THE NEUTRALITY OF THE REAL EQUILIBRIUM ALTERNATIVE FINANCING OF GOVERNMENT EXPENDITURES

by

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I. INTRODUCTION

In this paper we show conditions under which the real equilibrium of an economy is independent of the method by which the government finances its expenditures. The framework we use is a state-preference, Arrow-Debreu model with complete markets, I consumers, J firms, and with a government that collects lump-sum taxes and issues interest-bearing debt and non-interest bearing money. We show the necessary and sufficient conditions under which the real equilibrium is neutral to changes in the financing policies of the government in this model. We then use this characterization of neutrality to investigate the circumstances under which shifts in financing between taxes and debt, debt and money, and money and taxes are neutral. We show that no economy can be neutral to all three types of financing shifts. Furthermore, our analysis shows that the conditions necessary to achieve any type of neutrality are very restrictive. Finally, we use the necessary and sufficient conditions for neutrality to investigate the likely effects of non-neutral shifts in government financing policies.

Whether alternative schemes governments choose to finance their expenditures can alter the real equilibrium of an economy is a much debated question in economics, and it has important policy implications. Policymakers act as if they believe that government's choice to finance its expenditures by money, debt, or taxes will affect the allocation of output between private consumption and investment, and possibly total output. Economists disagree on the effects of different financing policies. For instance, though some economists claim that the real equilibrium is neutral to shifts between debt financing and tax financing, the issue is far from settled. There is also much disagreement on the effect of money creation on the real interest rate and real investment.

Ricardo (1895), and more recently Barro (1974), and Mundell (1971) have shown that under fairly general conditions the real equilibrium is neutral to shifts in financing between debt and taxes. They show that if, for instance, the government shifts taxes from the present to the future and finances the ensuing current deficit by issuing debt, the real equilibrium will not change.[1]

An alternative shift in financing is between money and debt. The government can, for instance, finance more of its expenditures by issuing more money (and less debt) now. Thus it has to pay off less debt in the future. Such a financing shift implies a change in the growth rate of money. The effects of different money paths on the real equilibrium of an economy have been analyzed extensively, though not directly in relation to the financing decision of the government.[2] One important conclusion from this literature is that, if two otherwise identical economies differ only by the quantity of nominal assets, these economies will have identical real equilibria.[3] If, however, the growth rates of the money supplies (or nominal assets) diverge then, at the very least, the real cash balances held in the two economies will differ. Mundell (1963) and others have shown that if money is included in

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[1]Barro, Mundell and others have shown that this is true as long as there are no market imperfections and no corner solutions in intergenerational transfers. These results have not been explicitly derived for an economy with uncertainty.

[2]A short and incomplete list of major contributions to this debate includes I. Fisher (1930), Metzler (1951), Friedman (1968), Mundell (1963, 1971) Sidrauski (1967), Samuelson (1958), Dornbusch and Frenkel (1973), Fischer (1972), Brock (1975), Lucas (1975) and others.

[3]In most of the models that investigate the effects of a change in the quantity of money, money is the only nominal asset. Thus, if money differs by only a multiple, the real equilibrium remains unchanged. But as Tobin (1969) has shown, if the economy has positive net supplies of other nominal assets, the ratios of nominal assets must be preserved as well for the real equilibria to be identical.

wealth, and consumption is a function of wealth, then an anticipated higher growth rate of money implies a lower real rate of interest. This result derives from the underlying assumption that since consumers will hold fewer real balances they will substitute towards physical capital, which increases the quantity of the capital stock and lowers the real rate of interest.

On the other hand, Sidrauski (1967) has shown that in an economy composed of infinitely-lived utility maximizers who include real cash balances in their utility functions, neither the size of the capital stock nor the real interest rate will be affected by a change in the growth rate of money, in the long run. The real cash balances held, however, will change. He gets this result by assuming a fixed rate of time preference, identical consumers, a fixed supply of labor, and certainty.[4] None of these studies takes into account the impact of uncertainty or the effect of having individuals with differing tastes and initial endowments.

There are two interesting definitions of neutrality. We will say that a policy shift is "strongly neutral" if none of the elements of the vector of individual consumptions and real balances changes as a result of the policy shift. For the second definition, we will say that a policy shift is "weakly neutral" if only the vector of individual consumption remains unchanged as a result of the policy shift. This second definition permits real balances to change.

The model we use to examine the neutrality properties of alternative financing schemes takes into account uncertainty, differing tastes and initial endowments, and the government budget constraint. We show that financing

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[4]Various imperfections, such as lack of intergenerational contracts, as in Samuelson (1958), signal extraction problems as in Lucas (1975), or human capital markets inefficiencies as in Drazen (1978), may change the Sidrauski result.

shifts are strongly neutral if and only if these schemes don't change the present value of real taxes for each consumer. We also show that financing shifts are weakly neutral if and only if these schemes hold constant the sum of the present value of real taxes and of the real present value of the cost of holding money, for each consumer. These conditions hold only if taxes are lump-sum.

It follows that a policy of financing expenditures by borrowing more now and taxing more in the future is strongly neutral, since the path and the distribution of money are unchanged. In contrast, financing expenditures by issuing more money and less debt now (or vice versa) is not neutral, except for a special case. The reason is that shifts in monetary policy imply changes in the time path of money and changes in the opportunity cost of holding money. These shifts change real balances and the marginal utility of consumption in every state. Since the cost to consumers of holding money balances is changed, and since their marginal utility of consumption is changed, they must alter their consumption-saving decisions. In the special case where neutrality is achieved, the policy shift is only weakly neutral.

The third type of policy shift is between money and taxes. Wallace (1981) shows that a shift in monetary policy can always be compensated for by a shift in initial endowments, and the economy remains neutral. His result is derived from a consumption-loan model with incomplete markets, where money is the only government debt, and where the government invests the tax revenues in the same technology as the private sector. Our results are similar. We show that to obtain money-tax neutrality, the government must differentially adjust each consumer's taxes. But more importantly, we prove that the special case in which money-debt neutrality prevails precludes money-tax neutrality, so that no economy can be neutral to all three types of policy shifts.

The remainder of the paper is organized as follows: Section II describes the model, the definition of equilibrium, and the first-order conditions. Discussion of some of the crucial features of the model is postponed to Section III. Section IV contains the main neutrality propositions, and Section V analyzes the likely effects of non-neutral policy shifts. The concluding section summarizes the main results.

II. MODEL SPECIFICATION AND EQUILIBRIUM

A. Model Description

We analyze a one-good, two-period model, where uncertainty is introduced in the second period. The supply of labor is assumed fixed, and markets are complete. All consumers know the outcome for each possible state of the world in the second period, and the probability with which each state will occur; there are N such states. In the first period each of the I consumers comes into the marketplace with endowment of shares in any of the J firms, and through transactions at the equilibrium prices attains a utility-maximizing portfolio. The portfolio consists of shares, nominal riskless bonds, money balances, and current period consumption. Money pays no interest, while government bonds pay the competitive rate of interest. Each consumer pays current nominal lump-sum taxes that are independent of the portfolio/consumption choice. Though we describe a monetary economy in which all transactions must be made with money, there is no explicit modelling of the payments mechanism and the resulting demand for money. The demand for money is subsumed in each consumer's utility function.

In the second period, state $n = 1,\ldots N$ obtains, consumers collect the proceeds from their portfolios, pay the predetermined lump-sum taxes, and allocate the remainder between cash balances and consumption.

Each consumer maximizes the utility function:

$$(1a) \qquad U_i = U_i\{\frac{M_{in}}{p_n}, x_{in}\} \qquad n = 0, 1, \ldots N \quad (n = 0 \text{ is the current period})$$

subject to

$$(1b) \qquad \sum_j \bar{s}_i^j p^j + \sum_j \bar{s}_i^j [p_o f_o^j - p_o z^j] = p_o x_{io} + \sum_j s_i^j p^j + T_{io} + b_i B + M_{io} ,$$

and

$$(1c) \qquad \sum_j s_i^j p_n f_n^j + M_{io} + b_i(1 + r)B = p_n x_{in} + M_{in}; \qquad n = 1, \ldots N ,$$

where:

M_{in} are nominal money balances held by the i'th consumer in each state $n = 0, 1, \ldots N$.

p_n is the money price of the good in each state, $n = 0, 1, \ldots N$.

x_{in} is the consumption of the i'th consumer in each state, $n = 0, 1, \ldots N$.

s_i^j is the proportion of shares of the j'th firm owned by the i'th consumer. $\bar{s}$ indicates first-period endowment of shares.

p^j is the value of the shares of the j'th firm in the current period, after it pays dividends.

f_o^j is the output of the j'th firm in the current period.

z^j are the resources that each firm plans to use for production in the second period ($\bar{z}$ is the resource endowment).

(Thus $p_o[f_o^j - z^j]$ are the net dividends paid out by the j'th firm.)

T_{in} are nominal taxes to be paid by the consumers in each of the n states, $n = 0, 1, \ldots N$. $T_{in} \gtrless 0$, allowing for net government transfers.

b_i is the share of government bonds owned by the i'th consumer.

B is the value of outstanding government bonds.

r is the rate of interest on nominally riskless debt.

Each firm possesses a different technology of production and it will generally produce different amounts of the same output in each state. Firms are endowed with inputs $\bar{z}^j$ and produce f_o^j in the current period (n = 0) which uses up the inputs. They sell f_o^j output in the marketplace, and they distribute net proceeds $p_o[f_o^j - z^j]$ to initial shareholders. Firms purchase inputs z^j in the marketplace at n = 0, and produce $f_n^j(z^j)$ (n = 1,...N) in the second period.

Total available output from the firms is,

(2) $\sum_j f_o^j(\bar{z}^j)$ in the first period, and

$\sum_j f_n^j(z^j)$ in the n'th state of the second period (n = 1, ...N).

The government consumes real goods in the current period and has no consumption in the second period. This assumption simplifies the algebra and has no impact on any of the conclusions. The government undertakes no investment in either period. It finances its current consumption by current nominal taxes T_o, current issues of money M_o', and current issues of debt B that pay the competitive nominal rate of return r. It guarantees total nominal payment of (1+r)B in every state n > 0; government bonds are riskless in nominal terms.

The government also announces lump-sum taxes for each individual consumer, for all states n > 0. These taxes are independent of consumers' portfolio and consumption choices. The condition we assume implicitly to hold is that every consumer can attain some consumption after taxes, in all states n > 0. The government budget constraints are:

(3) $$p_o g = B + T_o + M_o', \qquad B(1+r) = M_n' + T_n; \qquad n = 1,...N ,$$

where M_n' is the preannounced new money issue in each state in the second period. The announced taxes for each consumer are

(4a) $$T_i = [T_{io}, T_{i1}, \ldots T_{iN}] \qquad i = 1, \ldots I ,$$

and tax revenues are given by

(4b) $$T_n = \sum_i T_{in} \qquad n = 0, 1, \ldots N.$$

B. Equilibrium

We shall call $e^* = \{(x_i^*, m_i^*, b_i^*, s_i^*), (p_o^*, p_n^*), (z^{*j}), (M_n^*, T_{in}^*)\}$ an <u>equilibrium</u> in this economy if the following conditions hold:

i. consumers maximize the utility of their consumption and real money holding (defined by 1a), subject to the budget constraints (1b) and (1c);

ii. each firm j has chosen z_j^* so as to maximize the state-dependent net present value of production: $h^j - z^j$; and

iii. the money markets (5a), security markets (5b), and goods markets (5c) are in equilibrium, and the government budget constraint (8) is satisfied.

The equilibrium conditions are:

<u>Money Markets</u>:

(5a) $$M_o = \sum_i M_{io} = M_o' \; ; \qquad M_n = \sum_i M_{in} = M_o' + M_n' \qquad n = 1, \ldots N ,$$

where M_n is total money supply for each state (n > 0).

<u>Bond and Stock Markets</u>:

(5b) $$\sum_i b_i = 1; \qquad \Sigma_i s_i^j = 1.$$

Goods Markets:

(5c) $$\sum_i x_{io} = \sum_j f_o^j - \sum_j z^j - g, \qquad \Sigma_i x_{in} = \Sigma_j f_n^j; \qquad n = 1,\ldots N.$$

Consumers maximize their utility by taking their endowments $\bar{s}_i^j$, market prices p_n, p^j, r, and announced firm plans z^j as given. They decide on their consumption in the current period (n = 0), x_{io}, and on their portfolios, which consist of money, bonds, and shares, M_{io}, b_i, s_i^j. In the second period (n = 1,...N), they decide on their money balances M_{in}, and they consume the proceeds of their portfolios, x_{in}.

Define real money balances, real taxes, and the price of securities relative to the money price of the consumption good as,

(6) $$m_{in} = M_{in}/p_n, \qquad t_{in} = T_{in}/p_n, \qquad h^j = p^j/p_o; \qquad n > 0.$$

Consumers solve the following two-period problem (from 1a):

$$\max U_i = \max_{s_i^j, b_i, M_{io}, M_{in}} U_i \{m_{io}, \{m_{in}\}, x_{io}, \{x_{in}\}\}; \qquad n = 1,\ldots N,$$

subject to (1b) and (1c). Manipulating the first order conditions and the budget constraints (see Appendix I), we get the following results that will be useful in the remainder of this paper:

(7a) $$\frac{\partial U_i/\partial m_{io}}{\partial U_i/\partial x_{io}} = \frac{r}{1+r},$$ (the marginal rate of substitution between current real money balances and current consumption is an increasing function of the nominal interest rate)[5]

[5]It is important to point out that the impact of the level and the uncertainty associated with second period nominal money balances is contained only in the nominal interest rate, r. Fama and Farber (1979) also show that because money balances can be hedged completely with nominal bonds no uncertainty terms affect the demand for money, beyond those that determine the nominal interest rate.

(7b) $$\frac{\partial U_i/\partial x_{in}}{\partial U_i/\partial x_{io}} = q_{in}.$$ (the marginal rate of substitution between current and future consumption)

The q_{in}s are the i'th consumer's implicit state prices. Since financial markets are assumed to be complete each consumer can rearrange portfolio choices until implicit state prices equal the market state price, so $q_{in} = q_n$ for all i. Furthermore, we show in Appendix I that,

(7c) $$\frac{\partial U_i/\partial m_{in}}{\partial U_i/\partial x_{io}} = q_n,$$

(7d) $$\frac{\partial U_i}{\partial m_{in}} = \frac{\partial U_i}{\partial x_{in}},$$

(7e) $$\sum_n q_n \frac{p_o}{p_n} = \frac{1}{1+r}.$$ (the per $ money price of a nominally riskless bond)

We also show that the government budget constraint is

(8) $$g = t_o + \sum_n q_n t_n + m_o\left(\frac{r}{1+r}\right) + \sum_n q_n m_n,$$

the present value of real taxes, the real present value of the cost of holding money, and the present value of second period real cash balances held by consumers must equal government consumption g.

III. CONSIDERATIONS REGARDING THE MODEL SPECIFICATION

Since our task is to characterize conditions of neutrality, we have specified the model in a way that eliminates some obvious sources of non-neutrality.

The most important source of non-neutrality eliminated from the model concerns taxes. It is well known that taxes levied as a function of the consumers' decision variables will affect those decisions. For instance, income taxes affect the marginal conditions of intertemporal allocation of consumption and labor. A shift of taxes from the present to the future means higher tax rates in the future and a different intertemporal allocation of resources, even if the present value of taxes collected remains the same. Only changes in lump-sum taxes could possibly be neutral. Our specification is formally equivalent to one in which the government levies both proportional and lump-sum taxes, but where shifts in financing affect only the lump-sum taxes.

Endowments of either nominal money or nominal bonds are a source of non-neutrality, and they are both excluded from the model. Any policy that results in a different first-period money price of the good will change the real value of nominal asset endowments. Except under the most restrictive assumptions, individuals will no longer choose the same consumption and portfolio allocations, and the equilibrium will not be neutral. Analyzing an economy where some endowments are nominal is equivalent to analyzing unanticipated policy changes in a multiperiod model. Since the analysis here is confined to anticipated policies, we exclude nominal endowments.[6]

We do not specify a payments technology from which money holding emerges as a consumer and firm response to economic incentives. Though it may be a preferable setting in which to study the effects of changes in monetary policy, models of this type described in the literature depend on ad hoc

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[6]An alternative specification is to endow consumers with positive real money. The difficulty with this specification is that we must assume real balances in the new equilibria always exceed the aggregate real balance endowments. The chosen specification can be viewed as a special case where real money endowments are zero.

assumptions about asset availability or payments technologies.[7] Since there is no simple and generally accepted model in which money holding emerges as an economic response, we have chosen to include real money balances in the utility functions of consumers. One advantage of this formulation is that real balances are sensitive to the nominal interest rate, a common presumption among monetary theorists.

Also, we have not included real balances in the production functions. Such an inclusion makes the production possibility frontier sensitive to real balances, and therefore to the nominal interest rate.[8] Thus, any changes in monetary policy that affect the nominal interest rate would be non-neutral. This is a well-known result, on which we will not elaborate.

Finally, a discussion of the nature of risk in this model is in order. The states in the second period are defined over any possible outcomes of production plans, of taxation, and of money creation. The production of each firm, second period taxes, and new money issues are uncertain. Consumers take into account all these uncertainties when formulating their portfolio

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[7] Money holding emerges as an economic response in intergenerational models under certainty if no other asset is available [Samuelson (1958)] or in the case of uncertainty, if no other nominally riskless asset is available [Wallace (1981), Boonekamp (1978)]. Note that, even in these models, money would be dominated by the existence of a nominally riskless bond. Ad hoc restrictions requiring that transactions with money are costless but those without money are infinitely costly, coupled with the assumption that consumers believe that money will have value in the future (including the last time period), will allow money holding to emerge as an economic response.

[8] There is an apparent lack of symmetry in the results of putting money in the production function instead of the utility function. If money is included in the production function, any changes in real balance holdings will change the amount of output for the same inputs. However, if it is included in a utility function with consumption but without leisure no such change occurs. The shift in the production frontier shows up as a shift in the level of utility, but since the level of utility is not typically included in the specification of the real equilibrium, it appears as if the equilibrium has not changed.

decisions in the first period. In the second period the uncertainty is resolved, consumers receive their share of the value of each firm and their share of the repayment of the government debt, pay their share of taxes, decide on their money holdings, and purchase the consumption good.

IV. THE NEUTRALITY OF FINANCING POLICIES:

In this section we explore the conditions under which alternative financing policies of the fixed real government expenditures are neutral. We discuss two separate definitions of neutrality. Denote a financing policy by $\pi = (T, M', B)$.[9] The equilibrium e will depend in general on the financing policies. Let

$$e(\pi^*) = \{(x_{in}^*, M_{in}^*, b_{io}^*, s_{io}^*), (p_n^*), (z^{j*})\}, \ (n = 0, 1, \ldots N)$$

be an equilibrium. Let $e(\hat{\pi})$ be an equilibrium with alternative financing policies.[10]

We shall say that policy $\hat{\pi}$ is strongly neutral with respect to π^* if $(\hat{x}_{in}, \hat{m}_{in}) = (x_{in}^*, m_{in}^*)$ $[i = 1, \ldots, I; n = 0, 1, \ldots, N]$. Strong neutrality requires that consumption and real cash balances of each individual remain unchanged in every state.

We shall say that policy $\hat{\pi}$ is weakly neutral with respect to π^* if $\hat{x}_{in} = x_{in}^*$; $[i = 1, \ldots, I; n = 0, 1, \ldots, N]$. Weak neutrality requires only that individual consumption vectors remain unchanged in every state. It does not require that real cash balances remain unchanged.

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[9] T denotes tax policy $[T_{in}]_{i=1}^{i=I}$, M' denotes monetary policy M_n', and B denotes a debt policy; n = 0, 1, ...N.

[10] If more than one equilibrium exists for a given financing policy, $e(\pi)$ will be taken to denote the set of equilibria.

All the results below apply to pairwise changes in policies. That is, in every case we examine the consequence of financing shifts between any two policies, while leaving the third fixed. Theorem 1 characterizes both strong and weak neutrality.

<u>Theorem 1.</u> Assume markets are complete. Let π^* give equilibrium $e(\pi^*)$.[11]

(i) An alternative policy $\hat{\pi}$ is <u>strongly neutral</u> if and only if (for all i),

(9) $$t^*_{io} + \sum_n q^*_n t^*_{in} = \hat{t}_{io} + \sum_n q^*_n \hat{t}_{in} \qquad i = 1, \ldots, I,$$

and real money balances are constant: $m^*_{in} = \hat{m}_{in}$ for all i and n.

(ii) An alternative policy $\hat{\pi}$ is <u>weakly neutral</u> if and only if there exists $\hat{M}_{io}$ for every i such that

(10) $$t^*_{io} + \sum_n q^*_n t^*_{in} + m^*_{io}\left[\frac{r^*}{1+r^*}\right] = \hat{t}_{io} + \sum_n q^*_n \hat{t}_{in} + \hat{m}_{io}\left[\frac{\hat{r}}{1+\hat{r}}\right] \text{, and}$$

(3), (5a), and (5b) hold.

<u>Remark:</u>

The intuitive interpretation of the Theorem is the following: <u>weak</u> neutrality requires that the sum of the present value of real taxes and the real present value of the cost of holding money in the first period must remain fixed for each individual. The first two terms on each side of (10) represent the present value of the real taxes of each consumer. The third terms represent the real resource cost of holding money, discounted to the beginning of the first period, $(\frac{1}{p_o})\frac{M_{io}r^*}{1+r^*}$. (10) is a necessary condition, because if it does not hold, consumers' budget constraints will be violated

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[11]In Appendix II we prove a slightly more general form of the Theorem which allows for the existence of multiple equilibria.

in $e(\hat{\pi})$ with the portfolio allocation of $e(\pi^*)$. It is a sufficient condition because if it holds, consumers' consumption choices in $e(\pi^*)$ will remain optimal in $e(\hat{\pi})$, since the marginal conditions on consumption are not changed. Strong neutrality requires that the vector of real cash balances remain unchanged. Since this can only be achieved by holding the interest rate fixed $m^*_{io}(\frac{r^*}{1+r^*}) \equiv \hat{m}_{io}(\frac{\hat{r}}{1+\hat{r}})$ under strong neutrality, which gives equation (9).

In Appendix II we show that if $\hat{\pi}$ is weakly (or strongly) neutral with respect to π^*, then $\hat{q}_n = q^*_n$, $n = 1, \ldots, N$.[12]

Corollary 1: If $e(\hat{\pi})$ is neutral with respect to $e(\pi^*)$ then,

$$(11)\quad g = \hat{t}_o + \sum_n q^*_n \hat{t}_n + \hat{m}_o\left(\frac{\hat{r}}{1+\hat{r}}\right) + \sum_n q^*_n \hat{m}_n,$$

and the government budget constraint is satisfied automatically.[13]

Proof: The first three terms of (11) remain unchanged as long as the condition of Theorem 1 is satisfied. Therefore, (11) may not hold only if the m_n's change in response to a financing change. But from (7b) and (7c) the m_n's will not change as long as the economy is neutral. q.e.d.

We now examine the implications of Theorem 1 for three kinds of financing shifts. The strategy is to assume neutrality and ask whether a change in

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[12]The proof requires that we assume a certain richness in the set of available production functions. In particular, the matrix $[\partial f_i/\partial z_i]$ must be at least of rank N.

[13]It is interesting to note that the government budget constraint contains the term $m_o(\frac{r}{1+r})$. This term is the revenue to the government from inflation. In the discussions of revenue-maximizing inflation [see Cagan (1956)], the government revenue is shown to be a function of the inflation rate. In our model it appears to be a function of the interest rate, because the alternative to issuing more money is issuing more debt, which has to be repaid with interest. Thus, the implicit revenue from issuing money comes from not having to pay interest on debt, rather than from the depreciation of privately held real cash balances.

financing policy is consistent with the conditions of the Theorem. If it is, the economy can remain in the same equilibrium, and the policy shift is neutral.

(a) Debt-Tax Neutrality

Consider first an inter-period, inter-state shift in real lump-sum taxes, offset by the necessary change in the amount of government debt. There are no changes in the money supply in any state. Using Theorem 1 it is easy to show that a shift in financing between debt and taxes is strongly neutral provided the present value of each consumer's taxes remains unchanged.

To see this note that it is always possible to find an alternative tax vector that satisfies equation (9). As long as the money supply is not altered in any state (by assumption), neutrality guarantees that if (9) holds, neither nominal cash balances nor the interest rate are altered. Thus, the money market equilibrium (5a) is unaffected, each consumer's real balances can remain unchanged and the strong neutrality conditions of Theorem 1 are satisfied. Thus, any debt-tax shift that satisfies (9) is strongly neutral.

The conditions under which the economy is neutral to tax shifts are restricted. First, taxes have to be lump-sum. Changes in taxes whose incidence is different than that of the lump-sum taxes may not be neutral. Second, the shift in taxes has to leave each consumer's budget constraint unchanged. A policy shift that will satisfy the government budget constraint (11) is given by

$$\hat{T}_{io} = (1-a)T^*_{io}, \quad \hat{T}_{in} = T^*_{in} + aT^*_{io}(1+r^*) \qquad i = 1, \ldots, I .$$

This policy shift involves a transfer of nominal taxes between the first period and each state of the second period, the transfer being effected at the current interest rate r^*.[14] This scheme also guarantees that the government budget constraint will be satisfied in every state, without changes in nominal money issued in any state. The neutrality of debt-tax shifts has been analysed by Barro (1974), Mundell (1971), and others.[15] Our analysis shows that their conclusion extends to a world with money, and with uncertainty.

Since our model allows for second-period uncertainty, it is possible to investigate the effects of an intra-state redistribution of taxes. It follows from Theorem 1 that a redistribution which changes only second-period taxes and which satisfies (9) will be strongly neutral as long as the tax shift satisfies the government budget constraint in every state, without changes in monetary policy. It is possible, however, for a tax shift policy to be strongly neutral even if it requires intra-state changes in money issues, as long as the overall effect is to hold the nominal interest rate constant. From Theorem 1 and the first order conditions it follows that if (9) is satisfied and $\hat{r} = r^*$, the tax shift policy will be strongly neutral. If the nominal interest rate changes, however, an intra-state redistribution of taxes cannot be strongly neutral. If such a redistribution satisfies (10), i.e., if $m_{io}(\frac{r}{1+r})$ remains constant for all consumers, then the policy will be weakly neutral. As we will show in Lemma 1, $m_{io}(\frac{r}{1+r})$ will be constant only if the form of the utility function is severely restricted.

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[14] An alternative policy that meets the condition of equation (9) is as follows. Let lump-sum taxes be a fixed proportion of initial endowments. Then a γ proportion change in first period taxes, for everyone, coupled with a $\frac{\gamma \sum q_n^* t_n^*}{t_o^*}$ change in the opposite direction in second period taxes will be strongly neutral.

[15] They conclude that in the absence of market imperfections an economy must be neutral to shifts between debt and tax financing.

(b) Debt-Money Neutrality

Consider next the case where the government issues less (more) debt and more (less) money in the first period, without altering any individual taxes. Since the present value of taxes stays fixed (by assumption), and since the value of bonds does not appear in any of the market clearing conditions or in the conditions of Theorem 1, any impact on the real equilibrium must come from changes in the supply of money.[16] Take the case of issuing less debt and more money in the first period—an open market operation. The government revenue from money creation in the first period is equal to the real cash balances held by consumers. In order for the government to receive more revenue from issuing money it must be that consumers, in the aggregate, voluntarily agree to hold more real cash balances in the first period.[17] Under neutrality, the real cash balances held depend only on the nominal interest rate (from 7a), and the nominal interest rate must be lower to induce consumers to hold more real balances. Furthermore, under weak neutrality, the price level in each state in the second period is proportional to the money supply in that state (see Appendix III). Therefore the expected money growth rate has to be lower for the nominal interest rate

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[16] If money paid a real rate of return independent of the rate of inflation, then there would be no redistribution effects from inflation and money supply changes would be neutral.

[17] If this model allowed money endowments, the government could always collect some additional revenue from money creation, because it could cause existing money balances to depreciate sufficiently so that consumers decide to purchase the additional money. As was pointed out in the previous section, this case is equivalent to an unanticipated change in monetary policy.

to be lower.[18]

Since $t_{io} + \sum_n q_n t_{in}$ is held constant by assumption, a necessary and sufficient condition for weak neutrality (from 10) is

$$(12) \qquad m_{io}\left(\frac{r}{1+r}\right) = \text{constant.}$$

(12) says that each consumer's demand for money must be such that, while a decline in the growth rate of money induces him to hold more cash balances, he does so in a way that keeps the cost of holding money constant. Lemma 1 derives a condition on the utility function necessary for weak money-debt neutrality.

<u>Lemma 1.</u>

Let e* be an equilibrium for given government policies $\pi^* = (t^*_{in}, M^{*\prime}_n, B^*)$, n = 0, ...N, and let $e(\hat{\pi})$ be an alternative equilibrium for policies $\hat{\pi} = (t^*_{in}, \hat{M}'_n, \hat{B})$. If $\hat{\pi}$ is weakly neutral with respect to π^*, then

$$(13) \qquad \frac{\partial^2 U_i}{\partial x^*_{io}\partial m_{io}} = q^*_n \frac{\partial^2 U_i}{\partial x^*_{in}\partial m_{io}}, \qquad n = 1,\ldots,N.$$

<u>Proof:</u>

From the first order conditions,

$$(7b) \qquad \frac{\partial U_i/\partial x^*_{in}}{\partial U_i/\partial x^*_{io}} = q^*_n .$$

Since weak neutrality requires fixed q's, the partial derivative of (7b) with respect to m_{io} must be zero. Taking partials yields the condition

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[18]This analysis shares the insight provided by Wallace (1980) and Jones (1980), that financing a budget deficit with debt now and paying off the debt with money later is <u>more</u> inflationary than financing the deficit with money now. In the above example, financing the deficit with money in the current period implies a <u>lower</u> rate of inflation.

$$\frac{\partial^2 U_i}{\partial x^*_{in}\partial m_{io}} = q^*_n \frac{\partial^2 U_i}{\partial x^*_{io}\partial m_{io}} . \qquad \text{q.e.d.}$$

Lemma 1 shows the restrictions money-debt neutrality places on the form of the utility function. A separable utility function will satisfy equation (13) and therefore Lemma 1.

Since the collective cost of holding money is the same as the government's revenue from money, any announced monetary policy is feasible and neutral if (12), and therefore Lemma 1, is satisfied. The standard result in the literature on the neutrality of monetary policy is due to Sidrauski (1967). In a model with a representative, infinitely-lived consumer, and with no uncertainty, Sidrauski shows that monetary policy is weakly neutral in the long run. The consumer's utility function is not separable in money and goods, but is additive over time. Lemma 1 shows that Sidrauski's result does not generalize to the case of uncertainty. In Sidrauski's model, there is only one state whose price is β, the fixed time preference parameter. Certainty and time additivity of the utility function insure that Lemma 1 is satisfied for any such utility function. Thus, monetary policy is weakly neutral. In the case of uncertainty generally $q_n \neq \beta$ at least for some n, and that severely restricts the utility functions for which Sidrauski's result holds, under uncertainty.

If utility functions are separable in money and goods, then debt-money neutrality requires (from 12) that the utility function for money balances be of the form

(14a) $$U_i^m = k_i^o + k_i^1 \ln(m_i) ,$$

where k_i^1 is proportional to the i'th consumer's marginal utility of consumption, and k_i^o is a constant of integration. The demand for money function implied by (14a) is

(14b) $$m_{io} = k_i^1 \frac{(1+r)}{r} ,$$

and its interest elasticity is given by $\eta_r = - \frac{1}{1+r}$.[19] An implication of (14b) is that demand-for-money functions most often used in the literature--log-linear and semi-log--are not consistent with money-debt neutrality, because the elasticity they imply is constant or rises with inflation.

(c) Money-tax Neutrality

Both types of financing changes analyzed above hold the real present value of taxes constant for each individual. A third type of financing change involves changing the real taxes consumers pay, and issuing sufficient money in the first period to finance expenditures.[20] Wallace (1981) analyzes this case in the context of a consumption-loan model with incomplete markets, and where money is the only nominally riskless asset. He shows that for each monetary policy the government can find a set of initial endowments such that monetary policy is neutral.

From Theorem 1, weak neutrality requires that the change in the present value of taxes must be just offset by the change in the cost of holding the new quantity of first-period money balance, for each consumer. In principle it is possible to adjust each consumer's taxes (given the utility function) so as to satisfy the neutrality condition. Unlike the debt-tax case, it is not

[19]The absolute value of the elasticity is a declining function of the nominal interest rate. Real money balances decline with rising interest rates but the rate of decline falls. The limiting value of the real cash balances is proportional to the marginal utility of consumption.

[20]The additional money revenue must come in the first period because, under neutrality, the second period revenue from money is fixed at $\sum_m q_n^* m_n^*$.

possible to specify a simple tax rule that is consistent with neutrality. And unlike the debt-money case, it is not possible to specify a utility function which assures money-tax neutrality.

It is possible to show, however, that no economy can be neutral simultaneously to policy shifts between money and taxes, and between debt and money.

<u>Theorem 2</u>. Consider policies $\pi^* = (t^*_{in}, M'^*_n, B^*)$, $\hat{\pi} = (t^*_{in}, \hat{M}'_n, \hat{B})$ and $\tilde{\pi} = (\tilde{t}_{in}, \tilde{M}'_n, B^*)$ where $\sum_{n=0}^{N} q^*_n t^*_n \neq \sum_{n=0}^{N} q^*_n \tilde{t}_n$, $n = 0, \ldots, N$; $q_0 \equiv 1.0$.

(i) If policies of the form $\hat{\pi}$ are weakly neutral with respect to π^*, then there do not exist policies of the form $\tilde{\pi}$ which are weakly neutral with respect to π^*.

(ii) If policies of the form $\tilde{\pi}$ are weakly neutral with respect to π^* then there do not exist policies of the form $\hat{\pi}$ which are weakly neutral with respect to π^*.

<u>Proof</u>: If $\hat{\pi}$ is neutral with respect to π^* then $m_{io}(\frac{r}{1+r})$ = constant and $m_o(\frac{r}{1+r})$ = constant. From the government budget constraint it must be $\Sigma\, q^*_n \tilde{t}_n$ = constant. But this contradicts the definition of $\tilde{t}_n$. q.e.d.

Theorem 2 is a direct result of the neutrality conditions of Theorem 1. The weak neutrality conditions for money-debt neutrality (i.e. policies of type $\hat{\pi}$) guarantee that the cost of holding money is independent of the interest rate. Thus government revenue from money creation is fixed under money-debt neutrality. But if the present value of taxes is changed, then the cost of holding money and the revenue from money creation <u>must</u> change for neutrality to hold. Hence, no economy can be neutral to both money-tax and debt-money shifts.

V. NON-NEUTRAL FINANCING SHIFTS

Theorem 1 may be used to derive some comparative statics results for non-neutral exchanges between money and debt, given fixed government consumption g^*. We consider first a two-period economy, in which there is no uncertainty. Suppose that in such an economy the money demand function of some consumer i does not satisfy $m_{io} = (\frac{1+r}{r})$. It follows from Section IV.b., that no change in the government's monetary policy that keeps the net present value of real taxes the same can be neutral. Now suppose that, starting from some equilibrium e^*, the government plans a money-debt exchange in which the present value of real taxes will be kept the same for all consumers. Denote the resulting (new) equilibrium by $\hat{e}$. Since the government consumption of real goods has not changed, it follows from (8) that

$$g = \hat{t}_o + \hat{q}_1\hat{t}_1 + \hat{m}_o(\frac{\hat{r}}{1+\hat{r}}) + \hat{q}_1\hat{m}_1,$$

where the hats denote the equilibrium values in the new equilibrium $\hat{e}$, and where $\hat{q}_1$ is the price today of a unit of consumption tomorrow. If the present value of real taxes is the same in e^* and $\hat{e}$, then

$$\hat{m}_o(\frac{\hat{r}}{1+\hat{r}}) + \hat{q}_1\hat{m}_1 = m_o^*(\frac{r^*}{1+r^*}) + q_1^*m_1^*. \tag{15}$$

Below, we analyze the case where

$$\hat{m}_o(\frac{\hat{r}}{1+\hat{r}}) < m_o^*(\frac{r^*}{1+r^*}) , \tag{16a}$$

to illustrate the issues that arise. From (15) and (16a) it follows that,

$$\hat{q}_1\hat{m}_1 > q_1^*m_1^*. \tag{16b}$$

There are two possible ways in which (16b) can happen:

<u>Case 1</u>: $\hat{q}_1 > q_1^*$. By the first-order conditions for firm maximization, investment for all firms will be higher, $\hat{z}^j > z^{*j}$, and therefore aggregate first-period consumption will be lower at $\hat{e}$, $\sum_i \hat{x}_{io} = \hat{x}_o < x_o^*$. Similarly, future consumption will be higher, $\hat{x}_1 > x_1^*$. Since,

$$\hat{q}_1 = \left.\frac{\partial U_i/\partial x_{i1}}{\partial U_i/\partial x_{io}}\right|_{(\hat{x}_{io},\, \hat{x}_{i1})} \qquad \text{for every } i,$$

it must be the case that the marginal utility of future consumption must rise by more than the marginal utility of current consumption (or the marginal utility of x_0 must fall by less than the marginal utility of x_1)

$$\hat{x}_{io} \lessgtr x^*_{io} \qquad \text{if and only if} \qquad \hat{x}_{i1} \lessgtr x^*_{i1}.$$

From (7e) it also follows that

$$\hat{x}_{i1} \gtrless x^*_{i1} \qquad \text{if and only if} \qquad \hat{m}_{i1} \gtrless m^*_{i1}.$$

A shift in the equilibrium in this case implies both an aggregate shift towards more investment in the current period, and a redistribution of consumption (in the current and future periods) among consumers. When the rate of growth of nominal balances changes in a non-neutral economy, the total cost of taxes changes for each consumer, because of the change in his cost of holding money. The incidence of this cost is redistributed among consumers. Those whose total cost of taxes is lower have more resources in $\hat{e}$ than they did in e*, and they consume more now <u>and</u> later, while those whose total cost is higher at $\hat{e}$ consume less now <u>and</u> later.

<u>Case 2</u>: $\hat{q}_1 < q_1^*$. From (16b) it must be that $\hat{m}_1 > m_1^*$. This implies that some consumers will hold more money in the future, $\hat{m}_{i1} > m^*_{i1}$, and by (7e) they will consume more in the future as well, $\hat{x}_{i1} > x^*_{i1}$. On the other hand, since $\hat{q}_1 < q_1^*$, first-period investment must be less, $\hat{z}^j < z^{*j}$, which implies that $\hat{x}_1 < x_1^*$. One may be tempted to conclude that because of the apparent contradiction, this case is impossible. But the contradiction only occurs if

consumers are sufficiently similar, so that total second-period consumption has to increase. Otherwise, a fall in q's is consistent with equilibrium, because consumption is redistributed.

We conclude that, even in the case of certainty, it is not possible to find the direction in which the real equilibrium will move, without knowing more about consumer utility functions.

When uncertainty is introduced, the results are even less clear. The discussion above applies to each future state. If, for instance, revenue from money creation increases through a rise in the q's, $\sum_n \hat{q}_n \hat{m}_n > \sum_n q^*_n m^*_n$, it does <u>not</u> follow that all the q's rise. In contrast to the case of certainty, it is not possible to determine the direction of change of investment, even in this case.

There is, however, a general conclusion that can be drawn for both the certainty and uncertainty cases. Suppose those consumers who tend to save more in the current period have a demand-for-money schedule that is more sensitive to changes in the interest rate than an average schedule. When the interest rate rises, these consumers will be relatively better off—that is, the burden of paying for government expenditure will shift away from them. The amount of additional saving they do as a result will more than offset the decline in saving from those to whom the burden has been transferred. Investment will rise in the current period and future consumption will rise on average, barring peculiar configurations of state prices and initial endowments that will induce systematic wealth effects that impoverish these consumers who save more.[21]

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[21]Redistribution of investment among firms may be such that consumption may still be lower for some states.

Another interesting insight into the effect of changes in monetary policy can be gained by analyzing the model with a single representative consumer. This eliminates all the redistribution effects of the inflation tax. If the representative consumer has a demand-for-money function compatible with weak neutrality, as in (14b), then any monetary policy is neutral and feasible. If the demand-for-money function is not compatible with weak neutrality there is a unique monetary policy associated with a particular tax schedule that will satisfy all the equilibrium conditions, and it is the monetary policy of π^*. To see this, suppose the government announced financing policy $\hat{\pi} = (t_n^*, \hat{M}_n', \hat{B})$, different from $\pi^* = (t_n^*, M_n^{*\prime}, B^*)$, and that utility is separable in money and consumption, but not (14b). (The subscript i is dropped since we have a representative consumer.) Financing policy $\hat{\pi}$ will violate the government budget constraint for unchanged q's ($\hat{q}_n = q_m^*$ and $\hat{m}_n = m_n^*$). But the q's cannot change in this case, because a change in q's implies conflicting changes in the first period investment, and in the marginal utilities of consumption.[22] With many consumers, we showed that this apparent contradiction could be resolved through redistribution effects. Since there cannot be any redistribution with a representative consumer, monetary policy cannot be changed without changing taxes. As a result, if the government announces a change in monetary policy from π^*, it will find it impossible to balance its budget without changing the present value of taxes. Thus a shift in financing purely from money to debt is not possible. At the same time, if

[22]The contradiction is that when q's are higher investment must be larger, current consumption must be lower, and future consumption higher. But this implies that the marginal utility of future consumption should be lower and that of current consumption higher, implying lower q's. The reverse argument holds when q's are lower.

the utility function satisfies Lemma 1, any financing policy of lump-sum taxes, money and debt is neutral.

We have shown that, if consumers are identical, and

(i) have a utility function that satisfies Lemma 1 and equation (12), any monetary policy is neutral and feasible,

(ii) if utility satisfies Lemma 1 but not equation (12), a particular tax policy is compatible with only one monetary policy, but any mix of taxes and money consistent with the government budget constraint is neutral,

(iii) if utility does not satisfy Lemma 1, monetary policy is not neutral.

VI. CONCLUSION

We analyze alternative financing equilibria within the framework of a two-period, Arrow-Debreu, state-preference model, while holding real government expenditures fixed. The model allows for many consumers and firms, a fixed supply of labor, and one good. Markets are complete. Each consumer's utility is a function of the consumption and real cash balances vectors.

The government consumes g units of output in the first period and pays for it by collecting lump-sum taxes, and issuing money and debt in the first period. In the second period it pays its debt by collecting more taxes and issuing more money. All second-period outcomes are uncertain. Uncertainty is introduced through N possible states of the world that can occur in the second period.

The main conclusions are the following:

(1) a shift in financing between debt and taxes is strongly neutral if taxes are lump-sum and the shift leaves the present value of taxes the same for each consumer;

(2) a shift in financing between debt and money can be weakly neutral only under a restricted set of utility functions. Utility of money of the

form $U_i^m = k_i^0 + k_i^1 \ln(m_i)$ for each consumer, is sufficient for the economy to be neutral to debt-money shifts;

(3) a shift in financing betwen money and taxes is weakly neutral only if the government differentially adjusts each consumer's taxes to offset changes in his cost of holding money;

(4) no economy will exhibit both debt-money and money-tax neutrality.

Constraining the model first to no uncertainty and then to only one consumer yields interesting insights as to how uncertainty and differing consumers influence the results. As long as consumers are allowed to differ, it is not possible to predict the shift in the aggregate consumption/investment ratio, without more knowledge about the utility functions. Changes in financing policies will result in a redistribution of the burden of paying for the government expenditures. If the burden shifts away from those who save more, the consumption/investment ratio will fall. This result holds whether or not there is uncertainty.

If all consumers are identical there is a unique monetary policy associated with a particular tax policy, except for a special case. However, as long as utility functions satisfy the condition of Lemma 1, all combinations of monetary policies and tax policies that satisfy the government budget constraint are neutral. Separable utility functions satisfy Lemma 1. If utility functions do not satisfy Lemma 1, changes in monetary policy will not be neutral.

APPENDIX

(I) First-order conditions

The first order conditions of the maximization problem are:

(i) $$\frac{\partial U_i}{\partial m_{io}} + \sum_n \frac{\partial U_i}{\partial x_{in}} \left(\frac{p_o}{p_n}\right) = \frac{\partial U_i}{\partial x_{io}} ;$$

(ii) $$\frac{\partial U_i}{\partial m_{in}} = \frac{\partial U_i}{\partial x_{in}} ; \qquad n = 1, \ldots N ,$$

(iii) $$\frac{\partial U_i}{\partial x_{io}} = (1+r) \sum_n \frac{\partial U_i}{\partial x_{in}} \frac{p_o}{p_n} ; \text{ and}$$

(iv) $$h^j \frac{\partial U_i}{\partial x_{io}} = \sum_n \frac{\partial U_i}{\partial x_{in}} f_n^j .$$

The following results are immediate:

(a) To get equation (7e): from (iii),

(v) $$\sum_n q_n \frac{p_o}{p_n} = \frac{1}{1+r} .$$

(b) From (iv),

(vi) $$h^j = \sum_n q_{in} f_n^j$$

(c) To get equation (7a): substitute (iii) into (i) to get,

(vii) $$\frac{\partial U_i/\partial m_{io}}{\partial U_i/\partial x_{io}} = \frac{r}{1+r} .$$

(d) To get equation (8): from (3),

(viii) $$p_o(g - t_o) - M_o' - \frac{M_n' + T_n}{1 + r} = 0$$

Substituting (7e) and recognizing that $m_o' = m_o$ gives

(ix) $$g = t_o + m_o + \sum_n \frac{q_n}{p_n} (M_n' + T_n)$$

$$g = t_o + m_o + \sum_n \frac{q_n}{p_n} [M_n - M_o + T_n]$$

$$g = t_o + m_o + \sum_n q_n m_n - \sum_n q_n \frac{p_o}{p_n} m_o + \sum_n q_n t_n$$

and finally

(x) $$g = t_o + \sum_n q_n t_n + m_o\left(1 - \sum_n q_n \frac{p_o}{p_n} \right) + \sum_n q_n m_n$$

(xi) $$g = t_o + \sum_n q_n t_n + m_o \frac{r}{1+r} + \sum_n q_n m_n$$

(II) Proof of Theorem 1

Proposition II.1: Let $\bar{z} > 0$ be fixed and define

$$A = \{z = (z^1, \ldots, z^N) \mid z^j \geq 0 \text{ for every } j, \sum_j z_j = \bar{z}\} .$$

Furthermore suppose that f_n^j: $R_+ \to R_+^N$ are vectorial functions one of whose components is strictly concave for every j, j = 1, . . ., N, and define

$$F: A \to R^N$$

by

$$F(z) = \left(\sum_j f_1^j(z^j), \ldots, \sum_j f_N^j(z^j)\right) .$$

Then if z^{j*} maximizes

$$\sum_n q_n^* f_n^j(z_j^j) - z^j$$

for fixed (q_n^*) for each function j, and if $\sum_j z^{j*} = \bar{z}$, then

$z^* = (z^{1*}, \ldots, z^{N*})$ maximizes $\sum_n q_n^* F_n(z) - \overline{z}^*$.

Proof:

If not there exists $z \in A$ such that

$$\sum_n q_n^* F_n(z) - \overline{z} > \sum_n q_n^* F(z^*) - \overline{z} .$$

But this means that for at least one j

$$\sum_n q_n^* f_n^j(z^j) - z^j > \sum_n q_n^* f_n^j(z^{j*}) - z^{j*} ,$$

and this contradicts the assumptions of the proposition. QED

Proposition II.2: Let $z \in A$ maximize $\sum_n q_n^* F_n(z) - \overline{z}$. Then z is unique in A.

Proof: Since A is convex, the proof follows directly from the concavity of F.

QED

Proposition II.3: Suppose that the matrix $[df_n^j/dz_j]$ has rank N for all $z_j > 0$, $j = 1, \ldots, J$. Then $\hat{\pi}$ is weakly neutral => $\hat{q}_n = q_n^*$ for all $n = 1, \ldots, N$.

Proof: If $\hat{\pi}$ is weakly neutral it follows that

$$\hat{x}_o = \sum_i \hat{x}_{io} = \sum_i x_{io}^* = x_o^* \text{ and } \hat{x}_n = \sum_i \hat{x}_{in} = \sum_i x_{in}^* = x_n^*, \; n = 1, \ldots, N.$$

From this it is clear that $\hat{z} = \sum_j \hat{z}^j = \sum_j z^{j*} = z^*$ and that for every n,

$\sum_j f_n^j(\hat{z}^j) = \sum_j f_n^j(z^{j*})$. It now follows from Propositions 1 and 2 that $\hat{z}^j = z^{j*}$

for every j.

It remains to show that $\hat{q}_n = q_n^*$ for n = 1, . . ., N. This follows directly from the following proposition:

<u>Proposition II.4</u>: Let the matrix $[df_n^j/dz^j]$ have rank N. Then the vector $z^* = (z^{1*}, \ldots, z^{J*})$ such that z^{j*} maximizes

$$\sum_n q_n f_n^j(z^j) - z^j,$$

is a unique function of $(q_1, \ldots, q_N)$.

<u>Proof</u>: Suppose that $(\hat{q}_1, \ldots, \hat{q}_N) \neq (q_1^*, \ldots, q_N^*)$ and that $\hat{z}^j = z^{j*}$, where $\hat{z}^j$ maximizes

$$\sum_n \hat{q}_n f_n^j(z^j) - z^j ,$$

and z^{j*} maximizes

$$\sum_n q_n^* f_n^j(z^j) - z^j .$$

Without loss in generality we may assume that J = N. From the maximization properties of $\hat{z}^j$ and z^{j*} it follows that

$$\sum_n \hat{q}_n df_n^j/dz^j \Big|_{z^{j*}} = \sum_n q_n^* df_n^j/dz^j \Big|_{z^{j*}} .$$

By the assumption on the rank of $[df_n^j/dz^j]$ it now follows that $\hat{q}_n = q_n^*$, n = 1, . . ., N. QED

<u>Theorem II.1</u>: Suppose that the matrix $[df_n^j/dz_j]$ has rank N for all $z_j > 0$, j = 1, . . ., J. Then $\hat{\pi}$ is weakly neutral with respect to π^* if and only if:

(i) $\hat{q}_n = q_n^*$, $n = 1, \ldots, N$.

(ii) $t_{io}^* + \sum_n q_n^* t_{in}^* + m_{io}^* \frac{r^*}{1 + r^*} = \hat{t}_{io} + \sum_n q_n^* \hat{t}_{in} + \hat{m}_{io} \frac{\hat{r}}{1 + \hat{r}}$.

Proof:

Necessity: Proposition II.3 establishes the necessity of (i). To see (ii), write the i'th consumer's budget equations for equilibrium e* (for simplicity we omit the stars):

(10a) $$x_{io} = \sum_j \bar{s}_i^j(h^j+f^j-z^j) - \sum_j s_i^j h^j - t_{io} - b_i B/p_o - m_{io} ,$$

(10b) $$x_{in} = \sum_j s_i^j f_n^j - t_{in} + b_i(1+r)\frac{B}{p_n} + \frac{M_{io}}{p_n} - m_{in} .$$

Multiply (10b) by q_n and sum over n:

(10c) $$\sum_n q_n(x_{in}+m_{in}) + \sum_n q_n t_{in} = \sum_n q_n \sum_j s_i^j f_n^j + \sum_n q_n b_i(1+r) \frac{B}{p_n} + \sum_n q_n \frac{M_{io}}{p_n} .$$

By the first order conditions

$$\sum_n q_n \sum_j s_i^j f_n^j = \sum_j s_i^j h^j, \qquad 1 - \sum_n q_n \frac{p_o}{p_n} = \frac{1}{1+r} = \frac{\partial U_i/\partial m_{io}}{\partial U_i/\partial x_{io}} ,$$

$$\frac{M_{io}}{p_o} [1 - \sum_n q_n \frac{p_o}{p_n}] = \frac{\partial U_i/\partial m_{io}}{\partial U_i/\partial x_{io}} m_{io} = \sum_n q_n \frac{p_o}{p_n} \frac{M_{io}}{p_o} + \frac{\partial U_i/\partial m_{io}}{\partial U_i/\partial x_{io}} m_{io} ,$$

$$\sum q_n \frac{M_{io}}{p_n} = [1 - \frac{\partial U_i/\partial m_{io}}{\partial U_i/\partial x_{io}}] m_{io} .$$

So that (10c) becomes

(10d) $$\sum_n q_n(x_{in}+m_{in}) + \sum_n q_n t_{in} = \sum_j s_i^j h^j + m_{io}[1 - \frac{\partial U_i/\partial m_{io}}{\partial U_i/\partial x_{io}}] + b_i \frac{B}{p_o} .$$

Combine (10a) and (10d)

(10e) $$\sum_n q_n(x_{in}+m_{in}) + \sum_n q_n t_{in} + x_{io} = \sum_j s_i^j h^j + m_{io} - \frac{\partial U_i/\partial m_{io}}{\partial U_i/\partial x_{io}} + b_i \frac{B}{p_o}$$

$$+ \sum_j \bar{s}_i^j(h^j+f_o^j-z^j) - \sum_j s_i^j h^j - b_i \frac{B}{p_o} - t_{io} - m_{io} ,$$

(10f) $$\sum_n q_n(x_{in}+m_{in}) + x_{io} + t_{io} + \sum_n q_n t_{in} = - \frac{\partial U_i/\partial m_{io}}{\partial U_i/\partial x_{io}} m_{io}$$

$$+ \sum_j \bar{s}_i^j(h^j+f_o^j-z^j) .$$

This implies

(10g) $$\sum_n q_n x_{in} + x_{io} = - t_{io} - \sum_n q_n t_{in} - \frac{\partial U_i/\partial m_{io}}{\partial U_i/\partial x_{io}} m_{io} + \sum_j \bar{s}_i^j(h^j+f_o^j-z^j) - \sum_n q_n m_{in} .$$

By (7d) it follows

that $\hat{m}_{in} = m_{in}^*$, $\hat{x}_{in} = x_{in}^*$, $\hat{x}_{io} = x_{io}^*$, $\hat{h}^j = h^{j*}$, and $\hat{z}^j = z^{j*}$ under neturality. Using this result and writing $(\hat{t}_{io}, \hat{t}_{in}, \hat{m}_{io})$ on the right-hand-side of (10g) gives the desired result.

<u>Sufficiency</u>: By (i), $z^j = z^{j*}$ and $h^j = h^{j*}$. by completeness of the markets, we may find a portfolio $(\hat{s}_i^j, \hat{b}_i)$ for each consumer i which gives good consumption vector x_i^*, and real money balances in states 1, . . ., N, of $m_{i1}^*, \ldots, m_{iN}^*$.

Define $\hat{m}_{io}$ by the following equation:

$$\hat{m}_{io} = \frac{1+\hat{r}}{\hat{r}} \{\sum_j \bar{s}_i^j(h^{j*} + f_o^j - z^{j*}) - \hat{t}_{io} - \sum_n q_n^* \hat{t}_{in} - x_{io}^* - \sum_n q_n^*(x_{in}^* + m_{in}^*)\} .$$

Note that the vectors x_i^* and $(\hat{m}_{io}, m_{i1}^*, m_{i2}^*, \ldots, m_{iN}^*)$ fulfill the first-order conditions (7b)-(7e). In order to complete the proof, three things remain to be shown:

1. The vectors x_i^*, $(\hat{m}_{io}, m_{i1}^*, \ldots, m_{iN}^*)$ are feasible for consumer i.

To show this, note that summing $\sum_n q_n^* x_{in}$ and making the appropriate substitutions in the budget equation for x_{io} (much as was done in the proof of necessity) gives $\hat{m}_{io}$ as defined above.

2. The government budget constraint is fulfilled. It is sufficient to show that

$$g = \hat{t}_o + \sum_n q_n^* \hat{t}_n + \hat{m}_o \frac{\hat{r}}{1 + \hat{r}} + \sum_n q_n^* \hat{m}_n .$$

By (ii) it follows that

$$\hat{t}_o + \sum_n q_n^* \hat{t}_n + \hat{m}_o \frac{\hat{r}}{1 + \hat{r}} = t_o^* + \sum_n q_n^* t_n^* + m_o^* \frac{r^*}{1 + r^*} .$$

Since $\hat{m}_{in} = m_{in}^*$, it follows that the government budget constraint is fulfilled.

3. Markets are in equilibrium. This follows directly from the fact that markets were in equilibrium in e*.

This completes the proof of the theorem. qed.

<u>Theorem II.2</u>: Assume that markets are complete. Let π^* give an equilibrium $e(\pi^*)$. Then an alternative policy $\hat{\pi}$ is strongly neutral if and only if

$$t_{io}^* + \sum_n q_n^* t_{in}^* = \hat{t}_{io} + \sum_n q_n^* \hat{t}_{in} , \qquad i = 1, \ldots, I ,$$

and real money balances are constant: $m_{in}^* = \hat{m}_{in}$, for all i and N.

<u>Proof</u>: Note that $\hat{\pi}$ is strongly neutral if and only if it is weakly neutral and $m_{in}^* = \hat{m}_{in}$, $n = 0, 1, \ldots, N$. This implies that $\hat{\pi}$ is strongly neutral if and only if:

$$q_n^* = \hat{q}_n \text{ and } r^* = \hat{r} ,$$

where both of these conditions follow from the first-order conditions. Equation (ii) of the previous theorem now proves the above result. qed.

(III) Proposition III.1:

In the future period $(n > 0)$ the money price of x, (p_n) is proportional to the money supply M_n in each state if neutrality obtains and if (in the case of weak neutrality) the rank condition of Theorem II.1 holds.

Proof:

By neutrality $x_{in} = x_{in}^*$ and $q_n = q_n^*$, under the conditions of this proposition.

From (8-d)

$$\frac{\partial U_i/\partial m_{in}}{\partial U_i/\partial x_{io}} = q_n^* ,$$

so that $\partial U_i/\partial m_{in}$ and therefore m_{in} is constant.

The equilibrium condition is

$$\sum_i m_{in} = \frac{M_n}{p_n} = \text{constant.}$$

Therefore

$$\hat{M}_n = k_n M_n, \text{ and } \hat{p}_n = k_n p_n. \quad \text{qed.}$$

Proposition III.2:

The first period price level, p_o, is a function only of M_o and M_n, if neutrality obtains and if (in the case of weak neutrality) the rank condition of Theorem II.1 holds.

Proof:

From (8-a) and (8-c) it follows that

$$\text{(i)} \qquad \frac{\partial U_i/\partial m_{io}}{\partial U_i/\partial x_{io}} = 1 - \sum_n q_n \frac{p_o}{p_n} \, .$$

Let $\partial U_i/\partial x_{io} = \beta_i$ and $\partial U_i/\partial m_{io} = \gamma_i^{-1}(m_{io})$, $\quad \frac{\partial \gamma_i^{-1}}{\partial m_i} < 0$,

$$\gamma_i^{-1}(m_{io}) = \beta_i [1 - \sum_n q_n \frac{p_o}{p_n}] \, ,$$

$$\text{(ii)} \qquad m_{io} = \gamma_i(\beta_i [1 - \sum_n q_n \frac{p_o}{p_n}]) \, .$$

Since from Proposition 1, $p_n = k_n M_n$,

$$\text{(iii)} \qquad m_{io} = \gamma_i(\beta_i [1 - \sum_n q_n \frac{p_o}{k_n M_n}]) \, .$$

But since $\sum_i m_{io} = \frac{M_o}{p_o}$,

it follows that

$$\text{(iv)} \qquad \frac{M_o}{p_o} = \sum_i \gamma_i(\beta_i [1 - \sum_n q_n \frac{p_o}{k_n M_n}])$$

$$p_o = p_o(M_o, \{M_n\}) \text{ for } q_n = q_n^*, \; x_{io} = x_{io}^* \, . \quad \text{qed.}$$

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