Statistical Tests of Contingent Claims Asset-Pricing Models: A New Methodology

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A new methodology for statisticaly testing contingent claims asset-pricing models based on asymptotic statistical theory is developed. It is introduced in the context of the Black-Scholes-Merton option pricing model, for which some promising estimation, inference, and simulation results are also presented. The proposed methodology is then extended to arbitrary contingent claims by first considering the estimation problem for general Ito-processes and then deriving the asymptotic distribution of a general contingent claim which depends upon such an Ito-process.

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1. Introduction.

Since Black-Scholes (1973) and Merton (1973) introduced their now famous option-pricing model, their methodology has been applied to the pricing of a variety of other assets whose payoffs are contingent upon the value of some other underlying or "fundamental" asset. By assuming that the fundamental asset price process is of the Itô-type and that trading takes place continuously in time, the price of a contingent claim may often be derived by using the hedging and no-arbitrage arguments of Black-Scholes and Merton. Since the deduced pricing formulas are almost always functions of unknown parameters of the fundamental asset price processes, any empirical application of contingent-claims analysis must first consider the statistical estimation of fundamental asset price parameters.[1] In addition, since parameter-estimates are ultimately employed in the pricing formulas in place of the true but unknown parameters, the sampling variation of parameter-estimates will of course induce sampling variation in the estimated contingent-claims prices about their true values. The practical value of contingent claims analysis then depends critically on how parameter-estimation errors affect the accuracy of the resulting contingent-claims price estimator. Furthermore, some measure of the induced estimation error is required if the model is to be empirically tested. Indeed, although a number of papers have studied the discrepancies between estimated and observed prices for particular contingent claims, to date there have been few direct statistical tests of contingent claims models.[2] In a spirit similar to Gibbons' (1982) examination of the capital-asset pricing model, this paper proposes a new framework in which such tests may be performed and in which the accuracy of contingent-claims price estimates may be quantified statistically.

This new approach seems particularly fruitful for several reasons. Although it is introduced in the context of the Black-Scholes-Merton call-option pricing model, later sections show that the suggested methodology may be applied to any contingent claim for which the associated fundamental asset price parameters are estimated. Few additional assumptions beyond those common to all contingent claims models are required in order to apply the proposed methods. In addition, the results derived in this paper are computationally quite simple to implement. Furthermore, such a framework is well-suited to the standard tools of statistical inference, estimation, and forecasting. In fact, since the distribution of the contingent-claims estimator is derived in closed form, all the usual hypothesis testing and forecasting techniques may be applied to contingent claims analysis. This is achieved through the use of large-sample or asymptotic statistical theory which, essentially, consists of applying laws of large numbers and central limit theorems to otherwise intractable estimation and inference problems. By appealing to large-sample arguments, it is possible to derive explicitly the limiting distribution of highly nonlinear functions (such as the Black-Scholes-Merton formula) of fundamental parameter estimates.

There are several reasons why large-sample properties may be of more use to financial economists than existing small-sample results. First, due to the usual type of intractable nonlinearities, exact small-sample properties are in general quite difficult to derive.[3] However, it will be shown that corresponding large-sample properties are considerably more tractable. More importantly, large-sample statistical theory is particularly relevant for financial econometricians since financial data sets with over a thousand observations are not uncommon.[4]

In order to demonstrate the practical value of this new methodology and also to clarify the particular econometric issues at hand, Section 2 derives the large-sample properties of the Black-Scholes-Merton (BSM) call-option price estimator. The derived asymptotic statistics are then calculated using data for options written on three specific stocks and some simple hypothesis tests are performed. To explore the accuracy of the proposed estimators, some simulation evidence is presented in Section 3. In Sections 4 and 5 the methodology is developed in its most general form, and Section 6 concludes.

2. Estimation and Inference for the BSM Call Option Pricing Model.

Let S(t) denote the price of a stock at time t and let $F(S, E, r, \tau, \sigma^2)$ be the price of a corresponding call-option with exercise price E and time-to-maturity τ, where r is the interest rate on riskless (in terms of default) bonds and σ^2 is the variance rate of the underlying stock price process S(t). Under the assumptions of the BSM model, F is determined by the well-known formula:

$$F = S\Phi(d_1) - Ee^{-r\tau}\Phi(d_2) \tag{1a}$$

$$d_1 = \frac{1}{\sigma\sqrt{\tau}} \left[\ln\left(\frac{S}{E}\right) + (r+\tfrac{1}{2}\sigma^2)\tau\right] \tag{1b}$$

$$d_2 = d_1 - \sigma\sqrt{\tau} \tag{1c}$$

where Φ is the standard normal cumulative distribution function. Although the stock price, exercise price, time-to-maturity, and interest rate are observable without error, the variance σ^2 of the underlying stock is unknown. Recent studies have considered the implied standard deviations (variances) in call option prices under the assumption that investors price options according to the BSM model.[5] Indeed, it seems that these implicit volatilities may be better forecasts of future volatility than estimates derived from historical data. Because such an approach assumes that the BSM model obtains, actual tests of the model itself may be difficult to construct. In contrast to this approach, the following analysis takes as its starting point the assumption that the stock price process S(t) is the usual lognormal diffusion process given by:

$$\frac{dS}{S} = \mu dt + \sigma dw \tag{2}$$

The BSM model is not assumed to obtain, but instead forms the null hypothesis which is to be tested. Since an estimate $\hat{\sigma}^2$ of σ^2 may be obtained by using historical data, evaluating F at $\hat{\sigma}^2$ yields an estimate of the corresponding option price. Although the resulting option estimator is clearly not unbiased, it is consistent if the variance estimator is consistent. Consistency is a particularly desirable property since by definition a consistent estimator approaches the true value with probability one as the sample size grows. This is distinct from an unbiased estimator which, although is correct on average, may fluctuate considerably about its true value even in very large samples.[6]

Given a consistent estimator of the option price, a direct statistical test of the BSM model may then be constructed by comparing this estimate with the actual market price. Since the estimated price is subject to sampling variation, a measure of its "spread" is needed in order to perform a meaningful comparison. More formally, a test of whether or not the estimated option price differs significantly from the actual market price requires the calculation of the standard error about the estimated option price and the estimator's sampling distribution. In this section, the asymptotic distribution of the option price estimator is derived and is used to compare actual market prices with their BSM estimates.

Estimation of the stock price dynamics is considered first. Suppose that n+1 equally spaced observations of $S(t)$ are taken in the time interval $[0,T]$. Letting $h = T/n$, Rosenfeld (1980) has shown that the maximum likelihood (ML) estimator of σ^2 is given by:

$$\hat{\sigma}^2_{ML} = \frac{1}{T} \sum_{k=1}^{n} \left(X_k - \frac{1}{n} \sum_{j=1}^{n} X_j\right)^2 \tag{3}$$

where X_k is the log of the price-relative $\frac{S(kh)}{S\left((k-1)h\right)}$. Under mild regularity conditions, it is well-known that the general ML estimator is consistent, asymptotically normally distributed, and is efficient in the class of all consistent and uniformly asymptotically normal (CUAN) estimators.[7] In addition, the ML estimator of any well-behaved nonlinear function of a given parameter is simply the nonlinear function of the ML of that parameter. That is, the ML estimator $\hat{F}_{ML}$ of the option price F may be obtained by evaluating F at $\hat{\sigma}^2_{ML}$. Since F is a true ML estimator, it is also consistent, asymptotically normally distributed and minimum variance in the class of CUAN estimators.

Since the option estimator $\hat{F}_{ML}$ depends on the estimator $\hat{\sigma}^2_{ML}$, it is not surprising that the asymptotic distribution of $\hat{F}_{ML}$ is related to the asymptotic distribution of $\hat{\sigma}^2_{ML}$. It may easily be shown that $\hat{\sigma}^2_{ML}$ has the following asymptotic distribution:[8]

$$\sqrt{n}\,(\hat{\sigma}^2_{ML} - \sigma^2) \overset{A}{\sim} N(0,\ 2\sigma^4)\ . \tag{4}$$

Now consider the estimator $\hat{F}_{ML}$ as a function of $\hat{\sigma}^2_{ML}$, holding all other arguments fixed, i.e., $\hat{F}_{ML} = F(\hat{\sigma}^2_{ML})$. The asymptotic distribution of $\hat{F}_{ML}$ may then be derived by applying standard statistical limit theorems to the Taylor series expansion of $F(\hat{\sigma}^2_{ML})$ about the true parameter σ^2. More formally, consider:

$$\hat{F}_{ML} \equiv F(\hat{\sigma}^2_{ML}) = F(\sigma^2) + (\hat{\sigma}^2_{ML} - \sigma^2)\left[\frac{\partial F(\sigma^2)}{\partial \sigma^2} + R_n\right] . \tag{5}$$

Subtracting $F(\sigma^2)$ from both sides, multiplying by $\sqrt{n}$, observing that R_n converges in probability to 0 as n approaches infinity, and applying the Lemma stated in Appendix B yields the desired result:

$$\sqrt{n}\ (\hat{F}_{ML} - F) \overset{A}{\sim} N\left(0,\ 2\sigma^4\left(\frac{\partial F(\sigma^2)}{\partial \sigma^2}\right)^2\right) . \tag{6}$$

That is, for a sufficiently large number n of observations,[9] the sampling distribution of $\hat{F}_{ML}$ is normal with mean F and variance $\frac{2\sigma^4}{n}\left(\frac{\partial F(\sigma^2)}{\partial \sigma^2}\right) \equiv V_F$. Given the BSM pricing formula (1), the quantity V_F may be calculated explicitly as:

$$V_F = \frac{1}{2n} S^2\sigma^2\tau\phi^2(d_1) \tag{7}$$

where ϕ is the standard normal density function.

This expression is of interest for several reasons. In addition to providing a measure of option price estimators' dispersion in large samples, the analytic formula for V_F may be used to examine how changes in the underlying parameters affect the option estimates. In particular, empirical studies of call option prices have noted several patterns in the data. Macbeth and Merville (1979) observe that in-the-money call options are under-priced by the BSM formula and vice-versa for out-of-the money calls, and that the degree of mispricing is aggravated by the spread between stock and exercise price for most options. Black (1975), Merton (1976), and Gultekin, Rogalski, and Tinic (1982) observe essentially the opposite biases. To see whether such biases may be explained merely by sampling variation, consider the derivatives of V_F with respect to the stock and exercise prices and the time-to-maturity:

$$\frac{\partial V_F}{\partial S} = -\frac{1}{n} S\phi^2(d_1)d_2 \tag{8a}$$

$$\frac{\partial V_F}{\partial E} = \frac{1}{n}\frac{S^2}{E}\phi^2(d_1)\sigma\sqrt{\tau}\, d_1 \tag{8b}$$

$$\frac{\partial V_F}{\partial \tau} = \frac{1}{2n} S^2\phi^2(d_1)\sigma^2\Big[1 - \frac{1}{\sigma^2\tau}\Big([r + \tfrac{1}{2}\sigma^2]^2\tau^2 - [\ln\frac{S}{E}]^2\Big)\Big] \; . \tag{8c}$$

The following inequalities are easily established:

$$\frac{\partial V_F}{\partial S} \gtreqless 0 \quad \text{iff} \quad \frac{S}{E} \lesseqgtr e^{-(r-\frac{1}{2}\sigma^2)\tau} \equiv k_1 \tag{9a}$$

$$\frac{\partial V_F}{\partial E} \gtreqless 0 \quad \text{iff} \quad \frac{S}{E} \gtreqless e^{-(r+\frac{1}{2}\sigma^2)\tau} \equiv k_2 \; . \tag{9b}$$

Although obtaining a similar pair of equivalent inequalities for $\frac{\partial V_F}{\partial \tau}$ does not seem possible, a useful <u>sufficient</u> condition for the monotonicity of the derivative can be derived:

$$\frac{\partial V_F}{\partial \tau} > 0 \quad \text{if} \quad \frac{S}{E} < k_2 \text{ or } \frac{S}{E} > \frac{1}{k_2} \equiv k_3 \; . \tag{9c}$$

In Table 1, values of k_1, k_2, and k_3, have been tabulated for various times-to-maturity measured in weeks given an (annualized) interest rate of 10 per cent and an (annualized) standard deviation of 50 per cent.

TABLE 1 GOES HERE

Several observations may be made from the values in Table 1. Since the interval $[k_2, k_3]$ is fairly concentrated about 1.0, unless the option is very nearly at the money an increase in the time-to-maturity will increase the variance about the option price estimate. For example, if the stock price is

\$40 then options which are either in or out of the money by \$5 or more are more precisely estimated as the time-to-maturity declines. This may well explain Macbeth and Merville's (1979) finding that biases of in and out of the money options decrease as the time to expiration decreases. This would also support Gultekin, Rogalski and Tinic's (1982) observation that "In general, the [BSM] formula gives much less accurate estimates for long-lived options."

Another property of the option price estimator implied by the values in Table 1 is that, loosely speaking, if an option is deep in the money ($S/E > 1$) then as the exercise price increases, so will the variance about the option price estimate. If an option is deep out of the money ($S/E < 1$) then decreasing the exercise price increases the variance of the estimated option price. In other words, option price estimates exhibit more variation for either deep in or out of the money options as the exercise price shifts closer to the prevailing stock price. These statements may of course be made precise by computing the values k_1, k_2, and k_3 for particular options of interest.

The most direct application of the quantity V_F is in statistically testing the BSM model. In particular, consider the null hypothesis that the BSM model obtains. Letting $\overline{F}$ denote the observed market option price, this null hypothesis may be stated as:

$$H_0: \quad F(\sigma^2) = \overline{F} \,. \tag{10}$$

This hypothesis may then be tested by computing the statistic:

$$z \equiv \frac{F(\hat{\sigma}^2_{ML}) - \overline{F}}{\sqrt{V_F}} \,. \tag{11}$$

Since V_F depends upon the unknown parameter σ^2, a corresponding "t-statistic"[10] $\hat{z}$ may be calculated by using a consistent estimator $\hat{V}_F \equiv V_F(\hat{\sigma}^2_{ML})$

in place of V_F. Note that the resulting statistic is still asymptotically standard normal. The test is then performed by rejecting H_0 if $\hat{z}$ lies outside an acceptable range of 0 and accepting otherwise, where the range of acceptability is determined by the desired size of the test. If, for example, $\hat{z}$ fell outside the interval [-1.96, 1.96] then H_0 may be rejected at the 5% level. In addition, k%-confidence intervals about $\hat{F}_{ML}$ may also be constructed in the usual manner:

$$I_k = \left[\hat{F}_{ML} - m_{\frac{k}{2}}\sqrt{\hat{V}_F} \quad , \quad \hat{F}_{ML} + m_{\frac{k}{2}}\sqrt{\hat{V}_F}\right] \tag{12}$$

where $m_{\frac{k}{2}} \equiv \left|\Phi^{-1}\left(\frac{k}{200}\right)\right|$. Furthermore, confidence bands for the path of future option prices conditional upon projected future interest rates and stock prices may be obtained in a similar way, assuming that the estimated volatility is stationary over the forecast horizon. This may be of use to investors interested in constructing "worst case" scenarios for portfolios containing options.

Because the expression for V_F is analytically quite simple, computing standard errors for option price estimates requires little calculation beyond the estimation of the stock price volatility. For illustrative purposes, standard errors and the associated $\hat{z}$ statistics have been computed in Tables (2a), (2b), and (2c) for traded options on Litton, National Semiconductor, and Tandy stocks for January 12, 1979. These three stocks were chosen from a subset of five non-dividend paying stocks for which Rosenfeld (1980) estimated drift and variance coefficients according to the dynamics given by (2). In addition to their no-dividends property, Litton, National Semiconductor, and Tandy were chosen because they were trading in distinct cycles (March, February, January respectively). This was done merely to provide a complete cross-section of times-to-maturity. The estimates of the stocks' variances were obtained from Rosenfeld[11] (1980). They were estimated using 312 weekly observations from the period January 1973 to December 1978. The interest rate used was the (annualized) 26-week T-bill rate quoted on January 12, 1979 in the Wall Street Journal (9.443%).

TABLES 2a-c GO HERE

Note that, holding the exercise price constant, the standard error of every estimated option price in Tables 2a-c increases with an increased time-to-maturity. Also, whether or not an option is in or out of the money does not seem to be systematically related to whether it is underpriced or not. Of course, previous empirical studies have used a much larger set of options than the few considered here, so the lack of discernible patterns in Tables 2a-c is not conclusive.

In terms of testing the null hypothesis H_0 that the BSM model obtains, the $\hat{z}$-statistics seem to indicate that the data are inconsistent with H_0. For example, out of the eleven options written on Litton stock, only two estimates had standard errors outside the 1%-critical region and only one estimate had its standard error outside the 5%-critical region. However, caution must be exercised in interpreting this since for each stock, the tests are certainly not independent. Nevertheless, a simultaneous test of H_0 for all Litton options with a nine-week time-to-maturity results in rejection at the 5% level of significance.[12]

It is important to note that the above test of H_0 is in fact a joint test of the BSM option pricing model and of the associated stock-price dynamics. Rejecting H in this case may not necessarily imply that the BSM model does not obtain. However, because the BSM formula is so closely related to the particular form of the stock-price dynamics it is difficult to imagine a situation in which (1) obtains but (2) does not. In fact, Rosenfeld (1980) has tested the hypothesis that these three stocks follow the process (2) and rejects in favor of a combined lognormal diffusion and jump process. But in this situation, the model outlined in (1) does not obtain and must be modified along the lines Merton (1976) develops. In addition to the possibility of jumps, Rosenfeld (1980) and Marsh and Rosenfeld (1983) consider several other alternatives which may support the results in Tables 2a-c. Although it is not pursued in this paper, tests of such alternative hypotheses are readily constructed in the framework proposed here.

In addition to the obvious interest investors have in estimating option prices, the estimation of other quantities may also be of some importance. In particular, the hedge ratio H plays a prominent role in determining the overall riskiness of portfolios containing both stocks and options.[13] Given the BSM model (1), the hedge ratio is simply the derivative of the option price with respect to the stock price $H \equiv \frac{\partial F}{\partial S}$. Since H depends on the unknown parameter σ^2, an estimate $\hat{H}_{ML}$ of the true hedge ratio H may be obtained by evaluating H at the ML estimator $\hat{\sigma}^2_{ML}$ of the variance. Furthermore, since $\hat{H}$ is a true ML estimator of H it exhibits the usual properties of consistency, asymptotic normality, and efficiency in the class of all CUAN estimators of H. The asymptotic distribution of H may be readily calculated as outlined above and, after some manipulations, is given by:

$$\sqrt{n}\ (\hat{H}_{ML} - H) \overset{A}{\sim} N\left(0,\ {}^1\!/_2\, \phi^2(d_1)d_2^2\right) \ . \tag{13}$$

Although the asymptotic statistics for H are of less use for purposes of model-specification testing, they may be of considerable value <u>after</u> a particular form of the option pricing model has been validated by analysis on the option prices F. In such an instance, confidence intervals about H may easily be constructed using (13), thereby providing portfolio managers with upper and lower bounds for the hedge ratio. Confidence bands for the future path of hedge ratios conditional on projected future stock prices and interest rates may, as in the case of option prices, also be readily obtained. This framework allows investors to translate their forecasts of stock and bond prices into forecasts of hedge ratios with the added advantage of providing a simple measure of their forecasts' accuracy.

3. Simulation Evidence.

Although the empirical evidence presented in Section 3 is of interest in its own right, that analysis also illustrates the practical relevance of asymptotic statistical theory to the estimation of general contingent claims prices. Sections 4 and 5 demonstrate formally that this methodology may in fact be applied to any other contingent claim provided that its corresponding underlying fundamental asset price process may be estimated. However, an important isue which determines the usefulness of large-sample results is the number of observations required for those results to obtain. Unfortunately, no general guidelines exist so this issue must be resolved for each application individually. Nevertheless the increasing sophistication of statistical software coupled with the rapid decline of computer costs allow researchers to determine what constitutes a large sample for a particular estimator relatively easily.

In this section, a simple simulation study is conducted for the call-option price estimators proposed in Section 2. Each Monte Carlo experiment involves generating a time series for the stock price process with a given drift and variance rate using a random number generator, and then computing price estimates and corresponding asymptotic standard deviation estimates for hypothetical options written on that stock.[14] The estimated option price and asymptotic standard deviation may then be compared with their true values. This procedure is repeated 1000 times in order to deduce the finite sampling properties of the estimators. By varying the length of the stock price series generated for the 1000 replications and noting its effect upon the estimators' sampling behavior, it is possible to deduce the minimum number of observations required to insure that the associated asymptotic properties do obtain. By varying other parameters, it is also possible to study how the asymptotic approximation to finite-sample properties may be related to the terms of an option contract such as the time-to-maturity or the stock-price/exercise-price spread. Throughout the simulations, the following parameter values were assumed and held constant.

$$S = \$40$$

$$\sigma^2 = 0.5200 \text{ (annual)}$$

$$r = 0.1000 \text{ (annual)}$$

The simulations were carried out at the weekly frequency for which r and σ^2 were adjusted appropriately. Tables 3 and 4 summarize the finite-sampling properties of the option price and asymptotic variance estimators across the 1000 replications for various options and stock-price sample sizes.

TABLES 3a-c AND 4a-c GO HERE

Each table corresponds to experiments with hypothetical options of the same time-to-maturity TAU. Tables 3a, 3b, and 3c report simulation results for hypothetical options which are at the money and in and out of the money by \$5 for maturities 1, 13, and 26 respectively. Tables 4a-c display simulation results for options which are in and out of the money by \$15 with respectively 1, 13, and 26 weeks to go. Experiments with options of intermediate exercise prices, times-to-maturity other than 1, 13, and 26, and assorted interest rate/stock-price variance combinations were also conducted but since the results depicted in Tables 3 and 4 are generally confirmed in these other experiments, in the interest of brevity those results are not reported.[15]

Within each table, every row corresponds to a separate and independent experiment. Each experiment involves simulating a time series of stock prices of length N, computing the estimators $\hat{F}_{ML}$, $\hat{V}_F$, and test-statistic $\hat{z}$ for a particular hypothetical option, repeating this 1000 times, tabulating the subsequent sampling distribution for the estimators and $\hat{z}$, and finally testing the standard normality of $\hat{z}$. Although the estimators $\hat{F}_{ML}$ and $\hat{V}_F$ may also be checked for normality, for purposes of hypothesis testing and constructing confidence intervals the standard normality of the statistic $\hat{z}$ is more relevant. Of the many tests for departures from normality, only two are considered here. The first is the usual χ^2-test of goodness-of-fit which measures the "distance" between the hypothesized distribution function (normal) and the empirical distribution function. The second is the studentized range test which is more sensitive to departures from normality in the tails of the distribution. Since the primary use of $\hat{z}$ is in the testing of hypotheses, departures from normality in the tail areas are of more concern than differences in the center of the distribution. For this reason, the results of the studentized range test may be of more consequence than the χ^2-test. Both tests are performed and the results are given in the last two columns of each row.

Consider the entries in Table 3a. The first five rows comprise the simulation evidence for a call option with exercise price \$35 and one week to maturity. The second five rows correspond to the experiment of a call option with exercise price \$40 also maturing in one week, and the last five rows are results for a call with exercise price $45 and one week to go. The first column indicates the length of the stock-price series generated by the random number generator. The second, third, and fourth columns display respectively the true or population value of the option, the mean of the option estimator across the 1000 replications, and the bias in percentage terms. The standard deviation of the option estimate across the replications is given in parentheses under the option estimate. The fifth, sixth, and seventh columns present the true value, estimated value, and percentage bias of the asymptotic variance V_F respectively. The eighth column provides the mean and standard deviations of the $\hat{z}$-statistic over all the replications. In the last three columns, statistics which indicate how close $\hat{z}$ is to a standard normal variate are displayed. The first is the χ^2-test with the p-value given in parentheses below the test-statistic. The next column, labelled SKEW., displays the skewness coefficient of $\hat{z}$ across the replications and the last column, entitled STD. RG., presents the studentized range of $\hat{z}$.

As Boyle and Ananthanarayanan (1977) have shown, for an at-the-money call-option few observations are required in order to trivialize the bias of the option price estimator. The largest absolute price bias observed in Tables 3a-c where options are either at the money or in or out of the money by $5 is 0.64%. In addition, in Tables 3a-c the bias in estimating the asymptotic variance is also quite small, the largest being -1.95%. For most cases, both estimates were well within 1% of the true value. Note that, although on average the bias for both estimators decreases as the length of the stock-price series increases, the decrease is not monotonic. This is to be expected since each experiment is random and independent of the others and is subject to the usual sampling variation.

The biases for deep in or out of the money options, however, are quite large when the time-to-maturity is one week. Table 4a displays price biases of up to -31.41% and asymptotic variance biases of over -2200.0%. The extraordinary bias in the variance estimator may be due in part to loss in machine precision during the course of many calculations since the estimators' order of magnitude is extremely small (10^{-12}). Nevertheless this suggests that caution must be exercised in using these estimators for deep in or out of the money options just about to expire. However, Tables 4b and 4c show that as the time to maturity increases, the bias declines dramatically, the largest price bias being 1.24% and the largest variance bias being -2.43%.

Consider now the asymptotic behavior of the statistic $\hat{z}$. Under the null hypothesis that $\hat{z}$ is standard normal, the χ^2-test is performed for the 1000 replications of each experiment with 50 equiprobable categories yielding 49 degrees of freedom. From Tables 3 and 4, it seems that with a sample size of 100 weekly observations for stock-prices, the standard normality of $\hat{z}$ may be rejected at almost any level of significance. However, in most cases the null hypothesis of normality may be accepted at levels of 5% or smaller with 300 or more weekly observations of stock-price data. Nevertheless, it may be noted that the means of $\hat{z}$ are negative for almost all experiments. For the purposes of detecting skewness departures from normality, the skewness coefficient may yield a more powerful test than the χ^2-test. Under the null hypothesis that $\hat{z}$ is standard normal, the distribution of the sample skewness coefficient has been tabulated[16] and, for 1000 replications, the 90%-confidence interval is [-0.127, 0.127] and the 98%-confidence interval is [-0.180, 0.180]. It is clear that even in cases where the χ^2-test does not reject the null hypothesis of standard normality, the skewness coefficient is often outside the 98%-confidence interval. This indicates that the finite-sampling distribution of

$\hat{z}$ is skewed (to the left). However, if the "tail-behavior" of $\hat{z}$ is close to that of the standard normal, then the hypothesis tests based on $\hat{z}$ suggested in Section 2 are in fact appropriate. To measure possible departures from standard normality in the tails of the finite sample distribution of $\hat{z}$, the studentized range for each experiment may be compared with its tabulated distribution under the null hypothesis.[17] For 1000 replications, the 90%-confidence interval of the studentized range with 5% in each tail is given by [5.79, 7.33] and the 95%-confidence interval with 2.5% in each tail is [5.68, 7.54]. For the hypothetical options in Tables 3a-c, only in two cases do the computed studentized range fall outside the 90%-interval and only one of those is outside the 98% range. This suggests that, although the finite sample distribution of z may be skewed, its tail-probabilities match the standard normal's fairly closely. For purposes of testing the BSM model as specified by (1), the results seem to support the use of the $\hat{z}$-statistic as described in the previous section for options not too deep in or out of the money. Tables 4a-c show that for deep in the money options with 1 week to go, not even a sample size of 700 is sufficient to produce the asymptotic results for $\hat{z}$; both the studentized range and the χ^2 tests reject normality at practically any level of significance. However, for deep out of the money options with 1 week to go, sample sizes of 500 or more seem to be sufficient to render the tail behavior of $\hat{z}$ close to the standard normal's as measured by the studentized range. In this case, the previous caveat concerning possible inaccuracies due to loss in machine precision also applies so that caution must be applied in drawing inferences from Table 4a. The results in Tables 4b and 4c however show that once the time to maturity increases to 13 and 26 weeks, the tail behavior of the $\hat{z}$-statistic matches that of the standard normal even for deep in or out of the money options.

From the simulation evidence provided above, it may be concluded that if the call option pricing model (1) obtains, then for options which are not too deep in or out of the money and for deep in or out of the money options which are not just about to expire, using its asymptotic distribution for purposes of testing and inference may well be justified.

4. The Estimation of Generalized Itô Processes.

Since almost all contingent claims models assume that fundamental asset prices follow Itô-processes, in order to demonstrate that the methodology outlined in previous sections generalizes it is first necessary to consider the estimation problem for this class of stochastic processes. For expositional clarity we only consider the estimation problem for Itô processes with single jump and diffusion components. The extension to multiple jump and diffusion terms and vector Itô processes poses no conceptual difficulties but is notationally more cumbersome. Let X(t) be an Itô process with domain $\Omega \subset R$ satisfying the following stochastic differential equation:

$$dX = f(X, t; \alpha)dt + g(X, t; \beta)dW + h(X, t; \gamma)dN \qquad t \in [0, \infty) \tag{14}$$

where dW is the standardized Wiener process and dN is a Poisson counter (jump magnitude = 1), independent of dW, with intensity λ. There is clearly no loss of generality in assuming that the jump magnitude is unity since this is merely a normalization which may be subsumed by the coefficient function g. Suppose, however, the jump magnitude is stochastic. More generally, suppose that certain "parameters" in f, g, and h are in fact random variables. Without further information, there is of course little that can be done. If however it is posited that these random parameters are distributed according to a particular parametrizable probability law which is statistically independent of dW and dN, then the estimation procedure described in this section may still be applied. For example, if it is assumed that the jump magnitude is lognormally distributed with unknown parameters and is independent of dW and dN, these parameters may be estimated along with the other unknown parameters of f, g, and h as well. More will be said about this later.

In addition to assuming those conditions which insure the existence and uniqueness of the solution to (1)[18] we make the following additional assumptions:

(A1) Coefficient functions f, g, and h are known up to parameter vectors α, β, γ, λ respectively. The true but unknown parameters α_0, β_0, γ_0 and λ_0 lie in the interior of the compact parameter spaces A, B, Γ and Λ respectively. Let $\theta_0 \equiv (\alpha_0', \beta_0', \gamma_0', \lambda_0)'$ and let $\Theta \equiv A \times B \times \Gamma \times \Lambda$. The functions f, g, and h are twice continuously differentiable in (X, t) and three times continuously differentiable in θ.

(A2) n observations of X(t) are taken at times $t_1, t_2, \ldots, t_n$ <u>not</u> necessarily equally spaced apart, where $0 < t_1 < \ldots < t_n$. $X \equiv (X_1, X_2, \ldots, X_n)'$, where $X_i \equiv X(t_i)$, $i = 1, \ldots, n$. $X(t_0) \equiv X_0$ is known.

We may now state the estimation problem as: Given the observations X and the process dynamics (1), find the optimal estimator $\hat{\theta}$ of the true parameters θ_0. This, however, is still not a well-posed problem since we have not yet specified which class of estimators we are optimizing over nor have we stated the criteria by which we may compare alternative estimators. By restricting consideration to the class of consistent and uniformly asymptotically normal (CUAN) estimators, it has been shown that the ML estimator is optimal in the sense that it has the smallest variance of all other CUAN estimators. For this reason, ML estimation is the preferred approach. The ML estimator is obtained by considering the joint density function of the random sample X as a function of the unknown parameters and then finding that value $\hat{\theta}_{ML}$ which maximizes the joint density in θ. We now proceed to derive this joint density function which, when considered a function of the parameters θ given the data X, is called the joint-likelihood function.

Let $P(X_1, \ldots, X_n)$ denote the joint-distribution function of the random sample X, where the dependence of P on the unknown parameters θ and on $t_1, \ldots, t_n$ have been suppressed for notational simplicity. If we assume that:

(A3) P is absolutely continuous with respect to Lebesgue measure defined on the Borel sets of R^n for all θ,

then the existence and uniqueness of the associated joint density representation ρ of P is guaranteed. The joint density ρ may always be written as the following product of conditional densities:

$$\rho(X_1, \ldots, X_n) = \rho_1(X_1)\rho_2(X_2|X_1)\rho_3(X_3|X_2, X_1) \ldots \rho_n(X_n|X_{n-1}, \ldots, X_1) \ . \quad (15)$$

However, since X(t) is a Markov process[19] equation (2) reduces to:

$$\rho(X_1, \ldots, X_n) = \rho_1(X_1)\rho_2(X_2|X_1)\rho_3(X_3|X_2) \ldots \rho_n(X_n|X_{n-1}) \ . \quad (16)$$

If in addition, X(t) is time-homogeneous then the functional form of the transition density ρ_k only depends upon the time index k in terms of the time increment $t_k - t_{k-1}$ and not on t_k itself. In this case, the notation ρ_k should be interpreted as

$$\rho_k(X_k, t_k|X_{k-1}, t_{k-1}) \equiv \rho(X_k, \Delta t_k|X_{k-1}) \quad \text{where } \Delta t_k \equiv t_k - t_{k-1} . \quad (17)$$

If, for example, observations were then taken at equally spaced intervals of length h, then the ρ_k's are identical across time except for the starting values X_{k-1}. Of course, one of the greatest advantages of estimating continuous-time models is precisely that equally-spaced observations are <u>not</u>

necessary. Unless stated otherwise, we do not assume equally-spaced observations. For compactness of notation, we will write $\rho(X_k, t_k | k_{k-1}, t_{k-1})$ as ρ_k.

Given the functions f, g, and h, the joint density function $\rho(X)$ of the random sample X may be derived by solving the Fokker-Planck or forward equation for the transition densities ρ_k subject to any boundary conditions which may apply. For the Itô process (1) this relation is derived in Appendix A and is given by:

$$\frac{\partial}{\partial t} [\rho_k] = - \frac{\partial}{\partial X} [f\rho_k] + \tfrac{1}{2}\frac{\partial^2}{\partial X^2} [g^2\rho_k] - \lambda\rho_k + \lambda\tilde{\rho}_k \left|\frac{\partial}{\partial X} [\tilde{h}^{-1}]\right| \tag{18a}$$

such that

$$\tilde{h}(X, t; \gamma) \equiv X + h(X, t; \gamma) , \qquad \tilde{h}(\tilde{h}^{-1}, t; \gamma) \equiv X \tag{18b}$$

$$\tilde{\rho}_k \equiv \rho_k(\tilde{h}^{-1}, t) \tag{18c}$$

$$\rho_k(X, t_{k-1} | X_{k-1}, t_{k-1}) = \delta(X - X_{k-1}) . \tag{18d}$$

where $\delta(X - X_k)$ is the Dirac-delta function centered at X_{k-1}. Although the functional partial differential equation in (5) characterizes the transition densities hence the conditional likelihood functions, obtaining a closed form solution for ρ_k is generally quite difficult. By restricting the functional forms of f, g, and h however, it is possible to derive the transition densities explicitly. For example, if h = 0 (pure diffusion) and f and g satisfy the following reducibility condition:

$$\frac{\partial}{\partial X} \left[g \left\{ \frac{\frac{\partial}{\partial t} [g]}{g^2} - \frac{\partial}{\partial X} [\frac{f}{g}] + \tfrac{1}{2}\frac{\partial^2}{\partial X^2} [g] \right\} \right] = 0 \tag{19}$$

it may be shown[20] that there exists a transformed process Z(t) of X(t) for which the coefficient functions are independent of Z(t). That is, for some suitable change of variables $T[X(t)] \equiv Z(t)$, an application of Itô's lemma will yield:

$$dZ = p(t;\ \theta)dt + q(t;\ \theta)dW \tag{20}$$

In this case the transition density function for the transformed data is readily derived as:

$$\rho_k(Z,\ t) = \left[2\pi \int_{t_{k-1}}^{t} q^2 d\tau\right]^{-1/2} \exp\left[-\frac{\left(Z - Z_{k-1} - \int_{t_{k-1}}^{t} p dt\right)^2}{2\int_{t_{k-1}}^{t} q^2 d\tau}\right] . \tag{21}$$

For example, the transition density of the lognormal diffusion process (2) may be obtained by letting $Z(t) = \ln S(t)$ and computing dZ as:

$$dZ = (\mu - 1/2\,\sigma^2)dt + \sigma dW . \tag{22}$$

Since the coefficient functions of $Z(t)$ only depend upon time and the unknown parameters, by equation (21) the transition density of an observation is just:

$$\rho_k(Z,\ t) = \left[2\pi\sigma^2(t - t_k)\right]^{-1/2} \exp\left[-\frac{\left(Z - Z_{k-1} - (\mu - 1/2\,\sigma^2)(t - t_{k-1})\right)^2}{2\sigma^2(t - t_{k-1})}\right] . \tag{23}$$

Note that in the expression for the transformed process $Z(t)$ (equation (20)), both coefficient functions are dependent upon the parameter vector θ. Although for specific examples, such as the lognormal, the coefficient functions may be shown to be independent of particular subsets of θ, at this level of generality the dependence of both p and q on θ cannot be ruled out.

Given the transition densities ρ_k, the joint-likelihood and log-likelihood functions of the random sample X are given by:

$$L(\theta;\ X) = \prod_{k=1}^{n} \rho(X_k,\ t_k | X_{k-1},\ t_{k-1};\ \theta) \tag{24a}$$

$$G(\theta;\ X) = \sum_{k=1}^{n} \ln\rho(X_k,\ t_k | X_{k-1},\ t_{k-1};\ \theta) \equiv \sum_{k=1}^{n} \ell_k(X_k | X_{k-1};\ \theta)\ . \tag{24b}$$

Under assumptions (A) and mild regularity conditions, the ML estimator $\hat{\theta}_{ML}$ of θ_0 exists, is consistent, and is asymptotically efficient in the class of all CUAN estimators. That is,

$$\underset{n\to\infty}{\text{plim}}\ \hat{\theta}_{ML} = \theta_0 \tag{25a}$$

$$\sqrt{n}\ (\hat{\theta}_{ML} - \theta_0) \overset{A}{\sim} N\big(0,\ I^{-1}(\theta_0)\big) \tag{25b}$$

where the asymptotic covariance matrix $I^{-1}(\theta_0)$ is the inverse of the information matrix $I(\theta_0)$:

$$I(\theta_0) = -\lim \frac{1}{n} \sum_{k=1}^{n} E\left[\frac{\partial^2 \ell(X_k | X_{k-1};\ \theta_0)}{\partial\theta\partial\theta'}\right]\ . \tag{26}$$

The previously mentioned problem of random parameters may now be considered explicitly. If such random parameters are distributed independently of dW and dN according to a particular parametric family of distributions, then the following approach may be taken. First, a conditional joint-likelihood function for the sample X may be derived according to the procedure outlined above, conditioning upon fixed values of the random parameters. The unconditional likelihood function is then obtained by multiplying the conditional likelihood function by the particular marginal density or distribution functions of the random parameters. The maximization of the joint-likelihood function may then be performed for all parameters yielding ML estimates of the parameters determining the probability law of the random parameters in addition to ML estimates of the non-stochastic parameters. Note that, as a special case, such a specification may be employed to incorporate non-stationary parameters such as a time-varying drift and variance rates for the lognormal diffusion process.

For illustrative purposes, the likelihood functions of several particular processes are presented in examples 1, 2, and 3.

Example 1. (Ornstein-Uhlenbeck process)

Let $X(t)$ solve

$$dX = -\alpha_0 X dt + \beta_0 dW \qquad \alpha_0 > 0 \tag{27}$$

It may then be shown that the conditional-likelihood function is given by

$$\rho(X_k, t_k | X_{k-1}, t_{k-1}) = \left[\frac{\pi\beta_0^2}{\alpha_0}(1 - e^{-2\alpha_0 \Delta t_k})\right]^{-1/2} \exp\left[-\frac{\alpha_0 (X_k - X_{k-1} e^{-\alpha_0 \Delta t_k})^2}{\beta_0^2 (1 - e^{-2\alpha_0 \Delta t_k})}\right] \tag{28}$$

Example 2. (Diffusion with absorbing barrier)[21]

Let $X(t)$ be a Wiener process with drift, i.e.,

$$dX = \alpha_0 dt + \beta_0 dW \tag{29}$$

such that $X_0 > 0$ and suppose that $X(t) = 0$ is an absorbing state. In addition, let $X_1 > 0, \ldots, X_{n-1} > 0$, $X_n = 0$ so that absorption is realized in this sample some time between t_{n-1} and t_n. Then the likelihood-function for this sample would be the product of the conditional densities for observations X_1 to X_{n-1} where:

$$\rho(X_k, t_k | X_{k-1}, t_{n-1}) = [2\pi\beta_0^2 \Delta t_k]^{-1/2} \exp\left[-\frac{(X_k - X_{k-1} - \alpha_0 \Delta t_k)^2}{2\beta_0^2 \Delta t_k}\right], \quad k=1, \ldots, n-1 \tag{30}$$

multiplied by the distribution function of the first-passage time for observation X_n. Following Cox and Miller's (1973) derivation for the first-passage time distribution of a process with an absorbing barrier at $X = a > 0$, we calculate the distribution for the barrier at X = 0 to be:

$$P(\text{Absorption in } (t_{n-1}, t_n)) = \Phi\left[\frac{-X_{n-1}-\alpha_0\Delta t_n}{\beta_0\sqrt{\Delta t_n}}\right] + \exp\left[-\frac{2\alpha_0 X_{n-1}}{\beta_0^2}\right]\Phi\left[\frac{-X_{n-1}+\alpha_0\Delta t_n}{\beta_0\sqrt{\Delta t_n}}\right] \quad (31)$$

where Φ is the standard normal distribution function. Note that although X(t) may have been absorbed at any time between t_{n-1} and t_n, knowing that X(t) has been absorbed by time t_n is sufficient for computing ML estimates of the unknown parameters.

Example 3. (Combined lognormal diffusion and jump process)

Let X(t) be the following Itô process:

$$dX = \alpha_0 X dt + \beta_0 X dW + \gamma_0 X dN \quad (32)$$

By using the log-transformation Y = ln X and Itô's Lemma, the behavior of Y may be described by:

$$dY = (\alpha_0 - \frac{1}{2}\beta_0)dt + \beta_0 dW + \ln(1 + \gamma_0)dN \quad (33)$$

It may then be shown that the conditional-likelihood function is given by

$$\rho(Y_k, t_k | Y_{k-1}, t_{k-1}) = \sum_{j=0}^{\infty} \frac{e^{-\lambda}\lambda^j}{j!}\phi\left(\frac{Y_k - Y_{k-1} - \ln(1+\gamma_0)^j - (\alpha_0 - \frac{1}{2}\beta_0^2)\Delta t_k}{\beta_0\sqrt{\Delta t_k}}\right) \quad (34)$$

In practice, approximate ML estimates are numerically computed using only a finite number of terms from the above infinite series. Clearly such an approximation may be made as accurate as desired by including a suitably large number of terms in the estimation.

This example highlights an unlikely but potential problem with the ML estimation procedure. Since the practical value of maximum likelihood estimation depends critically on the smoothness of the likelihood function, clearly non-differentiable transition densities for X(t) must be ruled out. Although it may seem that the inclusion of a jump component might introduce some discontinuities in the corresponding likelihood function, it may be shown that as long as a diffusion component is present the likelihood function will be smooth. Heuristically, this is due to the fact that the convolution of two distributions is as smooth as the "smoother" of the two. If the domain of possible solutions to (19) is expanded to include all generalized functions (such as the delta-function), it may be readily shown that the conditional likelihood function of this example is simply the convolution of the normal and Poisson density functions.[22] Since the normal density function is infinitely differentiable, so is its convolution with the Poisson. Suppose, however, that the coefficient function g of the general process (14) is identically zero at the true parameter β_0. Although the solution to (19) still exists in the space of generalized functions,[23] it will not in general be differentiable hence the numerical implementation of the maximum likelihood procedure as discussed in this section clearly breaks down. In practice this issue will almost never arise since the probability that g = 0 is essentially zero. It is, however, of some importance if theoretical considerations imply that g is zero and one wishes to test this, since in such a situation under the null hypothesis the ML estimator as developed above is not defined.

5. The Asymptotic Distribution of General Contingent Claims Estimators.

Let F be the price of an arbitrary asset which is contingent upon the fundamental asset X(t). In particular, suppose F may be determined by the following known asset-pricing formula:

$$F = F(X,\ t,\ \eta;\ \theta_0)\ , \qquad F \text{ continuously differentiable in } \theta \tag{35}$$

where η is a vector of observables (e.g., interest rates, time to maturity, etc.) and θ_0 is the unknown true parameter vector associated with the fundamental asset price process X(t).

Given assumptions (A), the well-known "principle of invariance" states that the ML estimator of the contingent claims price F is simply

$$\hat{F}_{ML} = F(X,\ t,\ \eta;\ \hat{\theta}_{ML}) \tag{36}$$

where $\hat{\theta}_{ML}$ maximizes (24). Since $\hat{F}_{ML}$ is a true ML estimator of F, it is also consistent and asymptotically efficient in the class of all CUAN estimators of F. In addition, the asymptotic distribution of the estimator $\hat{F}_{ML}$ may be easily derived and, as shown in Appendix B, is given by:

$$\sqrt{n}\ (\hat{F}_{ML} - F) \overset{A}{\sim} N(0,\ V_0) \tag{37a}$$

$$V_0 \equiv \frac{\partial F(\theta_0)'}{\partial \theta}\ I^{-1}(\theta_0)\ \frac{\partial F(\theta_0)}{\partial \theta} \tag{37b}$$

Using (37), the usual forms of statistical inference may then be applied to the estimated contingent claims price. In particular, the model-specification test, confidence intervals, projected confidence bands, and other forms of statistical inference which were suggested in Section 2 for the BSM call option pricing model may also be applied to any other type of contingent claims model in a similar fashion.

6. Conclusion

In this paper we hope to have provided a general methodology for the estimation and testing of general contingent claims asset-pricing models by appealing to asymptotic statistical theory. Given the large-sample distribution of any contingent claims price estimator, the financial economist may bring to bear a considerable collection of statistical tools upon a variety of problems in model-specification testing and forecasting. Since what constitutes a "large sample" depends upon the particular estimator of interest, Monte Carlo studies must be performed on a case by case basis in order to determine the practical relevance of the proposed methods. The simulation results reported in Section 3 for the Black-Scholes and Merton call option pricing model suggest that for most call options, a large sample consists of between 300 and 500 weekly observations. Moreover, the costs of performing these simulation studies are quite small, certainly relative to their payoff but also in absolute magnitude. As an example, the costs of performing the simulations in Tables 3 and 4 did not exceed $25.00.

In addition to cost-effectiveness, another advantage of such large-sample results is tractability. The numerical estimation of the fundamental asset's parameters is a straightforward application of now standard maximum likelihood software packages such as BHHH, GQOPT, or TSP. In addition, part of the standard output of such packages is a consistent estimate of the inverse of the information matrix I^{-1}. Given this estimate, the asymptotic distribution of any corresponding contingent claim may then be derived by computing the derivative of its pricing formula with respect to the unknown parameters. For those contingent claims with tractable pricing formulas, expressions for their asymptotic distributions will also be tractable. The applicability of the proposed methods thus extends to practically all contingent claims models which are of theoretical interest since those are often ones for which pricing formulas may be determined explicitly. Although this approach seems quite promising, whether or not the application of these results to other contingent claims models will yield new insights can only be determined by further empirical investigations.

Appendix A - Derivation of the Forward Equation

Let X(t) solve the following stochastic differential equation:

(A1) $$dX = f(X, t; \alpha)dt + g(X, t; \beta)dW + h(X, t; \gamma)dN$$

where dW is the standard Brownian motion and dN is a Poisson counter with intensity λ and independent of dW. Let $\psi(X)$ be an arbitrary C^{∞} function. By Itô's Lemma[24] we have:

(A2) $$d\psi = \left[\psi_X f + \tfrac{1}{2}\psi_{XX} g^2\right]dt + \psi_X g dW + \left[\psi(X + h) - \psi(X)\right]dN .$$

where

$$\psi_X \equiv \frac{\partial \psi}{\partial X}, \qquad \psi_{XX} \equiv \frac{\partial^2 \psi}{\partial X^2} .$$

Define $D_{P,k}$ to be the Dynkin operator at time t_k, i.e., $D_{P,k} \equiv \frac{d}{dt} E_{t_k}[\cdot]$. Applying it to ψ yields:

(A3) $$D_{P,k}[\psi] = E_{t_k}\left[\psi_X f + \tfrac{1}{2}\psi_{XX} g^2\right] + \lambda E_{t_k}\left[\psi(X + h) - \psi(X)\right] .$$

Given assumption (A3), we may express $D_{p,k}[\psi]$ as the following integral:

(A4a) $$D_{P,k}[\psi] = \int_{\Omega} \left\{\psi_X f + \tfrac{1}{2}\psi_{XX} g^2 + \lambda\left[\psi(X + h) - \psi(X)\right]\right\}\rho_k(X, t)dX$$

(A4b) $$= \int_{\Omega} \left[-\psi \frac{\partial}{\partial X}(f\rho_k) + \tfrac{1}{2}\psi \frac{\partial^2}{\partial X^2}(g^2\rho_k) - \psi\lambda\right]dX + \lambda \int_{\Omega} \psi(X + h)\rho_k dX .$$

Let $Y \equiv \tilde{h}(X, t; \gamma) \equiv X + h(X, t; \gamma)$ be an <u>onto</u> map of Ω to Ω for all (t, γ) and suppose that $\left|\frac{\partial}{\partial X}(\tilde{h}) + 1\right| \neq 0$ for all (t, γ) and $X \in \Omega$. Then the Inverse Function Theorem guarantees the existence of $\tilde{h}^{-1}$ such that $X = \tilde{h}^{-1}(Y, t; \gamma)$. Using the change of variables formula, we have:

$$\text{(A5a)} \quad \int_{\Omega} \psi(X + h)\rho_k(X, t)dX = \int_{\Omega} \psi(Y)\rho_k\left(\tilde{h}^{-1}(Y, t; \gamma)\right) \left| \frac{\partial}{\partial Y}\left(\tilde{h}^{-1}(Y, t; \gamma)\right)\right| dY$$

$$\text{(A5b)} \quad = \int_{\Omega} \psi(X)\rho_k\left(\tilde{h}^{-1}(X, t; \gamma)\right) \left| \frac{\partial}{\partial X}\left(\tilde{h}^{-1}(X, t; \gamma)\right)\right| dX \; .$$

We then conclude that

$$\text{(A6)} \quad D_{p,k}[\psi] = \int_{\Omega} \left\{- \frac{\partial}{\partial X} (f\rho_k) + \tfrac{1}{2}\frac{\partial^2}{\partial X^2} (g^2\rho_k) - \lambda\tilde{\rho}_k \left| \frac{\partial}{\partial X} \tilde{h}^{-1}\right|\right\}\psi(X)\, dX \; .$$

Assuming that $\psi(X)\rho_k(X, t)$ is continuous on $\Omega \times [0, \infty)$, $D_{p,k}[\psi]$ may be calculated alternatively as

$$\text{(A7)} \quad D_{P,k}[\psi] = \frac{d}{dt} E_{t_k}[\psi] = \int_{\Omega} \psi(X) \frac{\partial}{\partial t} \left[\rho_k(X, t)\right]dX \; .$$

Equating (A7) and (A6) and noting that the equality obtains for arbitrary smooth functions ψ allow us to conclude that:.

$$\text{(A8)} \quad \frac{\partial}{\partial t} [\rho_k] = - \frac{\partial}{\partial X} [f\rho_k] + \tfrac{1}{2}\frac{\partial^2}{\partial X^2} (g^2\rho_k) - \lambda\rho_k + \lambda\tilde{\rho}_k \left| \frac{\partial}{\partial X} [\tilde{h}^{-1}]\right| \; .$$

Appendix B - The Asymptotic Distritution of the General Contingent Claim Estimator

The derivation of the general contingent claim's asymptotic distribution involves the application of basic results in large-sample statistical theory but for expositional purposes we present the derivation instead of simply quoting the results. For this purpose, we require the use of the following well-known lemma:

Lemma: If Y_n converges in distribution to Y and A_n and B_n converge in probability to a and b respectively, then $A_n + B_n Y_n$ converges in distribution to a + bY.

Proof: See Rao (1973).

Since $\hat{\theta}_{ML}$ is a maximum-likelihood estimator, $\sqrt{n}\,(\hat{\theta}_{ML} - \theta_0)$ converges in distribution to a random variable which is $N\left(0,\ I^{-1}(\theta_0)\right)$ where $I(\theta_0)$ is given in equation (26). Consider the ML estimator $\hat{F}_{ML} \equiv F(\hat{\theta}_{ML})$ of an arbitrary contingent claims price $F(\theta_0)$ where the dependence of F on (X, t, η) has been suppressed for notational compactness. Taking a Taylor expansion of F about the true parameter vector yields the following relation:

$$\text{(B1)} \qquad F(\hat{\theta}_{ML}) = F(\theta_0) + (\hat{\theta}_{ML} - \theta_0)'\left[\frac{\partial F(\theta_0)}{\partial \theta} + \frac{\partial^2 F(\tilde{\theta})}{\partial \theta \partial \theta'}\,(\hat{\theta}_{ML} - \theta_0)\right]$$

where the Hessian $\frac{\partial^2 F}{\partial \theta \partial \theta'}$ is evaluated at $\tilde{\theta} \equiv \theta_0 + \tau\bar{\theta}$ for some $\tau \in (0,\ 1)$ and $\bar{\theta} \in \Theta$ such that $\tilde{\theta}$ lies in an open set containing θ_0. Multiplying both sides of (B1) by $\sqrt{n}$ and rearranging yields:

$$\text{(B2)} \qquad \sqrt{n}\left(F(\hat{\theta}_{ML}) - F(\theta_0)\right) = \sqrt{n}\,(\hat{\theta}_{ML} - \theta_0)'\left[\frac{\partial F(\theta_0)}{\partial \theta} + R_n\right] .$$

Observing that the Hessian term R_n converges in probability to zero as $\hat{\theta}_{ML}$ converges in probability to θ_0 and applying the above lemma then produces the desired result:

$$\text{(B3)} \qquad \sqrt{n}\,(\hat{F}_{ML} - F) \overset{A}{\sim} N\left(0, \frac{\partial F'(\theta_0)}{\partial \theta}\, I^{-1}(\theta)\, \frac{\partial F(\theta_0)}{\partial \theta}\right) .$$

Footnotes

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[1]"Estimation" here is in the classical statistical sense, distinct from estimation in the filtering sense (see Davis and Marcus (1980)) in which properties of an unobserved process X(t) are deduced from observing a related process Y(t). Of course, observing X(t) without noise may be considered a special case of the general filtering problem.

[2]Papers by Black and Scholes (1972), Merton (1973), Black (1975), Macbeth and Merville (1979), and Gultekin, Rogalski, and Tinic (1982) have noted systematic differences between observed market prices of call options and prices obtained from the Black-Scholes formula but did not formally test whether such departures were statistically significant. Gultekin et. al. does consider how such biases change with the time-to-maturity although no formal explanation of their findings was proposed. However, several studies have considered testing the efficiency of options markets. In particular, Black and Scholes (1972), Galai (1977), and Finnerty (1978) have explored the possibility of excess returns resulting from observed options prices deviating from the Black-Scholes prices. Chiras and Manaster (1978) study possible excess returns generated by using implied standard deviations in the pricing formula. Whaley (1982) also uses implied standard deviations in examining various pricing formulas for calls on stocks with known dividends. Violations of certain boundary conditions by observed market prices have also been investigated by Galai (1978) and Bhattacharya (1983). Although many of these

empirical findings are quite striking, without some guidelines as to the statistical significance of observed deviations, hypothesis tests cannot formally be constructed. Even in Whaley's (1982) six regression tests of option valuation, since the linear regression equations are not determined by theoretical considerations there is no guarantee that the subsequent test statistics have a particular sampling distribution. In this paper, a new methodology based on asymptotic statistical theory is proposed in which the significance of such deviations may be quantified. In addition, the methodology may be applied to contingent claims other than options for which the standard tests of boundary conditions or market efficiency may be more cumbersome (as in the case of life-insurance or investment opportunities).

[3]See Boyle and Ananthanarayanan (1977) for the exact small-sample distribution of the call option price estimator given the usual small-sample χ^2-distribution for the variance estimator.

[4]For example, the Chicago Board of Trade Foundation provides a data base of daily open, high, low, and closing prices for twenty-six commodity futures for all delivery months or from the contracts' starting date for the years 1959-1969. Standard and Poor's COMPUSTAT Industrial Annual data base contains balance sheet and income statement data for approximately 2700 firms for the past ten years. The Center for Research in Security Prices' monthly stock returns file includes 2990 common stocks listed on the New York Stock Exchange and covers the period from December 1925 to December 1982.

[5]See for example Latane and Rendleman (1976), Chiras and Manaster (1978), Schmalensee and Trippi (1978), Manaster and Rendleman (1982), Whaley (1982), and Bhattacharya (1983).

[6]As an extreme example, consider a coin which has an unknown probability p of coming up "Heads" when tossed, where p is known to be between $^1/_4$ and

3/4. Toss the coin once and consider the estimator $\hat{p}$ which equals 1 if the coin comes up "Heads" and 0 if it comes up "Tails." Although this estimator is incorrect with probability one, it is in fact unbiased.

[7]For perhaps the weakest set of regularity conditions which insure consistency and asymptotic efficiency of maximum-likelihood estimators, see Huber (1967). As an example of a (stronger) set of sufficient conditions, assume the following (see Jorgenson (1982) and Section 4 of this paper):

(1) For all X, $\theta_1 \neq \theta_2$ implies $L(\theta_1; X) \neq L(\theta_2; X)$

(2) In some neighborhood of the true parameter value θ_0, we have:

(a) For all X and k, the first three derivataries of l_k exist.

(b) $-E\left[\frac{\partial^2 \ell_k}{\partial\theta\partial\theta'}\right] \equiv R_k(\theta)$, $R_k(\theta)$ exists, is finite and positive definite for all k.

(c) $\lim_{n\to\infty} \frac{1}{n} \sum_k R_k(\theta) \equiv R(\theta)$, $R(\theta)$ exists, is finite and positive definite for all k.

(d) The limit of n^{-2} times the sum of variances of an arbitrary linear combination of elements of the matrices $\left[\frac{\partial^2 \ell_k}{\partial\theta\partial\theta} + R_k(\theta)\right]$ is equal to zero as n approaches infinity.

(e) n^{-1} times the absolute value of the third derivative of ℓ_k is bounded above by a function with finite expectations and variance independent of the true parameter value. The limit of the average expectation of the third derivative as n approaches infinity exists and is finite. The average variance of an arbitrary linear combination of the elements of the third derivative exists and is finite.

(3) For any vector q and any scalar $\delta > 0$, $E\left[(q' \frac{\partial \ell_k}{\partial\theta})^{2+\delta}\right]$ exists and is finite, for all k.

[8]See Kendall and Stuart, (1973).

[9]It is assumed that h is constant as n increases so that T also increases. If instead T is kept constant while n increases and h decreases,

one of the regularity conditions will be violated. In particular, $\lim_{n\to\infty} \frac{1}{n} \sum_k R_k(\theta)$ is unbounded (see footnote 7 and Section 4).

[10] Of course, the distribution of $\hat{z}$ is not the Student's t since the numerator and denominator are not statistically independent. However, asymptotically it is normally distributed.

[11] I am grateful to Eric Rosenfeld for providing me with the variance estimates for these stocks.

[12] Specifically, using the Bonferroni correction for the simultaneous testing of five hypotheses at the 5% level, the appropriate critical values for a two-sided test is ±2.58 (corresponding to a tail probability of slightly less than $\frac{2.5}{5}$ %. For all five options, the associated $\hat{z}$-statistic falls within the critical region hence the simultaneous hypothesis may be rejected at the 5% level.

[13] I thank Krishna Ramaswamy for suggesting that I explore this issue.

[14] The random number generator used was the subroutine GGNQF in the IMSL software package. All computations were done in double precision FORTRAN on a Digital VAX 11/780.

[15] The complete set of simulation results are available from the author upon request.

[16] See Pearson and Hartley (1970), Table 34B.

[17] See Fama (1976), Table 1.9, p. 40.

[18] See Arnold (1974) chapter 6.

[19] See Kushner (1967).

[20] See Schuss (1980) chapter 4. Note that the particular transformation T may be derived explicitly by solving a specific differential equation given in chapter 4.

[21]For perhaps its first application in the econometrics literature, see Hausman and Wise (1983).

[22]Since dW and dN are independent and the coefficient functions do not depend on Y, we may consider the combined process as the sum of a diffusion process and a pure jump process. The probability law of γ may then be deduced by computing the convolution of the probability laws of the jump and diffusion components. Since the probability law of the diffusion component is absolutely continuous with respect to Lebesgue measure, it may be concluded that the probability law of Y is also absolutely continuous with respect to Lebesgue measure (see Chung (1974) chapter 6, problem 6) hence the conditional likelihood is well-defined. Letting V and Z represent the corresponding diffusion and jump processes respectively so that $dY = dV + dZ$, the conditional likelihood of Y may be obtained by calculating explicitly the convolution of $\rho_V(V, t)$ and $\rho_Z(Z, t)$ where:

$$\rho_V(V,\ t) = \left[2\pi\sigma^2 t\right]^{-1/2} \exp\left[\frac{(V - V_0 - \mu t)^2}{2\sigma_t^2}\right] \quad , \qquad \begin{array}{l} \sigma = \beta_0, \\ \mu = \alpha_0 - \tfrac{1}{2}\beta_0^2 \end{array}$$

$$\rho_Z(Z,\ t) = c \sum_{k=0}^{\infty} \frac{e^{-\lambda t}(\lambda t)^k}{k!} \delta(c\ Z - k) \quad , \quad c = \frac{1}{\ln(1+\gamma_0)}$$

where $\delta(cZ - k)$ is the delta-function. After some algebraic manipulations the convolution reduces to the conditional density given in Example 3.

[23]See Gel'fand and Shilov (1964) for the theory of generalized functions. As a simple example, consider the pure jump process $dX = dN$ which is simply a Poisson process with rate λ. Using the delta-function, the transition density of X may be expressed as:

$$\rho_X(X,\ t) = \sum_{k=0}^{\infty} \frac{e^{-\lambda t}(\lambda t)^k}{k!} \delta(X - k) \ .$$

In this case the forward equation given by (5) reduces to:

$$\frac{\partial}{\partial t}[\rho_X] = \lambda\left[\rho_X(X - 1, t) - \rho_X(X, t)\right] .$$

By taking the derivative of ρ_X with respect to time t, we have:

$$\frac{\partial}{\partial t}[\rho_X] = \sum_{k=0}^{\infty} \frac{-\lambda e^{-\lambda t}(\lambda t)^k}{k!} \delta(X - k) + \sum_{k=1}^{\infty} \frac{\lambda k e^{-\lambda t}(\lambda t)^{k-1}}{k!} \delta(X - k)$$

$$= -\lambda \sum_{k=0}^{\infty} \frac{e^{-\lambda t}(\lambda t)^k}{k!} \delta(X - k) + \lambda \sum_{k=0}^{\infty} \frac{e^{-\lambda t}(\lambda t)^k}{k!} \delta(X - 1 - k)$$

$$= \lambda\left[\rho_X(X - 1, t) - \rho_X(X, t)\right] .$$

[24]See Merton (1971) or Brockett (1984).

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TABLE 1. VALUES OF K_1, K_2, K_3 FOR VARIOUS TIMES-TO-MATURITY.

TAU	K_1	K_2	K_3
1.00	1.00057	0.99577	1.00425
5.00	1.00286	0.97904	1.02141
9.00	1.00515	0.96259	1.03887
13.00	1.00745	0.94641	1.05662
26.00	1.01496	0.89570	1.11645

$r = 0.100$ (annual)

$\sigma = 0.500$ (annual)

TABLE 2a. ESTIMATION RESULTS FOR LITTON CALL OPTIONS ($N = 312$, $\hat{\sigma}^2_{ML} = 0.00254$)

S	E	TAU	$\hat{F}_{ML}$ (Z-STAT)	$\sqrt{\hat{V}_F}$	$\bar{F}$
21.500	24.375	9.00	0.791 (-3.48)	6.0120E-02	1.000
21.500	15.000	9.00	6.768 (-2.68)	8.6664E-03	7.000
21.500	20.000	9.00	2.665 (-5.81)	5.7653E-02	3.000
21.500	25.000	9.00	0.647 (-2.96)	5.6121E-02	0.813
21.500	30.000	9.00	0.107 (-3.89)	2.0796E-02	0.188
21.500	24.375	22.00	-------	-------	b
21.500	15.000	22.00	7.302 (-22.06)	3.6421E-02	8.125
21.500	20.000	22.00	3.754 (-4.04)	9.1758E-02	4.125
21.500	25.000	22.00	1.665 (-2.05)	1.0232E-01	1.875
21.500	30.000	22.00	0.669 (-0.25)	7.5195E-02	0.688
21.500	24.375	29.00	-------	-------	b
21.500	15.000	29.00	-------	-------	a
21.500	20.000	29.00	4.226 (-3.81)	1.0477E-01	4.625
21.500	25.000	29.00	2.138 (-1.98)	1.1952E-01	2.375
21.500	30.000	29.00	-------	-------	b

a - Not traded. b - No option offered.

TABLE 2b. ESTIMATION RESULTS FOR N. SEMI. CALL OPTIONS ($N = 312$, $\hat{\sigma}^2_{ML} = 0.00746$)

S	E	TAU	$\hat{F}_{ML}$ (Z-STAT)	$\sqrt{\hat{V}_F}$	$\bar{F}$
23.375	15.000	5.00	8.516 (-63.41)	3.6902E-03	8.750
23.375	20.000	5.00	4.002 (5.48)	4.5971E-02	3.750
23.375	25.000	5.00	1.232 (2.39)	7.0585E-02	1.063
23.375	30.000	5.00	0.258 (3.57)	3.7205E-02	0.125
23.375	35.000	5.00	-------	-------	a
23.375	15.000	18.00	9.143 (-5.06)	4.5831E-02	9.375
23.375	20.000	18.00	5.494 (8.08)	1.0753E-01	4.625
23.375	25.000	18.00	3.049 (4.48)	1.3632E-01	2.438
23.375	30.000	18.00	1.608 (4.84)	1.2566E-01	1.000
23.375	35.000	18.00	0.825 (4.65)	9.6800E-02	0.375
23.375	15.000	25.00	9.518 (0.27)	6.7539E-02	9.500
23.375	20.000	25.00	6.120 (3.86)	1.2817E-01	5.625
23.375	25.000	25.00	3.769 (2.48)	1.5916E-01	3.375
23.375	30.000	25.00	-------	-------	b
23.375	35.000	25.00	-------	-------	b

a - Not traded. b - No option offered.

TABLE 2c. ESTIMATION RESULTS FOR TANDY CALL OPTIONS ($N = 312$, $\hat{\sigma}^2_{ML} = 0.00456$)

S	E	TAU	$\hat{F}_{ML}$	$\sqrt{\hat{V}_F}$	$\bar{F}$
28.500	22.500	1.00	6.039 (5.33E03)	5.4202E-05	5.750
28.500	25.000	1.00	3.560 (7.46)	4.1560E-03	3.250
28.500	30.000	1.00	0.266 (-1.95)	2.4046E-02	0.313
28.500	35.000	1.00	-------	-------	a
28.500	22.500	14.00	-------	-------	b
28.500	25.000	14.00	5.192 (2.20)	8.7359E-02	5.000
28.500	30.000	14.00	2.530 (0.26)	1.1495E-01	2.500
28.500	35.000	14.00	1.089 (0.92)	9.6560E-02	1.000
28.500	22.500	21.00	-------	-------	b
28.500	25.000	21.00	5.888 (-3.28)	1.1052E-01	6.250
28.500	30.000	21.00	3.321 (-1.28)	1.4002E-01	3.500
28.500	35.000	21.00	-------	-------	b

a - Not traded. b - No option offered.

TABLE 3a. SIMULATION RESULTS FOR OPTIONS AT THE MONEY AND IN/OUT OF THE MONEY BY $5 (TAU = 1).

N	F	MEAN $\hat{F}$ (STD. ER.)	% BIAS	V_F	MEAN $\hat{V}_F$ (STD. ER.)	% BIAS	MEAN $\hat{Z}$ (STD. ER.)	X^2-TEST (p-VALUE)	SKEW.	STD. RG.
(E = 35)										
100	5.2156	5.2145 (4.10E-02)	0.02	1.7752E-03	1.7916E-03 (6.98E-04)	-0.92	-0.2387 (1.09)	265.5 (0.00)	-0.803	6.61
300	5.2156	5.2151 (2.36E-02)	0.01	5.9175E-04	5.9285E-04 (1.33E-04)	-0.19	-0.1348 (1.01)	92.0 (0.00)	-0.490	6.95
500	5.2156	5.2172 (1.92E-02)	-0.03	3.5505E-04	3.6195E-04 (6.50E-05)	-1.95	-0.0087 (1.03)	62.8 (0.09)	-0.410	5.99
700	5.2156	5.2149 (1.57E-02)	0.01	2.5361E-04	2.5266E-04 (3.80E-05)	0.37	-0.1192 (1.00)	57.6 (0.19)	-0.125	6.91
(E = 40)										
100	1.6305	1.6297 (0.114)	0.05	1.2673E-02	1.2724E-02 (1.82E-03)	-0.40	-0.0811 (1.03)	75.6 (0.01)	-0.422	6.79
300	1.6305	1.6317 (6.44E-02)	-0.07	4.2244E-03	4.2372E-03 (3.42E-04)	-0.30	-0.0226 (0.99)	43.2 (0.71)	-0.151	6.14
500	1.6305	1.6300 (5.00E-02)	0.03	2.5346E-03	2.5355E-03 (1.59E-04)	-0.03	-0.0412 (0.99)	30.9 (0.98)	-0.061	6.33
700	1.6305	1.6285 (4.27E-02)	0.12	1.8104E-03	1.8072E-03 (9.68E-05)	0.18	-0.0747 (1.01)	61.7 (0.11)	-0.258	6.99
(E = 45)										
100	0.2580	0.2569 (6.04E-02)	0.44	3.7179E-03	3.7455E-03 (1.23E-03)	-0.74	-0.1924 (1.07)	108.5 (0.00)	-0.743	7.40
300	0.2580	0.2584 (3.48E-02)	-0.14	1.2393E-03	1.2474E-03 (2.35E-04)	-0.65	-0.0840 (1.01)	60.8 (0.12)	-0.349	6.93
500	0.2580	0.2584 (2.68E-02)	-0.14	7.4358E-04	7.4701E-04 (1.08E-04)	-0.46	-0.0591 (1.00)	70.5 (0.02)	-0.317	5.95
700	0.2580	0.2581 (2.34E-02)	-0.03	5.3113E-04	5.3247E-04 (6.75E-05)	-0.25	-0.0623 (1.04)	91.4 (0.00)	-0.416	6.66

TABLE 3b. SIMULATION RESULTS FOR OPTIONS AT THE MONEY AND IN/OUT OF THE MONEY BY $5 (TAU = 13).

N	F	MEAN $\hat{F}$ (STD. ER.)	% BIAS	V_F	MEAN $\hat{V}_F$ (STD. ER.)	% BIAS	MEAN $\hat{Z}$ (STD. ER.)	X^2-TEST (p-VALUE)	SKEW.	STD. RG.
(E = 35)										
100	8.7083	8.6958 (0.337)	0.15	1.1316E-01	1.1295E-01 (1.86E-02)	0.19	-0.1233 (1.03)	103.0 (0.00)	-0.354	6.98
300	8.7089	8.7152 (0.197)	-0.07	3.7719E-02	3.7892E-02 (3.62E-03)	-0.46	-0.0164 (1.02)	58.6 (0.16)	-0.275	6.46
500	8.7089	8.7120 (0.157)	-0.04	2.2631E-02	2.2688E-02 (1.73E-03)	-0.25	-0.0191 (1.05)	59.7 (0.14)	-0.118	5.88
700	8.7089	8.7126 (0.126)	-0.04	1.6165E-02	1.6204E-02 (9.94E-04)	-0.24	-0.0017 (1.00)	47.8 (0.52)	-0.164	6.24
(E = 40)										
100	6.1384	6.1146 (0.393)	0.39	1.5577E-01	1.5516E-01 (2.13E-02)	0.40	-0.1303 (1.02)	68.6 (0.03)	-0.341	7.03
300	6.1384	6.1326 (0.231)	0.09	5.1925E-02	5.1898E-02 (4.18E-03)	0.05	-0.0664 (1.02)	60.7 (0.12)	-0.215	6.92
500	6.1384	6.1352 (0.172)	0.05	3.1155E-02	3.1146E-02 (1.86E-03)	0.03	-0.0472 (0.98)	33.0 (0.96)	-0.243	6.27
700	6.1384	6.1345 (0.148)	0.06	2.2253E-02	2.2237E-02 (1.14E-03)	0.07	-0.0514 (1.00)	51.0 (0.39)	-0.320	6.62
(E = 45)										
100	4.2346	4.2075 (0.405)	0.64	1.6446E-01	1.6356E-01 (2.40E-02)	0.54	-0.1426 (1.03)	83.0 (0.00)	-0.402	6.95
300	4.2346	4.2275 (0.229)	0.17	5.4819E-02	5.4754E-02 (4.56E-03)	0.12	-0.0712 (0.98)	39.1 (0.84)	-0.015	6.38
500	4.2346	4.2230 (0.181)	0.27	3.2891E-02	3.2783E-02 (2.14E-03)	0.33	-0.0970 (1.01)	64.8 (0.06)	-0.283	6.40
700	4.2346	4.2397 (0.153)	-0.12	2.3494E-02	2.3552E-02 (1.30E-03)	-0.25	0.0054 (1.00)	48.9 (0.48)	-0.203	6.27

TABLE 3c. SIMULATION RESULTS FOR OPTIONS AT THE MONEY AND IN/OUT OF THE MONEY BY $5 (TAU = 26).

N	F	MEAN $\hat{F}$ (STD. ER.)	% BIAS	V_F	MEAN $\hat{V}_F$ (STD. ER.)	% BIAS	MEAN $\hat{Z}$ (STD. ER.)	X^2-TEST (p-VALUE)	SKEW.	STD. RG.
(E = 35)										
100	11.1468	11.1372 (0.472)	0.09	2.2810E-01	2.2824E-01 (3.38E-02)	-0.06	-0.0953 (1.01)	64.6 (0.07)	-0.419	6.74
300	11.1468	11.1386 (0.276)	0.07	7.6034E-02	7.5891E-02 (6.60E-03)	0.19	-0.0802 (1.01)	50.9 (0.40)	-0.244	6.00
500	11.1468	11.1490 (0.222)	-0.02	4.5621E-02	4.5689E-02 (3.19E-03)	-0.15	-0.0259 (1.04)	40.4 (0.80)	-0.135	7.68
700	11.1458	11.1461 (0.180)	0.01	3.2586E-02	3.2597E-02 (1.84E-03)	-0.03	-0.0319 (1.00)	59.6 (0.14)	-0.230	6.72
(E = 40)										
100	8.8266	8.8137 (0.561)	0.15	2.9320E-01	2.9344E-01 (4.05E-02)	-0.08	-0.0971 (1.06)	94.5 (0.00)	-0.422	6.45
300	8.8266	8.8503 (0.326)	-0.27	9.7734E-02	9.8438E-02 (7.88E-03)	-0.72	0.0339 (1.05)	97.0 (0.00)	-0.270	6.91
500	8.8266	8.8161 (0.245)	0.12	5.8640E-02	5.8534E-02 (3.54E-03)	0.18	-0.0740 (1.02)	67.1 (0.04)	-0.172	6.36
700	8.8266	8.8248 (0.198)	0.02	4.1885E-02	4.1889E-02 (2.05E-03)	-0.01	-0.0320 (0.97)	38.3 (0.86)	-0.077	6.75
(E = 45)										
100	6.9661	6.9359 (0.573)	0.43	3.2651E-01	3.2540E-01 (4.42E-02)	0.34	-0.1221 (1.02)	63.6 (0.08)	-0.276	6.86
300	6.9661	6.9685 (0.334)	-0.04	1.0884E-01	1.0904E-01 (8.59E-03)	-0.18	-0.0324 (1.01)	33.7 (0.95)	-0.144	6.59
500	6.9661	6.9648 (0.249)	0.02	6.5302E-02	6.5328E-02 (3.84E-03)	-0.04	-0.0340 (0.98)	56.7 (0.21)	-0.220	5.99
700	6.9661	6.9829 (0.222)	-0.24	4.6644E-02	4.6856E-02 (2.45E-03)	-0.45	0.0510 (1.03)	51.1 (0.39)	-0.137	7.03

TABLE 4a. SIMULATION RESULTS FOR OPTIONS IN/OUT OF THE MONEY BY $15 (TAU = 1).

N	F	MEAN $\hat{F}$ (STD. ER.)	% BIAS	V_F	MEAN $\hat{V}_F$ (STD. ER.)	% BIAS	MEAN $\hat{z}$ (STD. ER.)	X^2-TEST (p-VALUE)	SKEW.	STD. RG.
(E = 25)										
100	15.0458	15.0458 (4.29E-06)	0.00	1.6991E-12	4.0050E-11 (3.17E-010)	-2257.10	-13.2558 (236.07)	2525.3 (0.00)	-28.549	30.46
300	15.0458	15.0458 (1.26E-06)	0.00	5.6637E-13	2.3736E-12 (6.24E-12)	-319.09	-0.7866 (3.07)	1055.7 (0.00)	-7.566	14.71
500	15.0458	15.0458 (7.62E-07)	0.00	3.3982E-13	7.9271E-13 (1.39E-12)	-133.27	-0.5064 (1.70)	650.3 (0.00)	-3.431	11.61
700	15.0458	15.0458 (6.25E-07)	0.00	2.4273E-13	4.7517E-13 (7.09E-13)	-95.76	-0.4184 (1.67)	518.6 (0.00)	-4.925	15.99
(E = 55)										
100	0.0010	0.0013 (1.16E-03)	-31.41	7.7194E-07	1.7998E-06 (3.16E-06)	-133.16	-0.6481 (2.26)	744.5 (0.00)	-5.039	14.24
300	0.0010	0.0011 (5.56E-04)	-8.37	2.5731E-07	3.4515E-07 (3.27E-07)	-34.13	-0.3629 (1.40)	416.4 (0.00)	-2.167	7.91
500	0.0010	0.0011 (4.51E-04)	-9.52	1.5439E-07	2.0034E-07 (1.53E-07)	-29.76	-0.1458 (1.12)	191.9 (0.00)	-0.995	6.78
700	0.0010	0.0010 (3.49E-04)	-4.22	1.1028E-07	1.2704E-07 (7.84E-08)	-15.20	-0.1729 (1.07)	141.0 (0.00)	-0.828	7.04

TABLE 4b. SIMULATION RESULTS FOR OPTIONS IN/OUT OF THE MONEY BY $15 (TAU = 13).

N	F	MEAN $\hat{F}$ (STD. ER.)	% BIAS	V_F	MEAN $\hat{V}_F$ (STD. ER.)	% BIAS	MEAN $\hat{Z}$ (STD. ER.)	X^2-TEST (p-VALUE)	SKEW.	STD. RG.
(E = 25)										
100	16.0252	16.0250 (0.123)	0.00	1.4982E-02	1.5294E-02 (6.19E-03)	-2.08	-0.2230 (1.12)	135.3 (0.00)	-0.926	7.76
300	16.0252	16.0271 (6.77E-02)	-0.01	4.9940E-03	5.0580E-03 (1.12E-03)	-1.28	-0.0800 (0.98)	71.5 (0.02)	-0.409	7.21
500	16.0252	16.0244 (5.26E-02)	0.00	2.9964E-03	3.0004E-03 (5.18E-04)	-0.13	-0.0997 (0.99)	74.9 (0.01)	-0.406	6.28
700	16.0252	16.0268 (4.65E-02)	-0.01	2.1403E-03	2.1581E-03 (3.28E-04)	-0.83	-0.0433 (1.02)	72.8 (0.02)	-0.410	7.51
(E = 55)										
100	1.9301	1.9212 (0.341)	0.46	1.1033E-01	1.1040E-01 (2.62E-02)	-0.06	-0.1578 (1.10)	117.6 (0.00)	-0.643	6.06
300	1.9301	1.9275 (0.198)	0.13	3.6778E-02	3.6794E-02 (5.06E-03)	-0.04	-0.0862 (1.05)	63.3 (0.01)	-0.366	6.78
500	1.9301	1.9330 (0.148)	-0.15	2.2067E-02	2.2139E-02 (2.28E-03)	-0.33	-0.0322 (1.00)	51.0 (0.39)	-0.239	6.64
700	1.9301	1.9241 (0.122)	0.31	1.5762E-02	1.5709E-02 (1.34E-03)	0.33	-0.0902 (0.99)	60.1 (0.13)	-0.361	7.18

TABLE 4c. SIMULATION RESULTS FOR OPTIONS IN/OUT OF THE MONEY BY $15 (TAU = 26).

N	F	MEAN $\hat{F}$ (STD. ER.)	% BIAS	V_F	MEAN $\hat{V}_F$ (STD. ER.)	% BIAS	MEAN $\hat{Z}$ (STD. ER.)	X^2-TEST (p-VALUE)	SKEW.	STD. RG.
(E = 25)										
100	17.4116	17.4260 (0.262)	-0.08	6.5952E-02	6.7556E-02 (1.88E-02)	-2.43	-0.0918 (1.06)	95.4 (0.00)	-0.558	6.06
300	17.4116	17.4045 (0.150)	0.04	2.1984E-02	2.1881E-02 (3.57E-03)	0.47	-0.1332 (1.05)	79.8 (0.00)	-0.385	6.31
500	17.4116	17.4115 (0.111)	0.00	1.3191E-02	1.3210E-02 (1.58E-03)	-0.15	-0.0584 (0.97)	62.6 (0.09)	-0.181	7.02
700	17.4116	17.4117 (9.81E-02)	0.00	9.4218E-03	9.4351E-03 (1.00E-03)	-0.14	-0.0522 (1.02)	78.5 (0.00)	-0.212	6.42
(E = 55)										
100	4.3258	4.2721 (0.550)	1.24	3.0674E-01	3.0283E-01 (5.23E-02)	1.28	-0.1864 (1.03)	86.3 (0.00)	-0.325	6.28
300	4.3258	4.3178 (0.327)	0.18	1.0225E-01	1.0213E-01 (1.04E-02)	0.11	-0.0773 (1.03)	35.0 (0.93)	-0.118	6.20
500	4.3258	4.3232 (0.255)	0.06	6.1348E-02	6.1349E-02 (4.86E-03)	0.00	-0.0517 (1.04)	65.0 (0.06)	-0.283	7.22
700	4.3258	4.3272 (0.211)	-0.03	4.3820E-02	4.3864E-02 (2.88E-03)	-0.10	-0.0263 (1.01)	62.9 (0.09)	-0.069	6.48