A CRITICAL REEXAMINATION OF THE EMPIRICAL EVIDENCE ON THE ARBITRAGE PRICING THEORY

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Working Paper No. 12-82

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August 1982
Revised December 1982

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We owe a great deal to Mustafa N. Gultekin for his generous help in computer programming and for sharing his expertise in computer applications of factor analytic methods with us.

1. Introduction

The APT model as exposited by Ross (1976), (1977) and extended by Huberman (1981) has provided the basis of an extensive literature, as for example, Brown and Weinstein (1981), Chen (1982), Hughes (1982), Ingersoll (1982), Jobson (1982), Reinganum (1980), Roll and Ross (RR) (1980) and Shanken (1982), to mention but a few. It has also been treated in a more general context by Chamberlain and Rothschild (1981).

Our purpose here is not to comment on this collection of papers--many of which are unpublished--but to reexamine the evidence presented by RR and point out major pitfalls involved in the empirical methodology employed by them and others who have followed their lead.

RR claim that, on theoretical and more important on empirical grounds, arbitrage pricing theory (APT) is an attractive alternative to the capital asset pricing model (CAPM). The APT, it is argued, requires less stringent and presumably more plausible assumptions, is more readily testable since it does not require the measurement of the market portfolio, and may be better able to explain the anomalies found in the application of the CAPM to asset returns.

It is not clear to us that the APT assumptions are more plausible or more economically attractive. Moreover, without ascribing economic meaning to the factors on which the APT is based, it is difficult to see how the empirical implementation of arbitrage pricing theory might be useful either for explanatory or predictive purposes. The acceptability of the APT, like that of the CAPM or any other theory, ultimately depends on its ability to explain the relevant empirical evidence. Most of this paper will carefully reexamine the evidence adduced in the first comprehensive study of the APT by RR as well as in several other studies and will present new analyses of the contribution to the explanation of asset returns of a multiple-factor APT. This new material will be presented in Sections 2-8 and summarized in Section 9. Much of this subsequent analysis, it should be noted, raises questions not only about the RR and related empirical investigations of the APT but also about the testability of that theory in the present state of the art. However, before discussing this new material, it may be useful to mention here some of the empirical evidence that has been used in the evaluation of the validity and usefulness of the CAPM which has relevance also to the assessment of the APT and to the tests of that theory carried out by RR.

Both the CAPM and APT imply that only common or covariance (as distinguished from unique or variance) risks are relevant to the pricing of risky assets and, in conjunction with the assumption of homogeneous expectations, they both imply that investors will hold well-diversified portfolios. In fact, it has been shown in Blume and Friend (1978) and elsewhere that a very large proportion of individuals' stock portfolios and other asset holdings are highly undiversified and that the procedures individual investors claim to use in their risk assessments are much more frequently related to variance than to covariance measures.[1]

Regression tests of the contribution made by unique or residual variance measures of risk to the cross-sectional CAPM explanation of differences in *ex post* asset returns based on covariance or beta measures of risk have varied widely in their conclusions. The most recent analysis of this type by Friend and Westerfield (1981) which attempted to correct for deficiencies in earlier tests concluded that residual variance generally seemed fully as important (or in some instances as unimportant) as betas in explaining asset returns. The Friend-Westerfield paper, like virtually all earlier regression tests, found that very little variation in *ex post* returns on individual assets is explained by the CAPM and that the return-risk intercept is significantly different from the risk-free rate predicted by the Sharpe-Lintner version of that model.

The results of one other recent study of the CAPM are relevant to an assessment of the APT tests carried out by RR, especially since these authors stress the impossibility of measuring the market portfolio for testing the CAPM. A comprehensive analysis by Stambaugh (1981) confirms earlier findings that CAPM tests are not very sensitive to different specifications of the market portfolio obtained by adding other classes of assets to the broad stock indexes customarily used to represent the market.[2] However, this analysis does indicate that statistical inferences about the CAPM's validity are more sensitive to selection of the class of individual assets whose risk-return relations are being estimated. Tests of the APT, like those carried out by RR, may be fully as sensitive to the set of assets analyzed as the CAPM.

With this abbreviated background to the earlier empirical evidence on the CAPM relevant to the assessment of the potential usefulness of the APT and of the RR tests of that theory, this section concludes with a brief introduction to the RR he RR results which will be examined much more thoroughly in the subsequent sections. RR follow a two-step procedure in testing the APT. In the first step, expected returns and factor coefficients (loadings) are estimated from time-series data on individual stock returns. The second step uses these estimates of the factor loadings to test the cross-sectional pricing conclusions of the APT which implies that asset returns are a linear function of the factor coefficients, with the intercept predicted to be the risk-free rate. The second step is fairly straightforward but the first step requires both deriving a sample product-moment covariance matrix from a time-series of returns and then carrying out a maximum-likelihood factor analysis on the covariance matrix to estimate the number of factors and the matrix of factor loadings.

There are many problems involved in the RR estimation of the number of factors required to best explain the cross-sectional variation in asset returns. These problems largely (but not exclusively) reflect the need to break down the entire sample of assets being analyzed (in this case 1260 New York or American Stock Exchange stocks on a daily basis from July 3, 1962 to December 31, 1972) into much smaller groups in view of the size of the covariance matrix required for the entire sample and the limited processing capacity of the computer (RR actually use 42 groups of 30 stock each). As our subsequent analysis demonstrates, it is just not clear what interpretation is to be placed on the "factors" determined in each group by the empirical procedure followed by RR for testing the APT. Moreover, we also show that following this procedure the number of factors derived is an increasing function of the size of the group, so that for the same level of significance we find a two-factor model associated with a group of 15 securities, 3 factors for a group of 30 securities, 4 factors for 45 securities, and 7 factors for 60 securities, hardly a satisfactory state of affairs for a general theory. We also show that the RR procedure does not determine, as they seem to believe, whether each of the factors derived is priced but simply whether at least one of them is priced. Our own analysis suggests that it is only in about 20% of the sample groups tested that at least one of the common factors is priced.

One other part of the work by RR which should be mentioned at this point is their extensive analysis of the pricing of residual variance vs. covariance measures which they consider a critical test of APT. While we shall carry out our analysis of this issue later in the paper, we should point out that their rejection of residual variance is far from convincing even on the basis of their own results. RR first carry out a number of statistical tests whose results imply that residual variance is priced in the market but they note that the results may be attributable to complications caused by skewness in the returns distribution. They then carry out a test which they consider more satisfactory that permits them, at the 7.17 level of significance, to conclude that expected returns on individual assets are unaffected by "own" variances.[3] However, our analysis in this paper casts some doubt on the usefulness of both residual variance and common factors; in fact, once residual variance (or standard deviation) and skewness are introduced, risk premia are almost universally insignificant and residual variance does not fare much better.

2. The APT Model: Implications and the Nature of Empirical Tests

In order to establish notation and make this paper as self-contained as possible, we give a brief exposition of the APT model and discuss some aspects of its empirical implications.

The model begins by postulating the return generating function

$$r_{t\cdot} = E_{t\cdot} + f_{t\cdot}B + u_{t\cdot} \tag{1}$$

where $r_{t\cdot}$ is an m-element row vector containing the observed rates of return at time t on the m-securities under consideration; $E_{t\cdot}$ is similarly an m-element row vector containing the expected (mean) returns at time t. Finally,

$$v_{t\cdot} = f_{t\cdot}B + u_{t\cdot} \tag{2}$$

represents the error process at time t. It is an essential feature of the APT model that the error process has two components: the idiosyncratic component

$$u_{t\cdot}, \quad t = 1, 2, \ldots$$

and the common component

$$f_{t\cdot}B .$$

It is assumed that

$$\{u'_{t\cdot}, \quad t = 1, 2, \ldots\}$$

is a sequence of independent identically distributed (i.i.d.) random vectors with

$$E(u'_{t\cdot}) = 0, \ Cov(u'_{t\cdot}) = \Omega , \tag{3}$$

the covariance matrix Ω being <u>diagonal</u> and such that

$$0 < \omega_{ii} < \infty , \quad i = 1, 2, \ldots m . \tag{4}$$

Regarding the common component, we note that the form in which it is stated creates an identification problem, since neither $f_{t\cdot}$ nor B are directly observable. We (partly) eliminate this problem by specifying that

$$\{f'_{t\cdot}: \ t = 1, 2, \ldots\}$$

is a sequence of k-element i.i.d. random vectors with[4]

$$E(f'_{t\cdot}) = 0, \ Cov(f'_{t\cdot}) = I . \tag{5}$$

It is a consequence of the assertions above that

$$\{(r_{t\cdot} - E_{t\cdot})': \ t = 1, 2, \ldots\}$$

is a sequence of i.i.d. random vectors with

$$E[(r_{t\cdot} - E_{t\cdot})'] = 0, \ Cov[(r_{t\cdot} - E_{t\cdot})'] = B'B + \Omega = \Psi . \tag{6}$$

We further note that

$$\begin{aligned} \mathrm{Cov}(r_{ti}, r_{tj}) &= b'_{\cdot i} b_{\cdot j} && i \neq j \\ &= b'_{\cdot i} b_{\cdot i} + \omega_{ii} && i = j \end{aligned} \tag{7}$$

and that indeed the columns of B ($b_{\cdot i}$, which are k×1) contain information on the covariation of securities. Finally, since only Ψ can be estimated directly from the data, (6) shows that there is a further identification problem not eliminated by the assertion in (5) and the discussion of footnote 4.

For if B is a matrix satisfying (6) and (1) and even if $f_{t\cdot}$ obeys (5), then for any orthogonal matrix Q, QB also satisfies (6) and (1). Hence B can be identified only up to left multiplication by an orthogonal matrix.

Now, just what restrictions on empirical evidence are implied by the APT model? First, we should note that the proof of the crucial implication of APT requires the invocation of a strong law of large numbers (SLLN); hence, the universe of securities to which one seeks to apply the model must contain a sufficiently large number of them so that the invocation of the SLLN may be reasonably justified.

Secondly, the fundamental conclusion of APT requires that there exist a (k + 1)-element row vector, $c_{t\cdot}$, such that

$$E_{t\cdot} = c_{t\cdot} B^*, \qquad t = 1, 2, \ldots, T \tag{8}$$

where

$$B^* = \begin{pmatrix} e' \\ B \end{pmatrix}, \tag{9}$$

e being an m-element column of ones.

Thus, the no arbitrage condition characterizing equilibrium rates of return requires that if $x_{t\cdot}$ is such that

$$u_{t\cdot} x'_{t\cdot}$$

is an entity to which the SLLN applies and if $x_{t\cdot}$ belongs to the column null space of B^*, then E_t <u>must lie in the row space of B^*</u>.[5] If the number of securities (m) is sufficiently large, then at any desired degree of approximation we can rewrite (1) as

$$r_t = c_t B^* + f_t B + u_{t\cdot}, \qquad t = 1, 2, \ldots, T . \tag{10}$$

The restriction on empirical evidence imposed by (10) is rather stringent; in particular it requires that no other (relevant) economic/financial variables have any bearing on the determination of expected rates of return.

The empirical tests of APT carried out by RR and others are based on a two step factor analytic approach. Factor analytic methods are, in effect, suggested by the formulation in (1) and the composition of the covariance matrix in (6). In the first step one determines the number of factors (k) and estimates the elements of B and in the second stage, using the latter as the "independent variables," we estimate the vector $c_{t\cdot}$, whose elements have the interpretation that c_{ti} is the risk premium attached to the i^{th} factor, $i = 1, 2, \ldots, k$, while c_{to} is the risk-free rate, or possibly the return on a zero-beta asset.

The question often arises as to whether all (common) risk factors are priced or only a subset thereof. Before we close this section, we wish to address the methodological issues bound up with these concerns.

Thus, suppose T is sufficiently large so that these covariance or correlation matrices, Ψ, can be estimated with reasonable accuracy and by factor analysis we estimate B, say by $\tilde{B}$, and Ω, say by $\tilde{\Omega}$. Thus, we have implicitly estimated

$$\tilde{\Psi} = \tilde{B}'\tilde{B} + \tilde{\Omega} . \tag{11}$$

This completes the first stage; in the second stage for each t we may estimate

$$\tilde{c}'_{t\cdot} = (\tilde{B}^*\tilde{\Psi}^{-1}\tilde{B}^{*\prime})^{-1}\tilde{B}^*\tilde{\Psi}^{-1}r'_{t\cdot}, \qquad t = 1, 2, \ldots, T . \tag{12}$$

If the underlying error process admits of a central limit theorem, it is easy to show that asymptotically

$$\sqrt{T}\,(\tilde{c}_{t\cdot} - c_{t\cdot})' \sim N[0,\ (B^*\Psi^{-1}B^{*\prime})^{-1}] \ . \tag{13}$$

Thus, we may view the $\{\tilde{c}'_{t\cdot}:\ t = 1, 2, \ldots, T\}$, approximately, as drawings from a multivariate normal distribution with mean

$$c'_{t\cdot}:\ t = 1, 2, \ldots, T$$

and covariance matrix

$$(B^*\Psi^{-1}B^{*\prime})^{-1} \ .$$

Recalling, however, that B is identified by factor analytic procedures only to the extent of left multiplication by an orthogonal matrix, we are led to doubt the manner in which tests on individual elements of $c_{t\cdot}$ make any sense, or indeed whether tests of the fundamental proposition of APT are feasible in this context. General tests, see for example RR, involve the introduction of other explanatory variables and a test of the hypothesis that the corresponding coefficients are zero.

Specifically, we shall examine the question whether, in the context of the specification

$$r_{t\cdot} = c_{t\cdot}B^* + d_{t\cdot}P + v_{t\cdot} \tag{14}$$

where P is a vector of "extraneous" variables and B* is only identified to within left multiplication by

$$Q^* = \begin{bmatrix} 1 & 0 \\ 0 & Q \end{bmatrix} \tag{15}$$

and Q is orthogonal, it is possible to have unambiguous tests of significance on $d_{t\cdot}$, c_{to} and c_{ti}, $i = 1, 2, \ldots, k$. In order that we may examine these questions with as few extraneous issues as possible, we shall suppose that T is sufficiently large so that the estimates of B obtained by a factor analytic approach have negligible sampling variation so that we can deal with them as if they were their probability limits. Let B_o be the "true" parameter matrix as initially specified in (1). Then the output of the factor analytic procedure and therefore the set of explanatory variables in (14) is a matrix, say, $\overline{H}$, which is related to the true matrix H_o through

$$\overline{H} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & Q & 0 \\ 0 & 0 & I \end{bmatrix} H_o, \quad H_o = \begin{bmatrix} e' \\ B_o \\ P \end{bmatrix} \tag{16}$$

Note that H_o is unambiguously specified although B_o is unknowable; what is knowable is only the transformation

$$\overline{B} = QB_o$$

where Q is an arbitrary orthogonal matrix. Hence, the only unambiguous conclusions that may be derived from such an analysis must be conclusions "modulo" Q, i.e., conclusions that do not in any way depend on Q.

We have

Proposition 1: Consider the general model in (14) and suppose T is sufficiently large so that sampling variation may be ignored. Then,

$$\overline{h}'_{t\cdot} = \overline{Q}\tilde{h}^{o\,\prime}_{t\cdot}, \quad \overline{h}'_{t\cdot} = (\overline{H}\Psi^{-1}\overline{H}')^{-1}\overline{H}\Psi^{-1}r'_{t\cdot} \tag{17}$$

where

$$\overline{Q} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & Q & 0 \\ 0 & 0 & I \end{bmatrix}, \quad \tilde{h}^{o\,\prime}_{t\cdot} = (H_o\Psi^{-1}H_o')^{-1}H_o\Psi^{-1}r'_{t\cdot}\ .$$

Q is an orthogonal matrix of order k, I is the identity matrix of order s (s being the number of extraneous explanatory variables--the rows of P), $\overline{Q}$ is a nonsingular matrix of order s + k + 1, and $\overline{H}$, H_o are as in (16).

Proof: Obvious by noting that

$$\overline{Q}'\overline{Q} = I ,$$

i.e., $\overline{Q}$ is also an orthogonal matrix.

<u>Remark 1</u>: It is evident that the "risk free" or "zero-beta" rate is uniquely estimated since in the obvious notation

$$\overline{c}_{to} = \tilde{c}^{o}_{to} .$$

Moreover, the vector $d_{t\cdot}$ is also uniquely estimated since

$$\overline{d}_{t\cdot} = \tilde{d}^{o}_{t\cdot} .$$

On the other hand

$$(\overline{c}_{t1}, \overline{c}_{t2}, \ldots, \overline{c}_{tk})' = Q \begin{bmatrix} \tilde{c}^{o}_{t1} \\ \tilde{c}^{o}_{t2} \\ \tilde{c}^{o}_{tk} \end{bmatrix}$$

which states that in general

$$\overline{c}_{ti} \neq \tilde{c}^{o}_{ti}, \qquad i = 1, 2, \ldots, k .$$

Hence, the interpretation of the regression estimates $\overline{c}_{ti}$ as the "risk premium" for the i^{th} common risk factor is a serious overreaching of the empirical evidence.

Moreover, in RR as well as Hughes (1982) among others, many tests are carried out as to how many factors are being "priced"; in addition in RR tests are also carried out on the "significance" of individual extraneous variables as a "test" of the APT model. To what extent are the results of such tests unambiguous? This is in part answered by

Proposition 2: Under the conditions of Proposition 1 the covariance matrix of the estimator $\bar{h}'_{t\cdot}$ has the following properties:

1. the variance of $\bar{c}_{to}$ is exactly the variance of $\tilde{c}^{o}_{to}$, i.e., it does not depend on the matrix Q;
2. the covariance matrix corresponding to $\bar{d}_{t\cdot}$ is exactly the covariance matrix of $\tilde{d}^{o}_{t\cdot}$, i.e., it does not depend on the matrix Q;
3. the covariance matrix corresponding to the "risk premia" assigned to the common risk factors does depend on the matrix Q.

Proof: The covariance matrix of $\bar{h}'_{t\cdot}$ is evidently given by

$$(\bar{H}\Psi^{-1}\bar{H}')^{-1} = \bar{Q}(H_o\Psi^{-1}H_o')^{-1}\bar{Q} .$$

Thus the variance of $\bar{c}_{to}$ is simply the (1, 1) element of $(H_o\Psi^{-1}H_o')$ which is independent of Q. Specifically, it is

$$[e'\Psi^{-1}e - e'\Psi^{-1}H_2^{o\prime}(H_2^o\Psi^{-1}H_2^{o\prime})^{-1}H_2^o\Psi^{-1}e]^{-1}$$

where

$$H_2^o = \begin{bmatrix} B_o \\ P \end{bmatrix} .$$

For the second statement of the proposition we note that the corresponding covariance matrix is

$$[P\Psi^{-1}P' - P\Psi^{-1}B_o^{*\prime}(B_o^*\Psi^{-1}B_o^{*\prime})^{-1}B_o^*\Psi^{-1}P']^{-1}$$

which evidently does not depend on Q. For the last part of the statement we note that the relevant covariance matrix is

$$\Phi_{22} = Q[B_o\Psi^{-1}B_o' - B_o\Psi^{-1}P^{*\prime}(P^*\Psi^{-1}P^{*\prime})^{-1}P^*\Psi^{-1}B_o']^{-1}Q' \qquad (19)$$

where

$$\Phi_{22} = Q\Phi^o_{22}Q'$$

$$P^* = \begin{bmatrix} e' \\ P \end{bmatrix}$$

and Φ^o_{22} has the obvious meaning. This makes the dependence abundantly clear and completes the proof of the Proposition.

Remark 2: The results above should make it evident that tests on individual coefficients, c_{ti}, or subsets of risk premia coefficients are not unambiguous. To demonstrate this let us assume normality of the error terms so that the test statistics become unambiguously determined in their distribution. Such linear tests involve consideration of quantities like

$$A\bar{c}^{*\prime}_{t\cdot}$$

where

$$\bar{c}^*_{t\cdot} = (\bar{c}_{t1}, \bar{c}_{t2}, \ldots, \bar{c}_{tk})$$

and A is a suitable matrix with known elements. The usual test statistic then would obey

$$\bar{c}^*_{t\cdot}A'(A\Phi_{22}A')^{-1}A\bar{c}^{*\prime}_{t\cdot} = \tilde{c}^{o*}_{t\cdot}Q'A'(AQ\Phi^o_{22}Q'A')^{-1}AQ\tilde{c}^{o*\prime}_{t\cdot} \sim \chi^2_k \qquad (20)$$

where

$$\Phi^o_{22} = Q'\Phi_{22}Q \ .$$

Since in general

$$Q'A'(AQ\Phi^o_{22}Q'A')^{-1}AQ \neq A'\left(A\Phi^o_{22}A'\right)^{-1}A \ ,$$

a test of the hypothesis

$$A\bar{c}^{*\prime}_{t\cdot} = 0$$

is not equivalent to a test of

$$Ac^{o*\prime}_{t\cdot} = 0$$

where A is $r\times k$ $(r < k)$.

In the special case, however, when A is the identity matrix of order k (i.e., when we simultaneously test all risk premia), we find that the test statistic (20) reduces to

$$\tilde{c}^{o}_{t\cdot}*(\Phi^{o}_{22})^{-1}\tilde{c}^{o}_{t\cdot}*' \sim \chi^2_k$$

which is appropriate for testing the hypothesis

$$c^{o}_{t\cdot}*' = 0 .$$

Thus, in this context the crucial testable hypothesis is how many factors there are and whether none of them is priced, rather than whether some of them are priced and others are not.

3. A Critical Appraisal of the RR and Similar Empirical Tests

The APT model as explained earlier requires the set of securities to which the (rates of) return generating function in (1) applies to be "large," i.e., large enough so as to assure us that a SLLN applies. It has the important implication that

$$E_{t\cdot} = c_{t\cdot}B* .$$

There is no presumption of time stationarity with respect to the vector $c_{t\cdot}$ which describes the dependence of the mean (expected) rate of return vector on the rows of B*.

It is a practical necessity, and was explicitly assumed earlier, that the distributions of $f_{t\cdot}$ and $u_{t\cdot}$ be time stationary, i.e., it was explicitly assumed that their distribution (or at least their second moments) did not vary with t. At least this must be so over a sufficiently long period to permit the estimation of the relevant covariance matrix.

One significant limitation that is found in all papers attempting to estimate or test the APT model is that in the estimation of the relevant covariance matrix it is assumed that the mean return process is time invariant. This is so since the (sample) covariance matrix computed by any factor analytic software package is, barring instructions to the contrary,

$$S = \left(1/(T-1)\right) \sum_{t=1}^{T} (r_{t\cdot} - \bar{r})'(r_{t\cdot} - \bar{r}) \tag{21}$$

where

$$\bar{r} = (1/T) \sum_{t=1}^{T} r_{t\cdot} = (1/T) \sum_{t=1}^{T} E_{t\cdot} + \left(\frac{1}{T} \sum_{t=1}^{T} f_{t\cdot}\right)B + (1/T) \sum_{t=1}^{T} u_{t\cdot} .$$

If T (the time dimension of the sample) is large enough, which is usually the case in such applications, then of course

$$\frac{1}{T} \sum_{t=1}^{T} u_{t\cdot} \approx 0 , \qquad \frac{1}{T} \sum_{t=1}^{T} f_{t\cdot} \approx 0$$

so that

$$\bar{r} \approx \frac{1}{T} \sum_{t=1}^{T} E_{t\cdot}$$

but if $E_{t\cdot}$ is not time invariant, it is not clear that computing the covariance matrix in the manner of (21) makes a great deal of sense. Thus, this common practice that seldom receives any comment actually implies that the vectors $c_{t\cdot}$ must be time invariant!

A second important implication of the APT model as exposited above is that within any degree of (probabilistic) approximation desired the vector of returns of the m securities in question may be written as

$$r_{t\cdot} = c_{t\cdot}B^* + f_{t\cdot}B + u_{t\cdot}, \qquad t = 1, 2, \ldots, T \tag{22}$$

which means that if we are to subject the model to empirical testing, we ought to treat <u>all m securities symmetrically</u>.

It has been a practice initiated in the paper by RR and frequently imitated by others, say, Brown and Weinstein (1982), to divide the universe of securities into a number of subcategories (42 in the case of RR) and treat these subgroups as "cross sections" from a population in the manner one treats a sample of households in the context of a consumer expenditures survey. The analogy is, of course, quite appealing, which is the reason for its wide acceptance. It is, however, very misleading. What enables us to use the cross-sectional information of a consumer expenditures survey to infer something about the parameters that characterize that particular universe is that each individual in the cross-section has some fairly well-defined attributes which can be measured unambiguously and independently of how many individuals there are in the cross-section coupled with a presumption of parametric homogeneity among the entities of the relevant universe.

A reflection on the nature of the model as exhibited in (22) will disclose that if we partitioned the universe into 42 groups as RR do, then for the i^{th} group we should have

$$r_{t\cdot}^{(i)} = c_{t\cdot}^{(i)} B^*_{(i)} + f_{t\cdot} B_{(i)} + u_{t\cdot}^{(i)} , \qquad i = 1, 2, \ldots, 42$$

where $r_{t\cdot}^{(i)}$ consists of the first 30 elements of $r_{t\cdot}$, $r_{t\cdot}^{(2)}$ of the second thirty elements and so on. Similarly, $B_{(1)}$ consists of the first 30 columns of B, $B_{(2)}$ of the second 30 columns and so on. The same is true of $B^*_{(1)}$, $u_{t\cdot}^{(1)}$, $B^*_{(2)}$, $u_{t\cdot}^{(2)}$, etc.

The question arises whether each group can be dealt with in isolation with any degree of assurance regarding the reliability of the ensuing results. In their paper, RR and those who have followed their lead such as, for example, Hughes (1982) and Brown and Weinstein (1982) make only the weak caveat that while only a relatively small number of factors may be identified in each group, one must bear in mind that perhaps it may be different factors that correspond to different groups. While the disclaimer is in place and therefore protects the authors against any criticism of overreaching in their conclusions, it would appear that the remainder of the discussion completely ignores this point and proceeds as if, indeed, the same small number of factors is identified for each group and RR are particularly pleased that the number of significant factors extracted in each of their 42 groups (of alphabetically arranged 30 securities) ranges between three and five.

Unfortunately, the situation is far more grave than the literature has thus far allowed for. Treating each group of 30 securities as a cross-section and looking to the results from such an exercise for confirmatory evidence about the number of factors is not appropriate. Most importantly it should be stressed that, in general, what is the equivalent of the explanatory variables (attributes) for the 30 securities cannot be measured reliably <u>independently</u> of the issue of how many securities we treat simultaneously. This is so since those "explanatory variables" are given by the (sub) matrix $B^*_{(i)}$ and this cannot be measured reliably in a 30-securities context--for reasons we shall explain below. Contrast this to the consumer expenditures survey context in which each individual household's income, size, composition and other relevant socioeconomic attributes can be accurately ascertained independently of how many households there are in the sample (cross-section).

To understand the reason why $B^*_{(i)}$ cannot be measured reliably in the context employed by RR and others such as Brown and Weinstein (1982) for example, consider the model in (10) and suppose $r_{t\cdot}$ is stated in terms of standard deviates, i.e., we subtract from each r_{ti} its mean and divide by its standard deviation so that the matrix, S, in (21) is a correlation matrix, thus, conforming to the standard procedures in factor analysis computer software. Making allowance for this correction, the interpretation of Ψ in (6) is now that of a correlation matrix so that its diagonal elements are unity.

Partitioning Ψ in accordance with the RR scheme above we have

$$\Psi = [\Psi_{ij}], \qquad i, j = 1, 2, \ldots, 42 \tag{23}$$

so that Ψ_{ii} is the correlation matrix for the i^{th} group of 30 securities and Ψ_{ij}, $i \neq j$, the "cross correlation" matrix between securities in the i^{th} and j^{th} groups. If we subject all 1260 securities to factor analysis simultaneously, we shall obtain estimates of B and Ω, say $\tilde{B}$ and $\tilde{\Omega}$, obeying

$$\begin{aligned} \mathrm{diag}(S) &= \mathrm{diag}(\tilde{B}'\tilde{B} + \tilde{\Omega}) \\ [\tilde{\Omega}^{-1/2}(S - \tilde{\Omega})\tilde{\Omega}^{-1/2}]\tilde{\Omega}^{-1/2}\tilde{B}' &= \tilde{\Omega}^{-1/2}\tilde{B}'(\tilde{B}\tilde{\Omega}^{-1}\tilde{B}') \;. \end{aligned} \tag{24}$$

On the other hand, factor analyzing each of the 42 groups we obtain

$$\begin{aligned} \mathrm{diag}(S_{ii}) &= \mathrm{diag}(\hat{B}_i'\hat{B}_i + \hat{\Omega}_i) \\ [\hat{\Omega}_i^{-1/2}(S_{ii} - \hat{\Omega}_i)\hat{\Omega}_i^{-1/2}]\hat{\Omega}_i^{-1/2}\hat{B}_i' &= \hat{\Omega}_i^{-1/2}\hat{B}_i'(\hat{B}_i\hat{\Omega}_i^{-1}\hat{B}_i') \end{aligned} \tag{25}$$

for $i = 1, 2, \ldots, 42$.

In (24) and (25) we impose, respectively, the conditions that $(\tilde{B}\tilde{\Omega}^{-1}\tilde{B}')$ and $(\hat{B}_i\hat{\Omega}_i^{-1}\hat{B}_i')$, $i = 1, 2, \ldots, 42$ be <u>diagonal</u>. The procedure in (25) is essentially the RR procedure, and we may either dismiss it as irrelevant or we may rationalize it as assuming that there exist orthogonal matrices, Q_i, such that

$$\tilde{B}_i = Q_i\hat{B}_i, \qquad i = 1, 2, \ldots, 42 \tag{26}$$

and that while the $\hat{B}_i$, $\hat{\Omega}_i$ of (25) do not satisfy (24), there is still a well defined relationship between them as given by (26). If that is, indeed, the case, then certain aspects of the RR methodology will not be inappropriate, even if the procedure would not be the most efficient possible. Intuitively, this is not very likely to be true since we seem to be arguing that the characteristic vectors corresponding to the k largest characteristic roots of a matrix and its principal submatrices are related in the manner of (26). We also seem to be arguing that ignoring the off diagonal blocks constitutes a misspecification that entails no cost. Both assertions are in error. But since the impression is widespread, we produce a counter example. This, although referring to a special case, is sufficient to disabuse us of the notion that any general theorem exists that guarantees the validity of (26). Thus, if one alleges a relation like (26), then one has the responsibility of delineating the circumstances, if any, under which it holds.

Now consider the case $m = 4$, the partitioning in two groups and the extraction of one "factor." Suppose

$$S_{11} = \begin{bmatrix} 1 & 2^{-1/2}\varepsilon_1 \\ 2^{-1/2}\varepsilon_1 & 1 \end{bmatrix}, \quad S_{22} = \begin{bmatrix} 1 & 3^{-1/2}\varepsilon_2 \\ 3^{-1/2}\varepsilon_2 & 1 \end{bmatrix}, \quad S_{12} = 0$$

and (27)

$$\Omega_1 = \begin{bmatrix} 1/2 & 0 \\ 0 & 1 \end{bmatrix}, \quad \Omega_2 = \begin{bmatrix} 1/3 & 0 \\ 0 & 1 \end{bmatrix}.$$

We find

$$\Omega_1^{-1/2}(S_{11} - \Omega_1)\Omega_1^{-1/2} = \begin{bmatrix} 1 & \varepsilon_1 \\ \varepsilon_1 & 0 \end{bmatrix}$$

(28)

$$\Omega_2^{-1/2}(S_{22} - \Omega_2)\Omega_2^{-1/2} = \begin{bmatrix} 2 & \varepsilon_2 \\ \varepsilon_2 & 0 \end{bmatrix}$$

Finally

$$\Omega^{-1/2}(S - \Omega)\Omega^{-1/2} = \begin{bmatrix} 1 & \varepsilon_1 & 0 & 0 \\ \varepsilon_1 & 0 & 0 & 0 \\ 0 & 0 & 2 & \varepsilon_2 \\ 0 & 0 & \varepsilon_2 & 0 \end{bmatrix} \qquad (29)$$

The largest characteristic root and associated characteristic vector of the two matrices in (28) are, respectively,

$$\lambda_1^{(1)} = \frac{1 + \sqrt{1 + 4\varepsilon_1^2}}{2}, \; x_1^{(1)} = 1, \; x_2^{(1)} = \frac{2\varepsilon_1}{2 + \delta_1^2}, \; \delta_1^2 = \sqrt{1 + 4\varepsilon_1^2} - 1$$
$$\lambda_1^{(2)} = 1 + \sqrt{1 + \varepsilon_2^2}, \; x_1^{(2)} = 1, \; x_2^{(2)} = \frac{\varepsilon_2}{1 + \sqrt{1 + \varepsilon_2^2}} \qquad (30)$$

For the matrix in (29) the corresponding entities are

$$\mu_1 = 1 + \sqrt{1 + \varepsilon_2^2}, \; y_1 = 0, \; y_2 = 0, \; y_3 = 1, \; y_4 = \frac{\varepsilon_2}{1 + \sqrt{1 + \varepsilon_2^2}} \qquad (31)$$

Consequently (26) could imply that there exists an orthogonal matrix, Q_1, such that

$$\begin{bmatrix} 1 \\ \dfrac{2\varepsilon_1}{2 + \delta_1^2} \end{bmatrix} = Q_1 \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad (32)$$

which is clearly impossible. Thus, it is just not clear what it is that RR estimate in the context of the 30-security groups relative to the parameters specified in the model of (1). Nonetheless the general principle of testing the model by introducing potentially relevant explanatory variables is a valid one and the procedure provides a rather stringent test.

Recently, a paper by Jobson (1982) proposes an alternative test which, we think, is a considerably weaker one, without dealing with the difficiencies we pointed out in the RR procedure. The main idea embodied in Jobson capitalizes on a joint normality assumption for the group of assets to which APT is applied and requires a priori knowledge of the number of "factors," k. Given this fact, we could obtain any set of k linearly independent securities or portfolios and write

$$
\begin{aligned}
r_{t\cdot}^{(1)} &= c_{t\cdot}B_1^* + f_{t\cdot}B_1 + u_{t\cdot}^{(1)} \\
r_{t\cdot}^{(2)} &= c_{t\cdot}B_2^* + f_{t\cdot}B_2 + u_{t\cdot}^{(2)}
\end{aligned}
\tag{33}
$$

where $r_{t\cdot}^{(1)}$ is 1×k and contains the "basic" k securities such that B_1 is a nonsingular matrix. In another paper, one of the authors, Dhyrmes (1982), shows that the apparent simplicity of Jobson's procedure is illusory. First, it concedes part of the APT model's contentions by assuming that the "constant term," c_{to}, of the vector $c_{t\cdot}$ in (23) is known--specifically that it is the "risk free" rate as represented, say, by the interest on a Treasury bill of suitable maturity. Second, the simple version that involves solving for $f_{t\cdot}$ in the first equation of (33) and substituting in the second equation forces us to deal with an "error in variables" model without the ready availability of instrumental variables. Third, the likelihood ratio (LR) version of the test requires us to use portfolios rather than assets as the "basic" explanatory variables and moreover portfolios for which we can assert a SLLN relative to the idiosyncratic error terms, i.e., in (33) the first equation should be replaced by

$$
r_{t\cdot}G_1 = c_{t\cdot}B^*G_1 + f_{t\cdot}BG_1 + u_{t\cdot}G_1 \tag{33a}
$$

where G_1 is m×k such that

$$u_{t\cdot}G_1 = 0$$

with probability one. However, this is likely also to induce, in the limit, the singularity of

$$G_1'B'BG_1$$

thus limiting the usefulness of the approach. Fourth, and worst of all, the number of factors is assumed rather than determined empirically. Fifth, and finally, a zero constant term for Jobson's relations would prevail if the model is

$$r_{t\cdot} = \mu e' + v_{t\cdot}' \tag{33b}$$

where μ is a scalar and is equated to the "risk free" rate. Then proceeding as in the LR version would yield

$$r_{t\cdot}^{(2)'} = \alpha' + \beta' r_{t\cdot}^{(1)'} + \eta_{t\cdot}^{(2)'} \tag{33c}$$

where now

$$\alpha' = \mu\left(e^{(2)'} - \Psi_{21}\Psi_{11}^{-1}e^{(1)'}\right)$$

Consequently operating with the reduced returns vector $(r_{t\cdot} - c_{t0}e')$ and testing for the nullity of the constant vector in the reduced returns version of (33c) <u>could</u> simply be a test that in a model like (33b) μ is the "risk free" rate and otherwise convey no implications as to the empirical validity of the APT model.

Thus, whether or not we grant the RR approach of dealing with small groups their testing procedure is both simple and straightforward--and certainly constitutes a more stringent test than the one proposed by Jobson. Thus, in what follows we shall concentrate on a reexamination of the RR procedures.

4. A Partial Test for the Loss of Information Entailed by the RR Procedures

One may interpret the RR methodology as operating on the implicit assumption that the off diagonal blocks of the matrix in (23) obey

$$\Psi_{ij} = 0, \quad i \neq j, \quad i, j = 1, 2, \ldots, 42 \tag{34}$$

and extracting five factors from each group (or 210 in all) rather than five from the entire group.

While we cannot actually test this for the entire set of securities we can carry out a "test" for a set of 240 securities (corresponding to groups numbered 1, 4, 5, 8, 9, 10, 11, 41 following the RR classification scheme). The (log) likelihood function (LF) under H_0 as in (34) is given by

$$L_1 = -\frac{30nT}{2}\ln(2\pi) - \frac{T}{2}\sum_{i=1}^{n}\left[\ln\left|B_i'B_i + \Omega_i\right| - \mathrm{tr}\Psi_{ii}^{-1}S_{ii}\right] \tag{35}$$

where

$$\Psi_{ii} = B_i'B_i + \Omega_i, \qquad S_{ii} = \frac{1}{T}\sum_{t=1}^{T} r_{t\cdot}^{(i)\prime} r_{t\cdot}^{(i)}, \qquad i = 1, 2, \ldots, n$$

(and as required in most computer software the return on the j^{th} security, r_{tj}, is stated as a standardized deviate).

Maximizing under H_0, and extracting five "factors" per group yields the RR results. In the particular case under consideration we find

$$\max_{H_0} L_1 = -\frac{30nT}{2}\left[\ln(2\pi) + 1\right] - \frac{T}{2}\sum_{i=1}^{n}\ln\left|\hat{B}_i'\hat{B}_i + \hat{\Omega}_i\right| \tag{36}$$

Under the alternative, the LF is

$$L_2 = -\frac{30nT}{2}\ln(2\pi) - \frac{T}{2}\ln\left|B'B + \Omega\right| - \frac{T}{2}\mathrm{tr}\Psi^{-1}S$$

where now

$$S = \frac{1}{T}\sum_{t=1}^{T} r_{t\cdot}' r_{t\cdot}$$

Maximizing under the alternative and still extracting five "factors" yields the RR results as would be obtained for the entire set of 240 securities. In particular, we find

$$\max_{H_1} L_2 = -\frac{30nT}{2}\left[\ln(2\pi) + 1\right] - \frac{T}{2}\ln\left|\tilde{B}'\tilde{B} + \tilde{\Omega}\right| \tag{37}$$

Now had we extracted only five "factors" from all 240 securities in the likelihood function of (35), as a nested set of hypotheses would require, we would find a maximized value for the LF in (35), say L^*, obeying

$$L_1^* \leq \max_{H_0} L_1$$

Thus, if we treat

$$\ln\lambda = \max_{H_0} L_1 - \max_{H_1} L_2 = -\frac{T}{2}\left[\sum_{i=1}^{n} \ln\left|\hat{B}_i'\hat{B}_i + \hat{\Omega}_i\right| - \ln\left|\tilde{B}'\tilde{B} + \tilde{\Omega}\right|\right]$$

as the likelihood ratio test statistic, we would obtain a value for λ which is larger than the correct one. Hence

$$-2\ln\lambda = T\left[\sum_{i=1}^{n} \ln\left|\hat{B}_i'\hat{B}_i + \hat{\Omega}_i\right| - \ln\left|\tilde{B}'\tilde{B} + \tilde{\Omega}\right|\right] \tag{38}$$

would yield a value which is smaller than the correct one. Hence, if we reject on the basis of (38), we would certainly reject on the basis of the correct LR test statistic. Now, in the present case

$$n = 8, \qquad T = 2196$$

$$\sum_{i=1}^{8} \ln\left|\hat{B}_i'\hat{B}_i + \hat{\Omega}_i\right| = -15.49002$$

$$\ln\left|\tilde{B}'\tilde{B} + \tilde{\Omega}\right| = -24.16200$$

Hence

$$-2\ln\lambda = 20,049 \tag{39}$$

and we remind the reader that, for the correct test, the statistic corresponding to (39) would be larger. The number of parameters properly estimated under H_o is 390; the number under H_1 is 1440. Hence, informally we may argue that the true LR test statistic is asymptotically chi square with degrees of freedom equal to

$$r = 1440 - 390 = 1050$$

Since r is rather large, we can use the approximation

$$\frac{\chi_r^2 - r}{\sqrt{2r}} \sim N(0,1)$$

In this case this leads to

$$\frac{-2 \ln \lambda - 1050}{\sqrt{2100}} \approx \frac{20,049 - 1050}{46} = 413$$

which clearly amounts to a rejection. Since, if estimation were done properly under H_o,

$$\frac{-2\lambda - 1050}{\sqrt{2100}}$$

would have been larger than 413, it seems unambiguous that the RR procedure cannot be rationalized in terms of ignoring information of little value. Indeed, if we were to accept the proposition that

$$\Psi_{ij} = 0 \qquad i \neq j \ ,$$

then since the assignment of securities to groups is arbitrary we would conclude that there are no common risk factors thus denying the *raison d'être* of the APT model.

5. How Many Factors Are There?

Notwithstanding the criticism of the basic RR approach developed in the earlier section, we deem it important to reexamine the results obtained by them with a view to reassessing the evidence presented this far on behalf of the empirical validity of the APT model. This is done for two reasons. First, it is an important scientific axiom that potentially important new empirical results be subjected to the test of replication; this is particularly important in the case of the work by RR since while their findings are very provocative, their data exhibit a rather vexing missing observations problem. Second, a reexamination of the evidence presented by them would certainly permit us to examine in an empirical context some of the issues we had raised earlier, and either confirm or raise doubt about certain purported empirical regularities obtained in the basic work by RR.

The first question we shall examine in this section is how many factors can be said to characterize the return generating process for securities traded on the New York and American stock exchanges. This question was certainly raised by RR who assert (Table II, p. 1088) that in about 88% of their groups (about 37 out of 42) "the probability that no more than five factors are needed to explain returns" is higher than .5, and (Table III, p. 1092) that when c_{to} is not taken as known, in 95% of the groups three or fewer factors have associated "risk premium significant at the 95% level."

As we have pointed out in section 2 one cannot test unambiguously the "significance" of individual risk premia although one can test unambiguously the null hypothesis

$$c^*_{t\cdot} = 0$$

where

$$c_{t\cdot} = (c_{to}, c^*_{t\cdot})$$

i.e., $c^*_{t\cdot}$ is the vector of risk premia. Thus, the important issue in this research is how many factors there are rather than how many "priced" factors there are. RR and others who followed their lead, such as Brown and Weinstein

(1982), Chen (1982) among others, settle on five factors. Apparently it has not occurred to previous researchers that, in the process of establishing "the number of factors" by the conventional likelihood ratio test (asymptotically a chi-squared test), how many factors are found may very well depend on the number of securities one considers.[6] Table 1 presents chi-square tests on one to five factors for each of the 42 groups in the RR analytical framework. Our results differ appreciably from those obtained by RR (Table II, p. 1088). Thus, if we interpret the "probability that no more than five factors are needed to explain returns" as the p-value associated with the statistic involved in testing for a 5-factor decomposition, the comparison is as follows:

p-value	.9	.8	.7	.6	.5	.4	.3	.2	.1	0
RR	38.1	16.7	7.1	2.4	12.0	2.4	4.8	4.8	9.6	2.4
Ours	0	12.0	9.5	4.8	16.7	11.9	11.9	9.5	7.1	16.6

In the above the entry below each p-value, say .5, is to be interpreted as the percent of groups with p-values in the interval [.5,.6). Thus, for example, our results for p-value equal to zero show that seven groups (16.6%) have a p-value which indicates that a five factor decomposition is inadequate at the 10% level of significance, while RR find only 1 such group (2.4%). The difference in the findings may be attributed either to the very large number of missing observations for some securities in the RR sample or to the greater precision of our computer software (SAS) or both. The differences in sample coverage between our set and that of RR amount only to 21 firms--and these and other data differences are discussed more fully in the Appendix. To examine the issue of "how many factors there are" we consider the case of an expanding "universe" of securities and we give these results in Table 2. Thus, if one considers groups of, say, 15 securities, then only one or two factors may be "found;" if one considers 30 securities, then two or three may be "found." This becomes plausible if one understands the operational significance of the test, instead of concentrating solely on the purely abstract and synthetic concept of factors. What the test does (see, for example, Morrison (1967), p. 269ff) is to test the hypothesis that after the extraction of k roots of the appropriate matrix the remaining roots are equal--and presumably small. Thus, looking at the 15×15 reduced correlation matrix entailed in the use of 15 security groups, it would not be surprising to find only one to two "distinct" characteristic roots; as we enlarge the scope of the investigation by dealing with, say, 30 security, 45 security, 90 security or 240 security groups, we should not be surprised if we encounter more distinct characteristic roots. This is well illustrated in Table 2 which gives the chi-squared statistics and p-values associated with a given number of factors for (overlapping) groups of 15, 30, 45, 60, and 90 security groups. We also give the p-value associated with five factors in the case of a group consisting of 240 securities.

We remind the reader that the p-value is the probability that a (chi-squared) statistic at least as large as the one obtained would be realized if the null hyopthesis is true, i.e. if there are most k factors--i.e. if the remaining m-k characteristic roots are the same. Presumably if we operate at the 10% level of significance, we should not tolerate a p-value greater than .1; this is so since if the p-value is say .4 it would mean, if the null hypothesis is true (i.e. there are at most k factors), then the probability of obtaining a statistic at least as large as the one obtained is .4. But at a 10% level of significance the acceptance region would include this particular statistic value.[7] In fact, the testing procedure for such models entails the acceptance of the k+r factor model, $r \geq 1$, given that a k-factor model is accepted. Thus, a proper phrasing of the testing procedure is that we seek the smallest (integer) value k such that the associated p-value of the test statistic obtained is equal to or greater than the desired level of significance.

Choosing a level of significance at .1, we see from Table 2 that for the group of 15 securities we have at most a two-factor model; for the group of 30 securities (containing the initial 15) we have a three-factor model; for the group of 45 securities we have a four-factor model; for the group of 60 securities a seven-factor model and so on. While these results have been obtained with a certain set of 240 securities, we have no reason to believe that, aside from singularity problems, the same phenomenon will not manifest itself with another group of 240 securities. The interesting question is at what level will the number of factors stabilize so that adding more securities to the universe will not change the number of factors conventional testing procedures will produce. It is incredibly expensive, however, to pursue this line of research and we have not done so.

We do, however, report in Table 3 the estimates of the factor loadings from a group of 240 securities and those obtained if we factor analyzed its eight constituent groups of 30 securities each. Due to space limitations we only present this comparison for the first 60 securities. Interestingly, for any given security, estimates of the first factor loading do not change much as the universe expands. Estimates of the factor loadings for the remainder of the factors, however, change dramatically as the "universe" expands.

The import of this aspect of our work, then, is that the empirical finding in RR that the return generating process may be adequately characterized by a five factor scheme is shaky both on its own grounds due possibly to missing observations and on the basis of very deficient methodology. It is then still an open question as to how many factors give an adequate characterization, but it is almost certain that there are more than five.

6. Is the Intercept of the Cross-Sectional Regressions the Daily Risk-Free Rate?

One important implication of the APT, which it shares with other capital asset pricing models, is that the "constant" term in the relation

$$r_{t\cdot} = c_{t\cdot}B^* + f_{t\cdot}B + u_{t\cdot}$$

i.e., the term c_{to}, corresponds to the risk-free rate, or at least a zero beta asset.[8] As we pointed out earlier, the operational procedure for obtaining $c_{t\cdot}B^*$ is time invariant, which would argue strongly that $c_{t\cdot}$ is time invariant. In turn this could imply that the "risk-free" rate is also time invariant--which is rather far-fetched and questionable. Despite this and the other reservations expressed earlier regarding the testability of the APT in the manner suggested by RR, we proceeded to carry out a test of this particular set of implications. We felt it particularly appropriate since RR carry out a test on the equality of intercept terms for adjacent groups only, instead of a test on equality of intercept terms for all groups.

Once the matrix B_i of factor loadings for the i^{th} group has been estimated on the basis of a five-factor model, we obtain GLS estimators by

$$\tilde{c}_{t\cdot}^{(i)'} = (\tilde{B}_i^*\tilde{\Psi}_{ii}^{-1}\tilde{B}_i^{*'})^{-1}\tilde{B}_i^*\tilde{\Psi}_{ii}^{-1}r_{t\cdot}^{(i)'} \qquad i = 1, 2, \ldots, 42 \qquad (40)$$

$$\text{where } \tilde{B}^* = \begin{bmatrix} e' \\ \tilde{B}_i \end{bmatrix}, \qquad \tilde{\Psi}_{ii} = \tilde{B}_i'\tilde{B}_i + \tilde{\Omega}_i \qquad t = 1, 2, \ldots, T$$

If daily returns are normal and if the sample size T on the basis of which $\tilde{B}_i$ and $\tilde{\Omega}_i$ are estimated is large--which it is in the present context--then we would expect that, approximately,

$$\tilde{c}_{t\cdot}^{(i)\prime} \sim N[c_{t\cdot}^{\prime}, (B_i^* \tilde{\Psi}_{ii}^{-1} B_i^{*\prime})^{-1}] \tag{41}$$

The important thing to realize here is that the covariance matrix is time invariant. Hence, we can treat the $\tilde{c}_{t\cdot}^{(i)}$ as "observations" from a population with mean $c_{t\cdot}$ and a constant covariance matrix. Hence, defining

$$z_{t\cdot}^{(i)} = \tilde{c}_{t\cdot}^{(i)} - \tilde{c}_{t\cdot}^{(1)} , \qquad i = 2, 3, \ldots, 42 \tag{42}$$

we have that under the APT model, approximately,

$$z_{t\cdot}^{(i)\prime} \sim N(0, K_{ii}) \tag{43}$$

where K_{ii} is an appropriate time invariant covariance matrix. Extracting the first element therefore, we find

$$z_{to}^{(i)} \sim N(0, K_{oo,i}) , \qquad i = 2, 3, \ldots, 42$$

In general, $z_{to}^{(i)}$ is correlated with $z_{to}^{(j)}$ but their covariance is also time invariant. Thus, let

$$z_{to}^* = \left(z_{to}^{(2)}, z_{to}^{(3)}, \ldots, z_{to}^{(42)}\right)$$

and observe that

$$z_{to}^{*\prime} \sim N(0, Q_o) ,$$

where Q_o is an appropriate time invariant covariance matrix. Clearly we can estimate the mean vector and covariance matrix by[9]

$$\bar{z}_o^{*\prime} = \frac{1}{T} \sum_{t=1}^{T} z_{to}^{*\prime} , \qquad \tilde{Q}_o = \frac{1}{T} \sum_{t=1}^{T} (z_{to}^{*\prime} - \bar{z}_{to}^{*\prime})(z_{to}^* - \bar{z}_o^*) \tag{44}$$

and employ the test statistic

$$T \bar{z}_o^* \tilde{Q}_o^{-1} \bar{z}_o^{*\prime} \sim \chi_{41}^2 \tag{45}$$

to test the hypothesis that the intercepts of the 42 groups are equal.

In this instance, the test statistic turns out to be

$$T\bar{z}^*_o \tilde{Q}^*_o \bar{z}^*_o = 34.4$$

and thus the hyopthesis is accepted. This accords with the results of RR who interpret this finding as an endorsement of the APT model.

Now acceptance of such a hypothesis while confirming an implication of the APT model does not tell us very much; for example, this hypothesis would be accepted even if all or nearly all intercepts were zero. Such a situation would cast some doubts on the usefulness of the APT. Thus, next we tested the hypothesis that all intercepts are zero. This is done through the statistic

$$T\bar{c}^*_o \tilde{W}_o^{-1} \bar{c}^{*\prime}_o \sim \chi^2_{42} , \tag{46}$$

where

$$\bar{c}^*_o = \left(\bar{c}^{(1)}_o, \bar{c}^{(2)}_o, \ldots, \bar{c}^{(42)}_o\right) , \qquad \bar{c}^{(i)}_o = \frac{1}{T} \sum_{t=1}^{T} \tilde{c}^{(i)}_{to} \tag{47}$$

$$\tilde{W}_o = \frac{1}{T} \sum_{t=1}^{T} (\tilde{c}^*_{to} - \bar{c}^*_o)'(\tilde{c}^*_{to} - \bar{c}^*_o) , \qquad \tilde{c}^*_{to} = (\tilde{c}^{(1)}_{to}, \tilde{c}^{(2)}_{to}, \ldots, \tilde{c}^{(42)}_{to}) \tag{48}$$

The test statistic in this case is

$$T\bar{c}^*_o \tilde{W}_o^{-1} \bar{c}^{*\prime}_o = 67.8$$

and thus the hyothesis is rejected.[10]

However, rejection of such a hypothesis only means that there is at least one coefficient which can be said to be non-zero. To clarify this issue, we examine Table 5, which gives the mean intercepts and the corresponding "t-ratios" in the 42 groups. Even a casual perusal of the table shows that at conventional levels only 13 of the intercepts can be said to be non-zero with a bilateral test. Using a unilateral test we find 24 "significant" intercepts. Thus, we are not really violating the meaning of the empirical evidence if we state that at best (from the point of view of the APT model) the evidence is ambiguous and at worst that one of the implications of the model is contradicted by the empirical evidence. The (mean) risk free rate computed from the 7th root of weekly Treasury Bill rates is clearly positive and its standard deviation does not support the hypothesis of a zero daily risk free rate. In fact, the mean weekly rate on Treasury Bills over this period is .00084166 with standard deivation .0002344, which is clearly significantly different from zero. The associated mean daily rate computed as the 7th root of one plus the weekly rate minus one is .0001204 and is also, evidently, significantly different from zero. In this connection we should point out that applying the same tests for the equality of constant terms in a one factor model yields the statistic

$$\chi^2_{41} = 43.30$$

which similarly implies acceptance. The same conclusion would be reached if we used a two, three or four factor model.

Thus, even in the RR context, one important implication fails to hold with any degree of firmness. But the RR procedure is only one of many possible ways of testing the implications of the APT model.

In particular, given the RR rationale, there is no more reason why securities should be arranged alphabetically, rather than in some other way. We have therefore arranged securities by increasing mean returns and following RR we split the sample into 42 groups of 30 securities. The results of fitting a five-factor model are given in Table 6, which gives the means of the vector estimates, $\tilde{c}^{(i)}_{t\cdot}$, obtained for each t by generalized least squares. For the low mean return groups (say 1-14) the mean intercept is significant only twice at the 10% level, and twice at the 5% level. For the remaining groups, the results are overwhelmingly significant. Most importantly, we reject the hypothesis that the intercepts of the 42 groups are equal when securities are arranged by increasing mean returns. In this case, the test statistic shown in (45) is 71.49 and this clearly indicates the rejection of the hypothesis.

On the other hand, the risk-free rate interpretation of the intercept is also rather far-fetched for these groups. Thus the mean intercept in groups 15-42 ranges from 5 to about 16 times the actual daily Treasury Bill rate.

We are aware that certain biases may arise because of the way in which a sample is ordered.[11] On the other hand, we wish to demonstrate forcefully the point made earlier, viz., that there is no economic rationale for splitting the "universe" into 42 groups; alphabetical partitioning is as damaging to the econometric integrity of the results as any other arrangement. Ranking by mean returns makes the point very forcefully in a strikingly obvious manner. We also wish to point out that if we dealt simultaneously with <u>all</u> 1260 securities how we arranged observations (securities) within the sample is quite irrelevant. Thus, the RR methodology introduces an irrelevant factor (the partitioning of the universe) slight variations of which produce dramatic changes in the conclusion.

7. How Well Does the APT Model Explain Daily Returns?

In dealing with complex estimation procedures like those entailed by the APT model it is not straightforward to determine just what is the explanatory power of the model or alternatively what is a measure of the "goodness of fit." We have chosen to measure "goodness of fit" by the (mean) square of the correlation coefficient between "predicted" and actual rates of return within the sample period for each group. For more details on why this is a useful measure see Dhrymes (1978), chapter 2. We shall first give an account of the procedure and then discuss our findings.

We designate the estimator of the vector of coefficients, $c_{t\cdot}$, (involving the "riskfree" rate and risk premia) by

$$\tilde{c}^{(i)'}_{t\cdot} = (\tilde{B}^*_i \Psi^{-1}_{ii} \tilde{B}^{*'}_i)^{-1} \tilde{B}^*_i \tilde{\Psi}^{-1}_{ii} r^{(i)'}_{t\cdot} , \qquad \begin{array}{l} i = 1, 2, \ldots, 42 \\ t = 1, 2, \ldots, T \end{array} \tag{49}$$

within each "cross section," or group of 30 securities. Thus, we have a collection of T estimates for such coefficients for each group. Owing to the fact that generalized least squares procedures are employed in the estimators of (49), the usual R^2 is not very useful or meaningful.

Thus, we use the estimates in (49) to "predict" rates of return, by

$$\tilde{r}^{(i)'}_{t\cdot} = \tilde{B}^{*'}_i (\tilde{B}^*_i \tilde{\Psi}^{-1}_{ii} \tilde{B}^{*'}_i)^{-1} \tilde{B}^*_i \tilde{\Psi}^{-1}_{ii} r^{(i)'}_{t\cdot} = A_i r^{(i)'}_{t\cdot} \qquad \begin{array}{l} i = 1, 2, \ldots, 42 \\ t = 1, 2, \ldots, T \end{array} \tag{50}$$

Then we compute (over T observations) the square of the correlation coefficients between the actual and predicted values within each group, i.e., we compute

$$R^2_{ij} = \mathrm{Corr}\left[r^{(i)}_{tj}, \tilde{r}^{(i)}_{tj}\right] \qquad j = 1, 2, \ldots, 30$$

The statistics given in Table 6 refer to

$$R^{*2}_i = (1/30) \sum_{j=1}^{30} R^2_{ij} \tag{51}$$

Consequently, what we have in Table 6 are statistics that give a measure of the mean explanatory power or goodness of fit for the 30 securities in the i^{th} group. We have done this in the case where we have used only one factor and where we have used five factors.

Several observations are in order. First, clearly at least one of the remaining four factors contributes importantly to the explanation of returns in the context of the APT model. Thus, the typical R^{*2}_i for the five factor case is about twice the corresponding R_i^{*2} in the one factor case. Second, the typical correlation is of the order of .3 in the case of five factors and about .15 in the case of one factor.

The important conclusion from this is that even though factors (factor loadings) are not reliably estimated, still the evidence is very strong that there is more than one factor, though such factors may differ among groups. We have not attempted to obtain similar statistics with respect to the case of only two, three or four factors due to the cost entailed by such calculations.

8. Additional Tests of the APT Model: Do Five Factors Exhaust the Explanation of the Mean (Expected) Return Process? Do They Contribute Anything at All?

The results that help us answer these questions appear in Tables 4, 5 and 7 where we give the test statistics for testing the hypothesis that, in the standard five factor model employed in the RR context,

$$c_{t\cdot}^{*\prime} = 0 \tag{52}$$

The relevant statistics for this appear in column 8 of Table 4; the results show that in only 6 (30-security) groups is the risk premium vector "significantly" different from zero (3 at the 5% level and 3 at the 10% level). Thus, in the RR context the evidence of Table 4 suggests a very substantial failure for one of the crucial implications of the APT model. In the paper by RR there is no counterpart so a comparison in not possible. On the other hand in Table 7, column 12, we give the statistics for testing the hypothesis that the vector of risk premia is null in a formulation that includes extraneous variables, such as the standard deviation and skewness of individual returns. Our results show that in this case only in two groups do we find "significant" premia. This is in contrast to RR (Table IV, p. 1094) who report "significant" risk premia vectors in 12 groups (28.6%). From Table

5, column 8, we also see that when securities are arranged by mean returns in <u>no</u> group do we find significant risk premia vectors.

Thus, just how much explanation for asset returns is afforded by the APT model in the RR context is questionable and certainly at variance with the results they present. In part, the difference may be accounted for by the fact that for, say, the i^{th} group we take the "cross sectional" GLS regression estimator

$$\tilde{c}_t^{*(i)'}: \quad t = 1, 2, \ldots, T$$

to be a sequence of observations from a normal population with constant mean vector and covariance matrix. For simplicity, we shall omit the group superscript in the discussion below. We thus compute

$$\bar{c}^* = \frac{1}{T} \sum_{t=1}^{T} \tilde{c}_t^*$$

$$W = \frac{1}{T} \sum_{t=1}^{T} (\tilde{c}_{t\cdot}^* - \bar{c}^*)'(\tilde{c}_{t\cdot}^* - \bar{c}^*)$$

and the test statistic

$$T\bar{c}^* W^{-1} \bar{c}^{*\prime} \sim \chi_5^2$$

which is asymptotically chi-square with 5 degrees of freedom. RR, by contrast, use the statistic[12]

$$T\bar{c}^* \bar{W}^{-1} \bar{c}^{*\prime} \sim \chi_5^2$$

and

$$\bar{W}^{-1} = \tilde{B}_i \tilde{\Psi}_{ii}^{-1} [\tilde{\Psi}_{ii} - \phi_i ee'] \tilde{\Psi}_{ii}^{-1} \tilde{B}_i'$$

where $\phi_i = 1/(e'\Psi_{ii}^{-1}e)$. In the RR approach one relies heavily on the "truth" of one's assertions relating to the distributional aspects of the cross sectional GLS estimated coefficients. The approach we employ is more robust to departures from the assumptions underlying the RR procedures.

Turning now to the question as to whether the use of the (five) factor model exhausts the "explanation" of the factor return process, the relevant results are in Table 7 columns 10, 11, 12 and 13. In column 12 we give test statistics for the test of the null hypothesis in (52), when the factor premia are estimated in conjunction with other extraneous variables' coefficients--in this instance standard deviation and skewness of own returns. In only two (out of forty-two) groups is the null hypothesis of zero risk premia rejected at conventional significance levels. Incidentally, this result is at variance with a commonly held view among applied econometricians that with a sufficiently large sample size any (point) null hypothesis is rejected.

Interestingly enough, column 13 of Table 7, which gives the test statistics for the test that the coefficients of standard deviation and skewness are zero, shows that the null hypothesis is rejected at the five percent level in three cases (groups) and at the ten percent level in five (additional) cases (groups). Thus, in thirty-four out of forty-two groups standard deviation and skewness cannot be said to have any perceptible influence on the return generating process.

Overall, the implications of the test results reviewed in this section are not very favorable to the APT model, at least in the RR methodological context. Our results, also, do not fully accord with those of RR; in testing for extraneous variables, however, they have not used a sample of contiguous days. This is not likely to explain the difference in our results but further investigation of this issue may be warranted.

9. Conclusions

In this paper, we sought a reassessment of the APT model by methods more extensive than those employed by RR. Our findings may be summarized as follows. First, it is not clear how we can answer definitively the question of how many factors there are; RR claims there are three to five factors, a position adopted by much of the literature on the subject. Our results show most emphatically that how many factors researchers "find" depends on the "universe" of securities they investigate. Previous research, including RR, treat the groups of thirty securities each, which form the basic unit of investigation, as a "cross-section." The logic of the APT model and its empirical implementation, however, demand that we deal with the "universe" to which the APT model is held to apply. Thus, if we properly take the universe to be the 1260 firms listed on the American and NYSE, then the RR methodology is in grave error, since by increasing the number of securities in each group we get an increasing number of "significant" factors. Software technology allows now the handling of as many as 240 securities but the cost of the exercise prevented us from systematically investigating the issue of whether the number of factors "stabilizes" at, say, twelve or fifteen irrespective of the number of securities in the group. To proceed as RR do requires certain constraints on the basic covariance matrix which are empirically contradicted by the evidence of the RR sample. This basic issue casts some doubt on the testability of the APT by proper econometric procedures given the present state of the literature and computer technology.

Second, even setting aside this very basic objection, we adopted the RR methodology and sought to test another implication of the model, viz., that the intercept terms $c_{to}^{(i)}$ are, on average, the same in all groups. This implication is not rejected by the empirical evidence; on the other hand, the same evidence suggests that on average $c_{to}^{(i)}$ is insignificantly different from zero for nearly all groups. This, of course, runs contrary to the interpretation the APT model places on this coefficient. Moreover, the application of the RR procedure to a different grouping of the 1260 firms universe into sub-groups points to substantially different intercepts (and hence estimates of the risk-free rate) across sub-groups and to unreasonably high estimates of the risk-free rate.

Third, in terms of explanatory power, clearly a one-factor version of the APT model is distinctly inferior to a five-factor version--within the sample period--although, on the whole, the explanatory power of the model is modest.

Fourth, notwithstanding the point just made, proper testing procedures cast some doubt on whether risk premia for the five risk factors introduced by RR are significantly different from zero. This finding would tend to reduce the appeal of APT as an "explanation" of the asset return generating process.

Fifth, when (own) standard deviation and skewness are introduced into the asset return function, while generally yielding insignificant coefficients, they turn out to be "significant" at least as frequently as the factors suggested by RR.

Finally, neither RR nor we have tested the predictive ability of the factor coefficients (i.e., betas) of the APT model beyond the sample period.

Thus, the evidence on the usefulness of this model is at best mixed and further work is needed to probe more deeply into its implications including the relative time stationarity of the empirical results. In a subsequent paper, we shall examine a number of such issues.

Footnotes

[1]Individuals own somewhere between one-half and two-thirds of all New York Stock Exchange stock (and a higher proportion of other stock).

[2]Roll's evidence (1977) against the reliability of the market proxies used in testing the CAPM consists of a demonstration that it is possible to construct a market proxy that supports the Sharpe-Lintner model even though this proxy has a .895 correlation with the market proxy used in one well-known test which resulted in a rejection of that model. A paper by Friend, Westerfield and Ferreira (1980) suggests that the computed tangent portfolio used by Roll, representing some unknown combination of assets, may be replete with short positions, which would hardly qualify as a legitimate approximation of a market portfolio.

[3]In an earlier version of their paper they present the results of the cross-sectional analysis of returns regressed on factor loadings, residual variance and skewness. That working-paper shows a significant effect of residual variance on asset returns even after skewness as well as factor covariance are held constant--a result consistent with that obtained in the Friend-Westerfield paper referred to earlier. Thus, it is difficult to understand the concluding rationalization in their published paper for not presenting these results, viz., "such methods would be biased against finding a true effect of the standard deviation, if one exists."

[4]Note that if $\mathrm{Cov}(f'_{t\cdot}) = \Phi$, $\Phi > 0$ otherwise arbitrary, then $f_{t\cdot}B$ is indistinguishable from $f^*_{t\cdot}B^*$ where, for arbitrary nonsingular C,

$$f^*_{t\cdot} = f_{t\cdot}C, \qquad B^* = C^{-1}B \ .$$

[5]See Huberman (1981) for a precise definition of arbitrage and a concise yet elegant exposition of the APT model. Also see Connor (1981) for conditions or economies under which equation (10) becomes an exact equality rather than an approximation.

[6]Factor analytic methods have been frequently used in the finance literature. In fact, the dependency between number of factors and number of securities was noted by Meyer (1973) without a rigorous explanation. See Elton and Gruber (1981), Chapter 6, for a good summary of the applications of factor analysis in finance and for an extensive literature survey.

[7]To be precise, let ξ be the test statistic, which is chi-squared with r degrees of freedom. If the level of significance is 10%, then the acceptance region is defined by

$$\text{Probability } (\xi \leq t_{.10}) = .9$$

where $t_{.10}$ is the boundary of the acceptance region defined by the specified level of significance. Hence

$$\text{Probability } (\xi > t_{.10}) = .1$$

both statements under H_o. The p-value that is associated with a given statistic s is then

$$\text{Probability } (\xi \geq s \mid H_o) = \text{p-value}$$

Hence, in order to "accept" a hypothesis at, say, the 10% level of significance, the test statistic obtained must have an associated p-value of at least .1. Of course, if the associated p-value is greater than .1, it may be the case that the hypothesis accepted contains "redundant" factors. For example, in Table 3 and for the case of 15 securities, the p-value associated with one factor is .0023, hence the one factor model should be rejected at the 5% significance level; the p-value for the two factor model is .4140, which means that this should be accepted; the p-value for the three factor model is .7676, which is also to be "accepted." On the other hand, this really contains one "redundant" factor.

[8]Under certain conditions, Ingersoll (1982) argues that the intercept in the APT could be a "zero beta" asset even though a risk-free asset exists. However, this would seem to imply that the market does price risk other than common or factor risk and that arbitrage pricing theory, unlike the CAPM, cannot explain the basic risk premium between risky and risk-free assets.

[9]Note that the mean of the daily coefficients $c_t^{(i)}$ can easily be estimated using mean returns data. If one strictly relies on the model and assumes time stationarity, one can also obtain the covariance matrix of the estimators from mean returns. We have consistently chosen to work with the daily coefficients, however, since this represents a procedure that is more robust to departures from stationarity.

[10]Since most readers are most familiar with the normal distribution, one may use the normal approximation employed earlier. This would yield in the present instance

$$(67.8 - 42)/\sqrt{84} \approx 2.8$$

while in the previous case we have

$$(34.4 - 41)/\sqrt{82} \approx -.72 \; .$$

Thus, in the first case we reject and in the second case we accept at the 10% significance level.

[11]It should be noted, however, that even for the intermediate groups of securities, with mean returns close to the average for all 1260 securities in the "universe," the intercepts were many times the Treasury Bill rate.

[12]This is appropriate when there are no extraneous variables; when there are, simply replace in the bracketted expression

$$\tilde{\Psi}_{ii} - \phi_i ee'$$

by

$$\tilde{\Psi}_{ii} - P^{*\prime}(P^* \Psi_{ii}^{-1} P^{*\prime})^{-1} P^*$$

where

$$P^* = \begin{bmatrix} e' \\ P \end{bmatrix}$$

and P is the matrix of observations on the extraneous variables.

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Table 1

CHI-SQUARE TEST OF THE HYPOTHESIS THAT k-FACTORS GENERATE DAILY SECURITY RETURNS

GROUP	k=1	k=2	k=3	k=4	k=5
1	572.35 (.0001)	435.41 (.0185)	372.63 (.1742)	318.38 (.5309)	269.37 (.8554)
2	625.20 (.0001)	533.30 (.0001)	431.20 (.0016)	353.49 (.1025)	293.56 (.5128)
3	600.66 (.0001)	478.63 (.0003)	412.96 (.0094)	358.90 (.0712)	311.20 (.2474)
4	533.87 (.0001)	463.37 (.0014)	389.74 (.0608)	328.31 (.3773)	282.68 (.6869)
5	572.90 (.0001)	447.90 (.0063)	394.82 (.0422)	342.01 (.2010)	286.17 (.6329)
6	873.95 (.0001)	659.04 (.0001)	452.08 (.0001)	362.26 (.0560)	309.89 (.2643)
7	680.86 (.0001)	508.27 (.0001)	422.93 (.0036)	348.93 (.1361)	288.34 (.5982)
8	618.97 (.0001)	507.65 (.0001)	438.51 (.0007)	389.00 (.0055)	341.08 (.0334)
9	769.93 (.0001)	554.97 (.0001)	458.27 (.0001)	378.64 (.0147)	306.71 (.3074)
10	566.94 (.0001)	470.16 (.0007)	376.42 (.1413)	308.53 (.6816)	266.05 (.8860)
11	611.09 (.0001)	472.36 (.0005)	399.48 (.0295)	333.67 (.3016)	280.32 (.7214)
12	1165.33 (.0001)	527.06 (.0001)	441.88 (.0005)	385.00 (.0082)	328.17 (.0894)
13	627.48 (.0001)	499.06 (.0001)	433.19 (.0013)	367.79 (.0368)	318.79 (.1632)
14	575.20 (.0001)	438.92 (.0139)	380.80 (.1091)	340.26 (.2202)	296.66 (.4619)
15	1142.34 (.0001)	542.31 (.0001)	460.81 (.0001)	401.62 (.0015)	343.24 (.0278)

Table 1 (continued)

GROUP	k=1	k=2	k=3	k=4	k=5
16	570.17 (.0001)	455.25 (.0031)	364.93 (.2556)	312.95 (.6157)	273.87 (.8062)
17	582.28 (.0001)	442.09 (.0106)	376.73 (.1389)	331.12 (.3367)	290.97 (.5554)
18	973.38 (.0001)	469.26 (.0007)	405.56 (.0180)	340.86 (.2135)	293.93 (.5066)
19	604.29 (.0001)	476.84 (.0003)	409.25 (.0131)	350.44 (.1242)	306.75 (.3068)
20	2285.53 (.0001)	997.33 (.0001)	471.72 (.0001)	368.96 (.0335)	312.70 (.2291)
21	692.12 (.0001)	596.79 (.0001)	502.53 (.0001)	444.12 (.0001)	382.87 (.0004)
22	669.98 (.0001)	535.14 (.0001)	443.69 (.0004)	366.55 (.0405)	320.82 (.1445)
23	555.17 (.0001)	454.92 (.0032)	393.46 (.0466)	346.91 (.1533)	297.66 (.4457)
24	685.53 (.0001)	519.07 (.0001)	444.09 (.0004)	374.13 (.0218)	326.05 (.1032)
25	658.52 (.0001)	538.84 (.0001)	470.52 (.0001)	419.73 (.0002)	367.66 (.0025)
26	547.99 (.0001)	441.93 (.0107)	377.82 (.1304)	315.79 (.5716)	277.64 (.7586)
27	585.45 (.0001)	434.72 (.0196)	373.27 (.1683)	315.01 (.5837)	271.38 (.8345)
28	600.67 (.0001)	467.17 (.0009)	401.94 (.0243)	348.89 (.1364)	299.47 (.4166)
29	649.44 (.0001)	518.44 (.0001)	421.93 (.0040)	351.62 (.1154)	305.54 (.3241)
30	689.98 (.0001)	550.89 (.0001)	478.90 (.0001)	416.40 (.0003)	361.21 (.0051)
31	553.53 (.0001)	466.29 (.0010)	394.49 (.0433)	327.30 (.3923)	288.28 (.5992)

Table 2

TESTS OF THE GOODNESS OF FIT OF THE FACTOR MODEL FOR DAILY SECURITY RETURNS USING VARYING GROUP SIZES - CHI SQUARE TEST

Number of Factors	Number of Securities in a Group 15	30	45	60	90	240
1	132.6 (.0023)	572.4 (.0001)	1246.4 (.0001)	2318.7 (.0001)	5548.9 (.0001)	
2	78.0 (.4140)	435.4 (.0189)	1065.1 (.0001)	2057.9 (.0001)	4986.59 (.0001)	
3	54.5 (.7676)	372.6 (.1742)	958.41 (.0094)	1845.6 (.0001)	4501.7 (.0001)	
4	37.7 (.9165)	318.4 (.5309)	858.6 (.1461)	1697.4 (.0023)	4190.5 (.0001)	
5	28.5[a] (.9132)	269.4 (.6554)	776.5 (.4785)	1603.3 (.0133)	3962.8 (.0001)	30756.2[b] (.0001)
6		230.3 (.9617)	711.2 (.7290)	1502.1 (.0762)	3776.7 (.0003)	
7		199.3 (.9869)	658.9 (.8403)	1409.4 (.2301)	3606.3 (.0061)	
8		165.6 (.9985)	610.6 (.9067)	1320.7 (.4735)	3460.0 (.0369)	
9		139.7 (.9999)	558.3 (.9660)	1247.4 (.6377)	3321.7 (.1299)	
10		117.4 (.9999)	570.4 (.9882)	1159.88 (.8691)	3188.9 (.3091)	

Figures in the first line are the chi-squared values. Figures in the parentheses indicate the p-value associated with the statistic, i.e., the probability that the test statistic (under the null hypothesis) will assume a value at least as large as the statistic obtained in this particular test. Degrees of freedom can be computed as $\frac{1}{2}[(n-k)^2 - n - k]$, where n is the number of securities in the group and k is the number of factors. Only five factors are estimated for the group of 240 securities.

[a]It is not possible to extract more than 5 factors.

[b]Only five factors are estimated due to accelerating computer costs.

Table 3 (continued)

SECURITY	FACTOR 1		FACTOR 2		FACTOR 3		FACTOR 4		FACTOR 5	
	(a)	(b)	(a)	(b)	(a)	(b)	(a)	(b)	(a)	(b)
31	0.21205	0.21198	0.07090	0.01332	0.06219	0.00521	-0.02153	-0.01223	0.00524	0.05128
32	0.23807	0.25915	0.02371	0.07730	0.03052	0.16568	0.05748	-0.13780	0.01272	-0.07643
33	0.26478	0.29668	0.00696	0.18634	0.10602	-0.05850	-0.06638	-0.18642	-0.12960	0.04983
34	0.28912	0.31687	-0.03304	0.14878	0.10592	0.03991	-0.07752	-0.09832	-0.09863	0.02598
35	0.23751	0.27244	0.04911	0.09158	0.04311	0.16671	0.04915	0.13117	0.07946	0.04873
36	0.26254	0.27919	0.03167	0.02980	0.00398	0.05356	0.09937	-0.02001	-0.01669	0.05039
37	0.41241	0.40702	-0.36521	0.08224	-0.00582	-0.08784	0.07546	0.03160	-0.04735	0.06807
38	0.30784	0.29731	0.03028	-0.01583	0.08926	0.01638	-0.03049	0.01053	-0.00577	0.07849
39	0.32584	0.34266	0.02310	0.03574	0.00772	-0.21601	-0.07560	0.00520	-0.09633	0.00079
40	0.23206	0.26257	0.01510	0.11503	0.09286	0.09603	0.02584	0.03050	0.07580	-0.00539
41	0.21193	0.23441	-0.00166	-0.04631	0.07918	0.04014	0.10253	0.03320	0.00698	-0.13417
42	0.34305	0.37291	-0.01505	-0.00766	0.06114	0.07163	0.02607	-0.06447	0.04243	0.11717
43	0.24401	0.24909	0.04142	0.00853	-0.02423	-0.01114	0.13276	0.20230	-0.11286	0.04653
44	0.39762	0.44041	0.05887	0.04047	0.02390	-0.07102	0.04315	0.05332	-0.01232	-0.08036
45	0.28283	0.30192	0.03636	0.09511	0.11662	-0.03135	-0.11473	0.07821	-0.02905	0.06770
46	0.13576	0.12319	-0.03679	0.05422	0.03815	0.01119	0.09924	0.01295	-0.00831	-0.02105
47	0.27812	0.27343	0.01773	-0.06769	-0.00982	0.10950	0.08100	0.05530	0.02230	-0.05773
48	0.16851	0.19681	0.02790	0.02431	0.03237	0.12862	0.04847	0.02383	0.03735	-0.03807
49	0.22462	0.22965	0.00639	0.05360	-0.08456	0.02845	-0.08162	-0.09090	-0.07664	0.07176
50	0.35168	0.39336	-0.00676	0.13158	0.03084	-0.18638	0.06484	0.07740	-0.16676	-0.09556
51	0.19512	0.21469	0.01553	-0.02108	-0.06556	0.15959	-0.00852	0.03851	0.12456	-0.09492
52	0.20396	0.24635	0.03239	-0.03913	0.07650	0.09956	0.10694	-0.07547	0.00595	0.01579
53	0.50936	0.51384	0.05390	-0.20767	-0.08883	-0.04232	0.06991	0.05945	-0.02376	0.06582
54	0.51235	0.51515	0.07911	-0.21992	-0.17779	-0.01904	0.07804	-0.07762	-0.02102	0.03670
55	0.14234	0.14788	-0.02987	-0.03820	0.05303	0.03175	0.07368	-0.01377	-0.01225	0.11177
56	0.40374	0.39379	0.01991	-0.05442	-0.00689	-0.07360	-0.04476	-0.11685	0.00627	-0.02960
57	0.32436	0.30144	0.04618	-0.05421	-0.02671	-0.01982	0.00055	0.02976	-0.02102	-0.14782
58	0.40243	0.45506	0.02783	-0.02162	0.07134	0.01922	-0.00680	-0.07177	0.04908	-0.16761
59	0.22556	0.24618	0.02527	-0.02865	0.05926	0.02563	0.08455	0.03755	0.00653	0.10351
60	0.22458	0.28366	0.03544	0.08757	0.12825	0.07710	0.07703	0.16824	0.03885	0.01669

Column (a) contains the estimates of factor loadings for the kth factor k = 1, 2, ..., 5, from a group of 240 securities. Column (b) contains the estimates of the corresponding factors from groups of 30 securities, i.e., first 30 rows are the estimates from the first group, second 30 rows are the estimates from the second group, etc. Only the estimates of the factor loadings for the first sixty securities are shown here.

Table 4: TESTS OF SIGNIFICANCE FOR INTERCEPTS AND RISK PREMIA FOR 5-FACTOR MODEL USING ALPHABETICALLY RANKED GROUPS
7/12/1962 - 12/31/1972

Group	Statistic $\bar{c}_0$ a/ (1)	$\bar{c}_1$ (2)	$\bar{c}_2$ (3)	$\bar{c}_3$ (4)	$\bar{c}_4$ (5)	$\bar{c}_5$ (6)	$t(c_0)$ b/ (7)	χ^2 c/ (8)	Number of Observations (9)
1	.00032	.02906	-.00777	-.01885	.01954	.03965	1.56	1.987	2468
2	.000339	.01409	.02373	.04012	.01232	-.01387	1.44	2.551	2571
3	.00015	.02410	.02278	.03118	.07034	.04462	.86	4.594	2568
4	.00033	.04338	.00014	.02896	.08402	-.07534	1.30	9.869†	2575
5	.00022	.05741	.05473	.04038	.01538	-.00236	.76	6.122	2554
6	.00046**	.01674	.02821	-.01043	-.00619	-.07789	2.79	4.169	2405
7	.00023	.06399	.04183	-.03783	-.04399	-.01042	.91	8.110	2540
8	.00043*	.03022	-.04700	-.03291	.01554	-.03264	1.77	4.433	2553
9	.00043**	.02555	-.01606	.02111	.04431	.04152	2.21	3.935	2559
10	.00017	.07112	.03367	.04216	-.04987	-.00789	.70	8.859	2540
11	.00030*	.04828	-.01387	.01289	-.03399	.00652	1.67	4.096	2512
12	.00022	.01358	.07201	-.04364	-.00894	.06428	.88	11.959††	2575
13	.000345	-.01067	.04991	-.03986	.02182	-.03180	1.46	4.471	2536
14	.00015	.09109	-.04646	-.01407	.02251	.02635	.91	8.929	2547
15	.00017	.00330	.06522	-.03195	.00294	.00603	.58	4.329	2560
16	.00060**	-.00023	.09049	-.00043	-.02093	-.00129	2.06	5.970	2565
17	.00041	.01352	.00119	.03840	.00874	.04897	1.60	2.260	2547
18	-.00011	.01109	.15558	.01691	.05566	-.08433	-.29	10.737†	2446
19	.00033	.04910	.01900	.01450	-.00472	-.02273	1.53	2.517	2562
20	.00024	.03398	.04584	.05198	.03330	-.01970	1.15	4.963	2527
21	.00041**	.03379	-.00245	.03490	.00535	-.04920	2.20	3.588	2506
22	.00067**	-.01277	-.07062	.10889	-.08044	.06059	2.41	9.928†	2559
23	.00004	.01680	.10448	.04679	-.06551	-.08662	1.61	14.250††	2504
24	.00027	.02569	.01923	.02261	.01928	-.0609	.15	3.426	2575
25	.00018	.04813	.01171	-.04854	.02135	-.01703	.86	3.202	2563
26	.00056**	.03205	-.07018	.01329	.06546	.02552	2.48	7.448	2561
27	.00014	.06678	-.00318	-.01892	.01963	-.03929	.72	5.830	2510
28	.00046**	.03424	-.00961	-.00822	.00370	-.07811	2.51	6.057	2561
29	-.00014	.12947	-.03723	.06784	.00819	.03151	-.50	12.022††	2544
30	.00008	.06412	-.00678	.02018	.00510	.02000	.42	3.846	2545

Table 4 (continued)

Group	Statistic $\bar{c}_o$ a/ (1)	$\bar{c}_1$ (2)	$\bar{c}_2$ (3)	$\bar{c}_3$ (4)	$\bar{c}_4$ (5)	$\bar{c}_5$ (6)	$t(c_o)$ b/ (7)	χ^2 c/ (8)	Number of Observations (9)
31	.00037*	.03452	.02932	-.02035	-.02035	-.07711	1.67	6.575	2557
32	.00015	.08805	-.02079	.03852	-.03136	-.01862	.85	7.339	2492
33	.00031	.03286	-.00259	.02204	.01301	.00448	1.53	.855	2574
34	.00064**	.00090	-.01326	.05227	.00233	-.01953	3.04	1.976	2511
35	.00028	.06050	.00472	-.03068	.02146	-.09301	1.38	7.608	2546
36	.00037*	.01042	.03715	-.02369	-.04576	-.04488	1.85	3.265	2555
37	.00027	.04540	.05001	-.02108	-.06003	.04478	1.01	2.959	2076
38	.00012	.07880	.03941	-.01249	.01153	-.02106	.68	6.797	2563
39	.00011	.07281	-.00797	-.02980	-.07269	.06763	.38	7.304	1917
40	.00044**	.02028	-.01038	.01125	.00507	.06808	2.81	3.444	2508
41	.00032	.02727	.03385	.01141	.06001	.02950	1.61	3.476	2542
42	.00027	.01926	.04634	-.00777	.05096	-.04919	1.50	5.401	2546

a/ $\bar{c}_o$ through $\bar{c}_5$ are the arithmetic means of the daily cross sectional regression estimates using the GLS model $c'_{t\cdot} = (\tilde{B}^* \Psi^{-1} \tilde{B}^{*\prime})^{-1} \tilde{B}^* \Psi^{-1} r'_{t\cdot}$.

b/ $t(c_o)$ is the "t-ratio" for the intercept term. "t-ratio" is given by $\sqrt{T}(\bar{c}_o/s_o)$, where $\bar{c}_o = \frac{1}{T}\sum_{t=1}^{T} \tilde{c}_{to}$ and $s_o = [\frac{1}{T}\sum_{t=1}^{T} (\tilde{c}_{to} - \bar{c}_o)^2]^{1/2}$. * and ** indicate intercept terms which are significantly different from zero at 10 and 5 percent levels respectively. In case of a "t-test," this refers to a bilateral test; if a unilateral test is desired the critical points are 1.65 and 1.31 respectively for the 5 and 10 percent level of significance tests.

c/ χ^2 is the test statistic to test the null hypothesis that none of the risk premia is priced. The test statistic is distributed as chi-square with 5 degrees of freedom. The critical values for chi-square distribution with 5 degrees of freedom at 10 and 5 percent significance levels are 9.24 and 11.10 respectively. † and †† indicate groups for which the null hypothesis is rejected at 10 and 5 percent levels of significance respectively.

Table 5: TESTS OF SIGNIFICANCE FOR INTERCEPTS AND RISK PREMIA FOR 5-FACTOR MODEL USING GROUPS RANKED BY THE MEAN RETURNS OF SECURITIES 7/12/1962 - 12/31/1972

Group	Statistic $\bar{c}_o$ a/ (1)	$\bar{c}_1$ (2)	$\bar{c}_2$ (3)	$\bar{c}_3$ (4)	$\bar{c}_4$ (5)	$\bar{c}_5$ (6)	$t(c_o)$ b/ (7)	χ^2 c/ (8)	Number of Observations (9)
1	.00003	.00018	.01508	-.00860	-.00023	-.00351	.16	.194	2569
2	.00015	.00197	.00253	-.00471	.00453	-.00553	.86	.066	2567
3	.00018	.00369	.00646	-.00017	.00424	.00014	.74	.024	2541
4	.00022	.00479	.00153	.00045	-.00476	.00275	1.42	.041	2570
5	.00027	-.00385	-.00235	.00363	-.00499	-.01502	1.42	.075	2584
6	.00027	.00255	.00244	.00751	.00286	.00215	1.34	.048	2547
7	.00031	.00402	-.00123	-.00105	.00264	-.00623	1.75	.062	2574
8	.00028	.00851	-.00307	.00032	.01259	-.00914	1.37	.165	2572
9	.00037*	-.01111	-.02361	.00358	.00859	.01328	1.94	.298	2537
10	.00037	-.00280	.01031	.00528	.00654	.01193	1.59	.208	2565
11	.00039**	-.00359	.00055	-.00199	.00629	-.00607	2.35	.077	2563
12	.00039**	.00404	.00643	-.00160	-.00369	.00592	1.92	.053	2566
13	.00040*	.00636	-.00353	.01314	-.00322	.00451	1.86	.114	2554
14	.00048	-.00099	.00245	.00143	.00138	-.00850	2.12	.037	2575
15	.00053	-.00435	-.00382	.01302	-.00092	.00181	3.18	.152	2513
16	.00053**	-.00331	-.00412	.00503	-.00275	.00411	2.86	.046	2525
17	.00042	.01046	.01136	-.00023	-.00083	-.00841	1.53	.174	2512
18	.00050**	-.00548	-.00349	.00116	-.00125	.00649	2.57	.036	2568
19	.000542**	.00308	.00241	.00220	-.00539	-.00560	2.82	.053	2542
20	.00058**	.00719	.01403	-.00160	-.00143	-.00593	2.40	.274	2537
21	.00055**	.00964	.01176	.01253	-.00473	.00305	2.18	.308	2530
22	.00065**	-.00738	.01328	.01281	.00019	.01002	2.94	.335	2450
23	.00065**	.00145	-.01385	.00912	-.01080	-.00176	2.88	.347	2468
24	.00064**	.00216	.00426	.00316	-.00009	-.00608	3.51	.039	2543
25	.00068**	.00064	-.00371	-.00475	-.00188	-.01300	2.94	.104	2550
26	.00065**	.00708	-.00456	-.00420	.00547	.01010	3.45	.166	2568
27	.00068**	.00945	.00397	.00673	.00528	-.00591	2.28	.065	2557
28	.00069**	.00800	-.00207	.00067	.00009	-.01216	2.53	.105	2549
29	.00074**	.01064	.01183	.00085	.00801	-.00869	3.51	.242	2542
30	.00078**	.00397	.00281	.00386	.00082	-.00075	2.86	.052	2568

Table 5 (continued)

Group	Statistic $\bar{c}_o$ a/ (1)	$\bar{c}_1$ (2)	$\bar{c}_2$ (3)	$\bar{c}_3$ (4)	$\bar{c}_4$ (5)	$\bar{c}_5$ (6)	$t(c_o)$ b/ (7)	χ^2 c/ (8)	Number of Observations (9)
31	.00073*	.02549	.01270	.01604	-.01507	.00712	1.87	.303	2449
32	.00082**	.00503	.00441	.00007	-.01417	-.00054	3.11	.170	2536
33	.00088**	.00318	-.00784	.00634	-.00821	.00411	3.35	.133	2428
34	.00093**	.00095	.00456	-.00489	.00222	.00087	3.01	.029	2505
35	.00101**	-.00488	.01373	.01041	.01055	-.01007	4.46	.286	2509
36	.00113**	-.00183	-.00889	-.00487	-.01277	.00392	2.79	.076	2493
37	.00102**	-.00270	.00475	.00059	.00087	-.00667	2.44	.055	2520
38	.00109**	.00394	.00611	.00722	-.00379	.00595	3.02	.060	2559
39	.00137**	.01429	.00822	-.01242	.00241	.00039	3.28	.132	2563
40	.00140**	-.00247	.00480	.00424	.00866	-.02160	3.58	.349	2517
41	.00146**	.00294	.00000	.00025	.00666	.00220	3.93	.041	2573
42	.00163**	.02367	-.01940	.00286	.00771	-.00184	3.53	.807	2516

a/ $\bar{c}_o$ through $\bar{c}_5$ are the arithmetic means of the daily cross sectional regression estimates using the GLS model $c'_{t\cdot} = (\tilde{B}^* \Psi^{-1} \tilde{B}^{*\prime})^{-1} \tilde{B}^* \Psi^{-1} r'_{t\cdot}$.

b/ $t(c_o)$ is the "t-ratio" for the intercept term. "t-ratio" is given by $\sqrt{T}(\bar{c}_o/s_o)$, where $\bar{c}_o = \frac{1}{T}\sum_{t=1}^{T}\tilde{c}_{to}$ and $s_o = [\frac{1}{T}\sum_{t=1}^{T}(\tilde{c}_{to} - \bar{c}_o)^2]^{1/2}$. * and ** indicate intercept terms which are significantly different from zero at 10 and 5 percent levels respectively. In the case of the "t-test", this refers to a bilateral test; if a unilateral test is desired the critical points are 1.65 and 1.31 respectively for the 5 and 10 percent levels of significance tests.

c/ χ^2 is the test statistic to test the hypothesis that none of the risk premia is priced. The test statistic is distributed as chi-square with 5 degrees of freedom. The critical values for chi-square distribution with 5 degrees of freedom at 10 and 5 percent significance levels are 9.24 and 11.10 respectively.

Table 6: SUMMARY STATISTICS FOR THE SQUARED CORRELATION COEFFICIENT BETWEEN REALIZED DAILY RETURNS AND THE FORECASTS BY ONE- AND FIVE-FACTOR MODELS

Group No.	1-Factor Model		5-Factor Model	
	Mean	Std. Dev.	Mean	Std. Dev.
1	.1626	.0933	.3069	.1381
2	.1802	.0859	.3184	.1833
3	.1588	.0809	.2843	.2614
4	.1617	.0952	.3013	.1290
5	.1789	.1048	.3135	.1650
6	.1536	.0681	.2933	.2164
7	.1524	.0794	.2960	.1813
8	.1734	.0695	.3105	.1259
9	.1711	.0925	.3130	.1581
10	.1619	.0957	.2994	.1483
11	.1482	.0811	.2919	.1329
12	.1649	.0816	.3034	.1740
13	.1518	.0988	.2883	.1810
14	.1637	.0791	.3033	.1493
15	.1695	.1236	.3035	.1753
16	.1768	.0913	.3126	.1621
17	.1903	.1076	.2368	.1453
18	.1675	.1023	.2991	.1983
19	.1540	.0905	.2928	.1515
20	.1529	.0774	.2936	.2423
21	.1685	.0814	.3052	.1702
22	.1712	.0828	.3081	.1373
23	.1533	.0852	.2914	.1754
24	.1693	.0911	.3018	.1842
25	.1672	.0793	.3056	.1677
26	.1612	.0690	.3026	.1007
27	.1524	.0897	.2957	.1458
28	.1639	.0829	.2959	.1894
29	.1508	.1005	.2899	.1695
30	.1547	.0791	.2959	.1462
31	.1560	.0824	.2943	.1678
32	.1496	.0749	.2971	.1571
33	.1617	.0841	.2970	.1856
34	.1459	.0859	.2882	.1728
35	.1655	.0872	.3042	.1480
36	.1580	.0866	.2880	.2190
37	.1588	.0973	.2957	.1731
38	.1460	.0684	.2842	.2041
39	.1467	.0822	.3019	.1832
40	.1828	.1088	.3212	.1830
41	.1720	.1050	.3091	.1634
42	.1486	.0825	.2879	.1962

Forecasts of daily returns are estimated by $\tilde{r}'_{t\cdot} = \tilde{B}^{*\prime}(\tilde{B}^{*}\tilde{\Psi}^{-1}\tilde{B}^{*\prime})^{-1}\tilde{B}^{*}\tilde{\Psi}^{-1}r'_{t\cdot}$. See equation (50) for definition of variables. Mean is the average of squared correlations between the forecast, $\tilde{r}_{t\cdot}$, and the realized daily return, $r_{t\cdot}$, for 30 securities in each group. Std. Dev. is the standard deviation of the squared correlation for the 30 securities in each group.

Table 7: SIGNIFICANCE TESTS OF STANDARD DEVIATION AND SKEWNESS OF DAILY SECURITY RETURNS AS AN ALTERNATIVE HYPOTHESIS TO 5-FACTOR APT MODEL
7/12/1962 - 12/31/1972

Group	STATISTIC													
	$\bar{c}_o$ a/	$\bar{c}_1$	$\bar{c}_2$	$\bar{c}_3$	$\bar{c}_4$	$\bar{c}_5$	$\bar{c}_6$	$\bar{c}_7$	$t(c_o)$ b/	$t(c_6)$	$t(c_7)$	χ^2_c c/	$\chi^2_{\sigma,k}$ d/	Number of Observations
	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	(12)	(13)	(14)
1	.00001	-.03783	-.07908	-.01371	-.00420	.01429	.03485	.00010	.04	1.41	.40	2.456	4.192	2468
2	-.00017	.00235	-.00961	.00857	-.05965	-.03645	.03865*	.00009	-.49	1.89	.41	2.178	5.932†	2571
3	-.00013	.03656	.00776	-.01452	.06596	.01809	.02348	.00015	-.56	1.57	1.02	4.332	3.931	2568
4	.00016	-.03912	-.02668	-.02482	.04715	-.05703	.02790*	.00024	.58	1.75	1.32	2.738	5.349†	2575
5	.00009	.00147	.02431	-.00108	.00853	-.06110	.03392	-.00027	.27	1.38	-1.21	2.245	2.663	2554
6	.00046**	.01532	.02302	-.01061	-.00669	-.07785	.00296	-.00003	2.10	.13	-.23	3.513	.054	2405
7	.00007	.03959	.01366	-.04117	-.04679	.01282	.01653	-.00005	.19	.81	-.29	3.954	.686	2540
8	.00035	.02473	-.02681	-.03594	.03054	-.02378	.00187	.00019	1.12	.08	.78	3.429	.783	2553
9	.00049**	.05007	-.00188	.02049	.04386	.04296	-.01380	.00009	2.19	-.83	.88	4.573	1.041	2559
10	.00016	.03880	.01806	.03115	-.06601	-.00243	.01694	-.00019	.57	.87	-1.11	3.969	1.412	2540
11	.00011	.00515	-.03769	.00521	-.03734	-.01791	.02288	.00004	.53	1.47	.31	1.936	3.427	2512
12	.00032	.02547	.10158	-.02315	-.00639	.07869	-.01650	.00000	1.07	-.77	.01	9.546†	.594	2575
13	.00006	-.01529	.04487	.00331	-.00123	.00331	.00935	.00023	.21	.51	1.50	1.005	4.807†	2536
14	.00008	-.07285	-.05287	-.04776	-.03221	.03541	.02288	-.00045	.36	.86	-1.46	6.761	2.251	2547
15	.00017	-.00678	.03384	-.04442	-.00291	-.00194	.01229	-.00007	.58	.61	-.25	2.023	.380	2560
16	.00028	.02331	.05987	.01510	-.01345	-.01167	-.00049	.00028	.67	-.02	1.12	1.745	1.390	2565
17	.00047	.03402	.03268	.05450	.01245	.05608	-.01680	.00022	1.53	-.79	1.33	2.578	2.213	2547
18	-.00073*	-.03615	.05279	-.01240	-.05942	-.12489	.07972**	-.00036	-1.65	2.51	-1.39	8.416	6.311††	2446
19	.00022	.05679	.02509	.02881	-.01251	-.03451	-.00182	.00019	.93	-.09	1.21	2.803	1.578	2562
20	.00004	.02563	.03358	.03710	.01617	-.04285	.01242	.00014	.15	.78	.73	3.753	1.350	2527
21	.00028	.01006	-.02361	.01270	.01619	-.05447	.02188	-.00017	1.29	1.40	-.65	2.393	2.006	2506
22	.00065*	-.01383	-.07108	.10832	-.07522	.05933	.00049	.00004	1.88	.02	.16	6.974	.029	2559
23	-.00005	.00615	.07042	.03423	-.06484	-.08953	.01824	-.00011	-.21	1.08	-.64	8.751	1.272	2504
24	.00015	.00961	.01333	.02624	.00522	-.04448	.00900	.00010	.66	.56	.39	1.323	.751	2575
25	-.00006	.02460	.00649	-.07010	-.01240	-.02746	.01814	.00006	-.19	.70	.28	2.675	.922	2561
26	.00051	.09075	-.06659	-.00845	.10648	-.00803	-.02824	.00051**	1.53	-1.02	2.10	11.349††	4.519	2563
27	.00051	-.02385	-.06570	-.03161	.01895	-.02429	.03354*	-.00001	.32	1.87	-.08	2.605	3.868	2510
28	.00024	.00765	-.07675	-.03080	.00530	-.08602	.03259	.00009	1.07	1.53	.66	5.927	5.926†	2561
29	-.00011	.01278	-.00711	-.01440	.05113	-.01208	.03532*	.00001	-.39	1.69	.05	1.892	3.722	2544
30	-.00016	.03900	-.03632	.03584	-.00207	-.01693	.01691	.00017	-.57	.72	.46	1.090	1.366	2545

Table 7 (continued)

Group	Statistic $\bar{c}_o$ a/	$\bar{c}_1$	$\bar{c}_2$	$\bar{c}_3$	$\bar{c}_4$	$\bar{c}_5$	$\bar{c}_6$	$\bar{c}_7$	$t(c_o)$ b/	$t(c_6)$	$t(c_7)$	χ^2_c c/	$\chi^2_{\sigma,k}$ d/	Number of Observations
	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	(12)	(13)	(14)
31	.00005	-.01467	.00712	-.07650	.02892	-.12455	.02882*	.00021	.20	1.66	1.34	8.555	7.076††	2557
32	.00007	-.05440	.01843	-.00050	-.03638	-.01186	.00961	.00021	.35	.48	1.15	2.980	2.013	2492
33	.00004	-.05363	-.07487	-.00650	.01200	.07720	.03483	.00030	.14	1.47	1.40	3.160	6.073	2574
34	.00029	-.04330	.01769	.00395	.01076	-.02522	.02099	.00022	1.51	1.01	.98	1.089	3.995	2511
35	.00016	.00121	-.02797	-.04815	-.01024	-.12319	.01615	.0026	.55	.62	1.07	8.427	3.541	2546
36	-.00001	.00225	.02570	-.01805	-.02191	-.07575	.02822*	.00004	-.05	1.86	.61	3.042	4.697†	2555
37	.00010	.02402	.04291	-.04103	-.06405	.05175	.01228	.00010	.35	.46	1.15	3.438	3.420	2076
38	-.00002	.02564	.02344	.00550	.02246	-.01185	.01913	-.00014	-.09	1.10	-.76	2.314	1.418	2563
39	.00011	.07328	.00209	-.02705	-.08171	.00608	-.00272	.00010	.28	-.09	.39	7.674	.219	1917
40	.00024	-.00121	-.01832	.00615	.00497	.03024	.01898	-.00001	.98	1.06	-.02	.650	1.267	2508
41	.00020	.00574	.03139	.00415	.05180	.00824	.01689	-.00010	.78	.98	-.55	1.477	1.166	2542
42	.00023	.01782	.03529	-.02173	.04517	-.04707	.00405	.00004	1.21	.18	.19	3.733	.087	2546

a/ $\bar{c}_o$ through $\bar{c}_7$ are the arithmetic means of the daily cross sectional regression estimates using the GLS model $c'_{t\cdot} = (\tilde{B}^{**}\Psi^{-1}\tilde{B})^{-1}\tilde{B}^{**}\Psi^{-1}r'_{t\cdot}$. $B^{**'}$ is $[e:B':\sigma:k]$ which is the augmented matrix of factor loadings with unit vector, (e), standard deviation (σ), and skewness (k) of the securities over the sample period (i.e., for the i^{th} security, $\sigma_i = [\frac{1}{T}\sum_{t=1}^{T}(\tilde{r}_{ti} - \bar{r}_i)^2]^{1/2}$), c_o is the intercept term, c_1 through c_5 are the regression coefficients for the factor loadings, and c_6 and c_7 are the coefficients for the standard deviation and skewness.

b/ $t(c_o)$ is the "t-ratio" for the intercept term and $t(c_6)$ and $t(c_7)$ are the "t-ratios" for the coefficients of the standard deviation and skewness, respectively; the "t-ratio" for the i^{th} coefficient is given by $\sqrt{T}(\bar{c}_i/s_{ii})$, where $\bar{c}_i = \frac{1}{T}\sum_{t=1}^{T}\tilde{c}_{ti}$ and $s_{ii} = [\frac{1}{T}\sum_{t=1}^{T}(\tilde{c}_{ti} - \bar{c}_i)^2]^{1/2}$. * and ** indicate regression coefficients which are significantly different from zero at 10 and 5 percent significance levels respectively. In the case of "t-test" this refers to a bilateral test. If a unilateral test is desired, the critical points are 1.65 and 1.31 respectively for the 5 and 10 percent levels of significance tests.

c/ χ^2_c is the test statistic to test the null hypothesis that none of the risk premia is priced. The test statistic is distributed as chi-square with 5 degrees of freedom. The critical values for chi-square distribution with 5 degrees of feedom at 10 and 5 percent significance levels are 9.24 and 11.10 respectively.

d/ $\chi^2_{\sigma,k}$ is the test statistic to test the null hypothesis that c_6 and c_7 are not different from zero. It is distributed as chi-square with 2 degrees of freedom. The critical values for chi-square distribution with 2 degrees of freedom at 10 and 5 percent significance levels are 4.61 and 5.99 respectively. † and †† indicate the groups for which the null hypothesis is rejected at 10 and 5 percent significance levels respectively.

Data Appendix

The data in our study are almost an exact replica of the data used by RR.[1] Our data are described in Table 1A. Since RR is the first major attempt to test the APT, using the same data should facilitate a comparison of the results. Like RR we first ordered securities alphabetically into groups of 30 individual securities from CRSP Daily Return Files. We were not able to find 13 securities in the original RR data in the 1982 CRSP Daily Return files due to name changes.[2] We furthermore replaced 11 securities which had more than 110 missing observations.[3] This was essential because estimation of the correlation matrix requires simultaneous observations within each group. Furthermore, most of our tests involve joint tests across all of the groups. This requires exclusion of observations in all groups which correspond to missing observations in any one group.

The smallest sample size among all 42 groups was 2424 days out of a possible 2619 days. There were 8 groups with sample size less than 2550 days, and only two with less than 2500 days.

We also ranked securities with respect to mean returns, and then ordered them into groups of 30 individual securities. The first group, in this case, contains the 30 securities with the smallest mean returns over the 3 July 1962 - 31 December 1972 period. And the forty-second group contains the securities with the largest mean returns in the same period. The reasons and

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[1]Richard Roll was kind enough to give us a complete list of companies in the RR study.

[2]The list we obtained only contained company names and ticker symbols.

[3]In the RR sample, there are several securities with more than 800, and one with more than 1400, missing observations. Using these securities would have eliminated much more than half of the observations in joint tests.

the impact of ordering securities in this fasion on empirical results are explained in detail in the text.

The properties of the daily stock returns have been subject to close empirical scrutiny recently. (See, for example, Keim (1982)). Several properties of the daily security returns, however, are worth mentioning here. First, there appears to be a high frequency of positive first order serial correlation. Second, the higher autocorrelations are predominantly negative and tend to be less frequent as lags are increased. A summary of autocorrelations is shown in Table 2A for 163 randomly selected securities.

Finally, another interesting feature of the daily stock returns is that there are only 300 securities among the total sample of 1260 securities with mean returns statistically distinguishable from zero over the sample period. Further description of daily stock returns and a list of companies are available from the authors.

Table 1A

DESCRIPTION OF THE DATA

Source:	Center for Research in Security Prices Graduate School of Business University of Chicago Daily Stock Returns Files
Selection Criteria:	(a) By alphabetical order of 42 groups with the size of 30 individual securities listed on the New York and American Stock Exchanges.
(b)	By first ranking with respect to the mean returns over the period 3 July 1962 - 31 December 1972, then we order securities by their ranked mean returns in group size of 30 individual securities. The first group contains the 30 securities with the smallest mean returns and the forty-second group contains the 30 securities with largest mean returns.
Maximum Sample Size Per Security:	2619 daily returns
Minimum Sample Size Per Security:	2509 daily returns
Number of Selected Securities:	1260

Table 2A

AUTOCORRELATIONS OF THE DAILY RETURNS OF THE RANDOMLY SELECTED 163 SECURITIES 7/3/1962 - 12/31/1972

Lag (Days)	Percent of Significant Autocorrelations* None	Positive	Negative
1	23	48	29
2	54	12	34
3	59	10	31
4	73	10	17
5	75	12	12
6	59	2	39
7	73	4	23
8	78	7	15
9	69	4	26
10	70	7	23
11	72	8	20
12	81	7	12
13	75	11	13
14	76	15	9
15	78	14	8
16	73	18	9
17	76	6	18
18	74	10	15
19	73	16	11
20	77	17	6
21	78	14	8
22	79	13	8
23	79	7	13
24	75	13	11

*Sample autocorrelation is at least two standard deviations to the left or to the right of its expected value under the hypothesis that the autocorrelation is zero. Minimum number of continuous observations per security is 2017. Autocorrelations are estimated by centering the observations around the mean.