NON-LINEAR PRICING SYSTEMS IN FINANCE

By

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INTRODUCTION

It is well-known that if financial markets are complete there can be no motive for "creative" corporate finance: Neither firm value nor consumer utility can be affected by the creation--by publicly traded firms--of new financial securities. Neither can the trading of such firms in the securities available to consumers affect their value. A careful examination of the logic of such "irrelevance" results reveals that the reason for the irrelevance of corporate financial activities in complete markets is not completeness *per se*, but rather the fact that the pricing system in such markets is linear. A linear pricing system has the following property: Let a and b be any two income streams and let V( ) denote the function which assigns market value to a given income stream. A pricing system is termed *linear* if $V(a) + V(b) = V(a + b)$. It is clear that a market in which the pricing system is linear must thus be one in which all financial activities relating purely to the division of existing income streams must be irrelevant.[1]

The study of relevant corporate financial activities must thus concentrate on markets in which non-linear pricing systems exist. In this paper I show that--under highly general conditions--non-linearity requires the existence of *position limits* on security holdings which are independent of the consumer budget constraint. These position limits can in principle be either limits on the total positive amount or the total negative amount (short sales) of any security which a consumer may hold. The first case--that of effective positive position limits--is sufficiently rare that we may assume such limits as exist to be ineffective. The second case--that of short-sales constraints--is widespread.[2] It is this case which I shall consider in this paper.

In Section 1 I shall show that if short sales are restricted in a quite general class of models of financial equilibrium, then the function V( ) will be sub-additive: $V(a) + V(b) \leq V(a + b)$. The reason for this is quite simple: If short sales are restricted, not all individuals will agree about the value of income streams. Each income stream will be sold to that individual who places the highest value on it; all other individuals will either agree with the purchaser or they will value the income stream at below its market value. (Note that if short sales were allowed, an individual who valued an income stream at less than its market value would short sell it; this would eventually lead to all individuals placing the same value on every income stream.) Denote the private valuation of individual i by $V_i(\ )$. In a market with restricted short sales,

$$(1) \qquad V(a) = \max_i V_i(a).$$

The sub-additivity of V( ) follows from this fact:

$$(2) \qquad V(a + b) = \max_i V_i(a + b) \leq \max_i V_i(a) + \max_i V_i(b) = V(a) + V(b).$$

Intuitively, what is happening here is that in splitting up the income stream a + b, we are able to sell a to its highest bidder and b to its highest bidder. We can do no worse in this case than if we sell off a and b together.

The argument outlined above is developed rigorously in Section 1 of this paper. Section 2 rephrases the argument to fit a standard, two-period, state-preference model which is used in the applications of Sections

3-5. These applications are the following: Section 3 considers the short-sales mechanism of American securities markets. I show that this mechanism is effectively a restraint on short sales which places limits on the non-linearities of the pricing system. The closed-end fund paradox is reconsidered in light of these limits. Section 4 deals with the pure capital structure question when short sales are constrained and shows that all consumers in such a market will prefer that firms have at least a minimal level of debt. Section 5 considers firm hedging in futures market when short sales are restricted.

1. NON-LINEARITIES AND SHORT-SALE CONSTRAINTS

A very large class of propositions in finance is derived from models which are roughly similar. In these models a typical consumer i chooses a portfolio of J securities, $s_i = (s_{i1}, \ldots, s_{iJ})$, in order to achieve a consumption vector $x_i(s_i)$. Given a security price vector $p = (p_1, \ldots, p_J)$, consumer i chooses s_i in order to maximize a concave utility function of consumption, $U_i(x_i(s_i))$, subject to a budget--and possibly other--constraints.

Write the i-th consumer's budget constraint by $g_{io}(s_i, p) \geq 0$ and assume that this constraint is bilinear in s_i and p; that is, the cost of the portfolio s_i is the dot product $ps_i = \sum_j s_{ij}p_j$. The budget constraint may, of course, include other terms which relate to the value of a consumer's initial portfolio or to a total wealth constraint; a specific example is given in Section 2.

Throughout the paper I shall assume that--except where limited by the budget constraint--there is no explicit constraint on the maximum size of any consumer's portfolio holding of a security. On the other hand, I shall wish to examine the effect of possibly other constraints on consumption and on portfolio positions. Label these constraints

$$g_{i\alpha}(s_i, p) \geq 0, \quad \alpha = 1, \ldots, N. \tag{3}$$

Constraints of interest in this paper are those which are explicitly or implicitly constraints on short sales, but other constraints are possible: If the consumer maximizes a state-dependent utility function, for example, the constraints $g_{i\alpha}$ may include constraints on minimum consumption in certain states.

Call a price vector p^* an <u>equilibrium price vector</u> if there exists, for each consumer i, a portfolio s_i^* having the following properties:

(E.1) $\sum_i s_i^* = \bar{s}$, where $\bar{s}$ is the vector of the total supply of securities.

(E.2) $g_{i\alpha}(s_i^*, p^*) \geq 0, \quad = 0, 1, \ldots, N$, for all consumers i.

(E.3) If $U_i(x_i(s_i)) > U_i(x_i(s_i^*))$ for some consumer i, then there exists $0 \leq \alpha \leq N$ such that $g_{i\alpha}(s_i, p^*) < 0$.

The last condition states that a portfolio which gives increased utility of consumption over s_i^* violates some constraint (budget or otherwise).

An equilibrium price vector p^* will be termed <u>linear</u> if when s and t are two portfolios such that $x(s) = x(t)$, then $p^*s = p^*t$. The following theorem gives necessary conditions for the existence of a linear pricing system.

<u>Theorem 1</u>: Suppose there exists a security j such that $x(\delta) > 0$ where is the portfolio $\delta = (\delta_1, \ldots, \delta_J)$

$$\delta_h = \begin{cases} 0 & \text{if } h \neq j \\ 1 & \text{if } h = j. \end{cases}$$ [3]

Then p^* is a non-linear pricing vector only if for all consumers i the conditions $g_{i\alpha}$ include (implicit or explicit) restrictions on short sales.

<u>Proof</u>:

Suppose p^* is non-linear and that there are no short-sale restrictions. With no loss in generality there exist two portfolios s and t such that

$x(s) = x(t)$ and $p^*s < p^*t$. Let s_i^* be consumer i's optimal portfolio and consider the following portfolio: $\tilde{s}_i$:

$$\tilde{s}_{ih} = s_{ih}^* + s_h - t_h, \quad h \neq j,$$

$$\tilde{s}_{ij} = s_{ij}^* + s_h - t_h + (p^*t - p^*s)/p_j^* .$$

The portfolio $\tilde{s}_i$ is both affordable and--assuming that the price of security j is positive (see next paragraph)--gives increased utility for every consumer i. This contradicts the assumption that s_i^* is an optimal portfolio for consumer i.

To see that p_j^* is positive: If not, every consumer can include unlimited amounts of security j in his portfolio, thereby increasing utility. This completes the proof. qed

Suppose that t and u are portfolios and that the portfolio s is the sum of t and u, $s = t + u$. Denote the prices of the three portfolios by p_s, p_t, p_u. A pricing system p^* will be termed <u>sub-additive</u> if for $s = t + u$,

$$p_s^* \leq p_u^* + p_t^* . \tag{4}$$

The following theorem shows that--in the class of models under consideration--a non-linear pricing system is necessarily sub-additive.

<u>Theorem 2</u>: Let p^* be an equilibrium pricing system. Then if p^* is non-linear, it is sub-additive.

<u>Proof</u>:

The proof is a direct result of the Kuhn-Tucker conditions (see Intrilligator, 1971, pp. 49ff). qed

2. SUB-ADDITIVITIES IN A TWO-PERIOD, STATE-PREFERENCE MODEL

In the examples which follow I shall consider a two-period, state-preference version of the model outlined in Section 1. Specifically, assume there to be two periods: today (labelled, when necessary, 0) and tomorrow (period 1). Any of M states of the world may occur in period 1, and security j is assumed to have a payoff of R_{jm} in state m of the world tomorrow, $m=1, \ldots, M$. Where convenient, $R_j = (R_{j1}, \ldots, R_{jM})$ will denote the payoff vector of security j in period 1. A consumer who purchases a security vector $s_i = (s_{i1}, \ldots, s_{iJ})$ of the J securities assures himself a return of

(5) $$x_{im} = \sum_j s_{ij} R_{jm}, \quad m = 1, \ldots, M,$$

in state m of the world.

Consumers enter period 0 with an initial endowment of securities. Denote consumer i's initial endowment by $\bar{s}_i = (\bar{s}_{i1}, \ldots, \bar{s}_{iJ})$. These initial endowments are traded for new portfolios s_i and for initial consumption x_{io}, where the budget constraint is given by

(6) $$x_{io} = \sum_j \bar{s}_{ij} p_j - \sum_j s_{ij} p_j \quad .$$

Consumers are assumed to maximize a state-dependent utility function

(7) $$U_i(x_{io}, x_{i1}, \ldots, x_{iM})$$

subject to (5), (6), and (possibly) short-sale constraints of the type

(8) $$s_{ij} \geq a_{ij} , \qquad -\infty < a_{ij} \leq 0.$$

It is important to note that while this particular model assumes agreement among consumers as to the number and classification of states and the returns of the securities in each state, it does <u>not</u> make any assumptions about the subjective consumer probabilities on the states; these are subsumed in each consumer's utility function.[4]

The Kuhn-Tucker conditions imply that if s_i^* is the optimal portfolio for consumer i given equilibrium prices p^*, then there exist <u>implicit state prices</u> $q^i = (q_1^i, \ldots, q_M^i)$ for each consumer such that

(9) $$q_m^i = \frac{\partial U_i/\partial x_{im}}{\partial U_i/\partial x_{im}} \quad \text{evaluated at } x_i^*,$$

and

(10) $$p_j^* \geq \sum_m q_m^i R_{jm}\text{, with equality holding if } s_{ij}^* > a_{ij} \ .$$

Another way of writing this last condition is to write

(11) $$p_j^* = \max_i \sum_m q_m^i R_{jm} \ .$$

Since the max function is sub-additive, it immediately follows that--with short-sale restrictions--an equilibrium pricing system must be sub-additive in a state-preference model.

3. THE AMERICAN SHORT-SALES MECHANISM

American stock markets allow short sales, but--as is well-known--these are not of the type modelled above. Specifically, a short seller in an American stock market cannot collect the proceeds of his short sale until the following period.[5] This short-sale mechanism may be modelled in a state-preference framework as follows:

Let j be one of the traded securities and assume that its price is p_j and its return vector R_j. A <u>short sale of j</u> is defined as the purchase of a security whose price today is 0 and whose return tomorrow is the vector $r_j = (p_j - R_{j1}, \ldots, p_j - R_{jM})$. Denote such purchases by t_{ij}. The American short-sales mechanism thus effectively doubles the number of securities, and the consumer's problem may be written

$$\max U_i (x_{io}, x_{i1}, \ldots, x_{iM})$$

subject to

(12)
$$x_{io} = \sum_j \bar{s}_{ij} p_j - \sum_j s_{ij} p_j ,$$
$$x_{im} = \sum_j s_{ij} R_{jm} + \sum_j t_{ij} (p_j - R_{jm}) ,$$
$$s_{ij}, t_{ij} \geq 0.$$

Now consider a consumer i who sells short security j (i.e., $t_{ij} > 0$). Then equation (10) holds with equality, and therefore

$$p_j = (1/\sum_m q_m^i)(\sum_m q_m^i R_{jm}). \tag{13}$$

If $\sum_m q_m^i < 1$, this means that $p_j > \sum_m q_m^i R_{jm}$, so that $s_{ij} = 0$. If, on the other hand, $\sum_m q_m^i > 1$, then (13) would imply that $p_j < \sum_m q_m^i R_{jm}$, so that we get a contradiction. Interpreting $\sum_m q_m^i$ as the inverse of one-plus-consumer i's-one-period lending (or borrowing) rate, this means that the American short-sales mechanism forces this rate to be positive.[5]

Closed-End Funds and the American Short-Sales Mechanism

Consider the above model with riskless borrowing and lending at the same interest rate.[6] The inverse of one-plus-the-one-period rate may thus be written $1/(1+r) = \sum_m q_m^i$, with the equality holding for all consumers.

Now consider any security j in the model for which the American short-sales constraint is in effect. It follows from equations (10) and (13) that if

(14) $$(1+r)\sum_m q_m^i R_{jm} > p_j > \sum_m q_m^i R_{jm},$$

for some consumer i, then i will neither buy the security nor short it; $s_{ij} = 0$ and $t_{ij} = 0$, the latter since $\sum_m q_m^i(p_j - R_{jm}) < 0$. Thus a consumer can consider a security overpriced and not desire to short sell it because of the structure of the American short-sales constraint.

The last paragraph can be used to derive a useful proposition about closed-end mutual funds: Consider a fund comprised of proportions λ_j of each of the j assets, and suppose that $\sum_j \lambda_j = 1$, $\lambda_j \geq 0$. Let p_f be the market price of the fund. By Theorem 2

(15) $$p_f \leq \sum_j \lambda_j p_j,$$

with equality holding if some consumer owns all of the fund's portfolios as individual shares. A lower bound on the fund's market price is

(16) $$p_f \geq \sum_j \lambda_j (\min_i \sum_m q_m^i R_{jm}) \geq \sum_j \lambda_j p_j/(1+r),$$

where the last inequality follows from (14). Thus, in a market with American short-sale restrictions the variation of a closed-end fund's price from its "net asset" value (i.e., $\sum_j \lambda_j p_j$) of up to the one-period interest rate can occur without inconsistencies.[7]

4. CAPITAL STRUCTURE WITH SHORT-SALE CONSTRAINTS[8]

Consider the model of Section 2 and look at a single firm j which has both debt and equity claims outstanding on its return vector $R_j = (R_{j1}, \ldots, R_{jM})$. Let D_j be the total payment on the firm's debt promised in each state; when bankruptcy is taken into consideration, the return vector to debtholders is given by

$$(17) \qquad R^d_{jm} = \begin{cases} D_j & \text{if } R_{jm} \geq D_j \\ R_{jm} & \text{otherwise} \end{cases} .$$

The return vector to equity holders given debt of D_j is

$$(18) \qquad R^e_{jm} = \begin{cases} R_{jm} - D_j & \text{if } R_{jm} \geq D_j \\ 0 & \text{otherwise} \end{cases} .$$

From Theorem 2 it follows that a firm j with some debt outstanding will never be valued at less than an equivalent firm which has no debt. To see this, consider any equilibrium in which firm j is as above and firm h has no debt but has $R_{hm} = R_{jm}$, m=1, . . ., M. Then

the value of j's debt $= p^d_j = \max\limits_i \sum\limits_m q^i_m R^d_{jm}$,

the value of j's equity $= p^e_j = \max\limits_i \sum\limits_m q^i_m R^e_{jm}$,

the value of firm h $= p_h = \max\limits_i \sum\limits_m q^i_m R_{jm}$.

It is immediately apparent that $p_h \leq p^e_j + p^d_j$. It does not follow from this, however, that firm j would necessarily increase its value when it issues more debt, nor that issuing more debt would be favored by consumers even if the firm's value were to rise as a result of the issuance of debt. To resolve the issue of consumer preferences, we differentiate consumer utility functions with respect to D_j, on the assumption that portfolios and implicit prices are constant.[9] The following theorem shows that--in a market with short-sale restrictions--prices can be found such that all consumers will favor increasing the firm's debt at least to the point where the debt becomes risky.

Theorem 3: Let p^* be an equilibrium in which firm j has debt D_j^*. Suppose that $D_j^* < \min_m R_{jm}$. Then there exist prices such that $\partial U_i/\partial D_j \geq 0$ for all consumers i.

Proof:

$$\frac{\partial U_i}{\partial D_j} = \frac{\partial U_i}{\partial x_{io}} \frac{\partial x_{io}}{\partial D_j} + \sum_m \frac{\partial U_i}{\partial x_{im}} \frac{\partial x_{im}}{\partial D_j} .$$

Assuming that all proceeds of additional debt sold are paid out to initial shareholders, it follows that[10]

$$\frac{\partial x_{io}}{\partial D_j} = -s_{ij}^* \frac{\partial p_j^e}{\partial D_j} - d_{ij}^* \frac{\partial p_j^d}{\partial D_j} + \bar{s}_{ij} \frac{\partial p_j^e}{\partial D_j} + \frac{\partial p_j^d}{\partial D_j} ,$$

$$\frac{\partial x_{im}}{\partial D_j} = s_{ij}^* \frac{\partial R_{jm}^e}{\partial D_j} + d_{ij}^* \frac{\partial R_{jm}^d}{\partial D_j} .$$

From (17) and (18) and the assumption that $D_j < \min_m R_{jm}$, it follows that

$$\frac{\partial R_{jm}^e}{\partial D_j} = -1 \quad \text{and} \quad \frac{\partial R_{jm}^d}{\partial D_j} = +1 , \quad m = 1, \ldots, M.$$

Assume that firm j's debt is priced using the implicit valuations of some consumer d (i.e., $p_j^d = \sum_m q_m^i D_j$), and set

$$\frac{\partial p_j^d}{\partial D_j} = \sum_m q_m^d \quad \text{and} \quad \frac{\partial p_j^e}{\partial D_j} = -\sum_m q_m^d .$$

Dividing $\partial U_i/\partial D_j$ through by $\partial U_i/\partial x_{io}$, we get that

$$\frac{\partial U_i}{\partial D_j} > 0 \text{ if and only if } -s^*_{ij}\frac{\partial p_j^e}{\partial D_j} - d^*_{ij}\frac{\partial p_j^d}{\partial D_j}$$

$$+ \bar{s}_{ij}\left(\frac{\partial p_j^d}{\partial D_j} + \frac{\partial p_j^e}{\partial D_j}\right) - s^*_{ij}\sum_m q_m^i + d^*_{ij}\sum_m q_m^i > 0.$$

Collecting terms, $\partial U_i/\partial D_j > 0$ if and only if

$$s^*_{ij}\{\sum_m q_m^d - \sum_m q_m^i\} + d^*_{ij}\{\sum_m q_m^i - \sum_m q_m^d\} +$$

$$+ \bar{s}_{ij}\{\sum_m q_m^d - \sum_m q_m^d\} > 0.$$

To see that the left-hand-side of the last expression is indeed non-negative for all consumers i, note that since $\sum_m q_m^d = \max_i \sum_m q_m^i$, the first bracketed term is non-negative. By the short-sale restrictions, the second bracketed term must be zero, since if $d^*_{ij} \neq 0$, $\sum_m q_m^i = \sum_m q_m^d$. This completes the proof. qed

The following example illustrates the intuition behind the theorem. Consider an economy with two future states and one firm which has a state-dependent revenue vector (3, 6). Suppose the firm currently has debt outstanding which promises a return of 1 independent of state. The payoffs to the debt and equity vectors of the firm will thus be:

debt: (1, 1); equity: (2, 5).

Let the equilibrium prices for the debt and equity vectors be .75 for the debt and 2.7 for the equity vector.

In order to allow for a graphical representation of the problem, suppose that all consumers maximize utility functions of first period consumption only. The consumer's maximization problem may thus be written:

$$
\begin{aligned}
&\max U(x_1, x_2) \\
&\text{s.t.} \\
&x_1 = 2s + d, \\
(19)\qquad &x_2 = 5s + d, \\
&2.7s + .75d = W, \\
&s, d \geq 0.
\end{aligned}
$$

In the above equations s and d represent the proportions of the firm's debt and equity r purchased by the consumer, and W is the value of the consumer's initial portfolio.

Figure 1 gives a graphical representation of the (x_1, x_2) possibilities afforded the consumer. The rays extending from the origin are the return vectors of the firm's debt and equity, and the line AB represents the locus of points which satisfy the budget constraint and the short-sale constraints. The indifference curves labelled 1, 2, and 3 indicate three typical optimal consumer portfolios. Consumer 1 is a non-corner maximizer, and divides his wealth equally between purchases of the bond and the equity vectors. Consumer 2 invests all of his wealth in the firm's equity and consumer 3 invests all of his wealth in the firm's debt. For each of the three consumers we may calculate the Kuhn-Tucker shadow prices for a unit of revenue in states 1 and 2. For consumer 1 these prices are derivable from the slope of the line segment AB by solving the equations

$$
(20)\qquad
\begin{aligned}
2q_1 + 5q_2 &= 2.7 \\
q_1 + q_2 &= .75.
\end{aligned}
$$

The solution to these equations yields $q_1 = .35$ and $q_2 = .4$. The implicit prices for consumers 2 and 3 are not derivable directly from the line AB but must be derived from their utility functions. It is clear, however, that consumer 2 places a higher implicit value on state 2 consumption than does consumer 1 and a lower implicit value on state 1 consumption. The reverse holds for consumer 3. Thus, for consumer 2:

shadow price for state 1 consumption $<$ .35,

shadow price for state 2 consumption $>$ .40.

Since consumer 2 purchases the equity of the firm, moreover, it follows from (11) that his implicit value for the firm's equity corresponds to the market value. Thus--denoting by r_1 and r_2 consumer 2's shadow prices--it must hold that $2r_1 + 5r_2 = 2.7$.

Finally, consumer 2 values the firm's bond at less than its market price. To see this, consider the line AC, whose slope is determined by consumer 2's implicit prices (Figure 2). Since point C is above B, and since this point corresponds to the amount of the firm's debt that the consumer could purchase were he to invest all his wealth in the firm's debt, it follows that at his shadow prices consumer 2 considers the debt to be overpriced. (Another way to see this is to consider what would happen if consumer 2 could short sell the firm's debt. In this case he would be able to choose a portfolio on the line AD in Figure 2, thus increasing his utility.)

Now consider what would happen if the firm increased its debt from 1 to 2. The new return vectors would be:

debt: (2, 2); equity: (1, 4).

In Figure 3 it may be seen that if the new return vectors are priced at the prices implicit in the old budget line AB (i.e., consumer 1's implicit prices), that consumer 2 is clearly better off than before, whereas the utility achievable by consumers 1 and 3 is unchanged. There exist, in fact, prices which would make all consumers better off; these are indicated by line CD in Figure 4.

5. CORPORATE HEDGING IN FORWARD MARKETS WHEN SHORT SALES ARE RESTRICTED

With some minor adaptation, the model of Section 2 may be used to establish that hedging by corporations in forward markets increases firm value. We assume the same basic state structure but add multiple commodities. Let there be H such commodities which are used for both consumption and production. Each of the J firms is assumed to own a stochastic production technology capable of producing one good.[11] Thus J=H, and firm j produces good j. Firms purchase input vectors today for production tomorrow; the input vector of firm j is denoted by $z_j = (z_j^1, \ldots, z_j^H)$, where z_j^h denotes the physical quantity of commodity h purchased by firm j. If firm j buys an input vector z_j today, its output tomorrow in state m will be determined by a function y_{jm}; the state-dependent output of firm j given its purchases of inputs z_j will thus be written

$$(21) \qquad y_j(z_j) = (y_{j1}(z_j), \ldots, y_{jM}(z_j)).$$

Denote the commodity price vector today and in state m of the world tomorrow by $p_m^c = (p_m^{c1}, \ldots, p_m^{cH})$, $m = 0, 1, \ldots, M$. Then the cost of purchasing the inputs of firm j will be $p_o^c z_j$ and the value of firm j's production in state m of the world tomorrow will be $p_m^c y_{jm}(z_j) = p_m^{cj} y_{jm}(z_j)$, where multiplication is taken to mean the vectorial dot product.

Denote the market value of firm j by V_j and consider first an economy without forward markets. Consumers who are initial owners of firm j (i.e. consumers i such that $\bar{s}_{ij} > 0$) are assumed to participate in the financing of the firm's inputs. They then sell their initial portfolios and purchase new portfolios s_i in order to achieve a consumption vector $x_i = (x_{io}, x_{i1}, \ldots, x_{iM})$, where

(22) $$x_{im} = (x^1_{im}, \ldots, x^H_{im}), \; m = 0, 1, \ldots, M.$$

The consumer's maximization problem may thus be written

(23)
$$\max U_i(x_i)$$
subject to
$$p^c_o x_{io} = \sum_j \bar{s}_{ij}(V_j - p^c_o z_j) - \sum_j s_{ij} V_j \, ,$$
$$p^c_m x_{im} = \sum_j s_{ij} p^c_m y_{jm}(z_j),$$
$$s_{ij} \geq 0.$$

Given commodity prices p^c and firm values $(V_1, \ldots, V_J)$, it may be shown that consumer i's implicit state prices are given by

(24) $$q^i_m = \frac{\partial U_i/\partial x^1_{im}}{\partial U_i/\partial x^1_{io}} \cdot \frac{p^{c1}_o}{p^{c1}_m} \, , \; m = 1, \ldots, M,$$

where the partial derivatives are evaluated at consumer i's optimal portfolio given the commodity and firm prices. The value of the firm is given by

(25) $$V_j \geq \sum_m q^i_m p^c_m y_{jm}(z_j), \; j = 1, \ldots, J,$$

with equality holding if $s_{ij} > 0$.[12]

Forward Contracts

A forward contract is a contract obliging the firm to deliver a fixed physical quantity of its product tomorrow irrespective of the state of the world; the purchaser of the contract promises to pay the producer a fixed price (not state-dependent) for his product. The price of the contract is set so that the current market value of the contract is zero.

Denote by α_j the quantity of good which firm j has promised to deliver at forward price β_j. The firm's revenues from this contract in state m may be written

$$\alpha_j(\beta_j - p_m^{cj}), \tag{26}$$

and the revenues of the purchaser of the contract will be

$$\alpha_j(p_m^{cj} - \beta_j). \tag{27}$$

It follows from the way we have defined the price β_j and from equation (11) that β_j is fixed such that

$$\max_i \sum_m q_m^i(p_m^{cj} - \beta_j) = 0, \tag{28}$$

where it is assumed that consumers cannot sell short in forward markets. Thus the contract will be sold to the individual whose evaluation of its proceeds is highest among all individuals, and the price β_j will be set so that this implicit valuation is zero.

Suppose firm j contracts to sell quantity α_j in a forward contract bearing price β_j. We may approximate the new price of the firm by finding the highest consumer valuation of the new revenues of the firm, using current consumer implicit valuations. Denoting this new price by $V_j(\alpha_j)$ and using (11) it follows that

$$V_j(\alpha_j) = \max_i \sum_m q_m^i\{ p_m^c y_{jm}(z_j) + \alpha_j(\beta_j - p_m^{cj}) \} \quad . \tag{29}$$

The next theorem shows that $V_j(\alpha_j)$ will never be less than V_j:

<u>Theorem 4</u>: $V_j(\alpha_j) \geq V_j$ for $\alpha_j \geq 0$.

Proof:

First note that it follows from (28) that β_j is fixed so that

$$\min_i \sum_m q_m^i(\beta_j - p_m^{cj}) = 0.$$

It follows that for any individual consumer e,

$$(30) \qquad \sum_m q_m^e(\beta_j - p_m^{cj}) \geq 0.$$

Now suppose that the maximum in (29) is obtained using the implicit prices of consumer f. Then letting e be any consumer for whom $s_{ej} > 0$, it follows from the definition of $V_j(\alpha_j)$ that

$$\begin{aligned}(31) \qquad V_j(\alpha_j) &= \sum_m q_m^f\{ p_m^c y_{jm}(z_j) + \alpha_j(\beta_j - p_m^{cj})\} \\ &\geq \sum_m q_m^e\{ p_m^c y_{jm}(z_j) + \alpha_j(\beta_j - p_m^{cj})\} \\ &= \sum_m q_m^e p_m^c y_{jm}(z_j) + \sum_m q_m^e \alpha_j(\beta_j - p_m^{cj}) \\ &\geq V_j \end{aligned}$$

The last inequality follows from (30) and from the fact that since $s_{ej} > 0$, there exists equality in equation (25). qed

Theorem 4 thus shows that any firm which sells in forward markets will not lower (and will, in most cases, raise) its market value. It is by now well-known that in imperfect capital markets the maximization of a firm's market value is not necessarily an objective desired by all of the firm's shareholders.[13] Thus, even though a firm may raise its market value by hedging on forward markets, it cannot yet be said unequivocally whether its shareholders will desire it to do so. To derive the conditions under which a firm's shareholders will desire it to hedge, take the derivative of a typical individual i's utility function with respect to α_j:

Theorem 5: If $\bar{s}_{ij} \geq s_{ij}$, then individual i will want firm j to engage in hedging.

Proof:

We wish to determine when

$$(32) \qquad \frac{\partial U_i}{\partial \alpha_j} = \frac{\partial U_i}{\partial x^1_{io}} \frac{\partial x^1_{io}}{\partial \alpha_j} + \sum_m \frac{\partial U_i}{\partial x^1_{im}} \frac{\partial x^1_{im}}{\partial \alpha_j}$$

$$= \frac{\partial U_i}{\partial x^1_{io}} \frac{1}{p^{c1}_o} (\bar{s}_{ij} - s_{ij}) \frac{\partial V_j(\alpha_j)}{\partial \alpha_j}$$

$$+ \sum_m \frac{\partial U_i}{\partial x^1_{im}} \frac{1}{p^{c1}_m} s_{ij}(\beta_j - p^{c1}_m) > 0 \qquad .$$

Dividing through by $\partial U_i / \partial x^1_{io}$ and using (31), it may be seen that the inequality in (32) holds if and only if

$$(33) \qquad (\bar{s}_{ij} - s_{ij}) \sum_m q^f_m(\beta_j - p^{cj}_m) + s_{ij} \sum_m q^i_m(\beta_j - p^{cj}_m) > 0.$$

Since $s_{ij} \geq 0$, the second term in (33) is always non-negative. Combining (30) with the knowledge that $\bar{s}_{ij} \geq s_{ij}$, it follows that the first term in (33) is also positive. This completes the proof. qed

Two remarks are in order about Theorem 5:

Remark 1: The increase (non-decrease) in the price of the firm as a result of its hedging operations established in Theorem 4 cuts two ways. On the one hand, it increases the wealth of initial shareholders; on the other hand, it makes new purchases of the firm's shares more expensive. Call the first of these effects the wealth effect and the second the consumption effect.

Then Theorem 5 may be interpreted as saying that corporate hedging is preferred by any firm shareholder for whom the wealth effect outweighs the consumption effect. If we conceive of the first period as representing a typical period in a multi-period model, and if we assume that most shareholders of the firm will make no change in their portfolios, then the conditions of Theorem 5 will always hold, and hedging will be preferred by all of the firm's shareholders. Another condition under which hedging will be preferred is if there are many firms which are similar to firm j. We may then assume that initial shareholders of firm j will set $s_{ij} = 0$; this may be done since they can always purchase shares in some other firm whose production function is the same as that of firm j and which does not engage in hedging operations.[14]

Remark 2: Theorem 5 gives only sufficient conditions for shareholders to prefer hedging. By manipulating equation (33), it follows that

$$s_{ij} < \frac{\bar{s}_{ij} \sum_m q_m^f(\beta_j - p_m^{cj})}{\sum_m (q_m^f - q_m^i)\ (\beta_j - p_m^{cj})} \tag{34}$$

is both a necessary and sufficient condition for consumer i to prefer corporate hedging. I have, however, been unable to find a satisfactory operational meaning for this equation.

A Numerical Example of Hedging

Consider an economy with two future states and three firms, each producing, with a state-dependent production function, one good. Suppose that firm 1 produces 4 units of its good in state 1 and 24 units in state 2, and suppose that the price of the good produced by firm 1 is .75 in state 1 and

.25 in state 2. The revenue of the firm is thus given by the vector

$$(4 \times .75,\ 24 \times .25) = (3,\ 6).$$

Without going into the same kind of detail, we shall assume that the revenue of the other two firms is given by the vectors

firm 2: (1, 1),

firm 3: (7, 2).

Suppose that--before any corporate hedging takes place--$V_1 = 4$, $V_2 = .75$, and $V_3 = 3$. A typical consumer's maximization problem is given by

$$\max U(x_1, x_2)$$

subject to

$$p_1^c x_1 = 3s_1 + s_2 + 7s_3,$$

$$p_2^c x_2 = 6s_1 + s_2 + 2s_3,$$

$$4s_1 + .75s_2 + 3s_3 = W,$$

$$s_1, s_2, s_3 \geq 0.$$

Figure 5 gives a graphical representation of the $(p_1^c x_1, p_2^c x_2)$ possibilities afforded the consumer. The rays extending from the origin are the revenue vectors of the firms and the solid lines--AB and BC--represent the optimal $(p_1^c x_1, p_2^c x_2)$ opportunities. Note that line AC is feasible but dominated by AB and BC. Note also that the absence of short sales means that the existence of three (linearly dependent) security return vectors in a two-state world is not inconsistent.[15] The optimal portfolios of two consumers are shown; each consumer invests in a portfolio containing only two securities.

Each of the line segments AB and BC corresponds to a unique set of implicit prices for a unit of revenue in states 1 and 2. The line segment AB corresponds to prices ($q_1 = .1667$, $q_2 = .5833$) and BC to prices ($r_1 = .3$, $r_2 = .45$).

Now let firm 1 hedge some of its production. Letting β represent the forward price, each unit of production hedged gives the firm the vector (β, β) instead of the commodity price vector $(.75, .25)$. The firm thus nets the vector $(\beta - .75, \beta - .25)$ from the hedging operation and the purchaser of the hedged goods will receive $(.75 - \beta, .25 - \beta)$.[16] An individual of type 1 will value this contract at implicit prices q_1 and q_2, and--since forward prices are set so that the net present value of the contract equals zero, individuals of type 1 will determine the amount they are willing to pay for the contract by solving

$$q_1(.75 - \beta) + q_2(.25 - \beta) = 0,$$

which gives $\beta = .3611$. Individuals of type 2 will solve for β using implicit prices r_1 and r_2; this results in $\beta = .4333$. The forward price is thus determined by individuals of type 2 and the resulting (new) revenue vector added to the market is shown in Figure 6.

What will be the value of firm 1 to consumers after the hedge? Denoting by α the quantity of the good hedged, we get (since firm 1 is priced at implicit prices q_1 and q_2) that the value of firm 1 becomes

$$q_1 \times \{ .75(4 - \alpha) + \alpha\beta \} + q_2 \times \{ .25(24 - \alpha) + \alpha\beta \} ,$$

which, substituting $q_1 = .1667$, $q_2 = .5833$, $\beta = .4333$, becomes

$$.1667 \times (3 - .3167\alpha) + .5833 \times (6 + .1833\alpha) = 4 + .0541\alpha > 4.$$

A value-maximizing hedge at current implicit prices would imply that the firm set α as high as possible. If the firm is constrained to hedge no more than it can supply to the purchaser of the hedge with certainty, and if its shareholders wish it to maximize value (see Theorem 5), then it should set $\alpha = 4$.

6. SUMMARY AND CONCLUSION

If short sales are restricted, the equilibrium pricing system will be sub-additive, so that value is increased by splitting up revenue streams. Furthermore, in a highly general framework, the source of non-linearities of the pricing system--if these exist--must be short-sale restrictions. This paper has examined the general theory of such non-additivities as well as three applications. It was shown that the short-sale mechanism of American securities markets is in fact a kind of short-sale restriction which places a limit on the non-linearity of the resulting equilibrium pricing system. In a re-examination of the importance of capital structure, it was found that all consumers will prefer that firms be levered at least up to the point of bankruptcy risk. Finally, in a market with short-sale restrictions, the creation of a new security--the forward sale by a firm of the product it produces--increases firm value.

FOOTNOTES

1. As far as I am aware, explicit recognition of this fact is to be found first in Beja (1967). Ross (1978) discusses the existence of linear pricing systems and their implications for financial theory in depth. Litzenberger and Sosin (1977) and Hart (1978) footnote the fact that market restrictions may cause non-linearities in the pricing system.

2. I am aware of only two cases in which limits on positive positions exist: First, it is, of course, impossible to buy more than all of the equity of a firm on the stock market. Second, most regulated futures markets in the United States have position limits. The first of these limitations is effective only when short sales are restricted; the second is widely known to be easily evadable.

3. By $x(\delta) > 0$ is meant that all coordinates of the vector are non-negative, with at least one coordinate positive.

4. The model of Section 2 is extremely flexible and can be used to model both the CAPM (cf. Baron 1979) and a version of the options pricing model (Ross 1976).

5. Note that different borrowing and lending rates <u>cannot</u> be modeled by assuming that the <u>same</u> security can be both bought and shorted (American). To model differential borrowing and lending rates, assume there exist two securities L and B, with

$$R_{Lm} = +1, \quad p_L = 1/(1+k_L), \text{ and position limits } \quad s_{iL} \geq 0, \; t_{iL} \equiv 0.$$

$$R_{Bm} = -1, \quad |p_B| = 1/(1+k_B), \quad \text{position limits} \quad s_{iB} \geq 0, \; t_{iB} \equiv 0.$$

Here k_L and k_B are taken to be the lending and borrowing rates, respectively. Suppose that $p_L > |p_B|$, i.e., the borrowing rate exceeds the lending rate. Then if $s_{iB} > 0$, $|p_B| = \sum_m q_m^i$ and if $s_{iL} > 0$, $p_L = \sum_m q_m^i$, and since $p_L > |p_B|$ no individual will borrow and lend simultaneously.

Note that I have forbidden (American) short sales in L and B in the paragraph above. If there exists a security with $R_m = +1$ and price p which may be both shorted (American) and bought long, then the one-period interest rate must be zero. To see this, suppose some consumer i shorts the security. Then it follows from (13) that

$$p = (1/\sum_m q_m^i)(\sum_m q_m^i) = 1.$$

6. In terms of the previous footnote, this means that $p_L = p_B$.

7. See Malkiel (1977) for a survey of the literature on closed-end fund discounts. Researchers of closed-end fund discounts have tended to ignore short-sale restrictions as a source of the discounts, with the exception of Litzenberger and Sosin, who mention them but do not research them empirically. Long (1978) has attributed market pricing inconsistencies for dividend paying and capital-gains paying stock to short-sale restrictions.

8. Parts of this section were first discussed in an unpublished working paper (Benninga 1980). The results have been extended by Benninga and Schneller (1981) to include the case of differential taxation of debt and equity.

9. This procedure is legitimate under one of two sets of assumptions: If the equilibrium (and hence implicit prices) is continuous for small changes in firm j's financial policy, then under the conditions of Theorem 3 the new equilibrium will be Pareto superior to the old. Another "legitimizing" assumption is that consumers use their current implicit prices to predict the effects of a change in corporate policy.

10. To account for the issuance of corporate debt, (5) and (6) have to be rewritten:

$$(5') \qquad x_{im} = \sum_j d_{ij}R^d_{jm} + \sum_j s_{ij}R^e_{jm},$$

$$(6') \qquad x_{io} = \sum_j \bar{s}_{ij}(p^e_j + p^d_j) - \sum_j d_{ij}p^d_j - \sum_j s_{ij}p^e_j .$$

11. This assumption is not strictly necessary for the analysis, but it allows for some notational simplification.

12. Equations (24) and (25) follow directly from the Kuhn-Tucker conditions. For a derivation see Benninga (1979).

13. See Leland (1974), Ekern and Wilson (1974), and Radner (1974). Under certain conditions, shareholders may not agree on what constitutes market value, but a voting solution may be possible; see Benninga and Muller (1979). Hart (1979) has shown that in large markets all shareholders will agree that market value is to be maximized, even though they may disagree what determines market value.

14. In a general equilibrium sense this strategy may be contradictory, since the shareholders of the firms which have the same production function as firm j may also decide to engage in hedging.

15. Students of the Lancasterian model in consumer economics have long known this to be true (cf., for example, Archibald and Rosenbluth (1975).

16. Note that the purchaser of the hedged commodity is not buying equity in firm 1; rather he is buying the price risk of the commodity produced by the firm, leaving the firm's shareholders with the quantity (production) risk.

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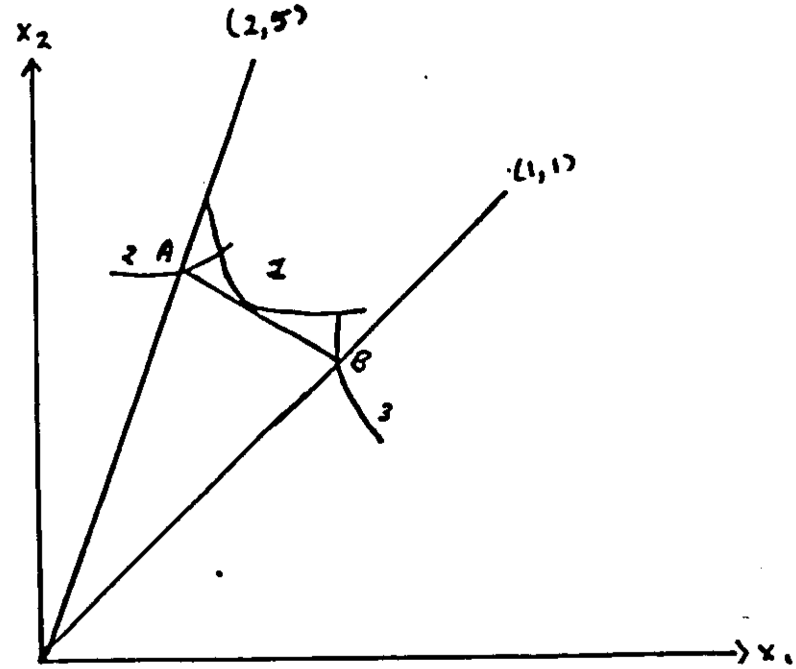
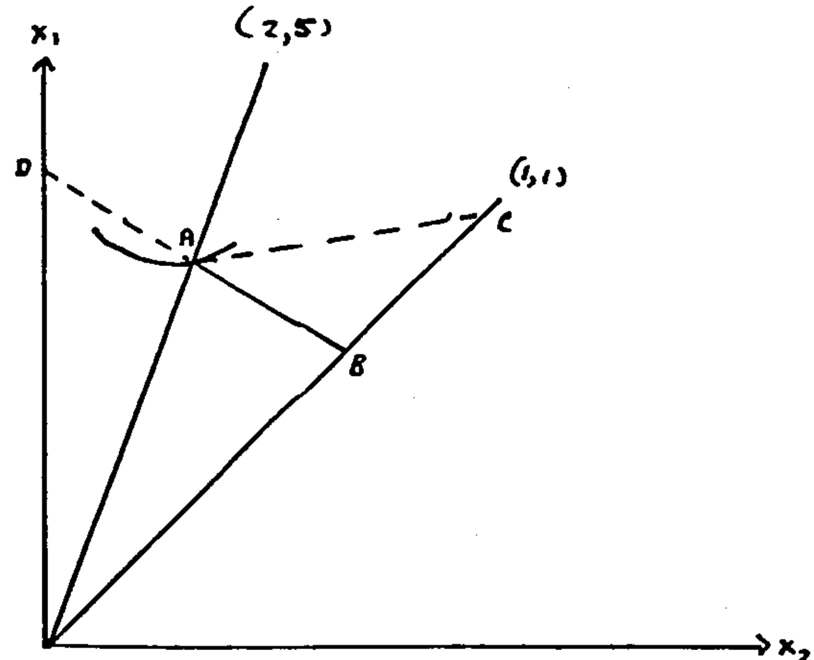


Figure 1



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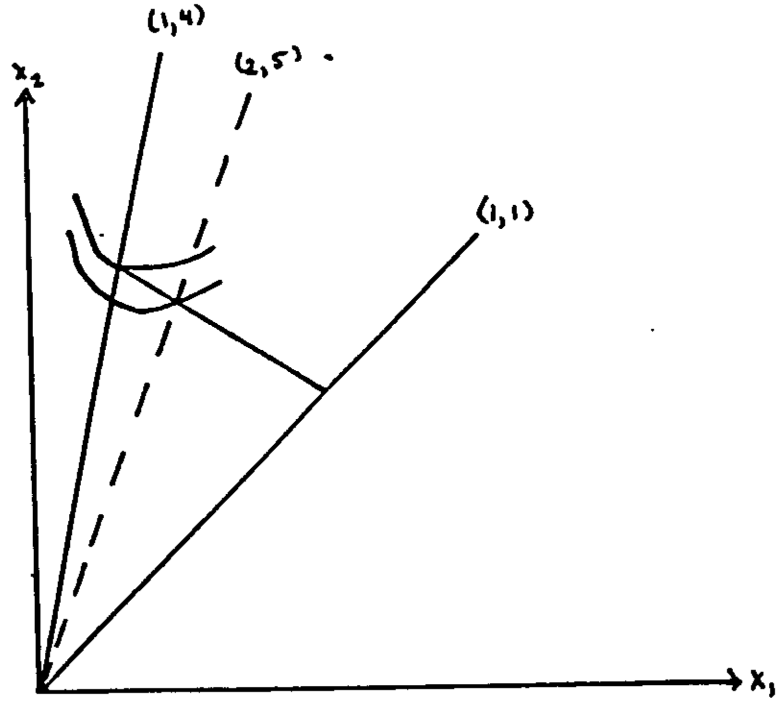

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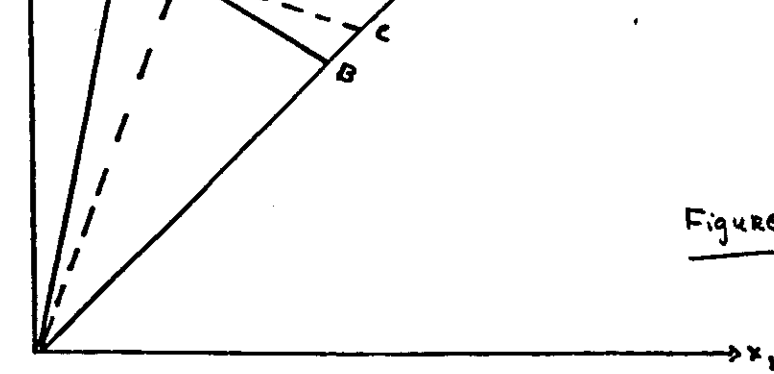

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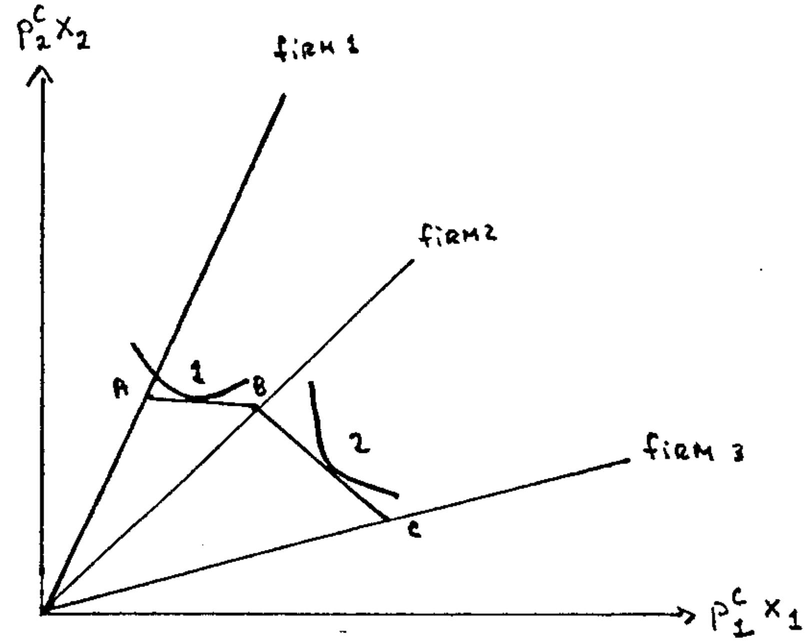

Figure 5

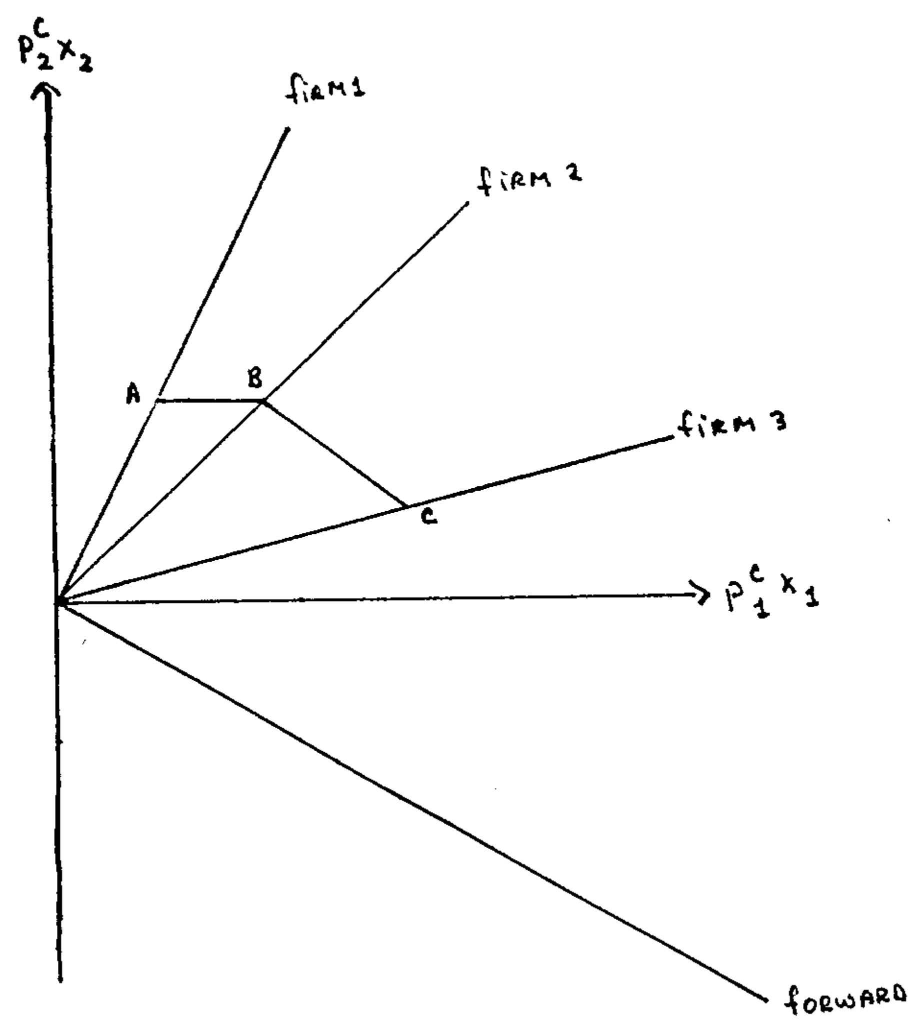

Figure 6