LIFETIME PORTFOLIO SELECTION AND INFORMATION

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ABSTRACT

The classical, finite horizon, consumption portfolio choice problem is reexamined, when the current return depends on the history of past observations. Optimal policies are characterized for the class of uncertainty/information structures, which result in an underlying Gaussian conditional distribution at any trading date. Decisions will be time dependent functions of wealth and of the mean and variance of the risky return.

For Bernoulli tastes, consumption is proportional to wealth but insensitive to information, whereas, in the isoelastic case, the proportionality coefficient is affected by new observations. The portfolio decision in both instances depends exclusively on information. For the logarithmic utility in addition, the optimal decision is myopic.

More informative structures in this latter case imply, under additional assumptions, a higher investment in the risky asset, for histories of observations resulting in a specified conditional mean. An increase in the latest observation has the same effect if it indicates a favorable shift in the mean risk faced. Finally, the dollar amount invested in the financial market remains unchanged in both instances.

I. INTRODUCTION

The dynamic choice of optimal consumption and portfolio strategies has been extensively discussed in the literature, in the case where returns are independently identically distributed over time. A general solution, as well as specific cases, (Bernoulli and Isoelastic utility functions) are described by Samuelson [11], when the investor trades at discrete points in time and has a bounded horizon. The infinite horizon consumption-saving problem is analyzed by Levhari and Srinivasan [5], when returns are multiplicative. Their main contribution is to provide a sufficient condition for optimality. An application to portfolio choice is made in the Isoelastic case, and special insights are gained under the additional assumption of lognormal risky returns.[1] In the same context, Hakansson [4] obtains closed form solutions for a class of utility functions including the power utility, the constant absolute risk aversion case, and the Logarithmic function. Finally, in [6], Merton characterizes optimal decisions when trading is continuous.

Investors dealing in financial markets constantly obtain new information (prices, etc.), in the light of which they proceed to rebalance their portfolios and adjust their consumption. While the results obtained in discrete time allow for an investigation of the effects of changes in riskiness (Levhari-Srinivasan [5]; Rothschild-Stiglitz [9]), they will only capture a situation where information is purely exogenous and the current resulting distribution does not depend on the past (by the temporal independence assumption). This paper seeks to study the case where information arises in an endogenous fashion, i.e., where it consists in the history of past returns. Since the past has an importance for the current appreciation of the risky opportunity, we simultaneously solve, though for a special class of processes, a system with explicit intertemporal connections.

In the first section the classical consumption-portfolio problem is adapted to the case where the environment is informative. The investor seeks to maximize lifetime utility conditional upon his knowledge, in an economy where the investment opportunity set includes a riskless and a single risky asset. The random payoff at any date is a monotonic transform of a basic underlying process. The class of uncertainty/information structures considered is characterized by a reproduction of the conditional nature of the underlying risk faced, which remains in the set of Gaussian distributions at any point in time. Thus even though its moments evolve in response to new observations, we build a form of stationarity in the model. Such a structure has the additional property that the conditional mean exclusively responds to information. The variance, which can be interpreted as a measure of informativeness (quality of the information collected), has a deterministic motion. Its evolution is a structural property.

The general solution for arbitrary Von Neuman-Morgenstern utility functions is developed in the second section. The vehicle of analysis is discrete-time dynamic programming in the spirit of Mossin [7]. As a result of the structure exposited above, the state at any trading date is completely summarized by the moments of the conditional distribution and by wealth. Optimal decisions will therefore depend on these three parameters and time.

Specific cases (Bernoulli and Isoelastic utility functions) are examined next. As expected, the consumption decision does not respond to information when tastes are logarithmic. In this case the nature of the risk faced is irrelevant for consumption purposes. As a consequence, the decision will be insensitive to information. The proportion of wealth invested in the risky asset is a time independent function of both the observations available and their quality (in the i.i.d. case this proportion is a constant). For isoelastic utilities, optimal consumption is proportional to wealth, but the proportionality coefficient evolves as the information set expands. The portfolio decision is nonmyopic, independent of wealth, but sensitive to information. If the horizon is far away, Samuelson [11] shows that, in general, hump saving takes place (i.i.d. returns). This result could not be proved, and presumably does not hold in our model, in which the amount saved depends in a non trivial way on the observation available and its quality.

In order to compare information structures, a dominance concept is introduced in Section V. A structure is "uniformly more informative" than a second one, if the quality of the information which results is "better" at any trading date, strictly better at least at one trading date. In the Bernoulli case, a uniformly more informative structure will induce a higher investment in the risky asset, for all histories of observations resulting in a specified conditional mean, and for transforms belonging to a particular class. An increase in the latest observation has the same effect, if it is associated with an increase in the mean of the current payoff (positive connection between consecutive returns). The total investment in the financial market in both instances, will remain unaffected. In the isoelastic case, the nonmyopic nature of decisions does not easily allow for a similar analysis.

II. The Model

The framework of this model is similar to the one exposited in Samuelson [11]. An investor with time additive preferences is faced with a resource allocation problem. At any trading date, he chooses a consumption level and a portfolio strategy so as to maximize his lifetime expected utility, conditional upon the information he possesses. The investment opportunity set consists of two assets: a riskless bond with a sure return r at each trading point, and a risky asset with a random payoff, $\tilde{x}_t$, at date t. The investor lives for exactly T periods and has full knowledge of his lifespan. Formally the problem faced can be written as:

$$\underset{\{\omega_t,\ c_t\}_{t=1}^{T-1}}{\text{Max}} \quad E\left\{ \sum_{t=1}^{T} \beta^{t-1} u[c_t] \;\middle|\; I_1 \right\} \tag{2.1}$$

S.t

$$W_{t+1} = (W_t - c_t)\left[(1 - \omega_t)r + \omega_t \tilde{x}_t\right] \qquad t \in [1;\ T) \tag{2.2}$$

$$0 \leqslant c_t \leqslant W_t \quad ; \quad \omega_t \geqslant 0 \qquad t \in [1;\ T) \tag{2.3}$$

$$\text{Prob}\left[(1 - \omega_t)r + \omega_t \tilde{x}_t \geqslant 0 \;\middle|\; I_1\right] = 1 \qquad t \in [1;\ T) \tag{2.4}$$

$$\omega_0;\ c_o \quad \text{given} \tag{2.5}$$

where W_t; c_t; ω_t respectively represent wealth, consumption and investment proportion in the risky asset, at trading date t. I_1 is the information set of the investor when he takes his first decision.

Expression (2.1) represents the standard lifetime optimization problem. In our model this maximization takes place conditional upon the information available, which will be detailed further. Equation (2.2) captures the dynamic evolution of wealth. The amount invested $(W_t - c_t)$ is split between the two assets bearing, respectively, r and $\tilde{x}_t$. The return on this investment constitutes wealth in the next period. While choosing an optimal consumption level, the investor is required to meet his budget constraint $(c_t \leqslant W_t)$. Negative consumption levels which have no economic meaning are not allowed. The capital market in this economy is developed to the point where borrowing at the riskless rate is available as long as it does not involve a possibility of bankruptcy, but where short selling of risky assets by traders is prohibited. The no-bankruptcy condition (2.4) can be compared to Hakansson's solvency constraint[2] (14) (Hakansson [4] p. 590). Finally the initial conditions are prespecified in (2.5).

The investor's utility is strictly increasing and strictly concave. In addition, $u'(0) = +\infty$. This assumption enables us to drop condition (2.3) since it implies that a corner solution for consumption is suboptimal.[3]

The information obtained by the decision maker is of an endogenous nature. It consists of the history of past returns and of the structure of the intertemporal connections. At time t, it is summarized in the set $I_t = \{x_s: 0 \leq s < t; \quad \Phi; \quad W_t\}$ where Φ designates the characteristics of the process resulting in the history $x_s: 0 \leq s < t$. The return process for obtaining a simple and tractable structure is specified as follows:

Assumption II.1 (Uncertainty/Information Structure): $\tilde{x}_t = g(\tilde{\varepsilon}_t)$ where $g(\cdot)$ is a transform and $\tilde{\varepsilon}_t$ a Gaussian process, defined by:

(a) $g(\cdot)$ is a twice continously differentiable, strictly monotonic and positive function with domain $(-\infty; +\infty)$, satisfying

(a.1) Non-dominance: $\exists\, \varepsilon^* \in (-\infty; +\infty) \;/\; g(\varepsilon^*) = r$

(a.2) Growth condition: $\exists (K, L) \in R^{+*} \times R^* \;/\; g(\varepsilon) \leq K + e^{L\varepsilon} \quad \forall \varepsilon \in (-\infty, +\infty)$

(b) $\tilde{\varepsilon}_t = A\tilde{r}_t + \tilde{e}_t$ where $\tilde{r}_t = a\tilde{r}_{t-1} + \tilde{v}_{t-1}$ is a Markov process with Gaussian initial distribution $\tilde{r}_0 \rightarrow N(m_0; \sigma_0^2)$, and $\{\tilde{e}_t\}$, $\{\tilde{v}_t\}$ are independent stationary processes with respective Gaussian distributions $N(0; \sigma_e^2)$ and $N(0; \sigma_v^2)$. In addition $\{\tilde{e}_t\}$ and $\{\tilde{v}_t\}$ are independent of $\tilde{r}_0$ and unobserved. The parameters of the structure (A, a) as well as the distributions are known.

An example satisfying the conditions (a) is the exponential function $g(\varepsilon_t) = \exp\lfloor \alpha\varepsilon_t \rfloor$, $\alpha > 0$. In this case $\tilde{x}_t$ is lognormally distributed. For the class of transforms specified in assumption II.1. the observation of the sequence $x_s: 0 \leq s < t$ is equivalent to the knowledge of the history $\varepsilon_s: 0 \leq s < t$. Condition (a.1) stipulates that the risky return $\tilde{x}_s$ is not uniformly higher or lower than the riskless rate r. For some realizations of $\tilde{\varepsilon}_s$, the risky investment turns out to be more fruitful than the sure return; whereas the contrary holds for alternative realizations. A uniform dominance situation would unambiguously result in a corner solution. The second restriction (a.2) ensures the convergence of the objective function in program (2.1) (a proof is given in Appendix A-1). It requires that the transform be uniformly bounded above on a subset of the real line, and that its growth on the complementary subset be less than exponential. Examples of such transforms are graphed in Figure I. In the remainder of the paper, Θ denotes the set of transforms satisfying assumption II.1 a. Additionally Θ^+ (Θ^-) will represent the subset of strictly increasing (strictly decreasing) elements of Θ.

The return process $\tilde{\varepsilon}_t$ has two components. The first is a first order Markov process, and the second is an independently, identically distributed noise process. The observation of the realization of the random variable $\tilde{\varepsilon}_t$ at time t provides some information about the realization r_t. This information is incomplete since the noise component $\tilde{e}_t$ is not observable. The first component of the return at time t + 1 $(\tilde{r}_{t+1})$ depends partly on the previous realization r_t. The information gathered at time t can thus be used by the decision maker in order to form his conditional beliefs at time t + 1.

Given the specific Uncertainty/Information structure postulated, the conditional distribution of $\tilde{r}_t$, given the information set $I_t = \{\varepsilon_s: \; 0 \leq s < t; \; (a; A; \sigma_e^2; \sigma_v^2; m_0; \sigma_0^2); W_t\}$, is Gaussian with parameters $\hat{r}_t$ and γ_t. Furthermore its temporal evolution is as follows[4]:

$$\hat{r}_{t+1} = a\hat{r}_t + z_t(\varepsilon_t - A\hat{r}_t) \tag{2.6}$$

$$z_t = a\gamma_t A \left[A^2\gamma_t + \sigma_e^2\right]^{-1} \tag{2.7}$$

$$\gamma_{t+1} = a^2\gamma_t + \sigma_v^2 - A^2a^2\gamma_t^2\left[A^2\gamma_t + \sigma_e^2\right]^{-1} \tag{2.8}$$

$$\hat{r}_0 = m_0; \quad \gamma_0 = \sigma_0^2 \tag{2.9}$$

The initial distribution of the process $\{\tilde{r}_t\}$ also represents the initial beliefs of the investor, since he knows the objective distribution of $\tilde{r}_0$. In order to solve the problem (2.1) - (2.5), he combines this knowledge with the information provided by the realization ε_0.

The conditional variance, by definition, is the conditional mean square deviation of $\tilde{r}_t$ from $\hat{r}_t$. It constitutes a measure of the average conditional dispersion of the distribution around its mean $\hat{r}_t$. Hence it can be interpreted as a measure of the quality of the information obtained. A low conditional variance is indicative of a high quality and conversely. The dynamic nature of the model implies that the conditional distribution is revised over time. The variance, however, has a deterministic motion given by equation (2.8). The history of the process does not affect its current level, which depends exclusively on the parameters of the Uncertainty/Information structure. The quality of the information enjoyed by the investor is therefore a structural property with our specification.

Complete information corresponds to a case where the variance vanishes. Examination of equation (2.8) shows that such a situation cannot be reached as long as the process $\{\tilde{v}_t\}$ is nondegenerate. Indeed assume that there is no uncertainty about the initial level r_0(i.e. $\sigma_0^2 = 0$). The conditional variance which results at time 1 is $\gamma_1 = \sigma_v^2$, a positive number in the nondegenerate case. Thereafter γ_t keeps increasing until it reaches a steady state γ^* (Figure II). There exists a unique fixed point γ^*. Its level depends on the parameters of the structure. A main property of the steady state is its independence from the initial parameters $(m_0; \sigma_0^2)$. As long as γ_0 is below γ^*, the quality of the information will deteriorate over time ($\{\gamma_t\}$ is an increasing converging series). On the contrary when γ_0 is above γ^*, the quality improves with time.

The conditional mean, which has the motion (2.6) depends directly upon the last observation made. It condenses the information stream into a single parameter and constitutes a sufficient statistic[5] for the conditional distribution.

III. The General Solution

The interpretation of the problem exposited in Section II is as follows. The investor acts in the first period on the basis of his information set $I_1 = \{\varepsilon_0;\ W_1;\ (a;\ A;\ \sigma_e^2;\ \sigma_v^2;\ m_0;\ \sigma_0^2)\}$, which is equivalent to the set $I_1' = \{\hat{r}_1;\ W_1;\ \gamma_1;\ (a;\ A;\ \sigma_e^2;\ \sigma_v^2)\}$. He knows that his next period's decision will be based on the new information set $I_2' = \{\hat{r}_2;\ W_2;\ \gamma_2;$ $(a;\ A;\ \sigma_e^2;\ \sigma_v^2)\}$, but at time 1 he can only act on the basis of I_1' under the postulate that future choices will be optimal with respect to the appropriate information set.

This dynamic choice problem can be broken down into a sequence of one period problems.[6] To proceed, let us start at the end of the planning horizon. The last decision is made in period T - 1 on the basis of $I_{T-1}' = \{\hat{r}_{T-1};\ W_{T-1};\ \gamma_{T-1}\}$. The resulting value function will depend on the information set. Thus the investor solves:

$$J_{T-1}[\hat{r}_{T-1};\ \gamma_{T-1};\ W_{T-1}] = \max_{\omega_{T-1};\ c_{T-1}} u[c_{T-1}] + \beta E\{ u[(W_{T-1} - c_{T-1})\ ((1 - \omega_{T-1})r + \omega_{T-1} g(\tilde{\varepsilon}_{T-1}))]\ |\ I_{T-1}'\} \tag{3.1}$$

$$\text{s.t} \qquad \omega_{T-1} \geq 0\ ;\quad \text{Prob}[(1 - \omega_{T-1})r + \omega_{T-1} g(\tilde{\varepsilon}_{T-1}) \geq 0\ |\ I_{T-1}'] = 1 \tag{3.2}$$

The necessary conditions for an interior solution to this program are:

$$\begin{cases} u'[c_{T-1}] = \beta E\{u'[W_T][(1-\omega_{T-1})r + \omega_{T-1}g(\tilde{\varepsilon}_{T-1})] \mid I'_{T-1}\} & (3.3) \\ 0 = E\{u'[W_T] \cdot [g(\tilde{\varepsilon}_{T-1}) - r] \mid I'_{T-1}\} & (3.4) \end{cases}$$

The maximand in (3.1) is strictly concave. The sufficient conditions in this case will be satisfied. By solving (3.3) (3.4) simultaneously, the optimal policies $(c^*_{T-1}; \omega^*_{T-1})$ are obtained. They clearly depend on the information set I'_{T-1}. Thus we can write $c^*_{T-1} = c^*_{T-1}(\hat{r}_{T-1}; \gamma_{T-1}; W_{T-1})$ and $\omega^*_{T-1} = \omega^*_{T-1}(\hat{r}_{T-1}; \gamma_{T-1}; W_{T-1})$.

The indirect utility function is obtained by plugging these solutions back into the maximand of program (3.1). The value function which results depends on $(\hat{r}_{T-1}; \gamma_{T-1}; W_{T-1})$. In addition by the envelope condition we obtain: $J_{W_{T-1}}[\hat{r}_{T-1}; \gamma_{T-1}; W_{T-1}] = u'[c^*_{T-1}(\hat{r}_{T-1}; \gamma_{T-1}; W_{T-1})]$

If we move one period backward in time, the new problem faced is:

$$J_{T-2}[\hat{r}_{T-2}; \gamma_{T-2}; W_{T-2}] = \max_{\omega_{T-2};\, c_{T-2}} u[c_{T-2}] + \beta\, E\{J_{T-1}[\hat{r}_{T-1}, \gamma_{T-1}; (W_{T-2} - c_{T-2})((1-\omega_{T-2})r + \omega_{T-2}g(\tilde{\varepsilon}_{T-2}))] \mid I'_{T-2}\} \quad (3.5)$$

$$\text{s.t} \qquad \omega_{T-2} \geq 0; \quad \text{Prob}[(1-\omega_{T-2})r + \omega_{T-2}g(\tilde{\varepsilon}_{T-2}) \geq 0 \mid I'_{T-2}] = 1 \quad (3.6)$$

The necessary conditions for an interior solution to this program are:

$$\begin{cases} u'[c_{T-2}] = \beta E\{J_{W_{T-1}}[\hat{r}_{T-1}; \gamma_{T-1}; W_{T-1}] \cdot [(1-\omega_{T-2})r + \omega_{T-2}g(\tilde{\varepsilon}_{T-2})] \mid I'_{T-2}\} & (3.7) \\ 0 = E\{J_{W_{T-1}}[\hat{r}_{T-1}; \gamma_{T-1}; W_{T-1}] \cdot [g(\tilde{\varepsilon}_{T-2}) - r] \mid I'_{T-2}\} & (3.8) \end{cases}$$

which admit the optimal policies $c^*_{T-2} = c^*_{T-2}(\hat{r}_{T-2}; \gamma_{T-2}; W_{T-2})$ and $\omega^*_{T-2} = \omega^*_{T-2}(\hat{r}_{T-2}; \gamma_{T-2}; W_{T-2})$, as solutions. When u is strictly concave it can be shown that J_{T-1} is also strictly concave.[7] Thus the first order conditions are also sufficient conditions.

This procedure can be repeated recursively. The strict concavity of the value function carries through, and the necessary and sufficient conditions for the optimization problem at time t are as (3.7) and (3.8) where t is substituted for the index T-2.

The interesting characteristic of this model is its Uncertainty/ Information structure. Even though the uncertainty is temporally linked, this connection arises "nicely." Indeed the information set at time t can be summarized in three variables. Two of them completely describe the conditional distribution, the third one results from the budget constraint. Optimal policies in this case exhibit a simple dependence on the past history of the economy: they depend on the triplet $(\hat{r}_t; \gamma_t; W_t)$. This feature is used in a later section, in order to analyze the impact of information on optimal decisions.

IV. Special Cases: Bernoulli and Isoelastic Utilities.

The Bernoulli and Isoelastic cases are of special interest, since optimal policies exhibit appealing economic properties, when returns are independently, identically distributed (Samuelson [11]). When information is conveyed by past returns in the way specified in Section II some of these properties remain unaffected. This section discusses these features, while the mathematical solutions appear in Appendices A-2 and A-3.

When the utility function is logarithmic, the optimal consumption level is proportional to wealth, and independent of both the observations available and the quality of the information. This solution turns out to be the same as in the case of independently, identically distributed (i.i.d) returns. In both instances the uncertainty does not matter for the consumption decision. A fortiori information about the random return is irrelevant.

If life is long (T large), consumption at an early age is small ($c_1^* = (1 - \beta) W_1$) but does not go to zero. As a consequence "hump saving" does not necessarily take place on average (Samuelson [11], p. 888 case [iii]).

The total investment in the capital market depends on wealth exclusively, but its breakdown between the risky and the riskless opportunity is independent of wealth. This property is shared by Samuelson's policies. The proportion ω^* however is not constant in our case. It will change, as a result of new observations, and of their quality. As time flows additional information becomes available and the investor updates the conditional probability density of future returns. A constant policy in this case is suboptimal, since it fails to adjust to the evolution of the distribution of the risky asset. This dependence of ω^* upon the conditional mean and the conditional variance has the property of time independence. It is the case that Bernoulli tastes, in the finite horizon economy, give rise to a similar portfolio problem at any date which is independent of future choices (myopic decision). The only modification is imposed by the evolution of the information set.

In the Isoelastic case, optimal consumption remains proportional to wealth. The coefficient of proportionality depends on the evolving parameters of the conditional distribution. In the limiting case where the parameter of risk aversion tends to zero (Bernoulli tastes), the coefficient becomes a function of the discount parameter β exclusively. The logarithmic solution is recovered.

The portfolio decision does not depend on the investor's wealth, but takes account of the changing appreciation of future returns ($\hat{r}_t$ and γ_t). The proportion of savings invested in the risky asset is nonmyopic: the current decision is a function of future optimal choices. The proportion ω_t^* in the i.i.d. case has the same general properties except for the fact that it does not depend upon the history of the economy.

The introduction of a temporal connection between risky returns (thus the introduction of "valuable" information for decision making) invalidates Samuelson's "hump saving" theorem. The amount which is consumed in each period depends critically on the observations available and on their quality. In the i.i.d case the proportion of wealth consumed is deterministic. As T becomes large initial consumption either goes to zero or to a limiting positive fraction. This results in hump saving in the first instance. In the second case hump saving is obtained only under an additional condition involving the risk corrected mean yield (Samuelson [11] p. 888). When past returns provide information about the current return the lifetime saving pattern depends on the information available in the first period. This relationship however, as well the dependency of initial consumption upon the investment horizon are not straightforward.

V. Information and Optimal Policies

In this section we attempt to assess the impact of information on optimal choices. Two types of effects will be investigated. The first one corresponds to the reaction of policies to higher observations. The second effect is the impact of "better" information on optimal consumption and portfolio choices.

Firstly we introduce some notation. Let $\chi[I]$ be the set of all Uncertainty/Information structures of the Gaussian type. Let $\chi[I(a;A;\sigma_e^2;\sigma_v^2)]$ be the subset of $\chi[I]$ consisting of the structures with identical parameters $(a, A, \sigma_e^2, \sigma_v^2)$. Two elements of this set differ in terms of their initial parameters m_o and σ_o^2. Finally define $\chi[I(a;A;\sigma_e^2;\sigma_v^2;m_0)]$ and $\chi[I(a;A;\sigma_e^2;\sigma_v^2;m_0;\sigma_0^2)]$ as the subsets of $\chi[I(a;A;\sigma_e^2;\sigma_v^2)]$ with respectively m_0 and $(m_0; \sigma_0^2)$ specified.

The comparative statics we intend to carry out require a criterion for comparison of Uncertainty/Information structures. To this end we introduce the following dynamic concept.

<u>Definition V.1</u>: Let $(I^1, I^2) \in \chi[I]^2$. I^1 is "uniformly more informative" than I^2 $(I^1 \underset{U.I.}{\gtrsim} I^2)$ if and only if $\gamma_t^1 \leq \gamma_t^2$ $\forall t \in [0; T]$, and $\exists t \in [0, T]$ such that $\gamma_t^1 < \gamma_t^2$. I^1 is "uniformly as informative" as $I^2 (I^1 \underset{U.I.}{\simeq} I^2)$ if and only if $\gamma_t^1 = \gamma_t^2$ $\forall t \in [0; T]$.

From an economic point of view $I^1 \underset{U.I.}{\gtrsim} I^2$ simply means that the quality of the information obtained using the structure I^1 is better than the one associated with I^2 at any date in the trading interval $[0; T]$, and that it is stricly better at one date at least.[8] For Gaussian processes in the subset $\chi[I(a;A;\sigma_e^2;\sigma_v^2)]$ a simple condition ensures dominance in the U.I. sense.

<u>Lemma V.2</u>: Let $(I^1, I^2) \in \chi[I(a;A;\sigma_e^2;\sigma_v^2)]^2$. $I^1 \underset{U.I.}{\gtrsim} I^2$ if and only if $\gamma_0^1 < \gamma_0^2$; $I^1 \underset{U.I.}{\simeq} I^2$ if and only if $\gamma_0^1 = \gamma_0^2$.

This statement can be easily checked by using Figure II. If $\gamma_0^1 < \gamma_0^2 < \gamma^*$ (γ^* is the fixed point for information structures in the class $\chi[I(a;A;\sigma_e^2;\sigma_v^2)]$), it is clear that $\gamma_t^1 < \gamma_t^2$ for all t in $[0; T]$. Similarly when $\gamma^* < \gamma_0^1 < \gamma_0^2$ it follows that $\gamma_t^1 < \gamma_t^2$ $\forall t \in [0, T]$. If the trading interval is large (T large), in the limit γ_t^1 and γ_t^2 converge to γ^* independently of the initial position. Finally $\gamma_0^1 = \gamma_0^2$ implies $\gamma_t^1 = \gamma_t^2$ $\forall t$. Thus for Gaussian structures in $\chi[I(a;A;\sigma_e^2;\sigma_v^2)]$, the initial conditional beliefs of the investor provide a convenient ranking. This property is used in the next theorem, which constitutes the main result of the paper.

<u>Theorem V.3</u>: (Bernoulli tastes)

(a) $\forall(I^1;\ I^2) \in \chi[I]^2 \ c_t^1 = c_t^2 \quad \forall t \in [1;\ T)$

(b) $\forall(I^1;\ I^2) \in \chi[I(a^+;A^+;\sigma_e^2;\sigma_v^2;m_0;\gamma_0)]^2$ (subset of $\chi[I(a;A;\sigma_e^2;\sigma_v^2;m_0;\gamma_0)]$ for which $A > 0$, $a > 0$); $\forall g \in \Theta^+$; $\forall t \in [1;\ T)$; if $\varepsilon_s^1 = \varepsilon_s^2$ $\quad s < t-1$ and $\varepsilon_{t-1}^1 > \varepsilon_{t-1}^2$ then $\omega_t^1 > \omega_t^2$.

(c) $\forall(I^1;\ I^2) \in \chi[I(a;A;\sigma_e^2;\sigma_v^2;m_0)]^2$; for all identical histories of the processes $\{\tilde{\varepsilon}_s^1\}_{s=0}^{t-1}$ and $\{\varepsilon_s^2\}_{s=0}^{t-1}$; $\exists\, g \in \theta^+$ such that $I^1 \underset{U.I.}{\gtrsim} I^2$ $\Longrightarrow \omega_s^1 > \omega_s^2$ $\forall s \in [t;\ T)$ for which ω_s^i lies in the interior of the constraint set and for all identical sample path of the processes $\{\tilde{\varepsilon}_s^1\}_{s=t}^{T-1}$ and $\{\tilde{\varepsilon}_s^2\}_{s=t}^{T-1}$.

The first statement above asserts that the optimal consumption level remains unaffected by the information structure. For Bernoulli tastes the consumption decision is independent of the nature of the risk faced in the financial market. It follows that information relative to this risk is irrelevant.

The second claim assesses the impact of observations on the portfolio decision. When A is positive an increase in the latest observation is indicative of a high realization of $\tilde{r}_{t-1}$. If a is positive the trader will have high expectations about the level of $\tilde{r}_t$ ($\hat{r}_t$ is large), which translates in a high appreciation of the potential return in period t ($g' > 0$ and $A > 0$). The risky investment becomes more attractive and induces the investor to shift resources away from the riskless opportunity. The total amount invested in the financial market however remains the same. The very same result holds true under the following permutations of assumptions: ($a < 0$, $A < 0$, $g \in \Theta^+$); ($a < 0$, $A > 0$, $g \in \Theta^-$) ($a > 0$, $A < 0$, $g \in \Theta^-$). The economic interpretation in these instances is as above.

The last proposition concerns the impact of the structure on optimal choices, for a given history of realizations. When an information structure is uniformly more informative, the investor receives a sharper assessement of future returns at any trading date. His reaction however depends crucially on the nature of the risk faced (the shape of the transform g). For "essentially" concave and increasing transforms, a "better" information results in an increased investment in the risky asset.

Proof of Theorem V.3:

Statement (a): By equation (A-2.13) in Appendix (A-2) the optimal consumption level is independent of the Uncertainty/Information structure. ◇

Statement (b)

1- ω_t^i $i = 1, 2$ are interior solutions: the history $\{\varepsilon_s^1 = \varepsilon_s^2$ $s < t-1$; $\varepsilon_{t-1}^1 > \varepsilon_{t-1}^2\}$ is equivalent to $\{\hat{r}_t^1 > \hat{r}_t^2$; $\gamma_t^1 = \gamma_t^2 \equiv \gamma_t\}$ for structures belonging to $\chi[I(a^+;A^+;\sigma_e^2;\sigma_v^2;m_0;\gamma_0)]$. It follows that the conditional density $\phi_t^1 \equiv \phi(r_t|\hat{r}_t^1;\gamma_t)$ is a rightward translation of the density $\phi_t^2 \equiv \phi(r_t|\hat{r}_t^2;\gamma_t)$ i.e. ϕ_t^1 dominates ϕ_t^2 in the "first order stochastic dominance" sense (Hadar-Russel [3]). Let $\Psi[\omega_t;\varepsilon_t]$ be defined by $\Psi[\omega_t;\varepsilon_t] \equiv [(1 - \omega_t)r + \omega_t g(\varepsilon_t)]^{-1}[g(\varepsilon_t) - r]$. It is easily checked that

$\Psi_\omega = -\psi^2 < 0 \quad \forall(\omega_t;\ \varepsilon_t) \in (-\infty;\ +\infty)^2$ and that $\Psi_\varepsilon = g'(\varepsilon_t)r\ [\ (1 - \omega_t)r + \omega_t g(\varepsilon_t)]^{-2} > 0 \quad \forall(\omega_t;\ \varepsilon_t) \in (-\infty;\ +\infty)^2$ for $g' > 0$. Here Ψ_ω and Ψ_ε denote respectively the partial derivatives of the function $\Psi[\omega_t;\varepsilon_t]$ with respect to ω_t and ε_t. Let ω_t^i represent the optimal choice given the information structure I^i. We have: $0 = E[\Psi[\omega_t^2;\varepsilon_t]\,|\,\hat{r}_t^2] < E[\Psi[\omega_t^2;\varepsilon_t]\,|\,\hat{r}_t^1]$. The first equality is the necessary condition for an optimum at time t given the Information structure I^2. The second inequality follows since Ψ increases in r_t $(A > 0)$ and ϕ^1 dominates ϕ^2 in the first order stochastic dominance sense. Combining this inequality with the fact that Ψ is decreasing in ω_t, we conclude that $\omega_t^1 > \omega_t^2$.

2- ω_t^2 lies on the boundary of the constraint set but does not satisfy the first order conditions: if $\omega_t^2 = \bar{\omega}_t$ ($\bar{\omega}_t$ is the maximum value of ω_t which satisfies the constraints), it is clear that $\omega_t^1 = \omega_t^2$; if $\omega_t^2 = 0$ then $\omega_t^1 > \omega_t^2$. Similarly if ω_t^2 only, satisfies the first order conditions, we obtain $\omega_t^1 > \omega_t^2$. ◇

<u>Statement (c)</u>: The proof will be conducted in four steps.

<u>Step 1</u>: We first show that the "informativeness" comparison between two structures belonging to $\chi[I(a;A;\sigma_e^2;\sigma_v^2;m_0)]$ is equivalent to the validity of the Rothschild-Stiglitz integral conditions.

<u>Definition V.4</u>: (Rothschild-Stiglitz [8]). Let G^1 and G^2 be two arbitrary distributions with identical domain $(-\infty;\ +\infty)$. $G^1 \underset{I}{\gtrsim} G^2$ (G^1 dominates G^2 in the integral conditions sense) if and only if $T(x) \equiv \int_{-\infty}^{x} [G^2(y)-G^1(y)]\,dy > 0$ $\forall x \in (-\infty;\ +\infty)$ and $\lim_{x \to +\infty} T(x) = 0$.

<u>Claim 1</u>: Let F_t^i represent the conditional distribution of $\tilde{r}_t^i$ given I_t^i $i = 1, 2$. $\forall(I^1, I^2) \in \chi[I(a;A;\sigma_e^2;\sigma_v^2;m_0)]^2$; if $\{\varepsilon_t^1 = \varepsilon_t^2 \quad \forall t \in [0;\ T]\}$ then: $I^1 \underset{U.I.}{\gtrsim} I^2 \Longrightarrow F_t^1 \underset{I}{\gtrsim} F_t^2 \quad \forall t \in [0;\ T]$.

Proof: By lemma V.2: $I^1 \underset{U.I.}{\gtrsim} I^2 \Longrightarrow Y_t^1 < Y_t^2 \quad \forall t \in [0,T]$ for I^1, I^2 belonging to $\chi[I(a;A;\sigma_e^2;\sigma_v^2;m_0)]^2$ (which is a subset of $\chi[I(a;A;\sigma_e^2;\sigma_v^2)]^2$) .For identical realizations up to time s (s = t-1) the conditional densities at time t have the same mean ($\hat{r}_t^1 = \hat{r}_t^2$). Thus F_t^2 is a mean preserving spread[9] of F_t^1. It follows that $F_t^1 \underset{I}{\gtrsim} F_t^2$ (Rothschild-Stiglitz [8]). The proof holds for all t in the trading interval [0; T].

Step 2: In this part we provide a criterion, which enables us to rank the conditional expectations of the function Ψ. Let H_t^i represent the conditional distribution of ε_t^i given the information set I_t^i. Let $\Theta_0^+ \equiv \{g \in \Theta^+ : \exists N \in R^{+*}$ $g/g' > N > 0\}$.[10] The following proposition is valid:

Claim 2: $\forall I^i \in \chi_I \quad i = 1, 2 ; \quad \forall g \in \Theta_0^+ , \quad \forall \omega_t$ which lies in the interior of the constraint sets associated with I_t^i i = 1, 2. If $T_t(+\infty) = 0$ then

$$\int_{-\infty}^{+\infty} \Psi_{\varepsilon\varepsilon}[\omega_t;\, y] T_t(y) dy \gtreqless 0 \Longleftrightarrow E\{\Psi[\omega_t;\, \tilde{\varepsilon}_t] | I_t^2\} \gtreqless E\{\Psi[\omega_t;\, \tilde{\varepsilon}_t] | I_t^1\} , \quad \text{where}$$

$$\Psi_{\varepsilon\varepsilon} \equiv \frac{\partial^2}{\partial \varepsilon^2} \Psi[\omega_t;\, \varepsilon_t] \text{ and } T_t(x) \equiv \int_{-\infty}^{x} \left(H_t^2(y) - H_t^1(y)\right) dy.$$

Proof: Let $S_t(x) \equiv H_t^2(x) - H_t^1(x)$. $E\{\Psi[\omega_t;\, \tilde{\varepsilon}_t] | I_t^2\} \lesseqgtr E\{\Psi[\omega_t;\, \tilde{\varepsilon}_t] | I_t^1\} \Longleftrightarrow$

$$\int_{-\infty}^{+\infty} \Psi[\omega_t;\, y] \; dS_t(y) \lesseqgtr 0 \quad \Longleftrightarrow \quad \Psi[\omega_t;\, \cdot] \; S_t(y) \Big|_{-\infty}^{+\infty} - \int_{-\infty}^{+\infty} \Psi_{\varepsilon}[\omega_t;\, y]$$

$$S_t(y) \quad dy \lesseqgtr 0 \Longleftrightarrow \int_{-\infty}^{+\infty} \Psi_{\varepsilon}[\omega_t;\, y] \; S_t(y) \; dy \gtreqless 0$$

The second equivalence above results from integration by parts. To obtain the final expression observe that $\lim_{y \to \pm\infty} S_t(y) = 0$ and that $\lim_{x \to \pm\infty} \Psi[\omega_t;\, x]$ is bounded when ω_t belongs to the interior of the constraint sets (For $g \in \Theta^+$, $\omega > 0 \Longrightarrow \omega + \frac{r}{g - r} > 0$ as $x \to +\infty$ and $(1 - \omega)r + \omega g(x) > 0 \Longrightarrow \Psi[\omega;\, x] < \infty$ as $x \to -\infty$). A second integration by parts results in $\Psi_{\varepsilon}[\omega_t;\, y]$. $T_t(y) \Big|_{-\infty}^{+\infty}$ -

$\int_{-\infty}^{+\infty} \Psi_{\varepsilon\varepsilon}[\omega_t; y]\, T_t(y)\, dy \gtreqless 0$. But $T_t(-\infty) = 0$ and by assumption $T_t(+\infty) = 0$. In addition since $g \in \Theta_0^+$ and ω_t is in the interior of the constraint set, $\Psi_\varepsilon[\omega_t; y]$ is bounded as $y \to \pm\infty$: $\Psi_\varepsilon[\omega_t; y] = [g'(y)]^{-1} r\big[(1-\omega_t)r/g'(y) + \omega_t g(y)/g'(y)\big]^{-2}$, thus if $g'(y)$ tends to infinity the last term is bounded above by $[\omega_t\, N]^{-2}$ which is finite ($N > 0$; $\omega_t > 0$); the case where $g'(y)$ is bounded as $y \to \pm\infty$ is straightforward. The result follows.

<u>Step 3</u>: Herein we show that the tail of the integral $\int_{-\infty}^{+\infty} \Psi_{\varepsilon\varepsilon}[\omega_t; y]\, T_t(y) dy$ can be made as small as desirable.

<u>Claim 3</u>: $\forall g \in \Theta^+$, $\forall \omega_t$ belonging to the interior of the constraint set, $\forall \delta > 0, \exists \mu^* \in (-\infty; -1) \,/\, \left| \int_{-\infty}^{\mu^*} \Psi_{\varepsilon\varepsilon}[\omega_t; y] T_t(y) dy \right| < \delta$.

<u>Proof</u>: $\forall g \in \Theta^+$, g' and g'' are bounded in $(-\infty; -1)$. $\Psi_{\varepsilon\varepsilon}[\omega_t; y] = r\,[(1-\omega_t) r + \omega_t g(y)]^{-3}\, \big[g''\big((1-\omega_t)r + \omega_t g(y)\big) - 2\omega_t (g')^2\big]$ will therefore also be bounded if ω_t lies in the interior of the constraint set. Let k be an upper bound for $|\Psi_{\varepsilon\varepsilon}[\omega_t; y]|$ on the subset $(-\infty; -1)$. Let μ be an arbitrary number in the interval $\big(-\infty; \mathrm{Min}(-1; -\sqrt{\gamma_t^1} + \hat{r}_t^1; -\sqrt{\gamma_t^2} + \hat{r}_t^2)\big)$. Denote by h_t^i the density function corresponding to H_t^i and by ϕ_N and Φ_N respectively the standard normal density and distribution functions. Since $\mu < -1$ we have:

$$\left| \int_{-\infty}^{\mu} \Psi_{\varepsilon\varepsilon}[\omega_t; y] T_t(y) dy \right| < \int_{-\infty}^{\mu} \left| \Psi_{\varepsilon\varepsilon}[\omega_t; y] \right| \cdot \left| T_t(y) \right| dy < k. \int_{-\infty}^{\mu} \left| T_t(y) \right| dy.$$

By definition $T_t(y) = \int_{-\infty}^{y} S_t(x) dx$. Thus $|T_t(y)| < \int_{-\infty}^{y} |S_t(x)| dx = \int_{-\infty}^{y} |H_t^2(x) - H_t^1(x)| dx < \int_{-\infty}^{y} [H_t^2(x) + H_t^1(x)] dx$. Since $H_t^i(x) = \Phi_N\Big(\frac{x - \hat{r}_t^i}{\sqrt{\gamma_t^i}}\Big) = 1 - \Phi_N\Big(-\frac{x - \hat{r}_t^i}{\sqrt{\gamma_t^i}}\Big)$, for $x < \mu$ we have $H_t^i(x) < \Big(-\frac{x - \hat{r}_t^i}{\sqrt{\gamma_t^i}}\Big)^{-1} \cdot \phi_N\Big(-\frac{x - \hat{r}_t^i}{\sqrt{\gamma_t^i}}\Big)$ (Woodroofe [12] p. 97). Hence $H_t^i(x) < \phi_N\Big(-\frac{x - \hat{r}_t^i}{\sqrt{\gamma_t^i}}\Big) = \phi_N\Big(\frac{x - \hat{r}_t^i}{\sqrt{\gamma_t^i}}\Big) = \sqrt{\gamma_t^i}\; h_t^i(x)$.

Thus for $y < \mu$: $| T_t(y) | < \sqrt{\gamma_t^1}\ H_t^1(y) + \sqrt{\gamma_t^2}\ H_t^2(y)$. By repeating this procedure we obtain $\int_{-\infty}^{\mu} | T_t(y) | dy \leq \gamma_t^1 H_t^1(\mu) + \gamma_t^2 H_t^2(\mu)$ and $| \int_{-\infty}^{\mu} \Psi_{\varepsilon\varepsilon}[\omega_t; y] \cdot T_t(y)\, dy | \leq k[\gamma_t^1 H_t^1(\mu) + \gamma_t^2 H_t^2(\mu)]$. Let δ be a positive number. Let $\mu^* \in (-\infty; \text{Min}(-1; -\sqrt{\gamma_t^1} + \hat{r}_t^1; -\sqrt{\gamma_t^2} + \hat{r}_t^2))$ be such that $H_t^i(\mu^*) < \frac{\delta}{2k\gamma_t^i}$ $i = 1, 2$ (such a number exists since $H_t^i(\mu) < \left(-\frac{\mu - \hat{r}_t^i}{\sqrt{\gamma_t^i}}\right)^{-1} \phi_N\left(-\frac{\mu - \hat{r}_t^i}{\sqrt{\gamma_t^i}}\right)$). The statement in claim 3 follows.

<u>Step 4</u>: We are now in a position to prove statement (c) of theorem V.3. Let $\Theta_1^+ \equiv \{g \in \Theta^+ : \exists\, x \in (-\infty; +\infty) \,/\, g''(z) < 0\ \ \forall z > x\}$ and let $\Theta_{1\mu}^+ = \{g \in \Theta_1^+ : g''(z) < 0\ \ \forall z > \mu\}$. Consider a transform g belonging to $\Theta_{1\mu}^+$. For $z > \mu$ $\Psi_{\varepsilon\varepsilon}[\omega_t; z] < 0$ since ω_t is in the interior of the constraint set. Let $\{\varepsilon_t^1 = \varepsilon_t^2\ \ \forall t \in [0; T]\}$. By claim 1, $I^1 \underset{U.I.}{\gtrsim} I^2 \implies F_t^1 \underset{I}{\gtrsim} F_t^2\ \ \forall t \in [0; T]$. This in turn implies $H_t^1 \underset{I}{\gtrsim} H_t^2\ \ \forall t \in [0; T]$ (i.e., $T_t(z) \geq 0\ \ \forall z \in (-\infty; +\infty)$, $\forall t \in [0; T]$). Consider the integral $\int_{-\infty}^{+\infty} \Psi_{\varepsilon\varepsilon}[\omega_t; y]\ T_t(y)dy$. It can be rewritten as $\int_{-\infty}^{\mu} \Psi_{\varepsilon\varepsilon}[\omega_t; y]\ T_t(y)dy + \int_{\mu}^{+\infty} \Psi_{\varepsilon\varepsilon}[\omega_t; y]\ T_t(y)dy$. The second term is clearly negative since $\Psi_{\varepsilon\varepsilon}[\omega_t; z] < 0\ \ \forall z > \mu$ and $T_t(z) \geq 0$. Denote by $\Theta_{1\mu}^{+*}$ the subset of $\Theta_{1\mu}^+$ for which the negative contribution of the second term outweighs the contribution of the first term. Let $\Theta_1^{+*} = \bigcup_{\mu} \Theta_{1\mu}^{+*}$. This set is non-empty by claim 3. (Choose μ small enough so that the contribution of the integral $\int_{-\infty}^{\mu}$ is negligible $\forall t \in [0, T]$). Take the intersection of Θ_1^{+*} with Θ_0^+, which is clearly non-empty. For all elements of this set claim 2 now applies and $E\{\Psi[\omega_t; \tilde{\varepsilon}_t] | I_t^2\} < E\{\Psi[\omega_t; \tilde{\varepsilon}_t] | I_t^1\}$. Evaluated at ω_t^2 the lefthand side of this inequality vanishes. Since Ψ is decreasing in its first argument the statement is proved. ◇

This completes the proof of theorem V.3. The effect of the quality of the information collected, as stated in Proposition (c), depends obviously on the assumption of a finite trading interval. If one focuses on large horizons, the optimal portfolio decisions associated with various information structures in $\chi[I(a,A,\sigma_e^2,\sigma_v^2,m_0)]$, will tend to converge to a common value.[11]

Corollary V.4: Let T be large. $\forall (I^1; I^2) \in \chi[I(a;A;\sigma_e^2;\sigma_v^2;m_0)]^2$, $\forall \{\varepsilon_t^i\}$ i = 1, 2 such that $\{\varepsilon_t^1 = \varepsilon_t^2 \quad t = 0, \ldots, T-1\}$, $\lim_{t \to T} \omega_t^1 \approx \lim_{t \to T} \omega_t^2$.

This statement follows from the fact that the quality of the information converges to a same fixed point for all structures in the set $\chi[I(a;A;\sigma_e^2;\sigma_v^2)]$. In the limit $\gamma_t^1 = \gamma_t^2 = \gamma^*$, which implies that $\omega_t^1 = \omega_t^2 = \omega_t^*$ for the same history of observations.

The theorem and its corollary are both based on the assumption of an identical history of observations for the two structures I^1 and I^2. In fact a slight strengthening of the statements holds.

Corollary V.5: The following propositions are true:

(a) $\forall (I^1, I^2) \in \chi[I(a^+;A^+;\sigma_e^2;\sigma_v^2;\gamma_0)]^2$, $\forall t \in [1; T)$, $\forall g \in \Theta^+$ if $\hat{r}_{t-2}^1 = \hat{r}_{t-2}^2$ and $\varepsilon_{t-1}^1 > \varepsilon_{t-1}^2$ then $\omega_t^1 > \omega_t^2$.

(b) $\forall (I^1; I^2) \in \chi[I(a;A;\sigma_e^2;\sigma_v^2)]^2$, for all histories of the processes $\{\tilde{\varepsilon}_s^1\}_{s=0}^{t-1}$ and $\{\tilde{\varepsilon}_s^2\}_{s=0}^{t-1}$, if $\hat{r}_t^1 = \hat{r}_t^2$ and $I^1 \underset{U.I.}{\gtrsim} I^2$, there exists a set of transforms $g(\cdot)$ such that $\omega_t^1 > \omega_t^2$ and $\omega_s^1 > \omega_s^2$ $s > t$ on all identical future sample paths for I^1 and I^2, for which the portfolio problems admit an interior optimal solution.

(c) $\forall$ $(I^1; I^2) \in \chi[I(a;A;\sigma_e^2;\sigma_v^2)]^2$; for all histories of the processes resulting in $\hat{r}_t^1 = \hat{r}_t^2$ at some limiting date t, the optimal portfolio choices ω_s^i $i = 1, 2$ $s > t$ will be "nearly" identical on all future identical paths of the processes $\{\tilde{\varepsilon}_s^1\}_{s=t}^T$ and $\{\tilde{\varepsilon}_s^2\}_{s=t}^T$.

This corollary asserts that the past histories of the two processes will not affect the future, if they result in the same conditional mean at time t. This property is intuitively expected since the conditional mean and variance are the only relevant parameters for portfolio choices. The past does not matter as long as it results in the same mean at time t. (The variance is unaffected in any case since it is deterministic). An additional consequence is that an infinite number of histories may give rise to the same decision at a specified date.

The analysis so far has focused on the logarithmic case. For isoelastic utilities, comparable results are difficult to obtain since current decisions depend on the future. A simple statement can however be made as to the reaction of the portfolio choice to the conditional mean at date T - 1. Specifically it can be shown that an increase in the conditional mean $\hat{r}_{T-1}$, will result in a higher proportion invested in the risky asset for information structures in the class $\chi[I(a^+;A^+;\sigma_e^2;\sigma_v^2;m_0;\gamma_0)]$ and if the transform g is strictly increasing. Assessing the reaction of decisions further back in time involves determining the direction of variations in the functions $a_t(\cdot)$.

VI. Conclusion

For investors with Bernoulli tastes we have been able to give an answer to the question which motivated this study. We have shown that Bernoulli traders react to favorable information by switching to a more aggressive investment strategy (the proportion invested in the risky asset increases). When the quality of the information obtained improves, the same behavior takes place, under additional assumptions on the transform function. This property is intuitively appealing since a better information translates in a reduction in the risk faced. (A similar result can be found in Levhari-Srinivasan [4]).

At first sight the analysis may appear severely constrained in terms of the risky payoff allowed. In fact quite a large number of distributions fit in the model since a variety of transforms satisfy our assumptions. The constraint is however effective in terms of the information structure (the nature of the underlying process). The reproduction of the conditional density over time, and its complete characterization through two parameters, is ultimately what allows for the analysis of the effect of information on optimal policies. A further investigation of the problem should seek to remove the Gaussian assumption and replace it by general conditions on the Markov structure of the environment.

FOOTNOTES

[1]When the risky assets are lognormally distributed, Levhari and Srinivasan proceed to show that an increase in the risk associated with an asset (i.e. an increase in its variance), ceteris paribus, results in a decrease in the proportion invested in this asset.

[2]Hakansson allows for a constant exogenous income stream at any point in time. Thus the solvency constraint, in his case, requires that the position taken at any trading date is covered by the net present value of the future non-capital income stream, with probability one.

[3]Assume that $c_t = 0$. The marginal contribution of an additional unit of consumption at time t is infinite. Increase now consumption at time t, and balance this change with a reduction in consumption at date t + 1, which leaves future opportunities unchanged. The net result is an increase in lifetime welfare.

[4]A proof of these statements is given in Astrom [1].

[5]It is a sufficient statistic in the sense that the knowledge of the history of observations $\{\varepsilon_s : s < t\}$ is equivalent, at time t, to the knowledge of the conditional mean $\hat{r}_t$.

[6]For additional comments on this method refer to [2, 4, 5, 7, 11].

[7]The concavity of the value function follows easily if one performs the change of variable $I = (W - c)\omega$, which represents the dollar investment in the risky asset.

[8]In the general class $\chi[I]$ it is clearly possible to find structures which cannot be ranked according to the U.I. ordering. These structures are characterized by a crossing property ($\exists t, t' \mid \gamma_t^1 < \gamma_t^2$ and $\gamma_{t'}^1 > \gamma_{t'}^2$), which is obtained by an appropriate choice of the parameters $(a^i, A^i, \sigma^2_{e_i} ; \sigma^2_{v_i})$ $i = 1, 2$.

[9]For Gaussian distributions it can be shown that an increase in the variance results in a collapse of the central part of the density and a symmetric enhancing of the tails, i.e., $\exists\,(x, x') \,/\, \phi^0(y) < \phi^1(y)$ $\forall\, y \in (x, x')$ and $\phi^0(y) > \phi^1(y)$ $\forall y \in (-\infty, x) \cup (x', +\infty)$. Here ϕ^0 and ϕ^1 represent respectively the densities associated with the variances σ_0^2, σ_1^2 $(\sigma_0^2 > \sigma_1^2)$.

[10]The exponential function satisfies this condition since for $g(x) = e^{tx}$, $t > 0$, we obtain $g(x)/g'(x) = 1/t > 0$.

[11]We do not try to compare the limits of the solutions to the finite horizon case, to the solutions obtained from the infinite horizon problem. In the latter case the model requires more structure. In particular the assumptions on the transform function need to be tightened in order to obtain a bounded program.

Appendix A-1: Boundedness of the Objective Function

Taking account of (2.2) - (2.4) we can write:

$$0 < c_t < W_t < W_1 \prod_{s=1}^{t-1} \lfloor (1 - \omega_s) r + \omega_s g(\tilde{\varepsilon}_s) \rfloor \qquad t = 2, \ldots$$

$$< W_1 \prod_{s=1}^{t-1} \lfloor r + \bar{\omega} g(\tilde{\varepsilon}_s) \rfloor \qquad t = 2, \ldots \qquad \text{(A-1.1)}$$

where $\bar{\omega}$ is a well defined positive and finite number which satisfies $\bar{\omega} = \text{Max } \{\omega_s\}_{s=1}^{T-1}$ s.t $\text{Prob}\lfloor (1 - \omega_s) r + \omega_s g(\tilde{\varepsilon}_s) > 0 \mid I_s \rfloor = 1 \quad s \in [1; T)$. By the growth condition (a.2) the R.H.S. of (A-1.1) in turn is bounded by $W_1 \prod_{s=1}^{t-1} \lfloor r + K\bar{\omega} + \bar{\omega} \exp[L\tilde{\varepsilon}_s] \rfloor$.

Hence $E_1 \sum_{1}^{T} \beta^{t-1} u[c_t] < u[W_1] + E_1 \sum_{t=2}^{T} \beta^{t-1} u\lfloor W_1 \prod_{s=1}^{t-1} (r + K\bar{\omega} + \bar{\omega} \exp[L\tilde{\varepsilon}_s]) \rfloor$

$$< u[W_1] + \sum_{t=2}^{T} \beta^{t-1} u\lfloor W_1 E_1 \prod_{s=1}^{t-1} (r + K\bar{\omega} + \bar{\omega} \exp[L\tilde{\varepsilon}_s]) \rfloor \qquad \text{(A-1.2)}$$

The last inequality follows from the concavity of the utility function. The expectation operator designates the expectation conditional upon the information set I_1. The proof is now complete if one notices that the exponential function has the Cauchy property: $\exp(x). \exp(y) = \exp(x + y)$, and that the moment generating function of a Gaussian distribution is well defined.

Appendix A-2: Bernouilli Case

At time T -1 the necessary conditions for an optimum are:

$$\begin{cases} c_{T-1}^{-1} = \beta E \left\{ \left[(W_{T-1} - c_{T-1}) \left((1-\omega_{T-1}) r + \omega_{T-1} g(\tilde{\varepsilon}_{T-1}) \right) \right]^{-1} \left[(1-\omega_{T-1}) r + \omega_{T-1} g(\tilde{\varepsilon}_{T-1}) \right] \middle| I'_{T-1} \right\} & \text{(A-2.1)} \\ 0 = E\left\{ \left[(W_{T-1} - c_{T-1}) \left((1-\omega_{T-1}) r + \omega_{T-1} g(\tilde{\varepsilon}_{T-1}) \right) \right]^{-1} \left(g(\tilde{\varepsilon}_{T-1}) - r \right) \middle| I'_{T-1} \right\} & \text{(A-2.2)} \end{cases}$$

$$\iff \begin{cases} W_{T-1} - c_{T-1} = \beta c_{T-1} & \text{(A-2.3)} \\ 0 = E\left\{ \left[(1 - \omega_{T-1}) r + \omega_{T-1} g(\tilde{\varepsilon}_{T-1}) \right]^{-1} \left(g(\tilde{\varepsilon}_{T-1}) - r \right) \middle| I'_{T-1} \right\} & \text{(A-2.4)} \end{cases}$$

Thus $c^*_{T-1} = \frac{1}{1 + \beta} W_{T-1}$ (A-2.5)

$$\omega^*_{T-1} = \omega^*_{T-1}(\hat{r}_{T-1}; \gamma_{T-1}) \equiv \omega^*(\hat{r}_{T-1}; \gamma_{T-1}) \quad \text{(A-2.6)}$$

$$J_{W_{T-1}}\left[\hat{r}_{T-1}; \gamma_{T-1}; W_{T-1}\right] = u'[c_{T-1}] = \left(\frac{1}{1 + \beta} W_{T-1}\right)^{-1} \quad \text{(A-2.7)}$$

In period T-2 the first-order conditions become

$$\begin{cases} c_{T-2}^{-1} = \beta(1 + \beta) (W_{T-2} - c_{T-2})^{-1} & \text{(A-2.8)} \\ 0 = E\left\{ \left[(1 - \omega_{T-2}) r + \omega_{T-2} g(\tilde{\varepsilon}_{T-2}) \right]^{-1} (g(\tilde{\varepsilon}_{T-2}) - r) \middle| I'_{T-2} \right\} & \text{(A-2.9)} \end{cases}$$

and $c^*_{T-2} = \frac{1}{1 + \beta + \beta^2} W_{T-2}$ (A-2.10)

$$\omega^*_{T-2} = \omega^*_{T-2}(\hat{r}_{T-2}; \gamma_{T-2}) = \omega^*(\hat{r}_{T-2}; \gamma_{T-2}) \quad \text{(A-2.11)}$$

$$J_{W_{T-2}}[\hat{r}_{T-2}; \gamma_{T-2}; W_{T-2}] = u'[c_{T-2}] = (1 + \beta + \beta^2) W_{T-2}^{-1} \quad \text{(A-2.12)}$$

The general solution at time t is:

$$c^*_t = (1 + \beta + \beta^2 + \ldots + \beta^{T-t})^{-1} W_t \quad \text{(A-2.13)}$$

$$\omega^*_t = \omega^*(\hat{r}_t; \gamma_t) \quad \text{(A-2.14)}$$

Appendix A-3: Isoelastic Case

In period T-1 c^*_{T-1} and ω^*_{T-1} solve:

$$\begin{cases} c_{T-1}^{\gamma-1} = \beta\, E\{(W_{T-1} - c_{T-1})^{\gamma-1}[(1-\omega_{T-1})r + \omega_{T-1}g(\tilde{\varepsilon}_{T-1})]^{\gamma} \mid I'_{T-1}\} & \text{(A-3.1)} \\ 0 = E\{[(1-\omega_{T-1})r + \omega_{T-1}g(\tilde{\varepsilon}_{T-1})]^{\gamma-1}[g(\tilde{\varepsilon}_{T-1}) - r] \mid I'_{T-1}\} & \text{(A-3.2)} \end{cases}$$

Define a_{T-1} as $\beta^{\frac{1}{\gamma-1}} \{E[(1-\omega_{T-1})r + \omega_{T-1}g(\tilde{\varepsilon}_{T-1})]^{\gamma} \mid I'_{T-1}\}^{\frac{1}{\gamma-1}}$

Then: $$c^*_{T-1} = \frac{a_{T-1}}{1 + a_{T-1}} W_{T-1} \tag{A-3.3}$$

$$\omega^*_{T-1} = \omega^*_{T-1}(\hat{r}_{T-1};\ \gamma_{T-1}) \tag{A-3.4}$$

$$J_{W_{T-1}}[\hat{r}_{T-1};\ \gamma_{T-1};\ W_{T-1}] = u'[c_{T-1}] = \left(\frac{a_{T-1}}{1 + a_{T-1}}\right)^{\gamma-1} W_{T-1}^{\gamma-1} \tag{A-3.5}$$

In the previous period the first order conditions are:

$$\begin{cases} c_{T-2}^{\gamma-1} = \beta E\{\left(\frac{a_{T-1}}{1 + a_{T-1}}\right)^{\gamma-1}(W_{T-2}-c_{T-2})^{\gamma-1}[(1-\omega_{T-2})r+\omega_{T-2}g(\tilde{\varepsilon}_{T-2})]^{\gamma} \mid I'_{T-2}\} & \text{(A-3.6)} \\ 0 = E\{\left(\frac{a_{T-1}}{1 + a_{T-1}}\right)^{\gamma-1}[(1-\omega_{T-2})r + \omega_{T-2}g(\tilde{\varepsilon}_{T-2})]^{\gamma-1}[g(\tilde{\varepsilon}_{T-2}) - r] \mid I'_{T-2}\} & \text{(A-3.7)} \end{cases}$$

Thus optimal solutions are:

$$c^*_{T-2} = \frac{a_{T-2}}{1 + a_{T-2}} W_{T-2} \tag{A-3.7}$$

$$\omega^*_{T-2} = \omega^*_{T-2}(\hat{r}_{T-2};\ \gamma_{T-2}) \tag{A-3.8}$$

$$J_{W_{T-2}}[\hat{r}_{T-2};\ \gamma_{T-2};\ W_{T-2}] = \left(\frac{a_{T-2}}{1 + a_{T-2}}\right)^{\gamma-1} W_{T-2}^{\gamma-1} \tag{A-3.9}$$

where $a_{T-2} \equiv \beta^{\frac{1}{\gamma-1}} [E\{\left(\frac{a_{T-1}}{1 + a_{T-1}}\right)^{\gamma-1}[(1-\omega^*_{T-2})r + \omega^*_{T-2}g(\tilde{\varepsilon}_{T-2})]^{\gamma} \mid I'_{T-2}\}]^{\frac{1}{\gamma-1}}$

In period t the general solution can be written as

$$c^*_t = \left(\frac{a_t}{1 + a_t}\right) W_t \tag{A-3.10}$$

$$\omega_t^* = \omega_t^*(\hat{r}_t;\ \gamma_t) \tag{A-3.11}$$

where a_t is recursively defined as:

$$a_t = \beta^{\frac{1}{\gamma-1}} \left[E\ \left\{\left(\frac{a_{t+1}}{1 + a_{t+1}}\right)^{\gamma-1}\left[(1 - \omega_t^*)r + \omega_t^* g(\tilde{\varepsilon}_t)\right]^{\gamma} \middle| I_t'\right\}\right]^{\frac{1}{\gamma-1}}$$

$$= a_t(\hat{r}_t;\ \gamma_t)$$

and ω_t^* solves: $0 = E\left\{\left(\frac{a_{t+1}}{1 + a_{t+1}}\right)^{\gamma-1}\left[(1 - \omega_t)r + \omega_t g(\tilde{\varepsilon}_t)\right]^{\gamma-1}\left[g(\tilde{\varepsilon}_t) - r\right] \middle|\ I_t'\right\}$

Figure I: Examples of transforms satisfying Assumption II.1

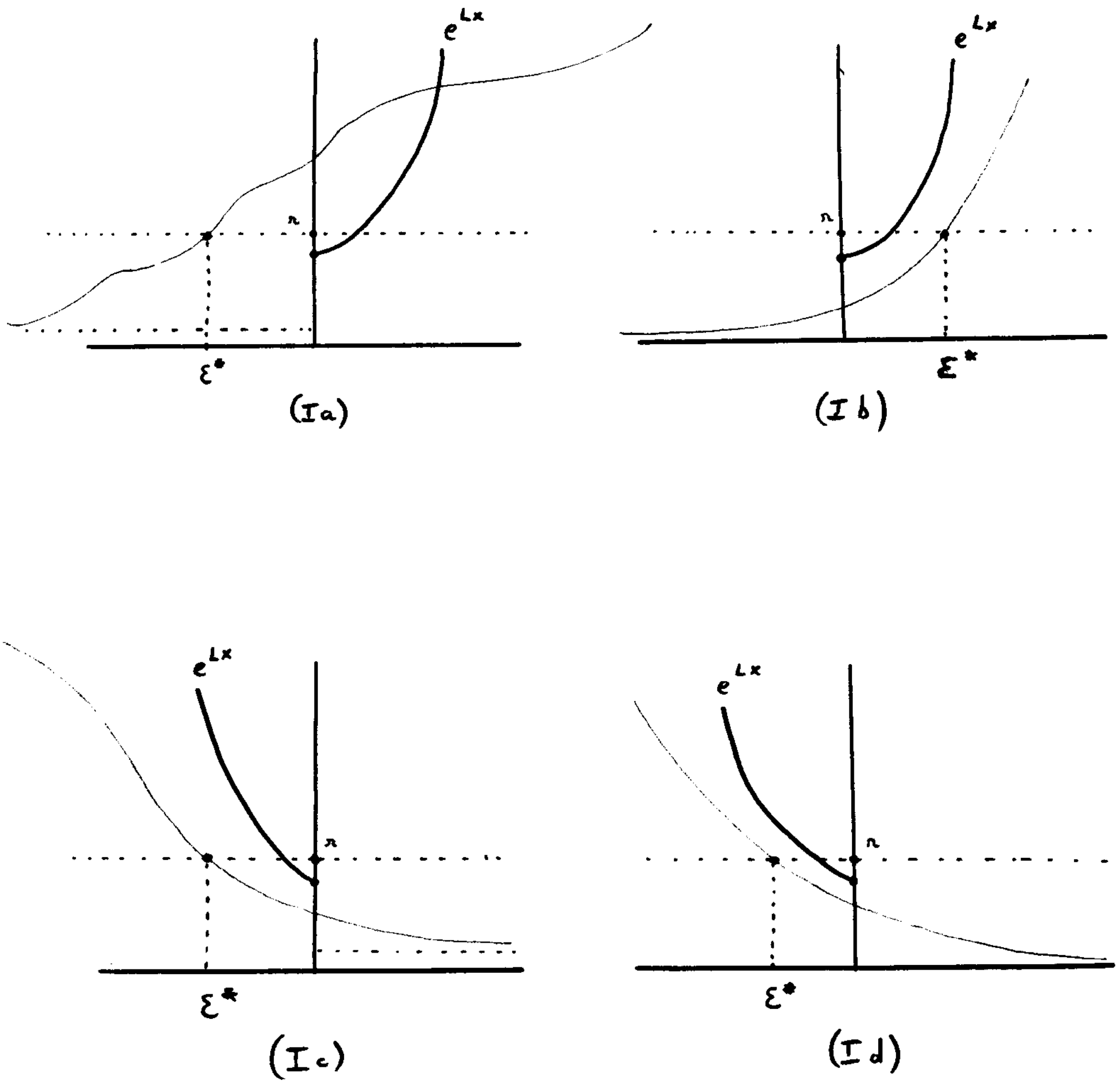
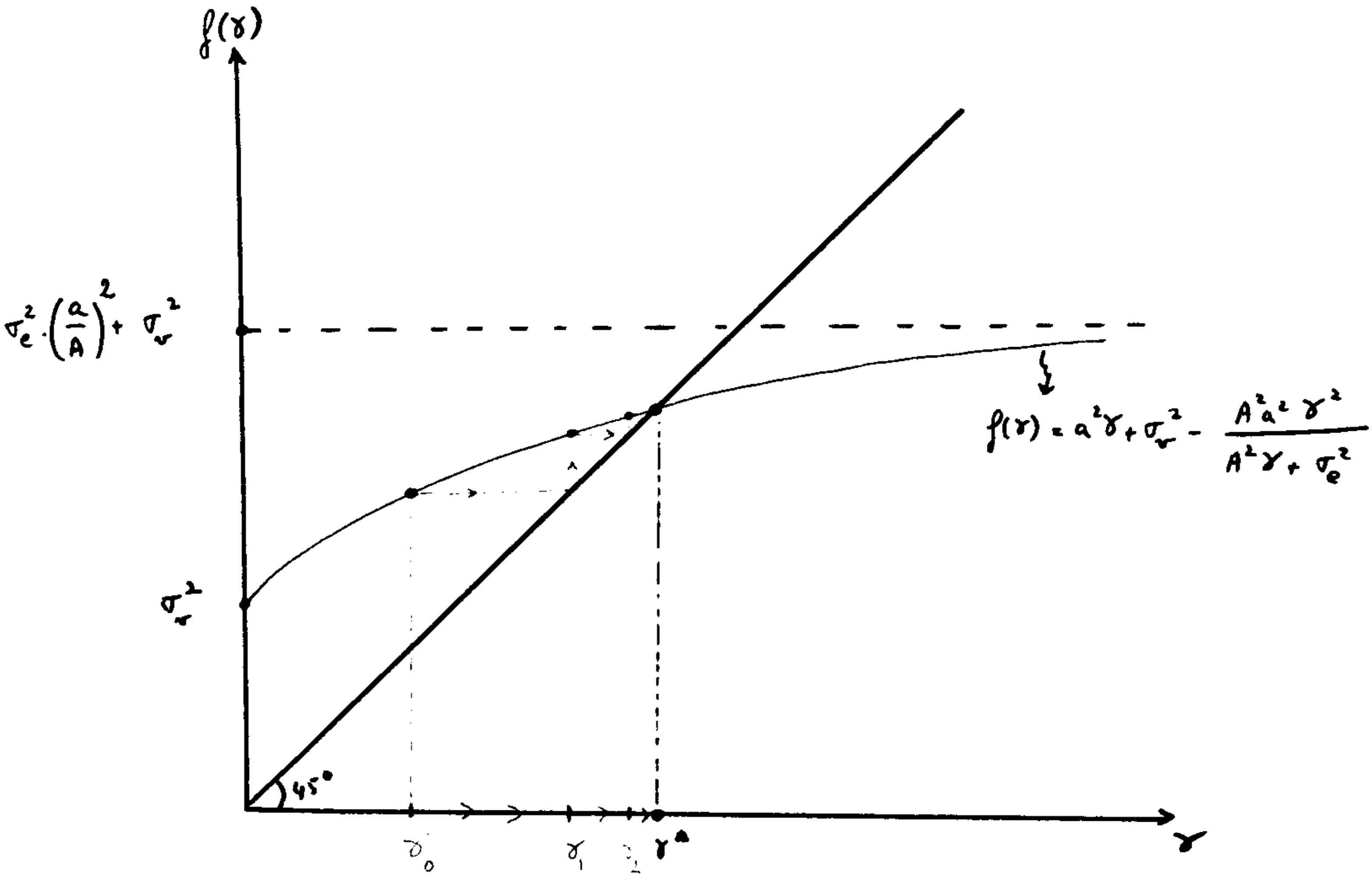


Figure II: Temporal adjustment of the conditional variance



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