SERIAL CORRELATION OF ASSET RETURNS AND OPTIMAL PORTFOLIOS FOR THE SHORT AND LONG TERM

by

Stanley Fischer & George Pennacchi

9-85

THE WHARTON SCHOOL

University of Pennsylvania

Philadelphia, PA 19104

March 1985

SERIAL CORRELATION OF ASSET RETURNS AND OPTIMAL PORTFOLIOS FOR THE SHORT AND LONG TERM

Stanley Fischer and George Pennacchi*

It is frequently said that an asset is a safe investment for the short term but not for the long term, or that an asset like gold is a good hedge against inflation in the long run but not in the short run.[1]

Such statements suggest that portfolio behavior should differ depending on the length of time for which assets are held. They can be interpreted by considering the serial correlation properties of asset returns. Suppose the (logarithm of the) return on an asset follows the first-order autoregressive process

(1) $x_t = \theta x_{t-1} + \varepsilon_t$

where ε_t is identically distributed and serially uncorrelated. Let σ^2 be the variance of ε_t, and therefore a measure of uncertainty about the return from holding the asset over one period.

Uncertainty about the return from holding the asset over more than one

---

*Department of Economics, MIT, and N.B.E.R.; and Department of Finance, University of Pennsylvania. This paper originated in research done for the NBER's Pensions Project. We are grateful to Michael Hamer for extensive comments and assistance on an earlier draft, and to Sudipto Bhattacharya, Fischer Black, Barry Goldman, Hayne Leland, Thomas McCurdy and Robert Merton for comments and discussion. Jeffrey Miron provided first class research assistance. Financial support from the National Science Foundation and Hoover Institution is acknowledged with thanks.

[1] Benjamin Klein (1976) drew attention to the issue in discussing the gold standard in the nineteenth century compared with the current monetary system. He argued there is less uncertainty about what the price level will be one year from now than there used to be (prices are more predictable in the short run) but more uncertainty about what the price level will be in the more distant future (prices are less predictable in the long run).

period depends on the autoregressive parameter θ. Looking at the variance of the *per period* return on the asset as it is held for longer periods, the asymptotic variance of the per period return is

$$(2)\quad \lim_{N\to\infty} \sigma^2(N) = \frac{\sigma^2}{(1-\theta)^2}, \qquad |\theta| < 1$$

where N is the number of periods for which the asset is held.

Using the variance of the per period return as a measure of risk, the riskiness of an asset will depend, through θ, on the length of time for which it is held. For instance, an asset with negative θ (as is claimed of gold in the nineteenth century) would be less risky if held for a long period than for a short period. Assets with positive serial correlation are more risky the longer they are held. Thus the notion that gold or any other asset has different risk properties for the long term than for the short term can be understood to refer to the serial correlation properties of the asset's returns. Table 1 presents some evidence indicating that returns on bills become more risky relative to stocks the longer the holding period.[2]

In this paper we investigate the question of whether optimal portfolios differ depending on the period for which they are held, with emphasis on the serial correlation properties of the asset returns. Early analysis of portfolios held for the short and long term focused on the effects of changes in the investor's *horizon* on the optimal portfolio (Mossin (1968), Samuelson (1969)). No systematic effects of the investment horizon on the optimal portfolio were found; indeed for the hyperbolic absolute risk aversion class of utility functions (which includes those with constant relative risk aversion) optimal portfolios were shown to be invariant to the length of horizon when asset returns are identically and independently distributed over time. Individuals with the

[2]Table 1 is updated from Fischer (1983).

same current wealth and same terminal utility of wealth function, belonging to the specified class of utility functions, would hold the same portfolios whether they were looking ahead one year or twenty.

In these analyses, all investors are assumed to have the same portfolio holding period, or interval of time between portfolio actions: they all rebalance their portfolios once a year, or monthly, or continuously. Goldman (1979) showed that changes in the portfolio holding period have systematic effects on portfolio composition. Working with utility functions of constant relative risk aversion, and with asset returns generated by diffusion processes, he proved that portfolios tend systematically to become less diversified as the holding period lengthens.[3] Goldman's contribution not only establishes that portfolio behavior differs depending on the length of time for which assets are held, but also shows that the key consideration is the portfolio holding period rather than the investor's horizon.

In this paper we therefore examine the effects of changes in the portfolio holding period on the optimal portfolio when asset returns are serially correlated. For the returns processes studied, the expected returns on assets change over time, thus changing investors' opportunity sets. The hedging terms made familiar from Merton's (1973) analysis of the effects of a changing opportunity set on portfolio demands therefore appear in portfolio behavior in our analysis as well. By using constant relative risk aversion utility functions we ensure that the investment horizon has no effect on the optimal portfolio.

The analytic results show that serial correlation of asset returns can have substantial effects on portfolio composition as the holding period changes, and can significantly change the nature of the results obtained by Goldman. Using

[3] That there must be some effect of the holding period on the optimal portfolio is implied by the fact that continuous time optimal portfolios differ from corresponding discrete time optimal portfolios.

aggregate nominal data we find little serial correlation of either stock or bill returns, though bill returns display substantially higher serial correlation when real data is used. Our calculated optimal portfolios turn out to show little sensitivity to the length of the holding period. We find also that hedging effects on portfolios are small. This is encouraging news for the use of the simple one-period CAPM as a good approximation for optimal pricing and portfolio decisions.

I. Preliminaries

1. The Dynamics of Asset Returns

There are two assets, at least one and perhaps both earning uncertain returns. Returns on the assets are defined by diffusion processes for the change in the asset's value:

$$(3)\quad \frac{dx_i}{x_i} = \alpha_i(t)dt + s_i dz_i \qquad i = 1,2$$

The expected rate of return per unit time, $\alpha_i(t)$, may itself follow a diffusion process of the type

$$(4)\quad d\alpha_i(t) = b_i(a_i - \alpha_i(t))dt + s_{i+2}dz_{i+2} \qquad i = 1,2$$

This is essentially equivalent to a discrete time first-order autoregressive process as can be seen in equation (7) below. Coefficients of correlation between variables dz_i and dz_j are denoted ρ_{ij}.

Asset 2 will be described as the (relatively) safe asset, or bonds, or bills. Asset 1 is stocks. Since all the results of interest can be obtained if only bill returns have a changing expected return, we assume henceforth that stock returns are identically distributed over time, with $\alpha_1(t) \equiv a_1$ and $s_3 = 0$.[4] The cumulation of bill returns in a portfolio held for a long period should be thought of as resulting from the continual rolling over of such a portfolio, as for instance in a money market mutual fund held for a period of years.

Given (3) and (4), the natural logarithms of $x_i(t)$ are normally distributed with[5]

$$(5)\quad E_0 \left[\ell n \frac{x_i(t)}{x_i(0)}\right] = (a_i - \frac{s_i^2}{2})\, t - (\frac{a_i - \alpha_i(0)}{b_i})\,(1 - e^{-b_i t}) \quad i = 1,2$$

(The second term is identically zero in the case of stocks under the assumption $s_3 = 0$.)

$$(6)\quad \mathrm{Var}\ (\ell n \frac{x_1(t)}{x_1(0)}) = s_1^2 t$$

[4]Enough information is provided for the reader to work out the more general formulation in which $s_3 \neq 0$. In Section III we briefly present optimal portfolios for the case where the expected return on stocks is equal to the short rate plus a constant risk premium. In this case $s_3 = s_4$.

[5]If $s_3 \neq 0$, then the covariance expression corresponding to (8) is:

$$\mathrm{cov}\left[\ell n \frac{x_1(t)}{x_1(0)}, \ell n \frac{x_2(t)}{x_2(0)}\right] = \rho_{12} s_1 s_2 t + \rho_{14} \frac{s_1 s_4}{b_2}\left[t - \frac{1-e^{-b_2 t}}{b_2}\right]$$

$$+ \rho_{23} \frac{s_2 s_3}{b_1}\left[t - \frac{1-e^{-b_2 t}}{b_1}\right]$$

$$+ \rho_{34} \frac{s_3 s_4}{b_1 b_2}\left[t - \frac{1-e^{-b_1 t}}{b_1} - \frac{1-e^{-b_2 t}}{b_2} + \frac{1-e^{-(b_1+b_2)t}}{b_1 + b_2}\right]$$

$$(7)\quad \text{Var}\ \left(\ln \frac{x_2(t)}{x_2(0)}\right) = s_2^2 t + \frac{s_4^2}{b_2^3}\ [b_2 t - 2(1-e^{-b_2 t}) + \frac{1}{2}(1-e^{-2b_2 t})]$$

$$+ \frac{2\rho_{24}s_2 s_4}{b_2^2}\ [b_2 t - (1-e^{-b_2 t})]$$

$$(8)\quad \text{Cov}\ \left(\ln \frac{x_1(t)}{x_1(0)}, \ln \frac{x_2(t)}{x_2(0)}\right) = \rho_{12}s_1 s_2 t + \frac{\rho_{14}s_1 s_4}{b_2^2}\ [b_2 t - (1-e^{-b_2 t})]$$

The changing relative riskiness of asset returns as it depends on the length of the period the assets are held can be seen using (6) and (7). In particular,

$$(9)\quad \lim_{\Delta t \to 0} \frac{\text{Var}\ [\ln \frac{x_1(\Delta t)}{x_1(0)}]}{\text{Var}\ [\ln \frac{x_2(\Delta t)}{x_2(0)}]} = \frac{s_1^2}{s_2^2}$$

Thus instantaneously the variance ratio is the ratio of the variances in equation (3). However,

$$(10)\quad \lim_{t \to \infty} \frac{\text{Var}\ [\ln \frac{x_1(t)}{x_1(0)}]}{\text{Var}\ [\ln \frac{x_2(t)}{x_2(0)}]} = \frac{s_1^2}{s_2^2 + \frac{2\rho_{24}s_2 s_4}{b_2} + \frac{s_4^2}{b_2^2}}$$

Asymptotically the variance ratio is given by (10).

The important point of (9) and (10) is that two assets may have very different relative risk characteristics depending on how long they are held. The change in relative riskiness between (9) and (10) depends on the sign of $(b_2\rho_{24}s_2+s_4)$. If this expression is positive, bills are more risky relative to stocks in the long run than in the short run.

A special case for which portfolio behavior will be examined below occurs when $s_2=0$, so that the instantaneous return on bills is known with certainty. However, with $s_4 \neq 0$, there is still uncertainty about returns on bills held for any finite period. For any given value of s_4, the bills are more risky the smaller the absolute value of b_2. For $b_2=0$, the expected real return on bills follows a random walk and the asymptotic variance ratio is zero.[6]

2. The Optimization Problem.

The individual maximizes the expected utility of terminal wealth, W_T, where the utility function is isoelastic:

$$(11)\quad U(W_T) = \frac{W_T^{\beta}}{\beta} \qquad \beta<1$$

with $\beta=0$ corresponding to the logarthmic utility function. This class of utility functions has the important property that the derived utility function in a dynamic optimization with portfolio rebalancing also belongs to this class, with the same β.[7]

The problem to be solved is now the same as that of Goldman (1979), expect for the different behavior of asset returns. For any given length of holding period, the investor will maximize the expectation of the derived utility of wealth function at the end of the holding period. In this isoelastic case, the derived utility function will be isoelastic with coefficient β. Thus ignoring inessential constants, in all cases the portfolio holder faces the problem

$$(12)\quad \underset{\{w\}}{\text{Max}}\ E[(we^{x_1} + (1-w)e^{x_2})^{\beta}/\beta] = E[(we^{x_1-x_2} + (1-w))^{\beta}e^{\beta x_2}/\beta]$$

[6]Nelson and Schwert (1977), Garbade and Wachtel (1978), and Fama and Gibbons (1982) fail to reject the hypothesis that the real interest rate follows a random walk. However, it is not credible that the real interest rate follow such a process, which implies the real rate is unbounded above and below.

[7]The inclusion of consumption possibilities up to time T does not affect results so long as the utility of consumption function is also isoelastic with coefficient β.

where w is the portfolio share in stocks and x_1 and x_2 are normally distributed:

(13) $x_1 \sim N(\mu_1, \sigma_1^2)$

$x_2 \sim N(\mu_2, \sigma_2^2)$

$\mathrm{cov}(x_1, x_2) \equiv \sigma_{12}$

It is useful to define

(14) $y \equiv x_1 - x_2 \sim N(\mu, \sigma^2)$

and then note[8]

(15) $$E_{x_1, x_2}[(we^{x_1-x_2} + 1-w)^\beta e^{\beta x_2}/\beta] = E_y[(we^y + 1-w)^\beta E_{x_2|y}(e^{\beta x_2})/\beta]$$

$$= \frac{1}{\beta}\int_{-\infty}^{\infty} (we^y + 1-w)^\beta e^{-(y-(\mu+\gamma))^2/2\sigma^2} dy$$

where

$\gamma \equiv \beta(\sigma_{21} - \sigma_2^2)$

This leads to the first-order condition

(16) $$0 = \int_{-\infty}^{\infty} (we^y + 1-w)^{\beta-1}(e^y-1)e^{-(y-(\mu+\gamma))^2/2\sigma^2} dy$$

$$= F(w; \mu, \gamma, \sigma^2, \ldots)$$

Concavity guarantees satisfaction of the second-order condition.

Equation (16) can now be used to study the effects on optimal portfoio composition of changes in parameters, including the investment horizon.

3. The Goldman Results.

Goldman analyzes the case in which $s_4=0$ so that both asset returns are

[8]Equation (15) is used by Goldman (1979) as his canonical form. Multiplicative constants are omitted or ignored where no damage results.

serially uncorrelated. In this case:

$$\mu = [a_1 - \frac{s_1^2}{2} - (a_2 - \frac{s_2^2}{2})]\tau$$

(17) $$\sigma^2 = [s_1^2 - 2s_{12} + s_2^2]\tau$$

$$\gamma = \beta[s_{12} - s_2^2]\tau$$

The first and second moments in (17) are in this case proportional to τ, the length of the holding period.

The focus of the analysis is the effects of the length of the holding period on the composition of the portfolio. In particular, does the lengthening of the holding period affect the composition of the portfolio? And second, what is the asymptotic behavior of the portfolio, where the investor has to choose at time zero a portfolio composition that is not subsequently revised (all returns are reinvested in the asset that generates them) and the holding period goes to infinity?

The portfolio composition from the standard Merton continuous time, continuous revision formulation plays an important role in the analysis. Denote by w(0) the portfolio that is optimal when the portfolio is revised continuously. Then

(18) $$w(0) = \frac{a_1 - a_2}{(1-\beta)\sigma^2(0)} + \frac{s_2^2 - s_{12}}{\sigma^2(0)}$$

where

$$\sigma^2(0) \equiv s_1^2 - 2s_{12} + s_2^2$$

As is well known, the portfolio demand can be decomposed into an excess return term (the first term on the right hand side of (18)) and a term that is the share of equity in the minimum variance portfolio (the second term on the right hand side of (18)).

The role played by $w(0)$ in the anal[illegible]is results from the presence of the term $(\mu+\gamma)/\sigma^2$ in (16). It can be shown that

$$(19) \quad \frac{\mu+\gamma}{\sigma^2} = (1-\beta)w(0) - \frac{1}{2}$$

Denoting by $w(\tau)$ the optimal portfolio when the holding period is of length τ, Goldman's results can be stated as

<u>G1</u> If $w(0) < 0$, $w(\tau) = 0$ for all τ

$w(0) > 1$, $w(\tau) = 1$ for all τ

$w(0) = 1/2$, $w(\tau) = 1/2$ for all τ.

<u>G2</u> If $0 < w(0) < 1/2$, then (i) $\frac{dw(\tau)}{d\tau} < 0$ and (ii) $0 < w(\tau) < 1/2$

$1/2 < w(0) < 1$, then (i) $\frac{dw(\tau)}{d\tau} > 0$ and (ii) $1/2 < w(\tau) < 1$.

These results state that as the holding period lengthens, the portfolio becomes less balanced. If the share of stocks is originally positive but less than one half, it tends to fall. If the share of stocks is originally greater than one half, the share tends to rise. There is in general antidiversification as the holding period lengthens.

<u>G3</u> If $1/2 - \max\left(0, \frac{-\beta}{2(1-\beta)}\right) < w(0) < 1/2 + \max\left(0, \frac{-\beta}{2(1-\beta)}\right)$

then $w(\infty) = 1/2 - \frac{1-\beta}{\beta}[w(0) - 1/2]$,

otherwise $w(\infty) = 0$ or 1.

Result G3 is that the portfolio tends asymptotically to plunge to complete specialization unless the original portfolio composition is nearly balanced. In that case, as Goldman explains, the distance from 1/2 is magnified by a factor of

$(1-\beta)/\beta$. For $\beta > 0$ (utility functions with less risk aversion than the logarithm) all portfolios plunge asymptotically unless $w(0) = 1/2$. As β tends to minus infinity, the portfolio tends to stay frozen at its original composition, which is the variance minimizing portfolio.[9]

The Goldman results are illustrated in Figure 1, which shows the asymptotic share of stocks, $w(\infty)$, as a function of $w(0)$ defined in (18). In Figure 1 it is assumed that $\beta < 0$; for $\beta > 0$, the $ww(\infty)$ locus becomes vertical at $w(0) = 1/2$, and then horizontal at $w(\infty)=1$. Proposition G2 asserts that the movement away from $w(0)$ to $w(\infty)$ is monotone with the length of the holding period.

4. The Effects of a Change in the Variance of Bill Returns.

As a preliminary to examining the effects of serial correlation of bill returns on the sensitivity of the portfolio to the length of the holding period, it is useful to analyze the effects of a change in the variance of bill returns on the optimal portfolio, $w(0)$, for the continuous portfolio rebalancing, no serial correlation problem.

From (18):

$$(20)\quad \frac{\partial w(0)}{\partial s_2} = -\frac{w(0)}{\sigma^2(0)}\frac{\partial \sigma^2(0)}{\partial s_2} + \frac{2s_2-\rho_{12}s_1}{\sigma^2(0)}$$

---

[9] G2 and G3 can equivalently be stated as:

G2': If $\mu + \gamma + \frac{\beta\sigma^2}{2} < 0$, then $\frac{\partial w(\tau)}{\partial \tau} < 0$ and $0 < w(\tau) < 1/2$

$\mu + \gamma + \frac{\beta\sigma^2}{2} > 0$, then $\frac{\partial w(\tau)}{\partial \tau} > 0$ and $1/2 < w(\tau) < 1$

G3': If $\beta < 0$ and $0 < -\frac{\mu+\gamma}{\beta\sigma^2} < 1$, then $w(\infty) = -\frac{\mu+\gamma}{\beta\sigma^2}$

Otherwise $w(\infty) = 0$ or 1.

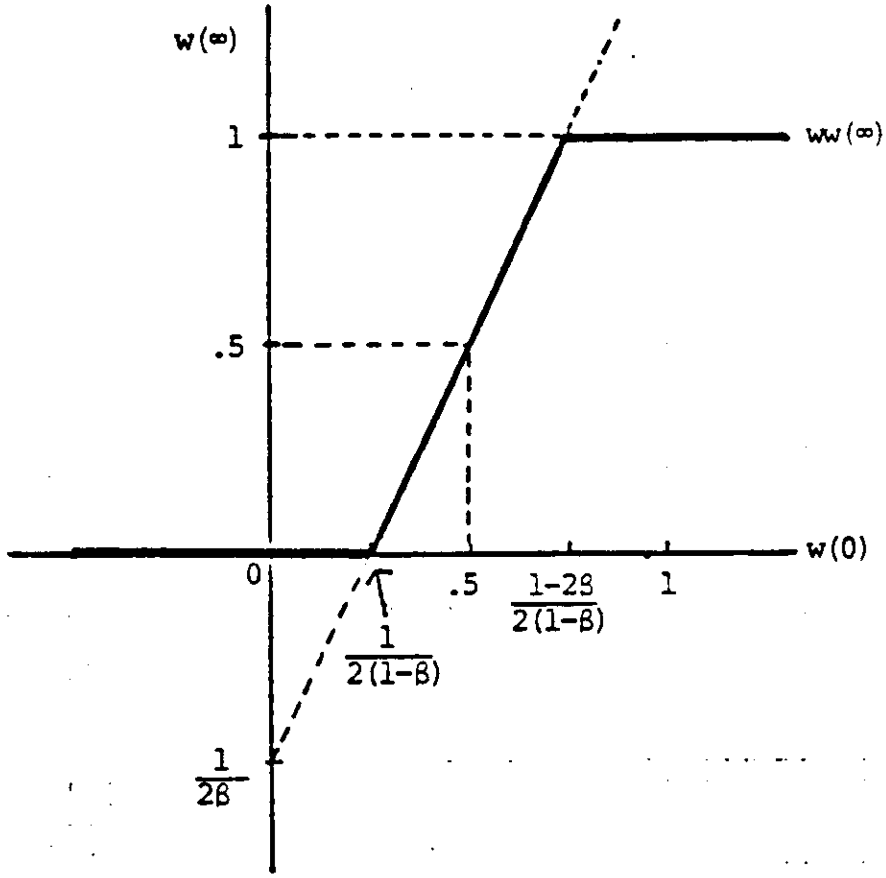

Figure 1: Asymptotic ($t \to \infty$) Portfolio Share of Stocks (for $\beta < 0$)

$$= \frac{2s_2(1-w(0))}{\sigma^2(0)} + \frac{\rho_{12}s_1(2w(0)-1)}{\sigma^2(0)}$$

The presumption would be that an increase in the variance of bill returns would, in this setting, increase the share of stocks in the portfolio. Provided that portfolio returns are uncorrelated ($\rho_{12}=0$), that is what happens so long as $0<w(0)<1$.

However, when $\rho_{12}\neq 0$, it is possible that an increase in the variance of bill returns reduces the share of stocks in the optimal portfolio. To focus on the ρ_{12} term in the second line of (20), assume that $s_2=0$ so that we are considering the effects of a change from bills being a perfectly safe to a slightly risky asset.

The effect of the change in s_2 then depends on the sign of $\rho_{12}(2w(0)-1)$. If $\rho_{12}>0$, then the individual will move away from stocks towards bills when the riskiness of bills rises, if $w(0)<1/2$. The reason is that an increase in s_2 increases the riskiness of the portfolio. Someone for whom $w(0)<1/2$ is sufficiently concerned about risk to be primarily in bills to begin with. When the portfolio becomes more risky he seeks the shelter of the relatively safer asset, which in this case is bills.

If $w(0)>1/2$, then the individual's response to the increase in the riskiness of bills and the existing portfolio is to move toward stocks. He thus moves in the direction of the more risky asset: his tendency to take a relatively risky position has already been signalled by the fact that his portfolio is predominantly in stocks.

If $\rho_{12}<0$, an increase in the riskiness of bills reduces the overall riskiness of the portfolio and responses are accordingly the reverse of those described in the preceeding two paragraphs. Thus in response to an increase in the riskiness of bills an individual may actually move his portfolio into bills. The direction of response depends on the factors set out in (20).

II. Asymptotic Portfolio Behavior When Returns are Serially Correlated.

We now examine the effects of increases in the length of the holding period on the optimal portfolio when bill returns are serially correlated. To avoid unnecessary complexity, s_2 is set equal to zero, implying that bill returns for the next instant are known with certainty. However, $s_4>0$, implying that future interest rates are not known with certainty.[10]

Referring back now to (13), (14), and (15), we have in this case

$$(21)\quad \mu = [a_1 - a_2 - \frac{s_1^2}{2}]\tau$$

$$\sigma^2 = \sigma_1^2 - 2\sigma_{14} + \sigma_4^2 = s_1^2\tau - 2\frac{s_{14}}{b_2}[\tau - \frac{1-e^{-b_2\tau}}{b_2}]$$

$$+ \frac{s_4^2}{b_2^3}[b_2\tau - 2(1-e^{-b_2\tau}) + \frac{(1-e^{-2b_2\tau})}{2}]$$

$$\gamma = \beta(\sigma_{14} - \sigma_4^2)$$

where σ_{14} and σ_4^2 are defined implicitly in the expression for σ^2, where it is assumed that $\alpha_2(0) = a_2$, and where it is understood that μ, σ^2, etc. are functions of the length of the holding period, τ.

Asymptotic portfolio behavior obtains as τ, the holding period, goes to infinity. The share of stocks in this portfolio is denoted $\hat{w}(\infty)$; the "$\wedge$" indicates that now there is serial correlation of asset returns. It is convenient to define

---

[10] These are the assumptions made by Merton (1973) in his examination of the effects of a changing opportunity set on CAPM.

(22) $\bar{\mu} = \mu/\tau = a_1 - a_2 - \frac{s_1^2}{2}$; $\bar{\sigma}_1^2 = \sigma_1^2/\tau = s_1^2$;

$$\bar{\sigma}_{14} = \lim_{\tau\to\infty} \frac{\sigma_{14}(\tau)}{\tau} = \frac{s_{14}}{b_2} \; ; \; \bar{\sigma}_4^2 = \lim_{\tau\to\infty} \frac{\sigma_4^2(\tau)}{\tau} = \frac{s_4^2}{b_2^2}$$

and

$$h = \bar{\sigma}_1^2 - \bar{\sigma}_{14} \; ; \; k = \bar{\sigma}_4^2 - \bar{\sigma}_{14}$$

Restating G3 in the form G3' of footnote 9:

G3': If $\beta < 0$ and $0 < -\frac{\mu+\gamma}{\beta\sigma^2} < 1$, then $\hat{w}(\infty) = -\frac{\mu+\gamma}{\beta\sigma^2}$

Otherwise $\hat{w}(\infty) = 0$ or 1.

Equivalently

(23) For $\beta < 0$:

if $\beta k < \bar{\mu} < -\beta h$, then $\hat{w}(\infty) = -\frac{\bar{\mu}-\beta k}{\beta(h+k)}$

if $\bar{\mu} < \beta k$, $\hat{w}(\infty) = 0$

$\bar{\mu} > -\beta h$, $\hat{w}(\infty) = 1$

We now take up in turn the analysis of asymptotic portfolios

(a) for $\bar{\sigma}_{14} = 0$, and (b) when $\bar{\sigma}_{14} \neq 0$

(a) When $\bar{\sigma}_{14} = 0$, there is zero correlation between changes in the expected return on bills and the return on stocks. Again working with w(0) from (18), we have

(23) $\hat{w}(\infty) = \dfrac{w(0)[(1-\beta)s_1^2] - \frac{s_1^2}{2} - \beta\bar{\sigma}_4^2}{-\beta[s_1^2 + \bar{\sigma}_4^2]}$ if $0 < w(\infty) < 1$, $\beta < 0$, and $\bar{\sigma}_{14} = 0$.

The $\hat{w}\hat{w}(\infty)$ schedule in Figure 2 describes the relationship (23). The schedule $ww(\infty)$ from Figure 1 is included for comparison. The effects of uncertainty about bill returns are reflected in the $\hat{w}\hat{w}(\infty)$ schedule lying above the $\hat{w}\hat{w}(\infty)$ schedule for all $w(0)$ for which the asymptotic portfolio is diversified when bill returns are certain.

As drawn, the $\hat{w}\hat{w}(\infty)$ schedule intercepts the vertical axis at positive $\hat{w}(\infty)$. The condition that produces the relationship shown in Figure 2 is that $\frac{s_1^2}{2} + \beta\bar{\sigma}_4^2 < 0$. This will happen either if the individual is very risk averse $((-\beta)$large) or if the variance of bill returns is high. As drawn, an individual who chooses to short stocks $(w(0) < 0)$ when the portfolio is instantly adjustable may hold positive amounts of stocks if the holding period is very long.

As risk aversion increases, we find

$$(24)\quad \lim_{-\beta\to\infty} w(\infty) = \frac{\bar{\sigma}_4^2}{s_1^2 + \bar{\sigma}_4^2}$$

which is just the variance minimizing share of stocks. Such an individual would have $w(0) > 0$ since instantaneously bills are riskless.

For $\beta > 0$, portfolios plunge asymptotically. Modifying G2' of footnote 9, we have that portfolios plunge to stocks if

$$(25)\quad \bar{\mu} - \beta k + \frac{\beta\bar{\sigma}^2}{2} > 0$$

where $\bar{\sigma}^2 = \lim_{t\to\infty} \frac{\sigma^2(\tau)}{\tau}$

Equivalently, for $\beta > 0$

$$(25)'\quad \hat{w}(\infty) = 1 \text{ if } \quad \bar{\mu} > (\beta/2)\,(\bar{\sigma}_4^2 - s_1^2)$$

$$= 0 \text{ if } \quad \bar{\mu} < (\beta/2)\,(\bar{\sigma}_4^2 - s_1^2)$$

Condition (25)' appears paradoxical in that for $\beta > o$ a large $\bar{\sigma}_4^2$, or variance of bill returns, apparently leads to plunging in stocks and vice versa.

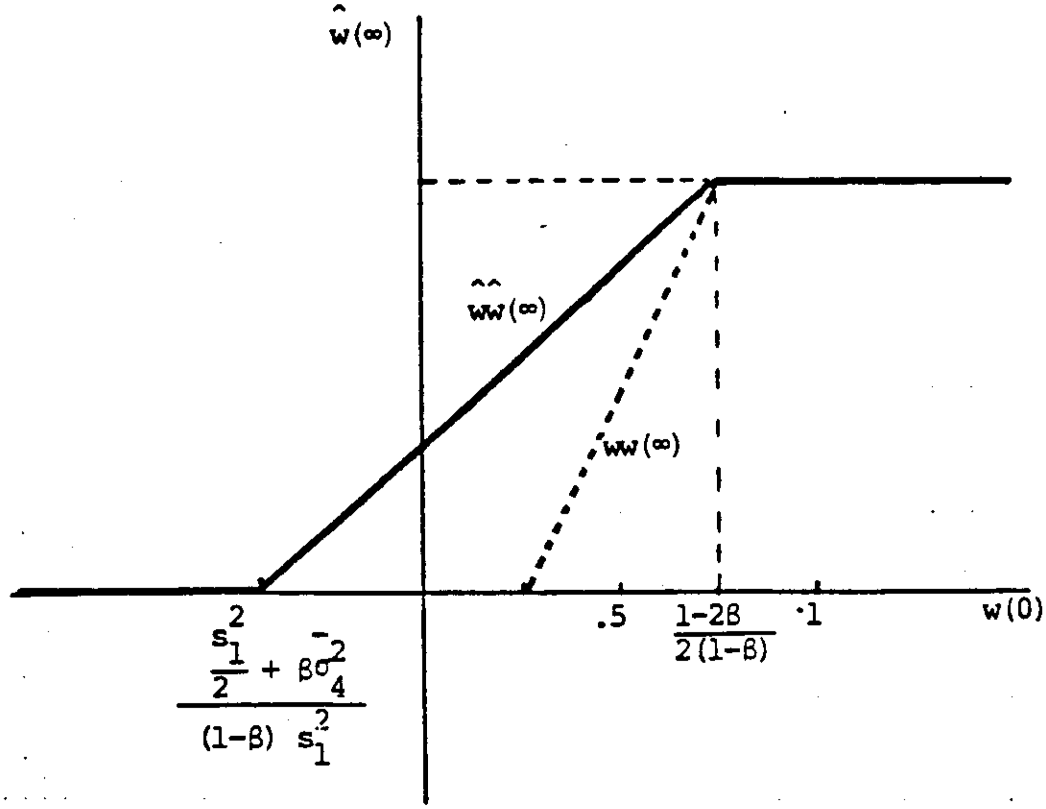

Figure 2: Asymptotic Portfolios When Bill Returns are Uncertain ($\bar{\rho}_{14}=0, \beta<0$)

However, note that the returns on stocks and bills are each log-normally distributed, implying that the expected excess return per period on stocks is

$$(26)\ \hat{\mu} = \bar{\mu} + \frac{s_1^2}{2} - \frac{\bar{\sigma}_4^2}{2}$$

Then (25)' can be rewritten as:

$$(25)''\ \text{For } \beta > 0: \quad \hat{w}(\infty) = 1 \text{ if } \mu > \frac{1-\beta}{2}(s_1^2 - \bar{\sigma}_4^2)$$

$$\hat{w}(\infty) = 0 \text{ if } \mu < \frac{1-\beta}{2}(s_1^2 - \bar{\sigma}_4^2)$$

Thus, holding the expected excess return on stocks constant, a smaller variance of stock returns or larger variance of bill returns tends to lead to plunging in stocks.

(b) When there is zero correlation between changes in the interest rate and stock returns, the effects of serial correlation of bill returns on asymptotic portfolios are almost entirely as expected. The serial correlation of bill returns makes bills a risky asset for the long term and tends to reduce their share in the optimal portfolio. Because asset returns are log-normal, though, an increase in the variance of bill returns increases the expected return on bills. When the individual is not very risk averse ($\beta > 0$), an increase in the variance of bill returns without any other parameter changing may increase the share of bills in the portfolio (from zero to one). However, if the expected return on bills is held constant through an offsetting change in α_2, then as (25)" shows, an increase in uncertainty about bill returns will not increase the share of bills.

Once correlation of bill and stock returns is introduced, some of the simplicity disappears. We start again with $\beta < 0$ and allow for $\bar{\sigma}_{14} \neq 0$ in

$$(23)' \quad \hat{w}(\infty) = \frac{w(0)[(1-\beta)s_1^2]\frac{s_1^2}{2} + \beta\bar{\sigma}_{14} - \beta\bar{\sigma}_4^2}{-\beta[s_1^2 - 2\bar{\sigma}_{14} + \bar{\sigma}_4^2]} \quad \text{if} \quad 0 < \hat{w}(\infty) < 1 \text{ and } \beta < 0$$

Now the relationship between $\hat{w}(\infty)$ and $w(0)$[11] depends on the sign of $\bar{\sigma}_{14}$. Figure 3(a) and 3(b) show the two possibilities. In both cases the schedules $\hat{w}\hat{w}(\infty)$ and $ww(\infty)$ from Figure 2 are included for comparison. When $\bar{\sigma}_{14} > 0$, the schedule $\bar{w}\bar{w}(\infty)$ (schedule for $\bar{\sigma}_{14} \neq 0$) is steeper than $ww(\infty)$, and there is a smaller range of $w(0)$ for which portfolios are diversified asymptotically (compared with Figure 2).

Comparing $\bar{w}\bar{w}(\infty)$ in Figure 3(a) with $ww(\infty)$, it is certainly the case that the portfolio plunges to stocks on $\bar{w}\bar{w}(\infty)$ for values of $w(0)$ below $(1-2\beta)/2(1-\beta)$, the critical value on $ww(\infty)$. For $(\sigma_4^2 - \sigma_{14}) < 0$, it is also true that the portfolio plunges to bills on $\bar{w}\bar{w}(\infty)$ for values of $w(0)$ above $1/2(1-\beta)$, the critical value on $ww(\infty)$.

Thus the effect of uncertainty about bill returns is certainly to drive portfolios that are predominantly in stocks further to stocks. However, when $(\bar{\sigma}_4^2 - \bar{\sigma}_{14}) < 0$, portfolios with small holdings of stocks may be driven further into bills as the variance of bill returns (or serial correlation of bill returns) increases. The explanation for this latter result is that when $\bar{\sigma}_4^2 - \bar{\sigma}_{14} < 0$, the addition of bills to the portfolio substantially reduces the

[11]Note we continue to use $w(0)$ from equation (18) as the comparison portfolio. However, when $\bar{\sigma}_{14} \neq 0$, it is no longer true that (18) gives the portfolio that would be held if there were continuous rebalancing. Rather, with $\bar{\sigma}_{14} \neq 0$, there is an additional hedging term in the demand function for the portfolio $\hat{w}(0)$, where "$\wedge$" indicates the presence of serial correlation of bill returns. (See Merton (1973)). For $\sigma_{14} = 0$, $\hat{w}(0) = w(0)$.

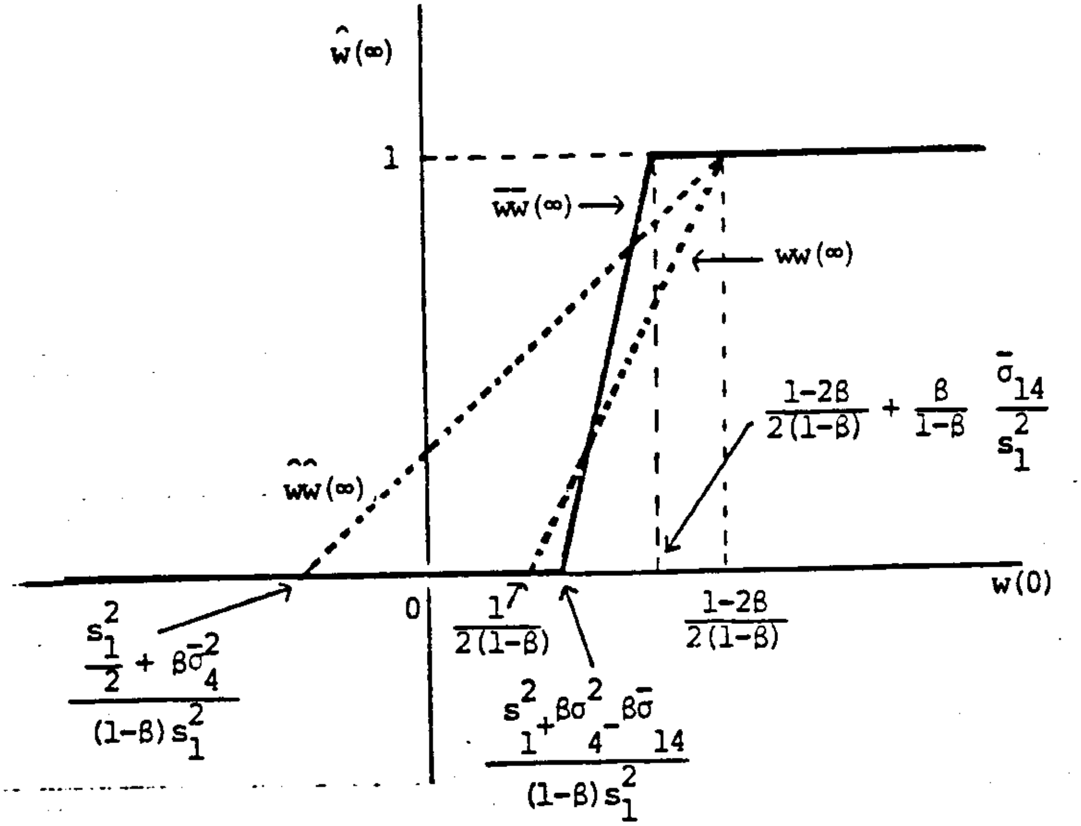

Figure 3(a): Asymptotic Portfolio With Uncertain Bill Returns ($\bar{\sigma}_{14} > 0$, $\beta < 0$)

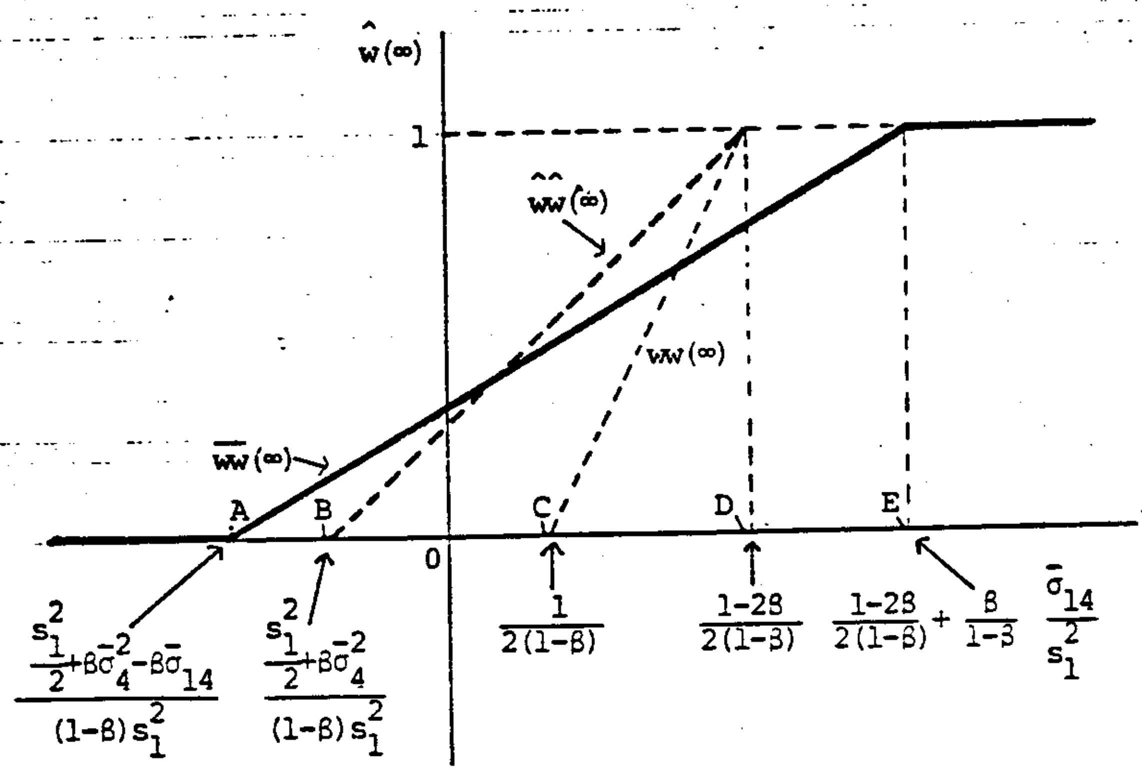

Figure 3(b): Asymptotic Portfolios With Uncertain Bill Returns ($\bar{\sigma}_{14} < 0$, $\beta < 0$)

overall riskiness of the portfolios. If $w(0)$ was low to begin with, then parameter values were such as to reflect relatively great concern about risk - which is reduced by adding bills to the portfolio.

Figure 3 (b) describes the asymptotic portfolio when $\bar{\sigma}_{14} < 0$, so that the expected return on bills is high when stock returns are low. This should be expected to lead to more diversification, which is precisely what it does. The relative positions of points A through E in Figure 3(b) are as shown. However B and A, or just B may be to the right of the origin. Further, point E is shown to the right of $w(0) = 1$, but it may be to the left.

The range between points A and E depends in part on the degree of risk aversion. For $(-\beta)$ large, A will be at a negative value of $w(0)$ and E will be at a value of $w(0)$ in excess of unity. In such a case the asymptotic portfolios are <u>more</u> diversified than $w(0)$. This is the opposite of Goldman's result. Similarly, the larger in absolute value is $\bar{\sigma}_{14}$, (for $\bar{\sigma}_{14} < 0$) the more likely is the asymptotic portfolio to be diversified for parameter values for which the portfolio $w(0)$ is not diversified.[12]

Turning again to asymptotic portfolios for $\beta > 0$, it turns out that the discussion of part (a) above applies exactly. Since these portfolios all plunge, diversification and covariance are not relevant here.

---

[12] The question arises of whether this result occurs because $w(0)$ and not $\hat{w}(0)$ is being used for comparison. The portfolio $\hat{w}(0)$, optimal under the asset dynamics of this section and with continuous rebalancing is:

$$w(0) = w(0) + \frac{K_2\bar{\sigma}_{14}}{(1-\beta)s_1^2K}$$

K_2/K is a "hedging coefficient", of the same sign as β. Thus for $\beta < 0$ and $\bar{\sigma}_{14} < 0$, $\hat{w}(0) > w(0)$. The values of $\hat{w}(\infty)$ for $w(0) < 0$ therefore certainly reflect portfolio diversification that would not occur even if the comparison were $w(0)$, but values of $\hat{w}(\infty) < 1$ for $w(0) > 1$ may not reflect diversification in excess of that occurring in the continued rebalancing problem.

(c) Figures 2 and 3, together with (25)" summarize the effects of uncertainty about bill returns on the asymptotic portfolio. The results are that increased uncertainty about bill returns increases the share of stocks when $\bar{\sigma}_{14} = 0$; that when $\bar{\sigma}_{14} > 0$, asymptotic portfolio diversification occurs over a more restricted range of $w(0)$; and that for $\bar{\sigma}_{14} < 0$, the uncertainty about bill returns over the long term increases the share of stocks for $w(0)$ small and decreases the share of stocks for $w(0)$ large.

For $\beta < 0$ and non-plunging portfolios, we can also summarize the above discussion by using (23) to study the effects of an increase in serial correlation, or reduction in b_2, on the asymptotic portfolio. Since b_2 always enters in the form s_4/b_2, we can consider the effects of an increase in serial correlation on the optimal portfolio by calculating $\frac{\partial \hat{w}(\infty)}{\partial s_4}$. From (23):

$$\bar{\mu} = (1-\hat{w}(\infty))\beta k - \hat{w}(\infty)\beta h$$

Thus

$$(27)\quad \frac{\partial \hat{w}(\infty)}{\partial s_4} = \frac{1}{b_2^2(h+k)}\left[2s_4(1-\hat{w}(\infty)) - \rho_{14}s_1 b_2(1-2\hat{w}(\infty))\right]$$

Comparing (27) with (20), we note that the effect of an increase in the serial correlation of bill returns (equivalently an increase in the variance of bill returns) on the asymptotic share of stocks is precisely the same as that of an increase in the variance of bill returns on the share of stocks in the portfolio problem with continuous rebalancing. Thus the ambiguities discussed following (20) apply here too: for $\rho_{14} = 0$, an increase in the variance of bill returns unabmiguously increases the share of stocks, (as in Figure 2) but for $\rho_{14} \neq 0$ it is quite possible that an increase in the variance of bill returns increases the share of bills. This would happen for instance, if $\rho_{14} > 0$ (Figure

3a) if $s_4 = 0$, and the portfolio is predominantly in bills to begin with, so $1-2\hat{w}(\infty) > 0$. Alternatively, if $\rho_{14} < 0$ (Figure 3b) and $\hat{w}(\infty)$ is large, an increase in s_4 may increase the share of stocks. Once more the simplicity of the Goldman results is lost.

III. Portfolio Behavior with Finite Revision Time Portfolios and Estimated Return Processes

In general, a closed form solution for the optimal portfolio weight, $\hat{w}(\tau)$, cannot be obtained when the revision time, τ, is finite, i.e., $0 < \tau < \infty$, and asset returns are serially correlated. However, given values for the coefficient of relative risk aversion, β, and the parameters of the asset returns' stochastic processes, equations (3) and (4), a numerical solution for $\hat{w}(\tau)$ can be found using equation (16). The form of the integrand of equation (16) is well suited for applying a Gauss-Hermite quadrature formula. After computing a value for the integral in (16) for given $\hat{w}(\tau)$, we can then iterate over values of $\hat{w}(\tau)$ until one is found that satisfies equation (16).

In this section we present calculated optimal portfolios $\hat{w}(\tau)$, based on estimated processes for stock and bill returns. We started by estimating processes (3) and (4), using weekly data over the period January 1978 to December 1983, a total of 312 weeks. As shown by Marsh and Rosenfeld (1983), using returns with a weekly observation interval provides accurate estimates of the continuous time model parameters of equations (3) and (4) when a discrete time approximation is used in the estimation process. However, since a price index series is not available on a weekly basis, equations (3) and (4) are assumed to describe nominal returns. For the expected rate of return on asset 2, bills, the annualized yield on outstanding 91 day Treasury Bills with approximately one week to maturity is used.[13] Continuing to assume that $s_2 = 0$ enables us to estimate a discrete time approximation to equation (4) of the form;

$$d\alpha_2(t) = \alpha_2(t) - \alpha_2(t-1) = b_2(a_2 - \alpha_2(t-1)) + \varepsilon_{4t} \tag{28}$$

where ε_{4t} is distributed $N(0, s_4^2)$ and serially uncorrelated over time.

For asset 1, stocks, daily returns from the Standard and Poors 500 Composite Index were aggregated into weekly returns. Attempts to estimate equations (3) and (4) by time varying parameter methods with no parametric restrictions led to highly inaccurate and unreasonable estimates of the parameters of the mean reversion process (4).[14] The difficulty in estimating the parameters of the expected return process for common stocks stems from the large relative magnitude of the stock's variance.[15] The dilemma is essentially that of the signal extraction problem where the signal (expected return) is small relative to the noise (variance) and hence difficult to identify.

Given these difficulties, we used two alternative models of returns, each a special case of (3) and (4). They are:

Model 1: $\alpha_1 = a_1$, a constant, as assumed in the text of Section II.

Model 2: Stock returns are serially correlated such that $\alpha_1 = \alpha_2 + a_1 - a_2$, i.e., the expected return on stocks is equal to the short term interest rate plus a constant spread, which might be interpreted as a risk premium. Under this assumption, from equation (17);

$$\sigma^2 = s_1\tau \tag{29}$$

---

[13]Data on average bid-ask rates were collected from the Wall Street Journal each Thursday (Wednesday if Thursday was a holiday) on Treasury bills with approximately one week to maturity.

[14]For example, the estimate of b_1 in equation (4) was -.2460 with an asymptotic standard error of .8082.

[15]See Merton (1980) on this issue.

$$\gamma = \beta \frac{s_{14}}{b_2^2}\left[b_2\tau - (1 - e^{-b_2\tau})\right]$$

and the asymptotic portfolio for this case is;

(30) $$\hat{w}(\infty) = \frac{w(0)(1 - \beta)\ s_1^2 - \frac{s_1^2}{2} + \beta \frac{s_{14}}{b_2}}{-\beta s_1^2}$$

if $0 < \hat{w}(\infty) < 1$ and $\beta < 0$. $w(0)$ is still given by equation (18). Note that while we assumed $s_3 = s_4 \neq 0$, the constant risk premium assumption implies s_3 does not enter the formula $\hat{w}(\infty)$ directly.

Model 1 Estimates: In this case, in which stock returns are assumed to be serially uncorrelated, the following estimates were obtained. Asymptotic standard errors are in parentheses.

$\hat{b}_2 = \underset{(.02046)}{.06926}$ $\hat{s}_4 = \underset{(.000437)}{.001545}$

$\hat{s}_1 = \underset{(.04432)}{.156643}$ $\hat{\rho}_{14} = \frac{\hat{s}_{14}}{\hat{s}_1\hat{s}_4} = \underset{(.05398)}{-.21586}$

The estimated value of b_2, 0.07, is sufficiently small (particularly for weekly data) that there is very little serial correlation of nominal bill returns. The covariance between changes in stock returns and the shifting mean of the interest rate is significantly negative. The standard deviation of the returns on stocks is one hundred times that of s_4; the shifts in the Treasury bill rate have very small (although statistically significant) variance.

Since the difference between the mean return on stocks and the long run expected return on bills, $a_1 - a_2$ can only be estimated with reasonable accuracy by using data over a long time period, an estimate of the spread was obtained from Ibbotson and Sinquefield (1982). The means of Standard and Poors stock returns and Treasury bill returns over the period 1926-1981 were $a_1 = .114$ and $a_2 = .031$, so we take the estimate of the spread to be .083.

Model 2 Estimates: For the alternative case in which stocks are serially correlated and $\alpha_1 = \alpha_2 + a_1 - a_2$ we have the following estimates ($\hat{b}_2$ and $\hat{s}_4$ are the same in the two models):

$$\hat{s}_1 = \underset{(.04433)}{.156667} \qquad \hat{\rho}_{14} = \frac{\hat{s}_{14}}{\hat{s}_1 \hat{s}_4} = \underset{(.05399)}{-.21513}$$

These are very similar to the Model 1 estimates of these same parameters.

Calculated Portfolios: We computed finite revision time portfolio weights using the point estimates for the returns processes of Models 1 and 2. Results for Model 1, with serially uncorrelated returns on stocks, are presented in Table 2 while Model 2 results appear in Table 3. (In the $w(x)$ expressions, x is measured in years.) The calculated portfolios are very similar in the two tables. The results are:

1. For coefficients of relative risk aversion of 0, -1 (the logarithmic utility function) and -2, the optimal portfolio weights are all equal to 1 and are not reported in the tables.[16]

2. The optimal portfolio weights change little with the length of the portfolio revision period. Most of the change occurs after $\tau = 10$ years.

---

[16] If bills and stocks were the only assets held in the portfolio, then we could estimate the coefficient of relative risk aversion by finding that value of β for which the calculated portfolio proportions were equal to the actual.

3. The Goldman plunging results are not applicable over the ranges seen in the tables. Indeed, with the serial correlation of bill returns, there are for high values of $-\beta$ movements towards diversification as the holding period lengthens. In all cases the $w(\tau)$ portfolio contains more stocks than the $w(0)$ portfolio. This is in accord with the Goldman anti-diversification results for values of β greater than -6; for $\beta < -6$, the Goldman anti-diversification and the greater relative riskiness of bills effect stressed in this paper work in the opposite direction. Asymptotically the increasing relative riskiness of bills effect dominates anti-diversification, but the change is not monotonic.

4. The difference between the $w(0)$ and $\hat{w}'(0)$ portfolios is zero to five places. This means that hedging effects on portfolio demands are negligible for the processes examined in this paper. That is not surprising given the small estimates of the serial correlation of bill returns. If such estimates are reliable, the one period CAPM provides a close approximation to the CAPM with a changing opportunity set.

An Alternative "Safe" Asset: We calculated optimal portfolios with one week bills replaced by 91 day bills. With asset 2 a 91 day bill, s_2 is not equal to zero. For this case equation (3) was estimated using weekly new issue auction yields on 91 day Treasury bills over the same period as before, January 1978 to December 1983. The expected return was assumed equal to a proportion of the short rate used previously in estimating (28) plus a constant. This assumption concerning the form of the stochastic process of Treasury bills is consistent with the term structure model of Vasicek (1977). The following point estimates were obtained for the case (Model 1) in which stock returns were assumed to be serially uncorrelated.

$\hat{s}_2 = \underset{(.0029211)}{.0103235}$ $\hat{\rho}_{12} = \underset{(.05419)}{.20655}$ $\hat{\rho}_{24} = \underset{(.04794)}{-.39137}$

The following alternative estimates were obtained for the case (Model 2) in which stock returns were assumed to be serially correlated.

$\hat{\rho}_{12} = \underset{(.05422)}{.20584}$ $\hat{\rho}_{14} = \underset{(.05399)}{-.21513}$

The model in which $s_2 \neq 0$ requires a straightforward modification of the formulas for μ, σ^2, and γ which can be made using equations (7) and (8).

We again calculated solutions for $\hat{w}(\tau)$ using equation (16). Table 4 presents the optimal stock holdings for the case in which stocks are assumed to be serially uncorrelated. Qualitatively, the results for this case for which the maturity of asset two has been extended are very similar to the results in Table 2 where asset two is the short (1 week) rate.

Essentially the same situation is found when stock returns are assumed to be serially correlated. The results in Table 5 give $\hat{w}(\tau)$ for serially correlated stock returns and asset two being 91 day bills. The monotonicity results of Goldman (1979) again do not hold. As in Tables 2 and 3 there is very little change in magnitude between $w(0)$ and $\hat{w}(\infty)$.

Estimation Using Real Asset Returns: Models 1 and 2 were re-estimated using real returns data and a monthly observation interval. Equations (3) and (4) are perhaps a more attractive returns generating process when returns are assumed to be in real terms rather than nominal. However, using real returns constrains us to use a monthly observation interval which may decrease the accuracy of the parameter estimates.

Treasury bill and common stock returns, deflated by the CPI, were obtained from the Ibbotson and Sinquefield bond file over the period 1926 to 1983, a total of 696 observations. Estimation of equation (4) for real bill returns yielded the following estimates.

$$\hat{b}_2 = \underset{(.03231)}{.47323} \qquad \hat{s}_4 = \underset{(.004053)}{.017501}$$

For the case in which real stock returns are serially uncorrelated (Model 1), the following estimates were computed.

$$\hat{s}_1 = \underset{(.04806)}{.20749} \qquad \hat{\rho}_{14} = \underset{(.03780)}{-.05872}$$

Similar estimates were obtained under the alternative assumption that real stock returns are correlated (Model 2).

$$\hat{s}_1 = \underset{(.04795)}{.20703} \qquad \hat{\rho}_{14} = \underset{(.03780)}{-.05889}$$

Using real monthly returns instead of nominal weekly returns results in bill returns having higher serial correlation ($\hat{b}_2 = .47323$), though the correlation between stocks and bills is smaller. Also, using a longer period (1926-83), we find that the standard deviations of real stock and bill returns, .207 and .018, respectively, are larger than the corresponding nominal stock and bill return standard deviation estimates over the 1978-83 period, .157 and .010.

Tables 6 and 7 give the optimal stock portfolio weight, $\hat{w}(\tau)$, for Model 1 and Model 2, respectively. The estimates in both of the Tables are quite similar. The larger estimated variance of stocks seems to have the effect of reducing the optimal portfolio weights compared to those estimated in Tables 2 and 5. However, there continues to be very little difference between the continuous revision portfolios for the non-serial correlation case, $w(0)$, and the serial correlation case, $\hat{w}(0)$. Also as in previous estimates, there is not generally monotonic anti-diversification as the revision period increases, though there continues to be little difference in optimal portfolio weights even out to a 10 year revision interval.

IV. Conclusions

The notion that portfolio behavior might depend on the length of time for which the portfolio is held is highly intuitive. Goldman (1979) showed that the relevant period is not the investor's horizon, but rather the portfolio revision period, the length of time for which the portfolio cannot be revised. Assuming serially uncorrelated asset returns Goldman proved an anti-diversification result, in which the portfolio becomes less diversified as the portfolio revision period increases.

When asset returns are serially correlated, the relative riskiness of assets is typically a function of the length of time for which they are held. We show in this paper how changes in the relative riskiness of assets interact with changes in the portfolio revision period to affect portfolios. The Goldman anti-diversification result no longer necessarily holds. Nor is the change in the portfolio any longer necessarily a monotonic function of the length of the portfolio revision period.

We estimated dynamic processes for bill and stock returns, and used them to calculate optimal portfolios as a function of the portfolio revision period. The most striking result was how little the portfolio proportions changed as the period lengthened. We did find in cases where the Goldman anti-diversification tendency conflicted with changing relative variances of asset returns, that the changing relative variances were asymptotically dominant.

Because the serial correlation of asset returns was estimated to be relatively low, there was very little difference between portfolios estimated with and without hedging demands. If our estimated processes are reasonably accurate, hedging demands and the errors made in assuming a one-period rather than multi-period CAPM are small.

It remains entirely possible that individual assets, like land and particular stocks, could display considerable serial correlation of returns despite the absence of significant serial correlation of asset returns at the aggregate level.

References

Fama, Eugene and Michael Gibbons (1982); "Inflation, Real Returns and Capital Investment", Journal of Monetary Economics, 9 (May): 297-324.

Fischer, Stanley (1983); "Investing for the Short and the Long Term", in Z. Bodie and J. Shoven (eds.); Financial Aspects of the United States Pension System, Chicago: University of Chicago Press.

Garbade, Kenneth and Paul Wachtel (1978); "Time Variation in the Relationship between Inflation and Interest Rates", Journal of Monetary Economics, 4 (November): 755-66.

Goldman, Barry M. (1979); "Anti-diversification, or Optimal Programs for Infrequently Revised Portfolios", Journal of Finance, 34 (May): 505-16.

Ibbotson, Roger G. and Rex A. Sinquefield, (1982), "Stocks, Bonds, Bills and Inflation: The Past and the Future," The Financial Analysts Research Foundation, Charlottesville, Virginia.

Klein, Benjamin (1976); "The Social Costs of the Recent Inflation: the Mirage of Steady 'Anticipated' Inflation", in Carnegie - Rochester Conference Series on Public Policy, Vol. 3, Institutional Arrangements and the Inflation Problem, Amsterdam: North-Holland.

Marsh, Terry A. and Eric R. Rosenfeld (1983); "Nontrading and Discrete Time Approximation Errors in the Estimation of Continuous Time Models in Finance", mimeo, MIT, November.

Merton, Robert C., (1973), "An Intertemporal Capital Asset Pricing Model," Econometrica, 41 (Sept.) 867-888.

________ (1980); "On Estimating the Expected Return on the Market", Journal of Financial Economics, 8: 323-61.

Mossin, J. (1968); "Optimal Multiperiod Portfolio Policies", Journal of Business 41: 215-29.

Nelson, Charles R. and William Schwert (1977); "On Testing the Hypothesis that the Real Rate of Interest is Constant", American Economic Review 67 (June): 478-86.

Samuelson, Paul A. (1969); "Portfolio Selection by Dynamic Stochastic Programming", Review of Economics and Statistics, 60 (August): 239-46.

Vasicek, Oldrich (1977) "An Equilibrium Characterization of the Term Structure," Journal of Financial Economics, 5, 117-188.

Table 1: Real Monthly Returns on Stocks and Bills

		Period					
		1926-1983			1948-1983		
		(1) Stocks	(2) Bills	(3) (1)/(2)	(1) Stocks	(2) Bills	(3) (1)/(2)
Mean return		.00687	.000145		.00656	.000340	
Variance of returns per month[(1)]							
Holding period in months:	1	.352	.00356	98.6	.163	.00119	137.0
	2	.411	.00553	74.3	.168	.00168	100.0
	4	.344	.00842	40.9	.162	.00284	57.0
	12	.372	.01712	21.7	.250	.00620	40.3
	60	(.259	.05023	5.2)	(.287	.00821	34.8)

Notes:

1. The variances should all be multiplied by .01.

2. Stock and bill returns are from the Ibbotson-Sinquefield File, Center for Research in Security Prices, University of Chicago. Real returns are calculated using the Consumer Price Index.

3. Parentheses in last row of table are a reminder that statistics are based on only eleven and seven data points respectively.

TABLE 2

Serially Correlated 1 Week Bills - Nominal Returns

Beta	w(0)	ŵ(0)	ŵ(.5)	ŵ(1)	ŵ(2)	ŵ(5)	ŵ(10)	ŵ(inf)
-3	0.84569	0.84569	0.84588	0.84607	0.84645	0.84645	0.84968	0.93547
-4	0.67655	0.67655	0.67684	0.67711	0.67765	0.67916	0.68140	0.71339
-5	0.56380	0.56380	0.56392	0.56404	0.56429	0.56503	0.56623	0.58014
-6	0.48325	0.48325	0.48322	0.48320	0.48319	0.48328	0.48373	0.49130
-7	0.42285	0.42285	0.42271	0.42258	0.42238	0.42206	0.42209	0.42785
-8	0.37586	0.37586	0.37565	0.37546	0.37515	0.37458	0.37443	0.38026
-9	0.3[illegible]828	0.33828	0.33802	0.33779	0.33741	0.33673	0.33653	0.34324
-10	0.3[illegible]52	0.30752	0.30724	0.30700	0.30659	0.30587	0.30572	0.31363

TABLE 3

Serially Correlated Stocks and 1 Week Bills - Nominal Returns

Beta	w(0)	ŵ(0)	ŵ(.5)	ŵ(1)	ŵ(2)	ŵ(5)	ŵ(10)	ŵ(inf)
-3	0.84537	0.84537	0.84631	0.84723	0.84903	0.85400	0.86114	0.99110
-4	0.67630	0.67630	0.67718	0.67805	0.67971	0.68427	0.69067	0.75098
-5	0.56358	0.56358	0.56421	0.56482	0.56600	0.56923	0.57378	0.60691
-6	0.48307	0.48307	0.48347	0.48386	0.48463	0.48681	0.[illegible]997	0.51086
-7	0.42268	0.42268	0.42292	0.42316	0.42364	0.42508	0.42733	0.44225
-8	0.37571	0.37572	0.37584	0.37597	0.37625	0.37720	0.37888	0.39080
-9	0.33815	0.33815	0.33819	0.33824	0.33839	0.33903	0.34035	0.35078
-10	0.30741	0.30741	0.30740	0.30740	0.30747	0.30791	0.30901	0.31876

Notes: 1. w(0) refers to the case of non-serially correlated stocks and bills.

2. ŵ(2) is the optimal portfolio for a two year revision period, etc.

TABLE 4

Serially Correlated 3 Month Bills - Nominal Returns

Beta	$w(0)$	$\hat{w}(0)$	$\hat{w}(.5)$	$\hat{w}(1)$	$\hat{w}(2)$	$\hat{w}(5)$	$\hat{w}(10)$	$\hat{w}(inf)$
-3	0.85601	0.85601	0.85614	0.85629	0.85658	0.85753	0.85924	0.94725
-4	0.68291	0.68291	0.68316	0.68340	0.68387	0.68518	0.68716	0.71661
-5	0.56751	0.56751	0.56759	0.56767	0.56784	0.56835	0.56922	0.58143
-6	0.48508	0.48508	0.48500	0.48493	0.48481	0.48463	0.48469	0.48997
-7	0.42326	0.42326	0.42306	0.42288	0.42257	0.42193	0.42153	0.42464
-8	0.37518	0.37518	0.37490	0.37465	0.37421	0.37331	0.37268	0.37565
-9	0.33671	0.33671	0.33639	0.33610	0.33559	0.33455	0.33385	0.33754
-10	0.30524	0.30524	0.30489	0.30457	0.30403	0.30297	0.30227	0.30706

TABLE 5

Serially Correlated Stocks and 3 Month Bills - Nominal Returns

Beta	$w(0)$	$\hat{w}(0)$	$\hat{w}(.5)$	$\hat{w}(1)$	$\hat{w}(2)$	$\hat{w}(5)$	$\hat{w}(10)$	$\hat{w}(inf)$
-3	0.85564	0.85564	0.85652	0.85738	0.85905	0.86369	0.87038	1.00000
-4	0.68262	0.68263	0.68347	0.68430	0.68589	0.69024	0.69636	0.75586
-5	0.56728	0.56728	0.56787	0.56845	0.56956	0.57259	0.57685	0.60831
-6	0.48489	0.48489	0.48526	0.48561	0.48631	0.48827	0.49112	0.50995
-7	0.42310	0.42310	0.42330	0.42350	0.42390	0.42512	0.42703	0.43969
-8	0.37504	0.37504	0.37512	0.37521	0.37541	0.37613	0.37747	0.38700
-9	0.33660	0.33660	0.33660	0.33661	0.33668	0.33708	0.33805	0.34601
-10	0.30514	0.30514	0.30509	0.30505	0.30504	0.30524	0.30599	0.31322

Notes: 1. $w(0)$ refers to the case of non-serially correlated stocks and bills

2. $\hat{w}(2)$ is the optimal portfolio for a two year revision period, etc.

TABLE 6

Serially Correlated 3 Month Bills - Real Returns

Beta	$w(0)$	$\hat{w}(0)$	$\hat{w}(.5)$	$\hat{w}(1)$	$\hat{w}(2)$	$\hat{w}(5)$	$\hat{w}(10)$	$\hat{w}(inf)$
-1	0.93724	0.93724	0.93670	0.93620	0.93542	0.93501	0.93772	1.00000
-2	0.62482	0.62482	0.62526	0.62587	0.62731	0.63165	0.63711	0.69295
-3	0.46862	0.46862	0.46873	0.46912	0.47026	0.47346	0.47595	0.47534
-4	0.37489	0.37489	0.37483	0.37512	0.37614	0.37892	0.38022	0.36654
-5	0.31241	0.31241	0.31230	0.31259	0.31366	0.31653	0.31756	0.30125
-6	0.26778	0.26778	0.26769	0.26803	0.26924	0.27241	0.27356	0.25773
-7	0.23431	0.23431	0.23427	0.23469	0.23606	0.23960	0.24105	0.22664
-8	0.20827	0.20827	0.20830	0.20880	0.21034	0.21428	0.21607	0.20333
-9	0.18745	0.18745	0.18755	0.18812	0.18983	0.19415	0.19629	0.18519
-10	0.17041	0.17041	0.17058	0.17123	0.17309	0.17777	0.18024	0.17069

TABLE 7

Serially Correlated Stocks and 3 Month Bills - Real Returns

Beta	$w(0)$	$\hat{w}(0)$	$\hat{w}(.5)$	$\hat{w}(1)$	$\hat{w}(2)$	$\hat{w}(5)$	$\hat{w}(10)$	$\hat{w}(inf)$
-1	0.94141	0.94141	0.94250	0.94349	0.94525	0.94935	0.95431	1.00000
-2	0.62760	0.62760	0.62900	0.63027	0.63251	0.63754	0.64311	0.70192
-3	0.47070	0.47040	0.47142	0.47203	0.47298	0.47451	0.47523	0.47146
-4	0.37656	0.37656	0.37688	0.37711	0.37735	0.37705	0.37544	0.35622
-5	0.31380	0.31380	0.31393	0.31398	0.31387	0.31273	0.31013	0.28708
-6	0.26897	0.26897	0.26902	0.26898	0.26874	0.26724	0.26428	0.24099
-7	0.23535	0.23535	0.23537	0.23532	0.23502	0.23343	0.23039	0.20806
-8	0.20920	0.20920	0.20923	0.20918	0.20889	0.20733	0.20436	0.18337
-9	0.18828	0.18828	0.18833	0.18830	0.18805	0.18659	0.18376	0.16416
-10	0.17116	0.17116	0.17125	0.17124	0.17105	0.16970	0.16704	0.14880

Notes:

1. $w(0)$ refers to the case of non-serially correlated stocks and bills
2. $\hat{w}(2)$ is the optimal portfolio for a two year revision period, etc.