THE PRICING OF CALL AND PUT OPTIONS ON FOREIGN EXCHANGE

By

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ABSTRACT

This paper derives exact pricing equations for American and European puts and calls on foreign exchange and discusses hedging strategy. Because every call option on foreign currency is simultaneously a put option on the domestic currency, an equivalence relation exists that allows the immediate derivation of put equations from the corresponding call formulas. The call and put pricing formulas are unlike the Black-Scholes equations for stock options in that there are two relevant interest rates, interest rates are stochastic, and boundary constraints differ. In addition, both American call and put options have values larger than their European counterparts.

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The Pricing of Call and Put Options on Foreign Exchange

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I. Introduction

This paper explores pricing relationships and develops pricing equations for European and American call and put options on foreign currency. An American call option is a security issued by an individual which gives its owner the right to purchase a given amount of an asset at a stated price (the exercise or striking price) on or before a stated date (the expiration or maturity date). For example, a call option on the British pound might give me the right to purchase £12,500 at 1.70 $/£ on or before the second Saturday in December 1983. A European call option is the same as the American call, except that it may be exercised only on the expiration date. In the previous example, only on the second Saturday in December 1983. An American put option is a security issued by an individual which gives its owner the right to sell a given amount of an asset at a stated price on or before a stated date. For example, a put option on the yen might give me the right to sell ¥6,250,000 at .004 $/¥ on or before June 13. A European put option is the same as the American put, except that it may be exercised only on the expiration date. In the preceding example, only on June 13.

Foreign currency options may be used for hedging or for speculation. Their use, and the consequent interest in their fair pricing, arises from two principal sources.

The first is organized trading in foreign currency options. Over the period December 1982 - February 1983 the Philadelphia Stock Exchange opened a

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market in American call and put options on five foreign currencies--the German mark, the British pound, the Swiss franc, the Canadian dollar, and the Japanese yen. The exercise price of each option is stated as the U.S. dollar price of a unit of foreign exchange, and the number of foreign currency units is one-half the contract size of the corresponding currency futures contract traded on the International Money Market of the Chicago Mercantile Exchange. Option contract sizes are £12,500; DM 62,500; Swiss franc 62,500; ¥6,250,000; and C$50,000. The expiration dates of the options are set to correspond to the March, June, September, December delivery dates on futures. Futures contracts expire on the second business day prior to the third Wednesday of each of these months, and option contracts expire on the second Saturday of each of these months. For each currency, options are opened with terms to maturity of 3, 6, and 9 months, corresponding to the March, June, September, December cycle. Exercise price intervals are $.05 for the £; $.02 for the DM, Swiss franc, C$; and $.0002 for the ¥. (If, for example, the spot price of the £ is 1.82 $/£ when June pounds open for trading, exercise prices are set at 1.80 $/£ and 1.85 $/£. If the £ then drops to $1.80, a new series with exercise prices of 1.75 $/£ is opened.)

The second source is the frequent appearance of foreign currency option features on bond contracts in the international bond markets. For example, take the case of a Japanese company which issues $1000 bonds at par in the Eurobond market. The coupon rate is 12% payable annually in U.S. dollars, and the bonds mature in May 1990. At maturity the bonds may be redeemed, at the owner's discretion, for dollars or for yen at an exchange rate of .005 $/yen. What value would you place on such a bond? Clearly the bond owner will opt for repayment of principal in yen if the spot price of yen is greater than .005 $/yen in May 1990. (For example, if the spot rate is .006 $/yen, she would redeem her bond for 1000/.005 = 200,000 yen and then sell the yen for (200,000)(.006) = $1200.) Thus the value of this Eurobond can be viewed as the sum of (1) the value of an ordinary $1000 bond with a 12% coupon, and (2) the value of a European call option on 200,000 yen, with an exercise price of .005 $/yen, and with an expiration date in May 1990. Thus, provided we can value European options on foreign currency, we can place a value on this Japanese currency-option bond.

Some previous studies have looked at foreign currency option pricing. Feiger and Jacquillat (1979) attempt to obtain foreign currency option prices by first pricing a currency option bond. They are not able, however, to obtain simple, closed-form solutions by this procedure. Stulz (1982) looks also at currency option bond pricing, but his paper is primarily concerned with the question of default risk on part of a contract, and it is not easy to grasp the fundamentals of foreign currency option pricing in the context of his more general investigation. Black (1976) looks at commodity options, and thus his results are relevant because foreign exchange is a commodity. However, they are not suitably general or detailed enough for someone primarily concerned with foreign currency options, as opposed to commodity options generally.

The organization of this paper is as follows. In terms of the selection of content, it is assumed that the reader has available the papers of Merton (1973) or Smith (1976) which review some elementary principles, or inequalities, in pricing options on common stock. Hence the present paper will only deal with features that are peculiar to foreign exchange options, and omit points that carry over unaltered from stock options. Section II sets out notation and assumptions. Section III derives some inequalities and equivalences for puts and calls on foreign exchange. In Section IV, some strong distributional assumptions are imposed, and exact pricing equations derived. Section V briefly considers the use of the equations for hedging and speculation, and Section VI concludes the paper.

II. Notation and Assumptions

Throughout this paper it will be assumed that contracts--whether spot or forward exchange, discount securities, or option contracts themselves--are default-free. In addition we will assume the absence of transaction costs, taxes, exchange controls, or similar factors. (Once the basic relationships have been established, these factors may be added back in where and when they are relevant.)

The symbols used will follow Smith's (1976) standardized notation, except for some minor variations that are tailored to the foreign exchange market. The notation is:

- $S(t)$ – the spot domestic currency price of a unit of foreign exchange at time t;
- $F(t, T)$ – the forward domestic currency price of a unit of foreign exchange, for a contract made at time t and which matures at time $t + T$;
- T – the time until expiration;
- $C(t)$ – the domestic currency price at time t of an American call option written on one unit of foreign exchange;
- $C^*(t)$ – the foreign currency price at time t of an American call option written on one unit of domestic currency;
- $c(t)$ – the domestic currency price at time t of a European call option written on one unit of foreign exchange;
- $c^*(t)$ – the foreign currency price at time t of a European call option written on one unit of domestic currency;
- $P(t)$ – the domestic currency price at time t of an American put option written on one unit of foreign currency;
- $P^*(t)$ – the foreign currency price at time t of an American put option written on one unit of domestic currency;

$p(t)$ – the domestic currency price at time t of a European put option written on one unit of foreign currency;

$p^*(t)$ – the foreign currency price at time t of a European put option written on one unit of domestic currency;

$B(t, T)$ – the domestic currency price of a pure discount bond with a maturity value of one unit of domestic currency at time $t + T$;

$B^*(t, T)$ – the foreign currency price of a pure discount bond with a maturity value of one unit of foreign exchange at time $t + T$;

X – the domestic currency exercise price of an option on foreign currency;

X^* – the foreign currency exercise price of an option on domestic currency.

III. Some Basic Relationships.

1. Consider the following two portfolio strategies undertaken at time t when the spot exchange rate is $S(t)$:

Strategy A: (1) Purchase for $c(S(t), X, t, T)$ a European call option, with an exercise price of X and which expires in T units of time, on 1 unit of foreign exchange.

(2) Purchase X domestic currency discount bonds, which mature in T units of time, at the current price $B(t, T)$.

Total Domestic Currency Investment: $c + XB$.

Strategy B: (1) Purchase one foreign discount bond, which matures in T units of time, at a domestic currency price of $S(t)B^*(t, T)$.

At time $t + T$, the spot exchange rate $S(t + T)$ will either be (i) less than X, or (ii) greater than or equal to X. The bonds will have values, in their respective currencies, of $B(t + T, 0) = 1$, $B^*(t + T, 0) = 1$. The call option will have a value $c(S(t + T), X, T, 0) = \max(0, S(t + T) - X)$. Therefore:

	Value of Strategy A	Value of Strategy B
$S(t + T) < X$	X	$S(t + T)$
$S(t + T) \geq X$	$S(t + T)$	$S(t + T)$

In either case, the payoff to strategy A will always be as good or better than strategy B. Hence the cost of A must, in economic equilibrium, be at least as great as that of B. Thus $c + XB \geq SB^* $ or $c \geq SB^* - XB$. Since an American call must be at least as valuable as its European counterpart (because it has all the same features plus the additional one that it can be exercised at any time), we get

$$C(S(t), X, t, T) \geq c(S(t), X, t, T) \geq S(t)B^*(t, T) - XB(t, T) \ . \qquad (1)$$

For example, if one-year Eurodollar deposits have an interest rate of 11.11% ($B(t, 1) = 1/1.1111 = .9$), one-year Euro-Swiss franc deposits have an interest rate of 5.26% ($B^*(t, 1) = 1/1.0526 = .95$), the spot dollar price of Swiss francs is $S(t) = .55$ \$/Swiss fr., then a 12-month American option on one Swiss franc with an exercise price of \$X = \$.50 will have a value

$$\begin{aligned} C(.55, .50, t, 1) \geq c(.55, .50, t, 1) &\geq (.55)(.95) - (.50)(.9) \\ &= \$.0725 \ . \end{aligned}$$

Since an American option can be exercised at any time, and must have a value at least as large as its immediate exercise value, we get the stronger inequality

$$C\left(S(t), X, t, T\right) \geq \max\left(0, S(t) - X, S(t)B^*(t, T) - XB(t, T)\right) \ . \qquad (1A)$$

2. Using the notation of Section II, we may write the Interest Parity Theorem as

$$F(t, T) = S(t) \frac{B^*(t, T)}{B(t, T)} \ . \qquad (2)$$

Substituting for $S(t)B^*(t, T)$ in equation 1, we obtain the relation

$$C(S(t), X, t, T) \geq c(S(t), X, t, T) \geq B(t, T)[F(t, T) - X] \ . \qquad (3)$$

The call option on 1 unit of foreign exchange must have a value at least as great as the discounted difference between the forward exchange rate and the exercise price. This assumes that interest parity holds. (Hence, in practical application $B(t, T)$ and $B^*(t, T)$ should be thought of as Eurocurrency securities, rather than as treasury bills.)

3. Here we derive a relationship between the prices of European calls and European puts. Consider the following two portfolio strategies:

Strategy A: (1) Buy, at a domestic currency price of $p(S(t), X, t, T)$, a put option on one unit of foreign currency, with exercise price X.

Strategy B: (1) Issue a foreign-currency-denominated discount bond at $B^*(t, T)$ and sell the foreign currency for $S(t)B^*(t, T)$.

(2) Buy X domestic discount bonds at a price of $B(t)$ each, for a total domestic currency amount of $XB(t)$.

(3) Buy, at a domestic currency price of $c(S(t), X, t, T)$, a European call option on one unit of foreign currency, with an exercise price of X.

Total domestic currency investment: $c - SB^* + XB$.

At time $t + T$ the spot exchange rate $S(t + T)$ will be such that either $S(t + T) < X$ or $S(t + T) \geq X$. In each case, since $p(S(t + T), X, T, 0) = \max(0, X - S(t + T))$, the strategies will have the payoffs:

	Value of Strategy A	Value of Strategy B
$S(t + T) < X$	$X - S(t + T)$	$X - S(t + T)$
$S(t + T) \geq X$	0	0

Since each strategy gives the same payoff, each must cost the same in equilibrium. Hence

$$p(S(t), X, t, T) = c(S(t), X, t, T) - S(t)B^*(t, T) + XB(t, T) . \qquad (4)$$

Thus the price of a European put is totally determined by the price of the corresponding European call, the spot exchange rate, and the prices of discount bonds denominated in the two currencies.

4. Using the Interest Parity Theorem, equation (2), and substituting into equation (4), we obtain

$$p(S(t), X, t, T) = c(S(t), X, t, T) + B(t, T)[X - F(t, T)] \quad (5)$$

The price of a European put differs from the price of the corresponding call by a factor which represents the discounted difference between the exercise price and the forward exchange rate.

5. If I have a call option on a foreign currency, written at an exercise price in terms of the domestic currency, then this same option is a put option on the domestic currency, written at an exercise price in terms of the foreign currency. For example, take the PHLX call option on 62,500 DM and suppose that the exercise price is .40 $/DM. This option gives me the right to buy 62,500 DM for (62,500)(.40) = $25,000. But at the same time it gives me the right to sell $25,000 for DM 62,500. Thus it is a put option on $25,000 with an exercise price of 1/.40 = 2.5 DM/$. This is a simple consequence of the fact an exchange rate has two sides. Hence, whether viewed as a call or a put, a contract must have the same domestic currency value.

In terms of the notation of Section II, this identity[1] may be written, for American option contracts:

$$C(S(t), X, t, T) = S(t)XP^*(1/S(t), 1/X, t, T) \quad (6)$$

(For example, if C is the value of an American call on 1 DM with an exercise price $X = $.40, then the same contract is a put option on $.40 with an exercise price of 1 DM. Hence it has the value of 40/100 of P*(1/S(t), 2.5, t, T), where P* is the DM value of a put option on $1 with an exercise price of 2.5 DM. But the dollar value of .40P* = XP* is just SXP*, which must be equal to C.)

For European option contracts we may write the identity as

$$c(S(t), X, t, T) = S(t)Xp^*(1/S(t), 1/X, t, T) . \tag{7}$$

For domestic-currency-valued puts on foreign exchange, we have

$$P(S(t), X, t, T) = S(t)XC^*(1/S(t), 1/X, t, T) \tag{8}$$

and

$$p(S(t), X, t, T) = S(t)Xc^*(1/S(t), 1/X, t, T) . \tag{9}$$

Notice that equations (6), (8) imply that if we obtain a pricing equation for American call options on foreign exchange, then we immediately get a pricing equation for American put options.

IV. Pricing Equations for Call and Put Options

Here we derive exact pricing equations for American and European calls and puts. To do this, some strong assumptions are imposed. Equation 1 shows that the value of an American call C will be a function of $S(t)B^*(t,T)$, $B(t,T)$, X, T, at least. The first assumption, then, is that C has the general functional form $C = C(S(t)B^*(t,T),B(t,T),X,T)$. Such a function would be subject to the boundary conditions

$$C(S(t+T),1,X,0) = \max\left[0,S(t+T)-X\right]; \tag{10A}$$

$$C(0,B(t,T),X,T) = 0; \tag{10B}$$

$$C(S(t)B^*(t,T),B(t,T),X,T) \geq \max\left[0,S(t)-X\right]. \tag{10C}$$

The first boundary condition is the terminal value of the call option, which has to be the greater of zero or the exercise value. The second boundary condition says that when the value of spot exchange is zero, the option to buy it has a zero value. The third boundary condition says the value of an American call can never be less than its immediate exercise value. (This third boundary condition does not apply in the case of a European call option.)

The second assumption has to do with the dynamics of S, B*, and B. Let dx, dy, dz denote Levy-Wiener processes with covariance matrix

$$\begin{bmatrix} 1 & \rho_{SB*} & \rho_{SB} \\ \rho_{SB*} & 1 & \rho_{B*B} \\ \rho_{SB} & \rho_{B*B} & 1 \end{bmatrix} dt ,$$

where $\rho_{ij} = \rho_{ij}(t,T)$ may be a known function of time (t) and the term to maturity of the bond (T). Then S, B*, B are assumed to follow the lognormal diffusion processes

$$\frac{dS}{S} = \mu_S(t)\ dt + \sigma_S(t)\ dx$$

$$\frac{dB^*}{B^*} = \mu_{B*}(t,T)\ dt + \sigma_{B*}(t,T)\ dy$$

$$\frac{dB}{B} = \mu_B(t,T)\ dt + \sigma_B(t,T)\ dz$$

On the basis of these assumptions we can define new variables dG, dW, as

$$\frac{dG}{G} = \frac{d(SB^*)}{SB^*} = (\mu_S + \mu_{B*} + \rho_{SB*}\sigma_S\sigma_{B*})\ dt + \sigma_S dx + \sigma_{B*} dy$$

$$= \mu_G(t,T)\ dt + \sigma_G(t,T)\ dw ,$$

and write the covariance matrix of dw, dz as

$$\begin{bmatrix} 1 & \rho_{GB} \\ \rho_{GB} & 1 \end{bmatrix} dt ,$$

where $\rho_{GB} = \rho_{GB}(t,T)$.

We know from equation 1 that the derived American call option value must satisfy the condition $C(t) \geq S(t)B^*(t,T) - XB(t,T)$. But that does not ensure that the boundary constraint in (10C) will not be violated, since $S(t)B^*(t,T) - XB(t,T) < S-X$ for sufficiently small B^*, given B. Thus we have to take explicit account of (10C) in deriving the pricing equation.

It is plausible to assume that the call option dynamics will be different away from the boundary than on the boundary. Thus we first consider the case that either $C(t) > S-X > 0$, or $C(t) > 0 > S-X$. We will assume that in this region, where the constraint (10C) is not binding, that C(t) is everywhere twice continuously differentiable. In order to derive the call option pricing equation for this case, we employ the strategy used by Merton (1973) in his variable-interest-rate version of the Black-Scholes model. We form a zero-wealth portfolio, the return to which is non-stochastic, or riskless. In economic equilibrium, the return to such a portfolio must be zero.

Now, applying Ito's lemma to the function $C(SB^*,B,X,T) = C(G,B,X,T)$, we get the option dynamics, for $C > S-X > 0$ or $C > 0 > S-X$,

$$dC = \frac{\partial C}{\partial G} dG + \frac{\partial C}{\partial B} dB + \frac{\partial C}{\partial T} dT$$

$$- \tfrac{1}{2}\left(\frac{\partial^2 C}{\partial G^2} G^2\sigma_G^2 + 2\frac{\partial^2 C}{\partial G \partial B} GB\rho_{GB}\sigma_G\sigma_B + \frac{\partial^2 C}{\partial B^2} B^2\sigma_B^2\right) dT$$

$$= \frac{\partial C}{\partial G} dG + \frac{\partial C}{\partial B} dB + \frac{\partial C}{\partial T} dT - \tfrac{1}{2}\phi dT$$

where ϕ represents the elements involving second derivatives, and the relation $dt = -dT$ has been used.

Let V be a portfolio composed of one option, b units of G, and e units of B:

$$V = C + bG + eB.$$

The dynamics of this portfolio are

$$dV = dC + b\ dG + e\ dB.$$

Choose b, e such that $b = -\frac{\partial C}{\partial G}$, $e = -\frac{\partial C}{\partial B}$. Then

$$dV = \left(\frac{\partial C}{\partial T} - \tfrac{1}{2}\phi\right)\ dT\ .$$

If the portfolio V uses no wealth, then in equilibrium it should yield a zero return. That is, if

$$V = C - \frac{\partial C}{\partial G} G - \frac{\partial C}{\partial B} B = 0 \qquad (11)$$

then $dV = 0$, which in turn implies that

$$\frac{\partial C}{\partial T} = \tfrac{1}{2}\phi\ . \qquad (12)$$

We look for a function C(G,B,X,T) that solves equations 11 and 12, and is also subject to the boundary conditions 10A, 10B.

It may be verified by direct substitution that a solution is

$$C(t) = S(t)B^*(t,T)N(d_1) - XB(t,T)N(d_2) \qquad (13)$$

where N(d) is the standard normal distribution

$$N(d) = \int_{-\infty}^{d} \frac{1}{\sqrt{2\pi}} \exp(-x^2/2)dx$$

and

$$d_1 = \frac{\ln(SB^*/XB) + \frac{\sigma^2}{2} T}{\sigma\sqrt{T}}$$

$$d_2 = \frac{\ln(SB^*/XB) - \frac{\sigma^2}{2} T}{\sigma\sqrt{T}}$$

$$\sigma^2 = \int_0^T \frac{1}{T}\left[\sigma_G^2(t+T-u,u) + \sigma_B^2(t+T-u,u) - 2\rho_{GB}(t+T-u,u) \cdot \sigma_G(t+T-u,u) \cdot \sigma_B(t+T-u,u)\right] du \ .$$

At this point, we divide states of the world into three cases.

<u>Case 1:</u> $SB^*N(d_1) - XBN(d_2) \geq \max(0,S-X)$.

<u>Case 2:</u> $S-X \geq \max(SB^*N(d_1) - XBN(d_2), 0)$.

<u>Case 3:</u> $0 \geq \max(S-X, SB^*N(d_1) - XBN(d_2))$.

In the first case, since there is no violation of the boundary constraint (10C), the assumed dynamics in the derivation of equation 13 are valid. Thus the zero-wealth, zero-return condition yields equation 13 as the value of C(t). In the second case, the assumption of no arbitrage opportunities implies C(t) = S-X, since C(t) must be at least as valuable as its immediate exercise value. In the third case, C(t) = 0, since a call option is a limited liability contract to the buyer of the call, so that it cannot have a negative value. Thus, in summary, pure arbitrage considerations yield as the value of the American call option

$$\boxed{C(t) = \max\left(0,\ S-X,\ S(t)B^*(t,T)N(d_1)-XB(t,T)N(d_2)\right) \ .} \qquad (14)$$

How does this differ from the value of an European call option? The European call does not have the boundary constraint (10C), but only (10A), (10B). Hence, the assumed dynamics for the zero-wealth hedge defined by equations 11 and 12 are always valid for a hedge formed from the European option. Therefore, the condition of no arbitrage opportunities yields as the value of the European call option

$$\boxed{c(t) = S(t)B^*(t,T)N(d_1) - XB(t,T)N(d_2) \ .} \qquad (15)$$

The equation for pricing American calls on foreign exchange, equation 14, differs from the Black-Scholes formula for pricing options on common stock in three respects. First, there are two interest rates, not one. These interest rates are represented in the current prices of discount bonds $B(t,T)$, $B^*(t,T)$. In the Black-Scholes model, money yields interest, but not stock. Hence interest is foregone if a call option is purchased, and there is no possibility of receiving interest if the option is exercised. For an option on foreign exchange, however, the situation is different. One acquires foreign exchange if the option is exercised, and one may then receive interest at the foreign interest rate. Second, the Black-Scholes model assumes a constant interest rate, and hence explicitly excludes covariation between movements in stock prices and interest rate movements. While this may be a reasonable simplification for the stock market, it is not appropriate for the foreign exchange market, where interest rate movements induce comovements in spot and forward exchange rates. Third, the asymmetry between interest payments on money (positive interest) and stock (zero interest, ignoring dividends) means that an American call option on stock is always worth more alive than dead, so that it will not be exercised prematurely. For American calls on foreign currency, however, the additional discount factor $B^*(t,T)$ in the expression $SB^*N(d_1) - XBN(d_2)$, means that the latter expression may be less than $S-X$. Thus, the omission of the term $S-X$ from the maximum of equation 14 could result in a violation of the boundary constraint of 10C. Overall, the payment of interest on the foreign currency, reflected in the price $B^*(t,T)$, reduces the value of the foreign currency option.

The American call option formula may be rewritten, using the Interest Parity relation $S(t)B^*(t,T) = F(t,T)B(t,T)$, as

$$\boxed{C(t) = \max\left(0,\ B(t,T)[F(t,T)N(d_1) - XN(d_2)],\ S(t)-X\right)} \qquad (16A)$$

where

$$d_1 = \frac{\ln(F/X) + \frac{\sigma^2}{2} T}{\sigma\sqrt{T}}$$

$$d_2 = \frac{\ln(F/X) - \frac{\sigma^2}{2} T}{\sigma\sqrt{T}}$$

and $\sigma^2 = \int_0^T \frac{1}{T} \sigma_F^2(t + T - u, u)du$ where $\sigma_F^2(t, T)$ is the instantaneous variance of $\ln(F(t, T))$. (Note that σ^2 here is exactly the same variable as in equation 13, as may be verified by applying Ito's lemma to the interest parity relation (2).)

What is remarkable about equation 16A is that the principal term, $B[FN(d_1) - XN(d_2)]$, does not involve the price of spot exchange, which is the underlying asset on which the option is written. This results because, given the current price of domestic currency discount bonds B, all of the relevant information concerning both the spot exchange rate and the foreign currency discount bond price that is necessary for option pricing is already reflected in the forward rate.

The value of the European call option is just

$$\boxed{c(t) = B(t,T)[F(t,T)N(d_1) - XN(d_2)] .} \tag{16B}$$

If we express the American call option value as a percentage of its exercise price, we get

$$\frac{C}{X} = \max\left(0,\ B(t,T)\ \frac{F(t,T)}{X}\ N(d_1) - N(d_2),\ \frac{S(t)}{X} - 1\right) . \tag{17}$$

As long as $\frac{C}{X} > \frac{S}{X} - 1$, we may write $\frac{C}{X}$ as a function of three variables: F/X, $\sigma\sqrt{T}$, B.

For a foreign-currency denominated call option $C^*(t)$ written on one unit of domestic currency, equation 14 becomes, upon the appropriate substitutions:

$$C^*(t) = \max\left(0, \frac{1}{S(t)} B(t,T)N(d_1^*) - \frac{1}{X}B^*(t,T)N(d_2^*), \frac{1}{S(t)} - \frac{1}{X}\right) \quad (18)$$

where

$$d_1^* = \frac{-\ln(SB^*/XB) + \frac{\sigma^2}{2} T}{\sigma\sqrt{T}} = -d_2$$

$$d_2^* = \frac{-\ln(SB^*/XB) - \frac{\sigma^2}{2} T}{\sigma\sqrt{T}} = -d_1 \ .$$

Thus, applying equation 8 to equation 18, we obtain the domestic-currency value of an American put on 1 unit of foreign currency as

$$\boxed{P(t) = \max\left(0, XB(t,T)N(d_1^*) - S(t)B^*(t,T)N(d_2^*), X-S(t)\right) \ .} \quad (19A)$$

The value of the European put on 1 unit of foreign currency is

$$\boxed{p(t) = XB(t,T)N(d_1^*) - S(t)B^*(t,T)N(d_2^*) \ .} \quad (19B).$$

As may be easily verified, Equation 19B, together with equation 15, satisfies the necessary relation between the prices of European calls and European puts derived in equation 4.

Equations 19A, 19B may be rewritten in terms of the forward rate as

$$P(t) = \max\left[0, B(t, T)[XN(d_1^*) - FN(d_2^*)], X - S(t)\right] \quad (20A)$$

$$p(t) = B(t, T)\left[XN(d_1^*) - FN(d_2^*)\right] \quad (20B)$$

where

$$d_1^* = \frac{-\ln(F/X) + \frac{\sigma^2}{2} T}{\sigma\sqrt{T}}$$

$$d_2^* = \frac{-\ln(F/X) - \frac{\sigma^2}{2} T}{\sigma\sqrt{T}} \ .$$

V. Use of the Pricing Formulas for Hedging or Speculation.

There are a wide variety of hedging relationships and speculative strategies that may be based on the option pricing formulas derived in the previous section. For the purpose of illustration, we will look at one hedging example and one speculative strategy.

Hedging the domestic currency value of a foreign bond. We will only consider the case of a long position in a foreign-currency bond, since the case of borrowing in a foreign currency is symmetrical. The domestic-currency value of the foreign-currency bond is $G = S(t)B^*(t, T)$. Then for a call option on the foreign-currency where $SB^*N(d_1) - SBN(d_2) \geqslant S - X \geqslant 0$ or $SB^*N(d_1) - SBN(d_2) \geqslant 0 \geqslant S - X$, the hedge is that previously described in equation 11. Equation 11 may be rewritten as

$$G + \left[\frac{\partial C}{\partial B}/\frac{\partial C}{\partial G}\right]B - \left[1/\frac{\partial C}{\partial G}\right]C = 0 \ . \quad (21)$$

Thus for each domestic-currency unit of the foreign-currency bond held long, the hedge is effected by buying $\left[\frac{\partial C}{\partial B}/\frac{\partial C}{\partial G}\right]$ units of the domestic currency bond and writing $\left[1/\frac{\partial C}{\partial G}\right]$ call options. Such a hedge would yield a change in wealth of zero, if the hedge were continually adjusted to reflect any change in the state variables.

The exact values of the partial derivatives used in effecting the hedge are, after some simplification,

$$\frac{\partial C}{\partial G} = N(d_1) \quad (21A)$$

$$\frac{\partial C}{\partial B} = -XN(d_2) \ . \quad (21B)$$

A similar hedge may be formed with the put option, when $XBN(d_1^*) - SB^*N(d_2^*) \geqslant X - S \geqslant 0$ or $XBN(d_1^*) - SB^*N(d_2^*) \geqslant 0 \geqslant X - S$. First, verify that

$$P - \frac{\partial P}{\partial G} G - \frac{\partial P}{\partial B} B = 0 \ . \tag{22}$$

Then for each domestic-currency unit of the foreign-currency bond held long, the hedge is brought about through the purchase of $[\frac{\partial P}{\partial B}/\frac{\partial P}{\partial G}]$ units of the domestic currency bond and writing $[1/\frac{\partial P}{\partial G}]$ put options. The values of the partial derivations are, after simplification,

$$\frac{\partial P}{\partial G} = -N(d_2^*) \tag{22A}$$

$$\frac{\partial P}{\partial B} = XN(d_1^*) \tag{22B}$$

and the values of d_1^*, d_2^* are given following equation 18.

<u>The mutual consistancy of option prices</u>. The option pricing formulas depend on six variables, five of which are observable. The variance rate involved in each formula must be estimated from past data in order to obtain a dollar figure for the price of the option. But it is not necessary to know the variance rate in order to price options relative to each other.

To price options relative to a given option, say a call option with exercise price X_1 and time to expiration T_1, take the five observable variables and the current market price C_1 of the given option, and solve the call option equation backwards to obtain the implied variance rate σ_1^2. Then, assuming that σ_1^2 is the correct variance rate for the currency in question, use σ_1^2 with the five observable variables to price all other call options with a different term to maturity and/or a different exercise price, and also to price all put options on the same currency. The prices obtained may be lower than or greater than the market prices of the other options. If, for example, a three-month option is used to calculate σ_1^2, and using σ_1^2 to price the corresponding six-month option yields a theoretical value larger than the market price of the six-month option, then we can say that six-month options are "underpriced" relative to three-month options, or that three-month options are "overpriced" relative to six-month options. The strategy, then, would be to buy six-month options and to write or sell three-month options. (The existence of transactions costs allows, of course, a certain amount of non-profitable inconsistancy in implied variance rates.) The profitability of such a strategy would depend on the relation between the assumptions employed in the derivation of the pricing formula and the actual variables that determine market prices.

VI. Conclusion.

This paper has explored a set of inequality-equality constraints on rational pricing of foreign currency options, and has developed exact pricing equations for American and European puts and calls. Simple arbitrage ensures that an American call option will have a value at least equal to its immediate exercise value. When this constraint is not binding, the assumption of lognormal diffusion processes allows us to set up a riskless hedge that uses no wealth, and which therefore must have a zero return in equilibrium. The construction of this hedge yields a partial differential equation whose solution is the European call option value and also the American call option value when the constraint imposed by the immediate exercise value of the option is not binding. Thus the simple assumption that a riskless portfolio that uses no wealth must have a zero return yields the American call value as the maximum of either the value given by solution of the partial differential equation or the value of immediate exercise. The put option equations are obtained immediately from the call equations through a set of symmetry relations derived from the fact that a call option on one currency, with an exercise price in terms of a second currency, is also a put option on the second currency, with an exercise price in terms of the first currency.

Footnotes

1. By contrast to the example, equations 6 to 9 are not purely identities, since use is also made of the fact that the option price is first degree homogeneous in the price of the underlying asset and the exercise price:

$$C(NS(t),NX,t,T) = NC(S(t),X,t,T), \text{ for } N > 0 ,$$

and similarly for the other options. Since 2 call options on 1 DM, each with an exercise price of X, give me the same privileges as a single option on 2 DM with an exercise price of 2X, first degree homogeneity ought to hold for rationally-priced foreign exchange options just as it does for stock options. See Merton (1973), Theorem 6, p. 147 and Theorem 9, p. 149 for discussion.

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