ON THE OPTIMALITY OF PORTFOLIO INSURANCE

by

Simon Benninga and Marshall Blume

Working Paper No. 16-84

THE WHARTON SCHOOL

University of Pennsylvania

Philadelphia, Pa 19104

ON THE OPTIMALITY OF PORTFOLIO INSURANCE

by

Simon Benninga and Marshall Blume*

Lastest Draft: October 8, 1984

*Finance Department, The Wharton School, University of Pennsylvania

We wish to thank Gunther Francke, Bruce Jacobs and Richard Rogalski for helpful discussions on this topic. All errors are, of course, ours.

I. Introduction

Portfolio insurance has always had an intuitive appeal to investors, particularly if the cost is not too great. What could be better than the possibility of making substantial sums of money with no chance of loss? Responding to this appeal, commercial organizations began selling insurance against investment loss in the United Kingdom in 1956 and in the United States in 1971. Gatto, Geske, Litzenberger, and Sosin (1980) provide an excellent and insightful analysis of the specific insurance plans that were offered to individuals by Harleysville Mutual Insurance Company and Prudential Insurance Company of America. Nowadays, several firms are marketing to institutions portfolio "insurance" techniques under such names as "Dynamic Asset Allocation/Protector Portfolio Management," "Portfolio Insulation," and "Portfolio Risk Control."

There are a number of ways to define portfolio insurance. Leland (1980) proposes one such definition in the context of a two-date world. Most of the analysis in this paper is consistent with his definition. At the first date, the investor has some wealth to invest so as to maximize his expected utility at the second date. An insured strategy is one in which the investor places part of his wealth in a portfolio of risky assets and uses his remaining wealth to buy a European put on that portfolio. If the value of the portfolio on the second date is less than the striking price of the put, he exercises the put; otherwise, he lets the put expire. Thus, he has insured his portfolio at the striking price of the put.

The purpose of this paper is to explore conditions under which an investor would utilize portfolio insurance as part of an overall strategy. Whether an invester would utilize portfolio insurance hinges, among other things, upon the completeness of the market. In the complete markets that Black and

Scholes postulate in deriving their option pricing model, the utility function that an investor would have to have is so bizarre that it is highly unlikely that any investor would purchase insurance. However, in some types of less complete markets, investors with plausible utility functions may find it desirable to buy insurance. The desirability of insurance thus hinges upon the way in which incompleteness is introduced into the markets of Black and Scholes. The paper concludes with an examination of some alternative definitions of portfolio insurance in a multi-date world.

II. Portfolio Insurance in a Two-Date Model

The development of the Black-Scholes option pricing model assumes that the returns on risky assets conform to Weiner processes and that there is continuous trading. Under their assumptions, there exists a trading strategy by which an investor can create an asset whose value will be indistinguishable from a European put. The investor can then use this so-called pseudo-put to insure a risky portfolio.

In the first part of the section, we define the required notation and show how to insure a portfolio. In the second part, we develop some propositions as to the investment strategy implicit in an insured portfolio. In the final part, we utilize these propositions to infer the characteristics of the von Neumann-Morgenstern utility function that would result in an investor insuring his risky portfolio.

A. Constructing the Two-Date Portfolio Insurance Policy[1]

As already pointed out, an investor can insure a portfolio of risky

---

[1] The problem discussed in this sub-section resembles that of pricing an equity-linked life insurance policy (see Brennan and Schwartz, 1976).

assets by purchasing a put on that portfolio with a striking price equal to the desired insurance level. Specifically, if the investor at time 0 has wealth W_0, he could insure his portfolio at K by using part of his wealth to purchase stock and the remainder to purchase a European put with striking price K. Let S_0 represent the investment in the stock, and P_0 the price of the put. In conformity with Black-Scholes, assume that the share price follows a Wiener process with expected return α and standard deviation σ, the risk-free rate is fixed at r, that there are no dividends, and that trading can take place continuously.

It is well known that a put can be replicated through the continuous revision of a portfolio with a long position in the risk-free asset and a short position in the stock. More specifically, the put can be duplicated by holding, for any $0 \leqslant t \leqslant 1$,

$P_t + N(-h)S_t$ of a risk-free discount bond maturing at $t = 1$,

$-N(-h)S_t$ of the stock,

where

$$\tau = 1 - t$$

$$h = \log(S_t/Ke^{-r\tau})/\sigma\sqrt{\tau} + {}^1\!/_2\,\sigma\sqrt{\tau}$$

$N(\ )$ is the standard cumulative normal distribution

$$P_t = -[S_t N(-h) - Ke^{-r\tau}N(\sigma\sqrt{\tau} - h)] \, .$$

This portfolio is "self-financing" and exactly replicates the value of the put over time.

Thus, even if there is no traded put on the stock, there exists a portfolio that replicates the put and thus provides the desired insurance policy. At any time t, the composition of an insured portfolio will be

$$S_t\left(1 - N(-h)\right) \qquad \text{of the stock,}$$

$$P_t + N(-h)S_t \qquad \text{of the riskfree asset.}$$

The proportion of the investor's wealth held in the risky asset at time t, ω_t, is given by

$$\begin{aligned}\omega_t &= \frac{S_t\left(1 - N(-h)\right)}{S_t\left(1 - N(-h)\right) + \left(P_t + N(-h)S_t\right)} \\ &= \frac{S_t\left(1 - N(-h)\right)}{S_t + P_t} \qquad (1) \\ &= \frac{S_t\left(1 - N(-h)\right)}{S_t\left(1 - N(-h)\right) + Ke^{-r\tau}N(\sigma\sqrt{\tau} - h)} .\end{aligned}$$

B. The Two-Date Model: The Behavior of the Portfolio Proportions Over Time

The portfolio policy implicit in a portfolio insurance scheme can be characterized by the behavior of ω_t over time. The first two propositions in this section set bounds for feasible levels of insurance. The next three propositions describe the behavior of ω_t from $t = 0$ to $t = 1$.

Proposition 1: If for any t, $0 \leqslant t < 1$, $K = W_t e^{r\tau}$, then $\omega_t = 0$.

Proof:

We shall prove the proposition for $t = 0$. If $K = W_0 e^r$,

$$\begin{aligned}W_0 &= S_0 + P_0 \\ &= S_0 + \left[-S_0 N(-h) - Ke^{-r}N(\sigma - h)\right] \qquad (2) \\ &= S_0\left(1 - N(-h)\right) + W_0 N(\sigma - h) .\end{aligned}$$

One solution of the equation is $h = -\infty$. Since the derivative of the right-hand side of (2) with respect to h is monotonic in h, this solution is the only solution. Substituting into (1) reveals that $\omega_t = 0$. QED

Proposition 2: If $K > W_t e^{r\tau}$, there is no feasible insurance policy.

Proof:

As before, it suffices to prove the proposition for $t = 0$. The proof follows directly from the monotonicity of the right-hand side of (2) used in the first proposition. Note that as K increases, h must decrease. But for $K = W_0 e^r$, $h = -\infty$. This is therefore the limit of feasibility. QED

Proposition 3: $W_t > Ke^{-r\tau}$ for all $1 > \tau > 0$, $1 > \tau > 0$, provided $K < W_0 e^{-r}$.

Proof:

It is well-known (see, for example, Jarrow and Rudd 1983, p. 219) that when $S_t = 0$, $P_t = Ke^{-r\tau}$. Furthermore, $\partial P/\partial S = -N(-h) > -1$. But the line $Ke^{-r\tau} - S$ has slope -1 (with S as the independent variable), and thus this line lies below P. Therefore

$$W_t = P_t + S_t > Ke^{-r\tau} - S_t + S_t = Ke^{-r\tau} .$$

QED

Proposition 4: If $K \leq S_t$, $\omega_t > 1/2$, $0 \leq t < 1$.

Proof:

Throughout the proof we drop the index on S_t. It follows from (1) that

$$\omega_t = \frac{1}{1 + \frac{K}{S} e^{-r\tau} \frac{N(\frac{1}{2}\sigma\sqrt{\tau} - r\sqrt{\tau}/\sigma - \log(S/K)/\sigma\sqrt{\tau})}{N(\frac{1}{2}\sigma\sqrt{\tau} + r\sqrt{\tau}/\sigma + \log(S/K)/\sigma\sqrt{\tau})}} \tag{3}$$

Consider the fraction which appears in the denominator of (3). As long as $K \leq S$,

$$1/2\,\sigma\sqrt{\tau} - r\sqrt{\tau}/\sigma \;\; - \log(S/K)/\sigma\sqrt{\tau} < 1/2\,\sigma\sqrt{\tau} + r\sqrt{\tau}/\sigma + \log(S/K)/\sigma\sqrt{\tau} \; .$$

Since in this case the whole denominator is less than 2, the proposition is proved. QED

It is tempting to hypothesize that when $K > S_t$, $\omega_t \leqslant 1/2$. A simple continuity argument shows, however, that this need not be true. By Proposition 4, $\omega_t > 1/2$ when $K = S$. Since ω_t is continuous in K and S, a small change in K/S will still leave $\omega_t > 1/2$.

The last proposition of this sub-section deals with the limiting distribution of assets as $t \to 1$, i.e., as we approach the end of the planning period.

Proposition 5: As $t \to 1$ (i.e., as $\tau \to 0$):

$$\lim_{t \to 1} \omega_t = \begin{cases} 1 & \text{if } S_t > K \\ 1/2 & \text{if } S_t = K \\ 0 & \text{if } S_t < K \end{cases}$$

Proof:

We may use equation (1) to calculate the limiting insurance portfolio proportions as $\tau \to 0$. To do this, write

$$h = \frac{\log(S_t/K)}{\sigma\sqrt{\tau}} + \frac{r}{\sigma}\sqrt{\tau} + 1/2\sqrt{\tau}$$

As $\tau \to 0$, we may differentiate among three cases:

Case 1: $S_t = K$. In this case $h \to 0$ as $\tau \to 0$, and $N(h) = N(-h) \to 1/2$.

Case 2: $S_t > K$. In this case $h \to \infty$ as $\tau \to 0$, and $N(h) \to 1$, $N(-h) \to 0$.

Case 3: $S_t < K$. In this case $h \to \infty$ as $\tau \to 0$, and $N(h) \to 0$, $N(-h) \to 1$.

The proof of the proposition now follows from (1). QED

C. The Implied Utility Function

Whether the insurance policy analyzed above is optimal depends, of course, upon the investor's utility function. The purpose of this subsection is to determine what type of utility function would dictate such an insurance strategy. The logic will be to assume that an investor has solved the usual expected utility maximization problem and that the solution is an insurance strategy. We shall then ask what the original utility function must have been to have obtained that solution.

Consider an investor who at each time t, $0 \leq t \leq 1$, chooses a portfolio composed of a proportion ω_t of a single risky asset and a proportion $1 - \omega_t$ of a riskless asset with return $r > 0$ so as to maximize his expected utility of end-of-period wealth. The investor's utility maximization problem may be written:

$$\max EU[W_1]$$

s.t.

$$W_t \geq 0,\ 0 \leq t \leq 1 ,$$

$$W_0 = \overline{W}_0 > 0$$

$$dW = W_t\{[\omega_t(\alpha - r) + r]dt + \omega_t \sigma(t) dZ\} ,$$

where dZ is a Weiner process. We make the usual assumptions that $U' > 0$ and $U'' < 0$.

Merton (1969), Friend and Blume (1975), and Ross (1975) show that an investor's coefficient of relative risk aversion C_t in an optimal strategy must satisfy

$$\omega_t = \frac{(\alpha - r)}{\sigma^2} \frac{1}{C_t} . \tag{4}$$

This relationship shows how the coefficient of relative risk aversion of an investor who chooses the two-date portfolio insurance strategy will vary over time and as a function of wealth. Namely, the coefficient of relative risk aversion will be given by

$$C_t = \frac{1}{\omega_t} \frac{(\alpha - r)}{\sigma^2} = \frac{S_t\big(1 - N(-h)\big) + Ke^{-r\tau}N(\sigma\sqrt{\tau} - h)}{S_t\big(1 - N(-h)\big)} \quad \frac{\alpha - r}{\sigma^2} . \tag{5}$$

Equation (5) and Propositions 4 and 5 imply that one-period insurance strategies can in general be optimal only for utility maximizers who have a non-constant coefficient of relative risk aversion. In the limit the coefficient of relative risk aversion may be infinite; this will occur, when as $t \to 1$, $S_1 < K$. More generally, for given K, $t < 1$, and other parameters, C_t decreases with increases in wealth. Thus, an investor must have decreasing proportional risk aversion function up to the exercise date of the put. This argument leads to the theorem:

Theorem 1: Under the assumptions made in the Black-Scholes model, the investor's utility function must have an unbounded coefficient of proportional risk aversion at some level of wealth if the investor finds it optimal to insure his total portfolio at some positive level.

Proof:

It follows from the previous discussion that with continuous revision of portfolios the full insurance strategy requires an unbounded coefficient of proportional risk aversion at the end of the period. QED

III. Less Complete Markets

Under the assumptions of the Black-Scholes model, the end-of-period utility function of an investor who insured his portfolio at some level would have to exhibit an unbounded coefficient of relative risk aversion below the insurance level and decreasing relative risk aversion above that level. The available empirical evidence is not consistent with this type of utility function, but rather points towards constant relative risk aversion over wide ranges of wealth.[1] Moreover, most of the empirical evidence suggests a coefficient of relative risk aversion of around two. If all of the Black-Scholes assumptions are made, an investor with a utility function displaying constant relative risk aversion would not insure a portfolio. However, in a world that is less complete than that of Black-Scholes, an investor might decide to insure a portfolio.

A. An Extreme Example

In this section, we examine this possibility. Throughout we assume that markets are segmented in the following way: One set of investors have available to them the Black-Scholes markets and can avail themselves of continuous trading opportunities. These investors price the assets in the market. In particular, we assume that there are enough of these investors to guarantee that options are priced at their Black-Scholes prices. Another group of investors can trade only at the beginning of the time period and face an incomplete market. A typical investor in this latter group will be assumed throughout the section to have constant proportional risk aversion.

Let us begin with an extremely incomplete market. For the moment, assume that the investor is prohibited, perhaps for some institutional reason, from investing in risk-free assets and that there is only one risky asset available to the investor. Although the investor cannot invest in risk-free assets, he

---

1 Cf. Friend and Blume (1975) for evidence in the U.S. and Morin and Suarez (1983) for evidence in Canada.

is permitted to purchase a put on the risky asset with any striking price that he sets. The put itself is priced by the Black-Scholes model.

In this highly stylized world, the only decision that the investor faces is the striking price of the put, and this decision must be made at the start of the period. Subsequently, we shall consider more realistic scenarios.

If the investor's initial wealth and the price of one share are each one dollar, he could buy one share of the risky asset with no insurance. Alternatively, he could buy a fraction of a share and a put on that asset with striking price K. If the price of a put on one share with striking price K is P, the investor would buy $1/(1 + P)$ share and the same fraction of the put. Thus, the investor's wealth at the end of the period, W_1, will be

$$W_1 = \max(S_1, K)/(1 + P) , \tag{6}$$

where S_1 is the value of one share at the end of the period.

Now, assume that the investor's utility function is of the constant relative risk aversion form

$$U(W_1) = \frac{1}{1 - \gamma} W_1^{1-\gamma} \tag{7}$$

with the coefficient of relative risk aversion γ taking on the values of 2, 4, 8, and 16. As indicated above, most of the empirical evidence points to the lower end of this range. Once one assumes specific values for the parameters of the Weiner process generating the risky return and the risk-free rate that is used in the Black-Scholes formula, it is possible to determine the optimal striking price for each coefficient of relative risk aversion through techniques of numerical analysis.

We use two sets of assumptions for the underlying parameters. The first set assumes that the Weiner process for the risky asset has a drift term of

0.15 and a standard deviation of 0.20. Together, these imply that the expected one-period return is 18.5 percent.[1] The drift term, or equivalently the continuously compounded rate of return, for the risk-free asset is 0.10 for an effective rate of 10.5 percent.

The second set assumes for the risky asset a drift term of 0.04 and a standard deviation of 0.20 for an expected one-period return of 6.2 percent. The continuously compounded rate of return for the risk-free asset is 1.0 percent for an effective rate of just over 1 percent. Some might view the first set as roughly approximating the nominal rates of today, and the second set as roughly approximating the real rates since the turn of the century.[2]

Using the first set of assumptions, an investor with a relative risk coefficient of 2.0 would realize his highest level of utility by purchasing a put with a striking price of 0.70 dollars.[3] The reader should recall that the

---

1 The end of period return is log normal. Therefore, one plus the expected one-period return is $\exp(0.15 + .5 \times .04)$, where exp is the exponential formula.

2 Cf., Stocks, Bonds, Bills and Inflation Yearbook (1926-1983).

e In calculating the expected utility for any specific strategy, the integral was evaluated from the drift term of the risky asset minus 5 standard deviations to the drift term plus five standard deviations. The interval from minus 5 to plus 5 standard deviations was divided into 100 equal subintervals. Increasing the number of subintervals to 500 produced no change in the expected utility values up to six places. The function to be integrated (the utility function times the density) was evaluated at the midpoint of each subinterval and then multiplied by the length of the subinterval. To minimize rounding errors, these resulting areas were summed alternately from the extremes, first using the smallest area on the left and then the smallest area on the right and so on.

If the strike price was more than 5 standard deviations below the drift term, the integral was calculated as if the strike price was zero. Thus, in the tables, any strike price of zero should really be interpreted as a strike price of more than 5 standard deviations below the drift term. Using any striking price between these two extremes would produce virtually the same expected utility, the same portfolio allocations up to the precision shown in the tables. Finally, a gradient procedure was used to determine the optimal strike price.

investor's wealth and one share of the risky asset are each assumed to be one dollar. Since the investor must pay a premium for the put, he could only buy a fraction of a full share. The insurance strategy requires that the investor purchase the same fraction of a put and the underlying stock. In this case, he would buy 99.9 percent of a share of stock and 99.9 percent of a put with exercise price of 0.70 dollars. Since the investor purchases less than a full put, the insurance level is somewhat less than 0.70 dollars.

In deriving these numbers, the computer program only considered striking prices in the range of zero to one dollar. The theoretical results from the previous section indicate that it is possible to insure a portfolio at a somewhat greater amount than the sum invested in the risky asset and that the maximum insurance level is determined by the rate of return on the risk-free asset. Nonetheless, the assumption that the maximum exercise price for a put can never exceed one dollar is a reasonable one for our purposes. First, purchasing some fraction of a put with a maximum striking price of one dollar is always feasible for any positive risk-free rate. Second, it seems in the spirit of an insurance policy not to insure one's wealth for more than its current value.

Thus, an investor in this particular case would choose to buy a put. An important question, especially for public policy, is how valuable is the availability of such a put to an investor. A measure of how valuable the put is to the investor is the amount that the investor would pay to have such a put available.

Following Goldman (1974), an answer to this question is obtainable as follows: Let $E\{U(W_1)\}$ be the expected utility of the end of period wealth associated with some arbitrary and generally non-optimal policy. For the moment, let this policy be an investment without insurance. If puts are valuable to an investor, he should be willing to pay some fraction of his end of period wealth to be able to purchase a put. Let π be the fraction of his wealth that he would pay. After paying the proportion π, the investor's expected utility with a put having an optimal striking price is $E\{U^*[(1 - \pi)W_1]\}$, where the superscript "*" indicates an optimal strategy. The value of π that equates the expected utility without a put to that with an optimal put is the maximum that an investor would pay to have the option of buying the put.

Equating these two expected utilities yields

$$E\{U^*[(1 - \pi)W_1]\} = E\{U(W_1)\}$$

or

$$E\{\frac{1}{1 - \gamma}[(1 - \pi)W_1]^{1-\gamma}\} = E\{\frac{1}{1 - \gamma} W_1^{1-\gamma}\} \tag{8}$$

$$(1 - \pi)^{1-\gamma} E[U^*(W_1)] = E[U(W_1)]$$

Finally solving for π gives the desired formula:

$$\pi = 1 - \left[\frac{E\{U(W_1)\}}{E\{U^*(W_1)\}}\right]^{\frac{1}{1-\gamma}} \tag{9}$$

An investor with a coefficient of relative risk aversion of 2.0 and facing the set of return assumptions corresponding roughly to today's nominal interest rates would be willing to pay up to 0.001 percent of his wealth in order to be able to buy an optimal put. Table 1 contains the results of these and similar calculations for values of the coefficient of relative risk ranging from two to sixteen for both sets of assumed return characteristics.

coefficient of relative risk aversion of two would use puts but would not find them very valuable. However, as the coefficient of relative risk aversion increases to four and beyond, the optimal striking price becomes the maximum obtainable and the value of the put to an investor increases dramatically.

B. A More Complete World

In the incomplete market assumed above, an investor with a reasonable utility function and under plausible return assumptions would utilize an insurance strategy by buying a put. Let us now examine what happens in a somewhat more complete world. Specifically, assume that the investor can, in addition to buying a put, lend money at the risk-free rate. As before, the investor cannot trade after the start of the period. The investor's decisions are what fraction of his wealth to invest in risk-free assets, the striking price of any put that he buys, and the proportion of his wealth to place in the risky asset.

Table 2 contains the optimal investment strategies for the various utility functions and for the two sets of return assumptions.[1] A most obvious feature of this table is that none of the optimal strategies involves the purchase of a put. The investor can achieve an optimal investment strategy using just the risky and risk-free asset. At least for these ranges of utility functions and return assumptions, the availability of a single put does not allow an investor to increase his expected utility.[2]

---

1 The optimal policy was calculated by a gradient search.

2 As can be seen from Table 2, the investor would like to short sell the put under the assumed distributional parameters. The computer calculated optimal decisions in a world in which the investor was allowed to short sell the same number of puts as he bought shares. As an example, under the lower return assumptions, an investor with a coefficient of relative risk aversion of r achieved maximum utility with the following portfolio: 38% in the risk-free asset, 62.002% in the risky asset, and a short (continued)

Moreover, given a choice as between adding a put or a risk-free investment, an investor would prefer the addition of the risk-free asset. From Table 1, an investor with a coefficient of relative risk aversion of 4 and facing the higher return assumptions would be willing to pay up to 1.47 percent of his wealth to have a put included in the market place, but 2.49 percent to have a risk-free asset included. The same qualitative conclusion applies to the lower return assumptions.

Of course, there are undoubtedly utility functions and return assumptions for which an investor would prefer to have a put rather than a risk-free asset. What is needed is a sufficiently great coefficient of relative risk aversion below some wealth level and a sufficient probability that the value of the risky asset be below that level. What this paper has demonstrated is that there exists reasonable utility functions and return assumptions for which an investor would not utilize an insurance strategy if a risk-free asset is available.

IV. Portfolio Insurance in Other Contexts

In this section we consider some other possible definition of portfolio insurance in complete markets.

A. The Multiple One-Period Model of Portfolio Insurance

Although the two-date model with the second date many years into the future may have some intuitive appeal for an individual, it probably has little intuitive appeal for most institutional investors. As an example, the spending plans of a college are often related to the size of the endowment.

---

sale of .62 of one put with exercise price of .59. The results were similar for other parameters. The general conclusion is that investors with constant proportional risk aversion of the types examined here are not greatly benefited by the ability either to buy or sell puts.

As the endowment increases, the college may often increase its financial commitments. Such a college would find little value in an insurance plan that insures against loss at some distant date.

However, an insurance policy that insures against loss at the end of a fiscal year may have some value to such a college. At the beginning of each fiscal year, the college might insure its endowment against loss and thus be confident that its budget plans could be implemented. At the beginning of the next year, the college might enter again into an insurance contract, insuring its endowment against loss, and so on through the years.[1]

More formally, the successive two-date model consists of successive non-overlapping two-date worlds in which the first date of one two-period world is the second date of the preceding two-period world. The insurance level in each two-date world is set as a proportion of wealth level achieved at the end of the prior two-period world.

In this model the portfolio strategy is "reset" at intervals $t = 0, 1, \ldots$. Thus the proportion of risky assets in the portfolio corresponds to

$$\omega_t = \frac{S_t\big(1 - N(-h)\big)}{S_t\big(1 - N(-h)\big) + Ke^{-r\tau}N(\sigma\sqrt{\tau} - h)} ,$$

$$\text{for } t = n + 1 - \tau,\ 0 < \tau \leq 1,\ n = 0, 1, \ldots .$$

The multiple one-period model (which most closely corresponds to the portfolio insurance plans currently marketed) thus involves <u>continuous intra-period</u> portfolio adjustment and <u>discontinuous inter-period</u> portfolio adjustment.

---

1 The legal aspects of the insurance policy may be problematic. See Pozen (1978).

The utility implications of this strategy are strange: The investor's coefficient of relative risk aversion behaves within periods as in the one-period case, but reverts at one-period intervals to some fixed value which depends on the initial insurance level chosen by the investor.

B. Portfolio Insurance With a Moving Horizon

An investor might have a fixed horizon such as five years at the end of which he insures his portfolio against any losses in excess of ϕW_0. As time marches on, the investor might continually revise his insurance so as to keep his horizon constant at, say, five years and the level of insurance constant at a proportion ϕ of his then current level of wealth.

Thus, he will continually be revising his insurance policy. This type of insurance might have some intuitive appeal to an investor, who has some distant horizon after which he is uncertain as to his specific desires, and as he moves through time, his horizon moves with him.

In this moving horizon model, the investor continually faces a two-date insurance problem. However, the two dates are moving over time and the insurance level is continually revised as a proportion of the investor's then current wealth.

In this case τ remains constant. Setting $\tau = T$, we thus find that

$$\omega_t = \frac{S_t\big(1 - N(-H)\big)}{S_t\big(1 - N(-H)\big) + Ke^{-rT}N(\sigma\sqrt{T} - H)}$$

where

$$H = \log(S_t/Ke^{-rT})/\sigma\sqrt{T} + \tfrac{1}{2}\sigma\sqrt{T} .$$

The use of H denotes the fact that the only time dependence in this model is caused by the change in the stock price; this follows since the horizon is fixed. Note also that in the moving horizon case the put portfolio is no longer necessarily self-financing—it may produce a profit or a loss.

In the special case where the insurance level at time t is a constant proportion of the then current wealth, the implied utility function exhibits constant proportional risk aversion. Thus, an investor with such a utility function can always be viewed as facing a dynamic portfolio insurance strategy, but in an almost trivial sense. Note that the actual portfolio implications for this case are diametrically opposite of those in the one-period model: In that case, the proportion of assets invested in the stock increases as the stock price goes up; in the moving horizon case where the insurance level is a constant fraction of current wealth, the proportion of assets invested in the stock does not change with changes in the stock price.

C. The Infinitesimal Horizon Model

Moving Infinitely Small Horizon Model: The investor has a multiperiod horizon and continuously revises his insurance level over the next instant of time to his current level of wealth. In this model, the wealth level of the investor will always increase monotonically with time.

This model parallels most closely the analogue to casualty insurance with instantaneous adjustments of the insurance level. As an example, an investor might insure his home against loss and whenever the value of the house changes, he changes his insurance level. In this case the investor insures himself against losses over an infinitesimal horizon. This is a special case of the previous model (the fixed horizon) and of the limiting cases discussed in Section II.b. We may determine the portfolio strategy involved by letting time approach zero. Suppose that the investor's desired level of insurance remains fixed as a proportion of the stock price. Denote this insurance level by ϕ, so that the investor consistently sets $K = \phi W_t$. It now follows from Proposition 5 that the following portfolio proportions will obtain for an infinitesimal horizon:

$$\omega_t = \begin{cases} 1 & \text{if } \frac{K}{S} < 1 \\ \frac{1}{2} & \text{if } \frac{K}{S} = 1 \\ 0 & \text{if } \frac{K}{S} > 1. \end{cases}$$

We may use (3) to relate the portfolio strategy and the coefficient of relative risk aversion. Combining (3) with the above portfolio proportions gives

$$C_t = \begin{cases} \frac{\alpha - r}{\sigma^2} & \text{if } \frac{K}{S} < 1 \\ \frac{\alpha - r}{\sigma^2} \frac{1}{\omega_t} = \frac{2(\alpha - r)}{\sigma^2} & \text{if } \frac{K}{S} = 1 \\ \infty & \text{if } \frac{K}{S} > 1. \end{cases}$$

Insurance policies of the type described are thus consistent with very few utility maximization models, suggesting that the infinitesimal horizon model is not a reasonable paradigm of investor's behavior.

V. Conclusion

The optimality of an insurance strategy in a portfolio-choice context depends on the completeness of markets. In complete markets with continuous rebalancing of portfolios, the implied utility functions are so peculiar that it is doubtful that any investor would want to follow a two-date insurance strategy. Insurance strategies may, however, be optimal in some types of incomplete markets. However, under at least one set of reasonable assumptions, the paper showed that an investor having the option of investing in the risk-free asset would not purchase a put.

Interestingly, an insurance strategy with a finite moving horizon was consistent with a constant proportional risk aversion function. Put another way, an investor who has a constant proportional risk aversion function in a continuous time framework with a finite moving horizon can be viewed as facing a specific type of insurance problem.

REFERENCES

Black, F. and M. J. Scholes. "The Pricing of Options and Corporate Liabilities." Journal of Political Economy 81 (1973), 637-59.

Blattberg, R. and Gonedes, N. "A Comparison of the Stable and Student Distributions as Statistical Models for Stock Price." Journal of Business 47 (1974), 244-80.

Brennan, M. J. and E. S. Schwartz. "The Pricing of Equity-Linked Life Insurance Policies with an Asset Value Guarantee." Journal of Financial Economics 3 (1976), 195-213.

Cass, D. and J. E. Stiglitz. "The Structure of Investor Preferences and Asset Returns, and Separability in Portfolio Allocation: A Contribution to the Pure Theory of Mutual Funds." Journal of Economic Theory 2 (1970), 122-60.

Friend, I. and M. E. Blume. "The Demand for Risky Assets." American Economic Review 65 (1975), 900-22.

Gatto, M. A., R. Geske, R. Litzenberger, and H. Sosin. "Mutual Fund Insurance." Journal of Financial Economics 8 (1980), 283-317.

Goldman, M. B., "A Negative Report on the 'Near Optimality' of the Max-Expected-Log Policy as Applied to Bounded Utilities for Long Lived Programs," Journal of Financial Economics 1 (1974), 97-103.

Jarrow, R. A. and A. Rudd. Option Pricing. Homewood, IL: Richard D. Irwin, 1983.

Leland, H. E. "Who Should Buy Portfolio Insurance?" Journal of Finance 35 (1980), 581-94.

Merton, R. C. "Lifetime Portfolio Selection under Uncertainty: The Continuous-Time Case." Review of Economics and Statistics 50 (1969), 247-57.

Morin, R.A. and A.F. Suarez, "Risk Aversion Revisited," Journal of Finance 3 (1983), 1201-16.

Pozen, R. C. "When to Purchase a Protective Put." Financial Analysts Journal, July/August 1978, 47-60.

Ross, S. A. "Uncertainty and the Heterogeneous Capital Good Model." Review of Economic Studies 42 (1975), 133-46.

________. "Options and Efficiency." Quarterly Journal of Economics 90 (1976), 75-89.

Stocks, Bonds, Bills and Inflation Yearbook (1926-1983), Chicago: R.G. Ibbotson Associates, Inc., 1984.

Table 1

Optimal Strategies Involving Puts and Risky Assets

Return Parameters %			Coefficient of Relative Risk Aversion	Optimal Strike Price	Portfolio Allocation %		Value of Right to Purchase Put (as a Percentage of Wealth)
Risk-free	Risky Drift	Risky Dispersion			Risky	Put	
10	15	20	2	0.70	99.9	0.1	0.001
			4	1.00*	96.4	3.6	1.47
			8	1.00*	96.4	3.6	6.01
			16	1.00*	96.4	3.6	16.74
1	4	20	2	0.92	96.8	3.2	0.25
			4	1.00*	93.1	6.9	2.79
			8	1.00*	93.1	6.9	8.54
			16	1.00*	93.1	6.9	20.31

* represents a constrained maximum.

Table 2

Optimal Strategies Involving Puts, Risky Assets, and Riskfree Assets

Return Parameters %			Coefficient of Relative Risk Aversion	Optimal Strike Price	Portfolio allocation %			Value of Right to Risk-free Asset (as a Percentage of Wealth)
Risk-free	Risky Drift	Risky Dispersion			Risk-free	Risky	Put	
10	15	20	2	0	12	88	0	0.06
			4	0	58	42	0	2.49
			8	0	78	22	0	9.29
			16	0	89	11	0	22.30
1	4	20	2	0	37	63	0	0.56
			4	0	69	31	0	3.70
			8	0	82	18	0	10.75
			16	0	92	8	0	23.69