimator is derived and is used to compare actual market prices with their BS estimates.

2.1. Estimation and Asymptotic Distribution of Call Option Prices.

Estimation of the stock price dynamics is considered first. Suppose that n+1 equally spaced observations of S(t) are taken in the time interval [0,T]. Letting $h = T/n$, Rosenfeld (1980) has shown that the maximum likelihood (ML) estimator of σ^2 is given by:

$$\hat{\sigma}^2_{ML} = \frac{1}{T} \sum_{k=1}^{n} \left(X_k - \frac{1}{n} \sum_{j=1}^{n} X_j\right)^2 \tag{3}$$

where X_k is the log of the price-relative $\frac{S(kh)}{S\left((k-1)h\right)}$. Under mild regularity conditions, it is well-known that the general ML estimator is consistent, asymptotically normally distributed, and efficient in the class of all consistent and uniformly asymptotically normal (CUAN) estimators.[3] In addition, the ML estimator of any well-behaved nonlinear function of a given parameter is simply the nonlinear function of the ML estimator of that parameter. That is, the ML estimator $\hat{F}_{ML}$ of the option price F may be obtained by evaluating F at $\hat{\sigma}^2_{ML}$. Since $\hat{F}_{ML}$ is a true ML estimator, it also exhibits the usual maximum likelihood properties cited above.

Since the option estimator $\hat{F}_{ML}$ depends on the estimator $\hat{\sigma}^2_{ML}$, the asymptotic distribution of $\hat{F}_{ML}$ is related to the asymptotic distribution of $\hat{\sigma}^2_{ML}$. It may easily be shown that $\hat{\sigma}^2_{ML}$ has the following asymptotic distribution:[4]

$$\sqrt{n}\ (\hat{\sigma}^2_{ML} - \sigma^2) \overset{A}{\sim} N(0,\ 2\sigma^4)\ . \tag{4}$$

Now consider the estimator $\hat{F}_{ML}$ as a function of $\hat{\sigma}^2_{ML}$, holding all other arguments fixed, i.e., $\hat{F}_{ML} = F(\hat{\sigma}^2_{ML})$. The asymptotic distribution of $\hat{F}_{ML}$ may then be derived by applying standard statistical limit theorems to the Taylor series expansion of $F(\hat{\sigma}^2_{ML})$ about the true parameter σ^2, yielding the desired result (see Rao (1973)):

$$\sqrt{n}\ (\hat{F}_{ML} - F) \overset{A}{\sim} N\left(0,\ 2\sigma^4\left(\frac{\partial F(\sigma^2)}{\partial \sigma^2}\right)^2\right) . \tag{5}$$

That is, for a sufficiently large number n of observations,[5] the sampling distribution of $\hat{F}_{ML}$ is approximately normal with mean F and variance $\frac{2\sigma^4}{n}\left(\frac{\partial F(\sigma^2)}{\partial \sigma^2}\right) \equiv V_F$. Given the BS pricing formula (1), the quantity V_F may be calculated explicitly as:

$$V_F = \frac{1}{2n} S^2 \sigma^2 \tau \phi^2 (d_1) \tag{6}$$

where ϕ is the standard normal density function.[6]

2.2. Historical Versus Implied Variance Estimators.

In contrast to estimating variances with historical data, several studies have indicated that variances implicit in options prices seem to be better estimators in several ways. Loosely speaking, this may be because historical estimates are "retrospective" whereas implicit estimates are "prospective." That is, since option prices are determined daily, all current information affecting (among other things) the volatility of the underlying stock price will be impounded in those prices. For example, new information which changes the current expectation of future volatilities will, in an efficient market, be reflected in observed option prices but will obviously not be evident from historical estimates.

Implicit in those studies are two critical assumptions. First, a specific option pricing model must be known to obtain. Second, the options markets must be known to be efficient. Under these two assumptions, implicit estimators are clearly preferred. However, the approach taken in this paper is fundamentally different. The only assumption required in order to apply the general methodology proposed in this paper involves the stochastic specification of the underlying asset price. In particular, it is assumed (in the general case of Section 4) that the price process may be described by a first-order nonlinear stochastic differential equation driven by both standard white noise and a Poisson component, i.e., general Itô processes. This, of course, is a special case of more general processes described by higher-order stochastic differential equations, which again may be considered special cases of still more general processes. But because empirically, asset-prices seem to be well represented by the class of Itô processes, assuming this particular form of dynamics may be justified to some extent. This paper then suggests a method of testing models based on this assumption. However, in computing implicit variances, a specific model has already been assumed to obtain. Therefore, using implicit variances in this study is inappropriate.[7] More specifically, a test of the BS model based on implied variances which presuppose that the BS model obtains will almost always confirm the model.

2.3. Analysis of the Asymptotic Variance V_F

The expression for V_F in equation (6) is of interest for several reasons. In addition to providing a measure of option price estimators' dispersion in large samples, the analytic formula for V_F may also be used to examine how changes in the underlying parameters affect the option estimates. As an example, consider the systematic biases of the BS prices noted in several empirical studies. Macbeth and Merville (1979) observe that in-the-money call options are under-priced by the BS formula and vice-versa for out-of-the money calls, and that the degree of mispricing is aggravated by the spread between stock and exercise price for most options. Black (1975), Merton (1976), and Gultekin, Rogalski, and Tinic (1982) observe essentially the opposite biases. To see whether such biases may be explained merely by sampling variation, consider the derivatives of V_F with respect to the stock and exercise prices and the time-to-maturity:

$$\frac{\partial V_F}{\partial S} = -\frac{1}{n} S\phi^2(d_1)d_2 \tag{7a}$$

$$\frac{\partial V_F}{\partial E} = \frac{1}{n}\frac{S^2}{E}\ \phi^2(d_1)\sigma\sqrt{\tau}\ d_1 \tag{7b}$$

$$\frac{\partial V_F}{\partial \tau} = \frac{1}{2n} S^2\phi^2(d_1)\sigma^2\left[1 - \frac{1}{\sigma^2\tau}\left([r + \tfrac{1}{2}\sigma^2]^2\tau^2 - [\ln \tfrac{S}{E}]^2\right)\right] . \tag{7c}$$

Let $k_1 \equiv e^{-(r - \frac{1}{2}\sigma^2)\tau}$, $k_2 \equiv e^{-(r + \frac{1}{2}\sigma^2)\tau}$, and $k_3 \equiv e^{(r + \frac{1}{2}\sigma^2)\tau} = \frac{1}{k_2}$.

The following inequalities are then easily established:

$$\frac{\partial V_F}{\partial S} \gtreqless 0 \quad \text{iff} \quad \frac{S}{E} \lesseqgtr k_1 \tag{8a}$$

$$\frac{\partial V_F}{\partial E} \gtreqless 0 \quad \text{iff} \quad \frac{S}{E} \gtreqless k_2 . \tag{8b}$$

Although obtaining a similar pair of equivalent inequalities for $\frac{\partial V_F}{\partial \tau}$ does not seem possible, a useful sufficient condition for the monotonicity of the derivative can be derived:

$$\frac{\partial V_F}{\partial \tau} > 0 \quad \text{if} \quad \frac{S}{E} < k_2 \text{ or } \frac{S}{E} > k_3 . \tag{8c}$$

In Table 1, values of k_1, k_2, and k_3, have been tabulated for various times-to-maturity measured in weeks, given an (annualized) interest rate of 10 per cent and an (annualized) standard deviation of 50 per cent.

TABLE 1 GOES HERE

Several observations may be made from the values in Table 1. Since the interval $[k_2, k_3]$ is fairly concentrated about 1.0, an increase in the time-to-maturity will increase the variance about the option price estimate unless the option is very nearly at the money. For example, if the stock price is $40 then options which are either in or out of the money by $5 or more are more precisely estimated as the time-to-maturity declines. This may well explain Macbeth and Merville's (1979) finding that biases of in and out of the money options decrease as the time to expiration decreases. This would also support Gultekin, Rogalski and Tinic's (1982) observation that "In general, the [BS] formula gives much less accurate estimates for long-lived options."

Another property of the option price estimator implied by the values in Table 1 is that, loosely speaking, if an option is deep in the money ($S/E > 1$) then as the exercise price increases, so will the variance about the option price estimate. If an option is deep out of the money ($S/E << 1$) then decreasing the exercise price increases the variance of the estimated option price. In other words, option price estimates exhibit more variation for either deep in or out of the money options as the exercise price shifts closer to the prevailing stock price. Of course, these statements may be made precise by computing the specific values of k_1, k_2, and k_3 for particular options of interest.

2.4. Statistical Tests of the BS Option Pricing Model.

The most direct application of the quantity V_F is in statistically testing the BS model. In particular, consider the null hypothesis that the BS model obtains. Letting $\overline{F}$ denote the observed market option price, this null hypothesis may be stated as:

$$H_0: \quad F(\sigma^2) = \overline{F} . \tag{9}$$

This hypothesis may then be tested by computing the statistic:

$$q \equiv \frac{F(\hat{\sigma}^2_{ML}) - \overline{F}}{\sqrt{V_F}} . \tag{10}$$

Since V_F depends upon the unknown parameter σ^2, a corresponding "t-statistic"[8] z may be calculated by using a consistent estimator $\hat{V}_F \equiv V_F(\hat{\sigma}^2_{ML})$ in place of V_F in equation (10). Note that the resulting statistic is still asymptotically standard normal. The test is then performed by rejecting H_0 if z lies outside an acceptable range of 0 and accepting otherwise, where the range of acceptability is determined by the desired size of the test. For example, if z falls outside the interval [-1.96, 1.96] then H_0 may be rejected at the 5% level. In addition, the usual forms of conditional forecasting and confidence interval calculations may be performed given the estimated variance.

2.5. An Empirical Example.

Because the expression for V_F is analytically quite simple, computing standard errors for option price estimates requires little calculation beyond the estimation of the stock price volatility. For illustrative purposes, standard errors and the associated z statistics have been computed in Tables 3a-c for traded options on Litton, National Semiconductor, and Tandy stocks for January 12, 1979. These three stocks were chosen from a subset of five non-dividend paying stocks for which Rosenfeld (1980) estimated drift and variance coefficients according to the dynamics given by (2). In addition to their no-dividends property, Litton, National Semiconductor, and Tandy were chosen because they were trading in distinct cycles (March, February, January respectively). This was done merely to provide a complete cross-section of times-to-maturity. The estimates of the stocks' variances were obtained from Rosenfeld[9] (1980). They were estimated using 312 weekly observations from the period January 1973 to December 1978. The interest rate used was the (annual) 26-week Treasury-bill rate quoted on January 15, 1979 in the Wall Street Journal (9.443%).

TABLES 2, 3a-c GO HERE

In Table 2, the estimated values of k_1, k_2, and k_3 for various times-to-maturity are presented for the three stocks. It is clear that since the estimates of k_2 and k_3 are both quite close to 1.0, inequality (8c) obtains for almost all options on the three securities. Note that, holding the exercise price constant, the standard error of every estimated option price in Tables 3a-c increases with an increased time-to-maturity. Also, whether or not an option is in or out of the money does not seem to be systematically related to whether it is underpriced or not. Of course, previous empirical studies have used a much larger set of options than the few considered here, so the lack of discernible patterns in Tables 3a-c is not conclusive.

The z-statistics seem to indicate that the data are inconsistent with the null hypothesis H_0 that the BS model obtains. For example, out of the eleven options written on Litton stock, only two estimates had standard errors outside the 1%-critical region and only one estimate had its standard error outside the 5%-critical region. However, caution must be exercised in interpreting this since for each stock, the tests are certainly not independent. Nevertheless, a simultaneous test of H_0 for all Litton options with a nine-week time-to-maturity results in rejection at the 5% level of significance.[10]

It is important to note that the above test of H_0 is in fact a joint test of the BS option pricing model and of the associated stock-price dynamics. Rejecting H_0 in this case may not necessarily imply that the BS model does not obtain. However, because the BS formula is so closely related to the particular form of the stock-price dynamics it is difficult to imagine a situation in which (1) obtains but (2) does not. In fact, Rosenfeld (1980) has tested the hypothesis that these three stocks follow the process (2) and rejects in favor of a combined lognormal diffusion and jump process. But in this situation, the model outlined in (1) does not obtain and must be modified along the lines Merton (1976) develops. In addition to the possibility of jumps, Rosenfeld (1980) and Marsh and Rosenfeld (1983) consider several other alternatives which may support the results in Tables 3a-c. Although it is not pursued in this paper, tests of such alternative hypotheses are readily constructed in the framework proposed here.

3. Simulation Evidence.

Although the empirical evidence presented in Section 2 is of interest in its own right, it also illustrates the practical relevance of asymptotic statistical theory to the estimation of general contingent claims prices. Section 4 demonstrates formally that this methodology may in fact be applied to any other contingent claim provided that its corresponding underlying fundamental asset price process may be estimated. However, an important isue which determines the usefulness of large-sample results is the number of observations required for those results to obtain. Unfortunately, no general guidelines exist so this issue must be resolved for each application individually. Nevertheless the increasing sophistication of statistical software coupled with the rapid decline of computer costs allow researchers to determine what constitutes a large sample for a particular estimator relatively easily.

3.1. Design of Experiments.

In this section, a simple simulation study is conducted for the call-option price estimators proposed in Section 2. Each Monte Carlo experiment involves generating a time series for the stock price process with a given drift and variance rate using a random number generator, and then computing price estimates and corresponding asymptotic standard deviation estimates for hypothetical options written on that stock.[11] The estimated option price and asymptotic standard deviation may then be compared with their true values. This procedure is repeated 1000 times in order to deduce the finite sampling properties of the estimators. By varying the length of the stock price series generated for the 1000 replications and noting its effect upon the estimators' sampling behavior, it is possible to deduce the minimum number of observations required to insure that the associated asymptotic statistics are adequate approximations. By varying other parameters, it is also possible to study how the asymptotic approximation to finite-sample properties may be related to the terms of an option contract such as the time-to-maturity or the stock-price/exercise-price spread. Throughout the simulations, the following parameter values were assumed and held constant:

$$S = \$40$$

$$\sigma^2 = 0.5200 \text{ (annual)}$$

$$r = 0.1000 \text{ (annual)}$$

The simulations were carried out at the weekly frequency for which r and σ^2 were adjusted appropriately.

3.2 Simulation Results.

Tables 4 and 5 summarize the finite-sampling properties of the option price and asymptotic variance estimators across the 1000 replications for various options and stock-price sample sizes.

TABLES 4a, b AND 5a, b GO HERE

Each table corresponds to experiments with hypothetical options of the same time-to-maturity TAU. Tables 4a, b report simulation results for hypothetical options which are at the money and in and out of the money by $5 for maturities 1 and 13 respectively. Tables 5a, b display simulation results for options which are in and out of the money by $15 with respectively 1 and 13 weeks to go. Experiments with options of intermediate exercise prices, times-to-maturity other than 1 and 13, and assorted stock price/interest rate/stock-price-variance combinations were also conducted but since the results depicted in Tables 4 and 5 are generally confirmed in these other experiments, in the interest of brevity those results are not reported here.[12]

Within each table, every row corresponds to a separate and independent experiment. That is, each experiment is based completely on newly generated data and uses no data generated in other experiments. Each experiment involves simulating a time series of stock prices of a given length (sample size), computing the estimators $\hat{F}_{ML}$, $\hat{V}_F$, and test-statistic z for a particular hypothetical option, repeating this 1000 times, tabulating the subsequent sampling distribution for the estimators and z, and finally testing the standard normality of z. Although the estimators $\hat{F}_{ML}$ and $\hat{V}_F$ may also be checked for normality, for purposes of hypothesis testing and constructing confidence intervals the standard normality of the statistic z is more relevant. Of the many tests for departures from normality, only two are considered here. The first is the usual χ^2-test of goodness-of-fit which measures the "distance" between the hypothesized distribution function (normal) and the empirical distribution function. The second is the studentized range test which is more sensitive to departures from normality in the tails of the distribution. Since the primary use of z is in the testing of hypotheses, departures from normality in the tail areas are of more concern than differences in the center of the distribution. For this reason, the results of the studentized range test may be of more consequence than the χ^2-test. Both tests are performed and the results are given in the last two columns of each row.

Consider the entries in Table 4a. The first five rows comprise the simulation evidence for a call option with exercise price $35 and one week to maturity. The second five rows correspond to the experiment of a call option with exercise price $40 also maturing in one week, and the last five rows are results for a call with exercise price $45 and one week to go. The first column indicates the length of the stock-price series generated by the random number generator. The second, third, and fourth columns display respectively the true or population value of the option, the mean of the option estimator across the 1000 replications, and the bias in percentage terms. The standard deviation of the option estimate across the replications is given in parentheses under the option estimate. The fifth, sixth, and seventh columns present the true value, estimated value, and percentage bias of the asymptotic variance V_F respectively. The eighth column provides the mean and standard deviations of the z-statistic over all the replications. In the last three columns, statistics which indicate how close z is to a standard normal variate are displayed. The first is the χ^2-test with the p-value given in parentheses below the test-statistic. The next column displays the skewness coefficient of z across the replications and the last column presents the studentized range of z.

As Boyle and Ananthanarayanan (1977) have shown, for an at-the-money call-option few observations are required in order to trivialize the bias of the option price estimator. The largest absolute price bias observed in Tables 4a, b where options are either at the money or in or out of the money by $5 is 0.64%. In addition, in Tables 4a, b the bias in estimating the asymptotic variance is also quite small, the largest being -1.95%. For most cases, both estimates were well within 1% of the true value. Note that, although on average the bias for both estimators decreases as the length of the stock-price series increases, the decrease is not monotonic. This is to be expected since each experiment is random and independent of the others and is subject to the usual sampling variation.[13]

The biases for deep in or out of the money options, however, are quite large when the time-to-maturity is one week. Table 5a displays price biases of up to -31.41% and asymptotic variance biases of over -2200.0%. This suggests that caution must be exercised in using these estimators for deep in or out of the money options just about to expire. However, the percentage bias is misleading in this case since the asymptotic variances are essentially zero and the estimator is virtually nonstochastic. Intuitively, this is simply due to the fact that the value of deep in or out of the money call options with short times-to-maturity do not depend significantly on the variance of the stock price process if the process is a pure diffusion. Since the simulated stock prices are constructed as lognormal diffusions, it is not surprising that with one week to go, the estimator for deep in and out of the money options has little asymptotic variation. In this case, a minute absolute difference between the theoretical and finite sample asymptotic variance can yield an extraordinary percentage bias.[14] Table 5b shows that as the time-to-maturity increases, the bias declines dramatically, the largest price bias being 0.46% and the largest variance bias being -2.08%.

3.3. Finite Sample Properties of the z-statistic.

Consider now the asymptotic behavior of the statistic z. Under the null hypothesis that z is standard normal, the χ^2-test is performed for the 1000 replications of each experiment with 50 equiprobable categories yielding 49 degrees of freedom. From Tables 4 and 5, it seems that with a sample size of 100 weekly observations for stock-prices, the standard normality of z may be rejected at almost any level of significance. However, in most cases the null hypothesis of normality may be accepted at levels of 5% or smaller with 300 or more weekly observations of stock-price data. Nevertheless, it may be noted that the means of z are negative for almost all experiments. For the purposes of detecting skewness departures from normality, the skewness coefficient may yield a more powerful test than the χ^2-test. Under the null hypothesis that z is standard normal, the distribution of the sample skewness coefficient has been tabulated[15] and, for 1000 replications, the 90%-confidence interval is [-0.127, 0.127] and the 98%-confidence interval is [-0.180, 0.180]. It is clear that even in cases where the χ^2-test does not reject the null hypothesis of standard normality, the skewness coefficient is often outside the 98%-

confidence interval. This indicates that the finite-sampling distribution of z is skewed (to the left). However, if the "tail-behavior" of z is close to that of the standard normal, then the hypothesis tests based on z suggested in Section 2 are in fact appropriate. To measure possible departures from standard normality in the tails of the finite sample distribution of z, the studentized range for each experiment may be compared with its tabulated distribution under the null hypothesis.[16] For 1000 replications, the 90%-confidence interval of the studentized range with 5% in each tail is given by [5.79, 7.33] and the 95%-confidence interval with 2.5% in each tail is [5.68, 7.54]. For the hypothetical options in Tables 4a, b, only in one case does the computed studentized range fall outside the 90%-interval. This suggests that, although the finite sample distribution of z may be skewed, its tail-probabilities match the standard normal's fairly closely. For purposes of testing the BS model as specified by (1), the results seem to support the use of the z-statistic as described in the previous section for options not too deep in or out of the money. Table 5a shows that for deep in the money options with 1 week to go, not even a sample size of 500 is sufficient to produce the asymptotic results for z; both the studentized range and the χ^2-tests reject normality at practically any level of significance. However, for deep out of the money options with 1 week to go, sample sizes of 500 or more seem to be sufficient to render the tail behavior of z close to the standard normal's as measured by the studentized range. The results in Table 5b however shows that once the time to maturity increases to 13 weeks, the tail behavior of the z-statistic matches that of the standard normal even for deep in or out of the money options.

From the simulation evidence provided above, it may be concluded that if the call option pricing model (1) obtains, then for options which are not too deep in or out of the money and for deep in or out of the money options which are not just about to expire, using its asymptotic distribution for purposes of testing and inference may well be justified.

4. The General Methodology.

Although the above analysis involved the BS option pricing model, this section demonstrates that the previously outlined methodology may be applied to virtually all contingent claims models. In fact, it will become clear that even if the contingent claims pricing function cannot be derived in closed-form, its asymptotic distribution is still normal with a limiting variance which may be estimated numerically.

4.1 Estimation of Itô Processes.

Since almost all contingent claims models assume that fundamental asset prices follow Itô-processes, it is first necessary to consider the estimation problem for this class of stochastic processes. For expositional clarity we only consider the estimation problem for Itô processes with single jump and diffusion components. The extension to multiple jump and diffusion terms and vector Itô processes poses no conceptual difficulties but is notationally more cumbersome. Let $X(t)$ be an Itô process with domain $\Omega \subset R$ satisfying the following stochastic differential equation:

$$dX = f(X, t; \alpha)dt + g(X, t; \beta)dW + h(X, t; \gamma)dN \qquad t \in [0, \infty) \tag{11}$$

where dW is the standardized Wiener process and dN is a Poisson counter (jump magnitude = 1), independent of dW, with intensity λ. There is clearly no loss of generality in assuming that the jump magnitude is unity since this is merely a normalization which may be subsumed by the coefficient function h.[17]

In addition to assuming those conditions which insure the existence and uniqueness of the solution to (11)[18] we make the following additional assumptions:

(A1) Coefficient functions f, g, and h are known up to parameter vectors α, β, γ, λ respectively. The true but unknown parameters α_0, β_0, γ_0 and λ_0 lie in the interior of the compact parameter spaces A, B, Γ and Λ respectively. Let $\theta_0 \equiv (\alpha_0', \beta_0', \gamma_0', \lambda_0)'$ and let $\Theta \equiv A \times B \times \Gamma \times \Lambda$. The functions f, g, and h are twice continuously differentiable in (X, t) and three times continuously differentiable in θ.

(A2) n observations of X(t) are taken at times $t_1, t_2, \ldots, t_n$ <u>not necessarily</u> equally spaced apart, where $0 < t_1 < \ldots < t_n$. $X \equiv (X_1, X_2, \ldots, X_n)'$, where $X_i \equiv X(t_i)$, $i = 1, \ldots, n$. $X(t_0) \equiv X_0$ is known.

We may now state the estimation problem as: Given the observations X and the process dynamics (11), find the optimal estimator $\hat{\theta}$ of the true parameters θ_0. By restricting consideration to the class of consistent and uniformly asymptotically normal (CUAN) estimators, it has been shown that the ML estimator is optimal in the sense that it has the smallest variance of all other CUAN estimators. For this reason, ML estimation is the preferred approach. The ML estimator is obtained by considering the joint density function of the random sample X as a function of the unknown parameters and then finding that value $\hat{\theta}_{ML}$ which maximizes the joint density in θ. We now proceed to derive this joint density function which, when considered a function of the parameters θ given the data X, is called the joint-likelihood function.

Let $\rho(X_1, \ldots, X_n)$ denote the joint-density function of the random sample X, where the dependence of ρ on the unknown parameters θ and on $t_1, \ldots, t_n$ have been suppressed for notational simplicity. The density ρ may always be written as the following product of conditional densities:

$$\rho(X_1, \ldots, X_n) = \rho_1(X_1)\rho_2(X_2|X_1)\rho_3(X_3|X_2, X_1) \ldots \rho_n(X_n|X_{n-1}, \ldots, X_1) . \quad (12)$$

However, since X(t) is a Markov process[19] equation (12) reduces to:

$$\rho(X_1, \ldots, X_n) = \rho_1(X_1)\rho_2(X_2|X_1)\rho_3(X_3|X_2) \ldots \rho_n(X_n|X_{n-1}) . \quad (13)$$

If in addition, X(t) is time-homogeneous then the functional form of the transition density ρ_k only depends upon the time index k in terms of the time increment $t_k - t_{k-1}$ and not on t_k itself. In this case, the notation ρ_k should be interpreted as

$$\rho_k(X_k, t_k|X_{k-1}, t_{k-1}) = \rho(X_k, \Delta t_k|X_{k-1}) \quad \text{where } \Delta t_k \equiv t_k - t_{k-1}. \quad (14)$$

If, for example, observations were then taken at equally spaced intervals of length h, then the ρ_k's are identical across time except for the starting values X_{k-1}. Of course, one of the greatest advantages of estimating continuous-time models is precisely that equally-spaced observations are <u>not</u> necessary. Unless stated otherwise, we do not assume equally-spaced observations. For compactness of notation, we will write $\rho(X_k, t_k|X_{k-1}, t_{k-1})$ as ρ_k.

Given the functions f, g, and h, the joint density function $\rho(X)$ of the random sample X may be derived by solving the Fokker-Planck or forward equation for the transition densities ρ_k subject to any boundary conditions which may apply. For the Itô process (11) the forward equation is derived in the appendix. Although the functional partial differential equation given in the appendix characterizes the transition densities (hence the conditional likelihood functions), obtaining a closed form solution for the ρ_k's is generally quite difficult. However, by restricting the functional forms of f, g, and h, it is often possible to derive the transition densities explicitly. For example, if h = 0 (pure diffusion) and f and g satisfy a certain "reducibility" condition, it may be shown[20] that there exists a transformed process Z(t) of X(t) for which the coefficient functions are independent of Z(t). That is, for some suitable change of variables $T[X(t)] \equiv Z(t)$, an application of Itô's lemma will yield:

$$dZ = p(t;\ \theta)dt + q(t;\ \theta)dW \tag{15}$$

In this case the transition density function for the transformed data is readily derived as:

$$\rho_k(Z,\ t) = \left[2\pi \int_{t_{k-1}}^{t} q^2 d\tau\right]^{-1/2} \exp\left[-\frac{\left(Z - Z_{k-1} - \int_{t_{k-1}}^{t} p dt\right)^2}{2\int_{t_{k-1}}^{t} q^2 d\tau}\right] . \tag{16}$$

The well-known lognormal diffusion process (2) is an example, for which the transformation T(x) is just lnX and $p = \mu - \frac{1}{2}\sigma^2$, $q = \sigma$.

Given the transition densities ρ_k, the joint-likelihood and log-likelihood functions of the random sample X are given by:

$$L(\theta;\ X) = \prod_{k=1}^{n} \rho(X_k,\ t_k | X_{k-1},\ t_{k-1};\ \theta) \tag{17a}$$

$$G(\theta;\ X) = \sum_{k=1}^{n} \ln\rho(X_k,\ t_k | X_{k-1},\ t_{k-1};\ \theta) \equiv \sum_{k=1}^{n} \ell_k(X_k | X_{k-1};\ \theta) . \tag{17b}$$

Under assumptions (A) and mild regularity conditions, the ML estimator $\hat{\theta}_{ML}$ of θ_0 exists, is consistent, and is asymptotically efficient in the class of all CUAN estimators. That is,

$$\operatorname*{plim}_{n\to\infty} \hat{\theta}_{ML} = \theta_0 \tag{18a}$$

$$\sqrt{n}\,(\hat{\theta}_{ML} - \theta_0) \overset{A}{\sim} N\left(0,\ I^{-1}(\theta_0)\right) \tag{18b}$$

where the asymptotic covariance matrix $I^{-1}(\theta_0)$ is the inverse of the information matrix $I(\theta_0)$:

$$I(\theta_0) = -\lim \frac{1}{n} \sum_{k=1}^{n} E\left[\frac{\partial^2 \ell(X_k | X_{k-1};\ \theta_0)}{\partial\theta\partial\theta'}\right] . \tag{19}$$

4.2. Two Examples.

For illustrative purposes, the likelihood functions of two particular processes are presented below.

Example 1. (Ornstein-Uhlenbeck process)

As an illustration of a general equilibrium characterization of the term structure of interest rates, Vasicek (1977) considers the specific process:

$$dX = \alpha(\gamma - X)dt + \beta dW \tag{20}$$

which is an Ornstein-Uhlenbeck process with steady-state mean γ. Because the increments of such a process are normally distributed, the conditional likelihood is particularly easy to derive and is given by:

$$\rho(X_k, t_k | X_{k-1}, t_{k-1}) = \left[\frac{\pi\beta^2}{\alpha}(1 - e^{-2\alpha\Delta t_k})\right]^{-1/2} \exp\left[-\frac{\alpha(X_k - X_{k-1}e^{-\alpha\Delta t_k} + \gamma[1 - e^{\alpha\Delta t_k}])^2}{\beta^2(1 - e^{-2\alpha\Delta t_k})}\right] \tag{21}$$

Example 2. (Diffusion with absorbing barrier)[21]

Although Black and Cox (1976) and Ho and Singer (1982) have derived valuation formulas for risky debt with various indenture provisions, to date they have not been empirically implemented. As a simple example of how corporate bankruptcy might be modelled and estimated, let $X(t)$ represent a firm's equity price at time t and suppose $X(0) > 0$ and that $X(t)$ follows arithmetic Brownian motion:

$$dX = \alpha dt + \beta dW \tag{22}$$

Furthermore, let $X = 0$ be an absorbing state so that if $X(t)$ reaches 0, it stays in that state thereafter, i.e., bankruptcy occurs. Now suppose that n observations of X are taken and that $X_1 > 0, \ldots, X_{n-1} > 0, X_n = 0$ so that bankruptcy occurs in this sample some time between t_{n-1} and t_n. Then the likelihood-function for this sample would be the product of the conditional densities for observations X_1 to X_{n-1} where:

$$\rho(X_k, t_k | X_{k-1}, t_{n-1}) = [2\pi\beta^2\Delta t_k]^{-1/2} \exp[-\frac{(X_k - X_{k-1} - \alpha\Delta t_k)^2}{2\beta^2\Delta t_k}], \; k=1, \ldots, n-1 \tag{23}$$

multiplied by the distribution function of the first-passage time for observation X_n. Following Cox and Miller's (1973) derivation for the first-passage time distribution of a process with an absorbing barrier at $X = a > 0$, the distribution for the barrier at $X = 0$ may be calculated to be:

$$P(\text{Absorption in } [t_{n-1}, t_n)) = \Phi[\frac{-X_{n-1} - \alpha\Delta t_n}{\beta\sqrt{\Delta t_n}}] + \exp[-\frac{2\alpha X_{n-1}}{\beta^2}]\Phi[\frac{-X_{n-1} + \alpha\Delta t_n}{\beta\sqrt{\Delta t_n}}] \tag{24}$$

where Φ is the standard normal distribution function. Note that although $X(t)$ may have been absorbed at any time between t_{n-1} and t_n, knowing that $X(t)$ has been absorbed <u>by</u> time t_n is sufficient for computing ML estimates of the unknown parameters. Given ML estimates of α and β, it is then possible to obtain estimates of the probability of bankruptcy within any given time interval for firms which are similar to the one which generated the original sample X. For example, it may be plausible to make inferences about the probability of default for a specific small savings and loan association using estimates based on data for a representative cross-section of small banks. Of course, the reliability of such bankruptcy forecasts depends on the relative impact of industry-effects versus idiosyncratic effects in triggering default and is an empirical question.

4.3. The Asymptotic Distribution of General Contingent Claims Estimators.

Let F be the price of an arbitrary asset which is contingent upon the fundamental asset X(t). In particular, suppose for now that F may be determined by the following known asset-pricing formula:

$$F = F(X,\ t,\ \eta;\ \theta_0)\ , \qquad F \text{ continuously differentiable in } \theta \tag{25}$$

where η is a vector of observables (e.g., interest rates, time to maturity, etc.) and θ_0 is the unknown true parameter vector associated with the fundamental asset price process X(t).

Given assumptions (A), the well-known "principle of invariance" states that the ML estimator of the contingent claims price F is simply

$$\hat{F}_{ML} = F(X,\ t,\ \eta;\ \hat{\theta}_{ML}) \tag{26}$$

where $\hat{\theta}_{ML}$ maximizes (17). Since $\hat{F}_{ML}$ is a true ML estimator of F, it is also consistent and asymptotically efficient in the class of all CUAN estimators of F. In addition, the asymptotic distribution of the estimator $\hat{F}_{ML}$ may be easily derived and is given by:

$$\sqrt{n}\ (\hat{F}_{ML} - F) \overset{A}{\sim} N(0,\ V_0) \tag{27a}$$

$$V_0 \equiv \frac{\partial F(\theta_0)}{\partial \theta}^T I^{-1}(\theta_0)\ \frac{\partial F(\theta_0)}{\partial \theta} \tag{27b}$$

Using (27), the usual forms of statistical inference may then be applied to the estimated contingent claims price. In particular, the model-specification test, confidence intervals, price forecasts, and other forms of statistical inference which were suggested in Section 2 for the BS call option pricing model may also be applied to any other type of contingent claims model in a similar fashion. In fact, even if it is not possible to solve in closed form the partial differential equation which yields the contingent claims pricing formula F, it may still be possible to calculate $\hat{F}_{ML}$ and its derivatives $\frac{\partial F(\hat{\theta}_{NL})}{\partial \theta}$ numerically.[22] The asymptotic distribution is then completely determined and the usual forms of statistical inference once again obtain. This is an example of a situation in which large-sample theory is the only practical form of statistical inference available since, in this case attempting to deduce the finite-sample distribution of the contingent claims estimator along the lines of Boyle and Ananthanarayanan (1977) would be prohibitively expensive.

5. Conclusion

In this paper we have provided a general methodology for the estimation and testing of general contingent claims asset-pricing models by appealing to asymptotic statistical theory. Given the large-sample distribution of any contingent claims price estimator, the financial economist may bring to bear a considerable collection of statistical tools upon a variety of problems in model-specification testing and forecasting. Since what constitutes a "large sample" depends upon the particular estimator of interest, Monte Carlo studies must be performed on a case by case basis in order to determine the practical relevance of the proposed methods. The simulation results reported in Section 3 for the Black-Scholes call option pricing model suggest that for most call options, a large sample consists of between 300 and 500 observations. Moreover, the costs of performing these simulation studies are quite small, certainly relative to their payoff but also in absolute magnitude. As an example, the costs of performing the simulations in Tables 4 and 5 did not exceed $25.00.

In addition to cost-effectiveness, another advantage of such large-sample results is tractability. The numerical estimation of the fundamental asset's parameters is a straightforward application of now standard maximum likelihood software packages. In addition, part of the standard output of such packages is a consistent estimate of the inverse of the information matrix I^{-1}. Given this estimate, the asymptotic distribution of any corresponding contingent claim may then be derived by computing the derivative of its pricing formula with respect to the unknown parameters. For those contingent claims with tractable pricing formulas, expressions for their asymptotic distributions will also be tractable. The applicability of the proposed methods thus extends to virtually all contingent claims models which are of theoretical interest since those are often ones for which pricing formulas may be determined either explicitly or numerically. Although this approach seems quite promising, whether or not the application of these results to other contingent claims models will yield new insights can only be determined by further empirical investigations.

Appendix - Derivation of the Forward Equation

Let $X(t)$ solve the following stochastic differential equation:

$$dX = f(X,\ t;\ \alpha)dt + g(X,\ t;\ \beta)dW + h(X,\ t;\ \gamma)dN \qquad \text{(A1)}$$

where dW is the standard Brownian motion and dN is a Poisson counter with intensity λ and is independent of dW. Let $\psi(X)$ be an arbitrary C^∞ function. By Itô's Lemma[23] we have:

$$d\psi = \left[\psi_X f + \tfrac{1}{2}\psi_{XX} g^2\right]dt + \psi_X g dW + \left[\psi(X + h) - \psi(X)\right]dN \ . \qquad \text{(A2)}$$

where

$$\psi_X \equiv \frac{\partial\psi}{\partial X}\ , \qquad \psi_{XX} \equiv \frac{\partial^2\psi}{\partial X^2}\ .$$

Define $D_{P,k}$ to be the Dynkin operator at time t_k, i.e., $D_{P,k} \equiv \frac{d}{dt} E_{t_k}[\cdot]$. Applying it to ψ yields:

$$D_{P,k}[\psi] = E_{t_k}\left[\psi_X f + \tfrac{1}{2}\psi_{XX} g^2\right] + \lambda E_{t_k}\left[\psi(X + h) - \psi(X)\right] \ . \qquad \text{(A3)}$$

Given assumption (A3), we may express $D_{p,k}[\psi]$ as the following integral:

$$D_{P,k}[\psi] = \int_\Omega \left\{\psi_X f + \tfrac{1}{2}\psi_{XX} g^2 + \lambda\left[\psi(X + h) - \psi(X)\right]\right\}\rho_k(X,\ t)dX \qquad \text{(A4a)}$$

$$= \int_\Omega \left[-\psi \frac{\partial}{\partial X}(f\rho_k) + \tfrac{1}{2}\psi \frac{\partial^2}{\partial X^2}(g^2\rho_k) - \psi\lambda\right]dX + \lambda \int_\Omega \psi(X + h)\rho_k dX \ . \qquad \text{(A4b)}$$

Let $Y \equiv \tilde{h}(X,\ t;\ \gamma) \equiv X + h(X,\ t;\ \gamma)$ be an <u>onto</u> map of Ω to Ω for all $(t,\ \gamma)$ and suppose that $\left|\frac{\partial}{\partial X}(\tilde{h}) + 1\right| \neq 0$ for all $(t,\ \gamma)$ and $X \in \Omega$. Then the Inverse Function Theorem guarantees the existence of $\tilde{h}^{-1}$ such that $X = \tilde{h}^{-1}(Y,\ t;\ \gamma)$. Using the change of variables formula, we have:

$$\int_\Omega \psi(X + h)\rho_k(X,\ t)dX = \int_\Omega \psi(Y)\rho_k\left(\tilde{h}^{-1}(Y,\ t;\ \gamma)\right)\left|\frac{\partial}{\partial Y}\left(\tilde{h}^{-1}(Y,\ t;\ \gamma)\right)\right|dY \qquad \text{(A5a)}$$

$$= \int_{\Omega} \psi(X)\rho_k\left(\tilde{h}^{-1}(X,\ t;\ \gamma)\right)\left|\frac{\partial}{\partial X}\left(\tilde{h}^{-1}(X,\ t;\ \gamma)\right)\right|dX \ . \qquad \text{(A5b)}$$

We then conclude that

$$D_{P,k}[\psi] = \int_{\Omega} \left\{-\frac{\partial}{\partial X}(f\rho_k) + \tfrac{1}{2}\frac{\partial^2}{\partial X^2}(g^2\rho_k) - \lambda\tilde{\rho}_k\left|\frac{\partial}{\partial X}\tilde{h}^{-1}\right|\right\}\psi(X)\ dX \ . \qquad \text{(A6)}$$

Assuming that $\psi(X)\rho_k(X,\ t)$ is continuous on $\Omega \times [0,\ \infty)$, $D_{P,k}[\psi]$ may be calculated alternatively as

$$D_{P,k}[\psi] = \frac{d}{dt}E_{t_k}[\psi] = \int_{\Omega} \psi(X)\frac{\partial}{\partial t}\left[\rho_k(X,\ t)\right]dX \ . \qquad \text{(A7)}$$

Equating (A7) and (A6) and noting that the equality obtains for arbitrary smooth functions ψ allow us to conclude that:

$$\frac{\partial}{\partial t}[\rho_k] = -\frac{\partial}{\partial X}[f\rho_k] + \tfrac{1}{2}\frac{\partial^2}{\partial X^2}(g^2\rho_k) - \lambda\rho_k + \lambda\tilde{\rho}_k\left|\frac{\partial}{\partial X}[\tilde{h}^{-1}]\right| \ . \qquad \text{(A8a)}$$

where

$$\tilde{h}(X,\ t;\ \gamma) \equiv X + h(X,\ t;\ \gamma)\ , \qquad \tilde{h}(\tilde{h}^{-1},\ t;\ \gamma) \equiv X \qquad \text{(A8b)}$$

$$\tilde{\rho}_k \equiv \rho_k(\tilde{h}^{-1},\ t) \qquad \text{(A8c)}$$

$$\rho_k(X,\ t_{k-1}|X_{k-1},\ t_{k-1}) = \delta(X - X_{k-1}) \ . \qquad \text{(A8d)}$$

and $\delta(X - X_k)$ is the Dirac-delta generalized function centered at X_{k-1}.[24]

Footnotes

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[1] Papers by Black and Scholes (1972), Merton (1973), Black (1975), Macbeth and Merville (1979), and Gultekin, Rogalski, and Tinic (1982) have noted systematic differences between observed market prices of call options and prices obtained from the Black-Scholes formula but did not formally test whether such departures were statistically significant. Gultekin et. al. does consider how such biases change with the time-to-maturity although no formal explanation of their findings was proposed. However, several studies have considered testing the efficiency of options markets. In particular, Black and Scholes (1972), Galai (1977), and Finnerty (1978) have explored the possibility of excess returns resulting from observed options prices deviating from the Black-Scholes prices. Chiras and Manaster (1978) study possible excess returns generated by using implied standard deviations in the pricing formula. Whaley (1982) uses implied standard deviations in examining various pricing formulas for calls on stocks with known dividends. Violations of certain boundary conditions by observed market prices have also been investigated by Galai (1978) and Bhattacharya (1983). Although many of these

empirical findings are quite striking, without some guidelines as to the statistical significance of observed deviations, hypothesis tests cannot formally be constructed. Even in Whaley's (1982) six regression tests of option valuation, since the linear regression equations are not determined by theoretical considerations there is no guarantee that the subsequent test statistics have a particular sampling distribution.

[2]As an extreme example, consider a coin which has an unknown probability p of coming up "Heads" when tossed, where p is known to be between $1/4$ and 3/4. Toss the coin once and consider the estimator $\hat{p}$ which equals 1 if the coin comes up "Heads" and 0 if it comes up "Tails." Although this estimator is incorrect with probability one, it is in fact unbiased. This rather contrived example illustrates the inadequacy of using unbiasedness as the sole criterion for choosing an estimator; its variance must also be considered.

[3]For perhaps the weakest set of regularity conditions which insure consistency and asymptotic efficiency of maximum-likelihood estimators, see Huber (1967).

[4]See Kendall and Stuart, (1973).

[5]It is assumed that h is constant as n increases so that T also increases. If, instead, T is kept constant while n increases and h decreases, one of the regularity conditions will be violated. In this case, the estimator need not approach the true parameter as n increases. For example, in estimating the parameters of a lognormal diffusion process, Merton (1980) and Rosenfeld (1980) observe that the accuracy of the drift rate estimator (as measured by its variance) does not increase with more frequent observations if T is fixed. This seeming contradiction to the asserted consistency of the ML estimators is resolved by observing that when T is fixed and n increases without bound, the regularity condition which requires that information matrix

approach a nonsingular matrix (see Rao (1973)) is violated. This, however, does not imply that ML is not applicable but rather that its asymptotic properties have been misinterpreted. Specifically, the asymptotic results used in proving the consistency of ML require that n increases without bound ceteris paribus, i.e., holding other parameters (μ, σ^2, h) constant. Note that if these parameters were in fact held constant, increasing n will indeed increase the accuracy of the drift estimator. This example thus illustrates the importance of checking the regularity conditions when applying asymptotic results to nonstandard situations, such as taking more frequent observations in a fixed time interval.

[6]Note that since Boyle and Ananthanarayanan (1977) have numerically determined finite-sample confidence intervals for option prices induced by corresponding confidence intervals for the variance estimator, the application of asymptotic theory to the BS model here is partly for expositional purposes. Nevertheless, several aspects of the large-sample approach may render it more useful than numerical determination of finite-sample properties even in this case. One significant advantage of asymptotic theory is that it is possible to derive analytic expressions for the limiting distribution with which general comparative static issues may be examined, as in Section 2.3.

However, the most obvious benefit of appealing to asymptotics is tractability. As an example, consider performing a two-sided hypothesis test for a particular option. In order to construct a finite-sample test, critical values for the distribution of the option price estimator must be determined by numerical integration. In contrast, an asymptotic test relies on the standard normal critical values. Furthermore, in situations where a contingent claim price function cannot be derived in closed form, numerically determining the finite-sample distribution may be computationally infeasible whereas the corresponding asymptotic distribution may still be derived with little difficulty.

Finally, the approach that Boyle and Ananthanarayanan propose does not extend readily to multi-parameter situations nor to cases in which the finite-sampling distribution of the fundamental asset-price parameter is unknown. For example, consider estimating the term structure of interest rates in Vasicek's (1977) model for the specific case in which the spot rate follows an Ornstein-Uhlenbeck process (20). Vasicek derives a closed-form expression for bond prices as a function of the parameters $\theta \equiv (\alpha, \gamma, \beta)$, hence estimates for bond prices may be computed by inserting the estimators $\hat{\theta} \equiv (\hat{\alpha}, \hat{\gamma}, \hat{\beta})$ in the pricing formula. To deduce the finite-sample distribution of bond prices then, the finite-sample distribution of the estimators is required. To this author's knowledge, there do not exist estimators $\hat{\theta}$ in this case for which the finite-sampling distribution is known. However, these parameters are easily estimated via maximum-likelihood and therefore have well-defined asymptoti distributions. The limiting distribution for bond price estimates is then readily determined as outlined in Section 4.

[7]An alternate explanation of why implied variances seem to perform better than historical estimates does have some implications for the proposed methodology. This involves the possibility of a nonstationary variance parameter. Suppose, for example, that the variance rate of a given stock grows linearly with calendar time, i.e., $\sigma^2 = \alpha + \beta t$. Assuming that the BS model obtains, computing the implied variance using current option prices will yield accurate estimates of σ^2 whereas computing historical estimates of σ^2 assuming it is constant will yield biased and inconsistent estimates. Thus if the BS model obtains, then the fact that implied variances are better predictors of future variances may indicate that the volatility is

nonstationary. This, however, does not vitiate the usefulness of asymptotic methods for statistical inference but simply suggests that there may exist a better stochastic specification for asset-price dynamics such as Cox's constant elasticity of variance process (for which the variance is related to the stock price level) or perhaps a process with a variance rate parametrized as above (in which case it is related to time). The appropriate specification for asset-price dynamics is an important issue in itself and will be addressed in future research. Indeed, asymptotic results are also useful for constructing specification tests for problems such as nonstationarity. However, the present study assumes that this issue has already been resolved and proceeds from there.

[8]Of course, the distribution of z is not the Student's t since the numerator and denominator are not statistically independent. However, asymptotically it is normally distributed.

[9]I am grateful to Eric Rosenfeld for providing me with the variance estimates for these stocks.

[10]Specifically, using the Bonferroni correction for the simultaneous testing of five hypotheses at the 5% level, the appropriate critical values for a two-sided test is ±2.58 (corresponding to a tail probability of slightly less than $\frac{2.5}{5}$ %. For all five options, the associated z-statistic falls within the critical region hence the simultaneous hypothesis may be rejected at the 5% level.

[11]The random number generator used was the subroutine GGNQF in the IMSL software package. All computations were done in double precision FORTRAN on a Digital VAX 11/780.

[12]The actual proportions of the z-statistics in the 5 and 10 percent tail regions (i.e., the actual size of 5 and 10 percent tests) were also tabulated

and generally agreed with the results in Tables 4 and 5. That is, in those experiments for which the studentized range test did not reject normality, the actual tail regions were not statistically different from the theoretical 5 and 10 percent values. The complete set of simulation results are available from the author upon request.

[13]To see this, consider several independent simulations for a sample size of 100. The bias of the estimators from each experiment will not be identical since the experiments are random. Indeed, the bias itself is a random viable and the actual value of the bias for a particular experiment is a realization of that random variable. Because the estimators are consistent, as the number of observations increase the bias approaches zero. That is, the distribution of the bias approaches a degenerate distribution (a Dirac delta-function) centered about zero. However, as n increases from say 100 to 300, the distribution of the bias still has some dispersion about zero (but less than the dispersion at $n = 100$). Therefore, it is possible that a particular realization of the bias at $n = 300$ is greater than the realization of the bias at $n = 100$. The same line of reasoning applies also to the χ^2-statistic which, on average, decreases with the sample size but does not do so monotonically due to sampling variation.

[14]I am grateful to Jay Shanken for raising this point.

[15]See Pearson and Hartley (1970), Table 34B.

[16]See Fama (1976), Table 1.9, p. 40.

[17]Suppose, however, the jump magnitude is stochastic. More generally, suppose that certain "parameters" in f, g, and h are in fact random variables. Without further information, there is of course little that can be done. If however it is posited that these random parameters are distributed according to a particular parametrizable probability law which is

statistically independent of dW and dN, then the estimation procedure described in this section may still be applied. For example, if it is assumed that the jump magnitude is lognormally distributed with unknown parameters and is independent of dW and dN, these parameters may be estimated along with the other unknown parameters of f, g, and h as well.

[18]See Arnold (1974) chapter 6.

[19]See Kushner (1967).

[20]See Schuss (1980) chapter 4 for a statement of the reducibility condition.

[21]For perhaps its first application in the econometrics literature, see Hausman and Wise (1983).

[22]Specifically, if the solution to the fundamental partial differential equation determining F exists, the ML estimate $\hat{F}_{ML}$ is obtained by numerically solving the p.d.e. with estimates of the parameters substituted into its coefficients as, for example, in Brennan and Schwartz (1980, 1982). The derivative $\frac{\partial F(\theta_{ML})}{\partial \theta}$ may be evaluated by perturbing the value of $\hat{\theta}_{ML}$ slightly, re-solving for $\hat{F}_{ML}$ numerically, and then computing the ratio of the change in $\hat{F}_{ML}$, i.e., $\frac{\partial F}{\partial \theta} \approx \frac{\Delta F}{\Delta \theta}$. The asymptotic variance is then readily computed. I am grateful to an anonymous referee for suggesting that this issue be explored.

[23]See Merton (1971) or Brockett (1984).

[24]See Gel'fand and Shilov (1964) for the theory of generalized functions. As a simple example, consider the pure jump process dX = dN which is simply a Poisson process with rate λ. Using the delta-function, the transition density of X may be expressed as:

$$\rho_X(X, t) = \sum_{k=0}^{\infty} \frac{e^{-\lambda t}(\lambda t)^k}{k!} \delta(X - k) .$$

In this case the forward equation given by (18) reduces to:

$$\frac{\partial}{\partial t}[\rho_X] = \lambda\left[\rho_X(X-1,\ t) - \rho_X(X,\ t)\right] .$$

which may be verified by taking the derivative of ρ_X with respect to time t:

$$\frac{\partial}{\partial t}[\rho_X] = \sum_{k=0}^{\infty} \frac{-\lambda e^{-\lambda t}(\lambda t)^k}{k!}\,\delta(X-k) + \sum_{k=1}^{\infty} \frac{\lambda k e^{-\lambda t}(\lambda t)^{k-1}}{k!}\,\delta(X-k)$$

$$= -\lambda \sum_{k=0}^{\infty} \frac{e^{-\lambda t}(\lambda t)^k}{k!}\,\delta(X-k) + \lambda \sum_{k=0}^{\infty} \frac{e^{-\lambda t}(\lambda t)^k}{k!}\,\delta(X-1-k)$$

$$= \lambda\left[\rho_X(X-1,\ t) - \rho_X(X,\ t)\right] .$$

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Table 1

Values of k_1, k_2, k_3 for various times to maturity τ (measured in weeks) where an annual interest rate of 10% and an annual standard deviation of 50% are assumed, corresponding to weekly values of $r = 0.00183$ and $\sigma = 0.06934$

τ	$k_1 \equiv e^{-(r-\frac{1}{2}\sigma^2)\tau}$	$k_2 \equiv e^{-(r+\frac{1}{2}\sigma^2)\tau}$	$k_3 \equiv e^{(r+\frac{1}{2}\sigma^2)\tau}$
1.00	1.00057	0.99577	1.00425
5.00	1.00286	0.97904	1.02141
9.00	1.00515	0.96259	1.03887
13.00	1.00745	0.94641	1.05662
26.00	1.01496	0.89570	1.11645

Table 2

Estimated values of k_1, k_2, k_3 for Litton, National Semiconductor, and Tandy stocks. The (annual) interest rate used is 9.443%, the average return on 26-week Treasury bills quoted in the January 15, 1979 Wall Street Journal.

Stock	τ[a]	$\hat{k}_1 \equiv e^{-(r-\frac{1}{2}\hat{\sigma}^2)\tau}$	$\hat{k}_2 \equiv e^{-(r+\frac{1}{2}\hat{\sigma}^2)\tau}$	$\hat{k}_3 \equiv e^{(r+\frac{1}{2}\hat{\sigma}^2)\tau}$
	1.00	1.00038	0.996157	1.00386
Litton	5.00	1.00190	0.980931	1.01944
($\hat{\sigma}^2$= 0.00423	9.00	1.00343	0.965938	1.03526
weekly)	13.00	1.00495	0.951174	1.05133
	26.00	1.00993	0.904732	1.10530
National	1.00	1.00200	0.994549	1.00548
Semiconductor	5.00	1.01003	0.973040	1.02771
($\hat{\sigma}^2$= 0.00746	9.00	1.01812	0.951996	1.05042
weekly)	13.00	1.02628	0.931407	1.07364
	26.00	1.05325	0.867520	1.15271
	1.00	1.00054	0.995994	1.00402
Tandy	5.00	1.00272	0.980130	1.02027
($\hat{\sigma}^2$= 0.00456	9.00	1.00490	0.964518	1.03679
weekly)	13.00	1.00709	0.949155	1.05357
	26.00	1.01423	0.900895	1.11001

[a] Times-to-maturity τ are measured in weeks.

TABLE 3a

Maximum-likelihood estimates of prices ($\hat{F}$) and asymptotic standard deviations ($\sqrt{\hat{V}_F}$) of Litton call options traded on January 12, 1979, and estimates of the corresponding $\hat{z}$-statistics ($\hat{z} \equiv (\hat{F} - \bar{F})/\sqrt{\hat{V}_F}$ where $\bar{F}$ is the observed market price of the option). The maximum-likelihood estimate for the variance of Litton stock is taken from Rosenfeld's (1980) study which used 312 weekly observations from January 1973 to December 1978, and is given by $\hat{\sigma}^2 = 0.00254$. The stock prices S, exercise prices E, and option prices $\bar{F}$ were obtained directly from the January 15, 1979 issue of the Wall Street Journal. The interest rate used was the average return on 26-week Treasury bills for which the same issue of the Wall Street Journal reported an annual rate of 9.443%.

S	E	τ[a]	$\bar{F}$	$\hat{F}_{ML}$ (Z-STAT)	$\sqrt{\hat{V}_F} \times 100$	$\hat{z}$[b] $\equiv \frac{\hat{F}-\bar{F}}{\sqrt{\hat{V}_F}}$
21.500	24.375	9.00	1.000	0.791	6.0120	-3.484
21.500	15.000	9.00	7.000	6.768	0.8666	-2.68
21.500	20.000	9.00	3.000	2.665	5.7653	-5.81
21.500	25.000	9.00	0.813	0.647	5.6121	-2.96
21.500	30.000	9.00	0.188	0.107	2.0796	-3.89
21.500	24.375	22.00	d	-------	-------	-------
21.500	15.000	22.00	8.125	7.302	3.6421	-22.06
21.500	20.000	22.00	4.125	3.754	9.1758	-4.04
21.500	25.000	22.00	1.875	1.665	10.2324	-2.05
21.500	30.000	22.00	0.688	0.669	7.5195	-0.25
21.500	24.375	29.00	d	-------	-------	-------
21.500	15.000	29.00	c	-------	-------	-------
21.500	20.000	29.00	4.625	4.226	10.4769	-3.81
21.500	25.000	29.00	2.375	2.138	11.9523	-1.98
21.500	30.000	29.00	d	-------	-------	-------

[a]Time to maturity is measured in weeks.

[b]Under the joint null hypothesis that the Black-Scholes-Merton options pricing model obtains and that the stock price follows a lognormal diffusion, the $\hat{z}$-statistic is asymptotically standard normal. Therefore, the null hypothesis may be rejected at the 5% level if the estimated $\hat{z}$-statistic falls outside the interval [-1.96, 1.96].

[c]Not traded.

[d]No option offered.

TABLE 3b

Maximum-likelihood estimates of prices ($\hat{F}$) and asymptotic standard deviations ($\sqrt{\hat{V}_F}$) of National Semiconductor call options traded on January 12, 1979, and estimates of the corresponding $\hat{z}$-statistics ($z \equiv (\hat{F} - \bar{F})/\sqrt{\hat{V}_F}$ where $\bar{F}$ is the observed market price of the option). The maximum-likelihood estimate for the variance of National Semiconductor stock is taken from Rosenfeld's (1980) study which used 312 weekly observations from January 1973 to December 1978, and is given by $\hat{\sigma}^2 = 0.00746$. The stock prices S, exercise prices E, and option prices $\bar{F}$ were obtained directly from the January 15, 1979 issue of the Wall Street Journal. The interest rate used was the average return on 26-week Treasury bills for which the same issue of the Wall Street Journal reported an annual rate of 9.443%.

S	E	τ	$\bar{F}$	$\hat{F}$	$\sqrt{\hat{V}_F} \times 100$	$\hat{z}^b \equiv \frac{\hat{F}-\bar{F}}{\sqrt{\hat{V}_F}}$
23.375	15.000	5.00	8.750	8.516	0.3690	-64.41
23.375	20.000	5.00	3.750	4.002	4.5971	5.48
23.375	25.000	5.00	1.063	1.232	7.0585	2.39
23.375	30.000	5.00	0.125	0.258	3.7205	3.57
23.375	35.000	5.00	c	-------	-------	-------
23.375	15.000	18.00	9.375	9.143	4.5831	-5.06
23.375	20.000	18.00	4.625	5.494	10.7533	8.08
23.375	25.000	18.00	2.438	3.049	13.6316	4.48
23.375	30.000	18.00	1.000	1.608	12.5656	4.84
23.375	35.000	18.00	0.375	0.825	9.6800	4.65
23.375	15.000	25.00	9.500	9.518	6.7539	0.27
23.375	20.000	25.00	5.625	6.120	12.8174	3.86
23.375	25.000	25.00	3.375	3.769	15.9162	2.48
23.375	30.000	25.00	d	-------	-------	-------
23.375	35.000	25.00	d	-------	-------	-------

[a]Time to maturity is measured in weeks.

[b]Under the joint null hypothesis that the Black-Scholes-Merton options pricing model obtains and that the stock price follows a lognormal diffusion, the $\hat{z}$-statistic is asymptotically standard normal. Therefore, the null hypothesis may be rejected at the 5% level if the estimated z-statistic falls outside the interval [-1.96, 1.96].

[c]Not traded.

[d]No option offered.

TABLE 3c

Maximum-likelihood estimates of prices ($\hat{F}$) and asymptotic standard deviations ($\sqrt{\hat{V}_F}$) of Tandy call options traded on January 12, 1979, and estimates of the corresponding z-statistics ($\hat{z} \equiv (\hat{F} - \bar{F})/\sqrt{\hat{V}_F}$ where $\bar{F}$ is the observed market price of the option). The maximum-likelihood estimate for the variance of Tandy stock is taken from Rosenfeld's (1980) study which used 312 weekly observations from January 1973 to December 1978, and is given by $\hat{\sigma}^2 = 0.00456$. The stock prices S, exercise prices E, and option prices $\bar{F}$ were obtained directly from the January 15, 1979 issue of the Wall Street Journal. The interest rate used was the average return on 26-week Treasury bills for which the same issue of the Wall Street Journal reported an annual rate of 9.443%.

S	E	τ^a	$\bar{F}$	$\hat{F}$	$\sqrt{\hat{V}_F} \times 100$	$\hat{z}^b \equiv \frac{\hat{F}-\bar{F}}{\sqrt{\hat{V}_F}}$
28.500	22.500	1.00	5.750	6.039	5.4202×10^{-3}	5.33×10^{-3}
28.500	25.000	1.00	3.250	3.560	0.4156	7.46
28.500	30.000	1.00	0.313	0.266	2.4046	-1.95
28.500	35.00	1.00	c	-------	-------	-------
28.500	22.500	14.00	d	-------	-------	-------
28.500	25.000	14.00	5.000	5.192	8.7359	2.20
28.500	30.000	14.00	2.500	2.530	11.4952	0.26
28.500	35.000	14.00	1.000	1.089	9.6560	0.92
28.500	22.500	21.00	d	-------	-------	-------
28.500	25.000	21.00	6.250	5.888	11.0523	-3.28
28.500	30.000	21.00	3.500	3.321	14.0015	-1.28
28.500	35.000	21.00	d	-------	-------	-------

[a] Time to maturity is measured in weeks.

[b] Under the joint null hypothesis that the Black-Scholes-Merton options pricing model obtains and that the stock price follows a lognormal diffusion, the $\hat{z}$-statistic is asymptotically standard normal. Therefore, the null hypothesis may be rejected at the 5% level if the estimated $\hat{z}$-statistic falls outside the interval [-1.96, 1.96].

[c] Not traded.

[d] No option offered.

TABLE 4a

Means, standard deviations, and percentage biases for the sampling distribution of the call option price estimator $(\hat{F})$ and its asymptotic variance estimator $(\hat{V}_F)$ and normality tests for the sampling distribution of the standardized call option price estimators $(\hat{z})$ for options with exercise prices E = 35, 40, 45 for stock price S = 40, and fixed time to maturity τ = 1 (week). For each option, three experiments were performed corresponding to sample sizes of 100, 300 and 500 respectively. All experiments involve 1000 replications and assume an (annual) interest rate of 10% (yielding a continuously-compounded weekly rate of 0.183%) and a (weekly) variance rate of 1%.

Sample Size	F (Theoretical Value)	$\bar{\hat{F}}$ (SD)	% BIAS($\bar{\hat{F}}$)[a]	V_F (Theoretical Value)	$\bar{\hat{V}}_F$ (SD)	% BIAS($\bar{\hat{V}}_F$)[b]	$\bar{\hat{z}}$ (SD)	χ^2-TEST[c] (p-VALUE)	SKEWNESS[d]	STUDENTIZED RANGE[e]
(E = 35)										
100	5.2156	5.2145 (4.10×10^{-2})	0.02	1.7752×10^{-3}	1.7916×10^{-3} (6.98×10^{-4})	-0.92	-0.2387 (1.09)	265.5 (0.00)	-0.803	6.61
300	5.2156	5.2151 (2.36×10^{-2})	0.01	5.9175×10^{-4}	5.9285×10^{-4} (1.33×10^{-4})	-0.19	-0.1348 (1.01)	92.0 (0.00)	-0.490	6.95
500	5.2156	5.2172 (1.92×10^{-2})	-0.03	3.5505×10^{-4}	3.6195×10^{-4} (0.65×10^{-5})	-1.95	-0.0087 (1.03)	62.8 (0.09)	-0.410	5.99
(E = 40)										
100	1.6305	1.6297 (0.114)	0.05	1.2673×10^{-2}	1.2724×10^{-2} (1.82×10^{-3})	-0.40	-0.0811 (1.03)	75.6 (0.01)	-0.422	6.79
300	1.6305	1.6317 (6.44×10^{-2})	-0.07	4.2244×10^{-3}	4.2372×10^{-3} (3.42×10^{-4})	-0.30	-0.0226 (0.99)	43.2 (0.71)	-0.151	6.14
500	1.6305	1.6300 (5.00×10^{-2})	0.03	2.5346×10^{-3}	2.5355×10^{-3} (1.59×10^{-4})	-0.03	-0.0412 (0.99)	30.9 (0.98)	-0.061	6.33
(E = 45)										
100	0.2580	0.2569 (6.04×10^{-2})	0.44	3.7179×10^{-3}	3.7455×10^{-3} (1.23×10^{-3})	-0.74	-0.1924 (1.07)	108.5 (0.00)	-0.743	7.40
300	0.2580	0.2584 (3.48×10^{-2})	-0.14	1.2393×10^{-3}	1.2474×10^{-3} (2.35×10^{-4})	-0.65	-0.0840 (1.01)	60.8 (0.12)	-0.349	6.93
500	0.2580	0.2584 (2.68×10^{-2})	-0.14	7.4358×10^{-4}	7.4701×10^{-4} (1.08×10^{-4})	0.46	-0.0591 (1.00)	70.57 (0.02)	-0.317	5.95

Table 4a (continued)

[a]The percentage bias is computed as 100 × (True F − $\hat{F}$)/(True F).

[b]The percentage bias is computed as 100 × (True V_F − $\hat{V}_F$)/(True V_F).

[c]The χ^2-test was performed using 50 equiprobable categories yielding 49 degrees of freedom. A p-value of 0.00 indicates that the actual p-value is less than 0.005.

[d]For 1000 replications, the tabulated 90 and 98 percent confidence intervals (equal weights in each tail) for the sample skewness statistic is [−0.127, 0.127] and [−0.180, 0.180] respectively (see Pearson and Hartley (1970)).

[e]For 1000 replications, the tabulated 90 and 95 percent confidence intervals (equal weights in each tail) for the sample studentized range statistic is [5.79, 7.33] and [5.68, 7.54] respectively (see Fama (1976)).

TABLE 4b

Means, standard deviations, and percentage biases for the sampling distribution of the call option price estimator ($\hat{F}$) and its asymptotic variance estimator ($\hat{V}_F$) and normality tests for the sampling distribution of the standardized call option price estimators ($\hat{z}$) for options with exercise prices E = 35, 40, 45 for stock price S = 40, and fixed time to maturity τ = 13 (weeks). For each option, three experiments were performed corresponding to sample sizes of 100, 300 and 500 respectively. All experiments involve 1000 replications and assume an (annual) interest rate of 10% (yielding a continuously-compounded weekly rate of 0.183%) and a (weekly) variance rate of 1%.

Sample size	F (Theoretical Value)	$\bar{\hat{F}}$ (SD)	% BIAS($\bar{\hat{F}}$)[a]	V_F (Theoretical Value)	$\bar{\hat{V}}_F$ (SD)	% BIAS($\bar{\hat{V}}_F$)[b]	$\bar{\hat{z}}$ (SD)	χ^2-TEST[c] (p-VALUE)	SKEWNESS[d]	STUDENTIZED RANGE[e]
(E = 35)										
100	8.7089	8.6958 (0.337)	0.15	1.1316×10^{-1}	1.1295×10^{-1} (1.86×10^{-2})	0.19	-0.1233 (1.03)	103.0 (0.00)	-0.354	6.98
300	8.7089	8.7152 (0.197)	-0.07	3.7719×10^{-2}	3.7893×10^{-2} (3.62×10^{-3})	-0.46	-0.0164 (0.02)	58.6 (0.16)	-0.275	6.46
500	8.7089	8.7120 (0.157)	-0.04	2.2631×10^{-2}	2.2688×10^{-2} (1.73×10^{-3})	-0.25	-0.0191 (1.05)	59.7 (0.14)	-0.118	5.88
(E = 40)										
100	6.1384	6.1146 (0.393)	0.39	1.5577×10^{-1}	1.5516×10^{-1} (2.13×10^{-2})	0.40	-0.1303 (1.02)	68.6 (0.03)	-0.341	7.03
300	6.1384	6.1326 (0.231)	0.09	5.1925×10^{-2}	5.1898×10^{-2} (4.18×10^{-3})	0.05	-0.0664 (1.02)	60.7 (0.12)	-0.215	6.92
500	6.1384	6.1352 (0.172)	0.05	3.1155×10^{-2}	3.1146×10^{-2} (1.86×10^{-3})	0.03	-0.0472 (0.98)	33.0 (0.96)	-0.243	6.27
(E = 45)										
100	4.2346	4.2075 (0.405)	0.64	1.6446×10^{-1}	1.6356×10^{-1} (2.40×10^{-2})	0.54	-0.1426 (1.03)	83.0 (0.00)	-0.402	6.95
300	4.2346	4.2275 (0.229)	0.17	5.4819×10^{-2}	5.4754×10^{-2} (4.56×10^{-3})	0.12	-0.0712 (0.98)	39.1 (0.84)	-0.015	6.38
500	4.2346	4.2230 (0.181)	0.27	3.2891×10^{-2}	3.2783×10^{-2} (2.14×10^{-3})	0.33	-0.0970 (1.01)	64.8 (0.06)	-0.283	6.40

Table 4b (continued)

[a]The percentage bias is computed as 100 × (True F − $\hat{F}$)/(True F).

[b]The percentage bias is computed as 100 × (True V_F − $\hat{V}_F$)/(True V_F).

[c]The χ^2-test was performed using 50 equiprobable categories yielding 49 degrees of freedom. A p-value of 0.00 indicates that the actual p-value is less than 0.005.

[d]For 1000 replications, the tabulated 90 and 98 percent confidence intervals (equal weights in each tail) for the sample skewness statistic is [−0.127, 0.127] and [−0.180, 0.180] respectively (see Pearson and Hartley (1970)).

[e]For 1000 replications, the tabulated 90 and 95 percent confidence intervals (equal weights in each tail) for the sample studentized range statistic is [5.79, 7.33] and [5.68, 7.54] respectively (see Fama (1976)).

TABLE 5a

Means, standard deviations, and percentage biases for the sampling distribution of the call option price estimator ($\hat{F}$) and its asymptotic variance estimator ($\hat{V}_F$) and normality tests for the sampling distribution of the standardized call option price estimators ($\hat{z}$) for deep in and out of the money options: exercise prices E = 25 and 55 for stock price S = 40, and fixed time to maturity τ = 1 (week). For each option, three experiments were performed corresponding to sample sizes of 100, 300 and 500 respectively. All experiments involve 1000 replications and assume an (annual) interest rate of 10% (yielding a continuously-compounded weekly rate of 0.183%) and a (weekly) variance rate of 1%.

Sample size	F (Theoretical Value)	$\bar{\hat{F}}$ (SD)	% BIAS($\bar{\hat{F}}$)[a]	V_F (Theoretical Value)	$\bar{\hat{V}}_F$ (SD)	% BIAS($\bar{\hat{V}}_F$)[b]	$\bar{\hat{z}}$ (SD)	χ^2-TEST[c] (p-VALUE)	SKEWNESS[d]	STUDENTIZED RANGE[e]
(E = 25)										
100	15.0458	15.0458 (4.29×10^{-6})	0.00	1.6991×10^{-12}	4.0050×10^{-11}) (3.17×10^{-10})	-2257.10	-13.2558 (236.07)	2525.3 (0.00)	-28.549	30.46
300	15.0458	15.0458 (1.26×10^{-6})	0.00	5.6637×10^{-13}	2.3736×10^{-12} (6.24×10^{-12})	-319.09	-0.7866 (3.07)	1055.7 (0.00)	-7.566	14.71
500	15.0458	15.0458 (7.62×10^{-7})	0.00	3.3982×10^{-13}	7.9271×10^{-13} (1.39×10^{-12}	-133.27	-0.5064 (1.70)	650.3 (0.00)	-3.431	11.61
(E = 55)										
100	0.0010	0.0013 (1.16×10^{-3})	-31.41	7.7194×10^{-7}	1.7998×10^{-6} (3.16×10^{-6})	-133.16	-0.6481 (2.26)	744.5 (0.00)	-5.039	14.24
300	0.0010	0.0011 (5.56×10^{-4})	-8.37	2.5731×10^{-7}	3.4515×10^{-7} (3.27×10^{-7})	-34.13	-0.3629 (1.40)	416.4 (0.00)	-2.167	7.91
500	0.0010	0.0011 (4.51×10^{-4})	-9.52	1.5439×10^{-7}	(2.0034×10^{-7}) (1.53×10^{-7})	-29.76	-0.1458 (1.12)	191.9 (0.00)	-0.995	6.78

[a]The percentage bias is computed as 100 × (True F − $\hat{F}$)/(True F). A value of 0.00 indicates that the actual percentage bias is less than 0.005 percent.

[b]The percentage bias is computed as 100 × (True V − $\hat{V}_F$)/(True V_F).

[c]The χ^2-test was performed using 50 equiprobable categories yielding 49 degrees of freedom. A p-value of 0.00 indicates that the actual p-value is less than 0.005.

[d]For 1000 replications, the tabulated 90 and 98 percent confidence intervals (equal weights in each tail) for the sample skewness statistic is [−0.127, 0.127] and [−0.180, 0.180] respectively (see Pearson and Hartley (1970)).

[e]For 1000 replications, the tabulated 90 and 95 percent confidence intervals (equal weights in each tail) for the sample studentized range statistic is [5.79, 7.33] and [5.68, 7.54] respectively (see Fama (1976)).

TABLE 5b

Means, standard deviations, and percentage biases for the sampling distribution of the call option price estimator ($\hat{F}$) and its asymptotic variance estimator ($\hat{V}_F$) and normality tests for the sampling distribution of the standardized call option price estimators ($\hat{z}$) for deep in and out of the money options: exercise prices E = 25 and 55 for stock price S = 40, and fixed time to maturity τ = 13 (weeks). For each option, three experiments were performed corresponding to sample sizes of 100, 300 and 500 respectively. All experiments involve 1000 replications and assume an (annual) interest rate of 10% (yielding a continuously-compounded weekly rate of 0.183%) and a (weekly) variance rate of 1%.

Sample size	F (Theoretical Value)	$\bar{\hat{F}}$ (SD)	% BIAS($\bar{\hat{F}}$)[a]	V_F (Theoretical Value)	$\bar{\hat{V}}_F$ (SD)	% BIAS($\bar{\hat{V}}_F$)[b]	$\bar{\hat{z}}$ (SD)	χ^2-TEST[c] (p-VALUE)	SKEWNESS[d]	STUDENTIZED RANGE
(E = 25)										
100	16.0252	16.0250 (0.123)	0.00	1.498×10^{-2}	1.5292×10^{-2} (6.19×10^{-3})	-2.08	-0.2230 (1.12)	135.3 (0.00)	-0.926	7.76
300	16.0252	16.0271 (6.77×10^{-2})	-0.01	4.9940×10^{-3}	5.0580×10^{-3} (1.12×10^{-3})	-1.28	-0.0800 (0.98)	71.5 (0.02)	-0.409	7.21
500	16.0252	16.0244 (5.26×10^{-2})	0.00	2.9964×10^{-3}	3.0004×10^{-3} (5.18×10^{-4})	0.13	-0.0997 (0.99)	74.9 (0.01)	-0.406	6.28
(E = 55)										
100	1.9301	1.9212 (0.341)	0.46	1.1033×10^{-1}	1.1040×10^{-1} (2.62×10^{-2})	-0.06	-0.1578 (1.10)	117.6 (0.00)	-0.643	6.06
300	1.9301	1.9275 (0.198)	0.13	3.6778×10^{-2}	3.6794×10^{-2} (5.06×10^{-3})	-0.04	-0.0862 (1.05)	63.3 (0.01)	-0.366	6.78
500	1.9301	1.9330 (0.148)	-0.15	2.2067×10^{-2}	2.2139×10^{-2} (2.28×10^{-3})	-0.33	-0.0322 (1.00)	51.0 (0.39)	-0.239	6.64

[a]The percentage bias is computed as 100 × (True F − $\hat{F}$)/(True F).

[b]The percentage bias is computed as 100 × (True V_F − $\hat{V}_F$)/(True V_F).

[c]The χ^2 − test was performed using 50 equiprobable categories yielding 49 degrees of freedom. A p-value of 0.00 indicates that the actual p-value is less than 0.005.

[d]For 1000 replications, the tabulated 90 and 98 percent confidence intervals (equal weights in each tail) for the sample skewness statistic is [−0.127, 0.127] and [−0.180, 0.180] respectively (see Pearson and Hartley (1970)).

[e]For 1000 replications, the tabulated 90 and 95 percent confidence intervals (equal weights in each tail) for the sample studentized range statistic is [5.79, 7.33] and [5.68, 7.54] respectively (see Fama (1976)).