ARBITRAGE PRICING WITH INFORMATION

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The Arbitrage Pricing Theory is extended to a setting where investors possess information about future asset returns. A no-arbitrage pricing restriction is obtained with arbitrage defined conditional on the investor's information. The restriction can be stated with either conditional or unconditional expected returns, but both versions of the restriction contain the factor loadings identified from the unconditional covariance matrix of returns.

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1. INTRODUCTION

Popular financial pricing models relate expected asset returns to covariances between asset returns and other variables. The underlying models of portfolio behavior implicitly assume that individuals use whatever information they have to assess means and covariances, but this conditioning on information is not often an explicit feature of the models. As a consequence, empirical tests typically compute estimates of unconditional means and covariances, i.e., estimates based solely on historical returns. Such a simplification conforms to the theory if, for example, any information received by investors is independent of future asset returns. In that case, conditional and unconditional moments are identical. If investors instead receive information that is correlated with future returns, then the unconditional distribution of returns may not be appropriate for testing the theory.

This study extends the Arbitrage Pricing Theory (APT) of Ross (1976, 1977) to a setting where investors possess information about future asset returns. Returns are assumed to obey a factor model unconditionally. That is, an "uninformed" econometrician, who possesses only the unconditional (historical) distribution of returns, identifies a factor model generating returns. It is not assumed that investors perceive this factor model when conditioning on their information. Nevertheless, no-arbitrage pricing restrictions are obtained, and these restrictions involve the factor loadings from the unconditional distribution of returns. The latter property is convenient for estimation and is consistent with previous empirical work on the APT. In fact, a special case of the model presented here is the original APT restriction on unconditional expected returns.

Section 2 describes the statistical setting, defines arbitrage, and derives a restriction on the expectations of an investor who perceives no arbitrage given his information set. A similar restriction follows for any subset of that information set. Section 3 discusses empirical implications and briefly considers applying the model to heterogeneous-information economies.

2. THE MODEL

Let r_i be the return on asset i, and for N assets let $\underline{r}'_N \equiv [r_1, \ldots, r_N]$. Define the mean and covariance matrix of the unconditional (marginal) distribution of returns:

$$\underline{\mu}_N \equiv E\{\underline{r}_N\} , \tag{1}$$

$$V_N \equiv E\{[\underline{r}_N - \underline{\mu}_N][\underline{r}_N - \underline{\mu}_N]'\} . \tag{2}$$

<u>Assumption 1</u>. V_N admits a K-factor model. That is,

$$V_N = \Lambda_N \Lambda'_N + D_N , \tag{3}$$

where Λ_N is N by K, D_N is diagonal with i^{th} diagonal element d_{ii}, and there is a $d < \infty$ such that

$$0 < d_{ii} < d \text{ for all } i. \tag{4}$$

The augmented loading matrix, $B_N \equiv (\underline{\iota}_N \;\; \Lambda_N)$, has rank K + 1 for sufficiently large N, where $\underline{\iota}_N$ is the unit N-vector.

The structure for V_N in (3) implies that, unconditionally, returns are generated by a factor model,

$$\tilde{r}_i = \mu_i + \lambda_{i1}\tilde{x}_1 + \ldots + \lambda_{iK}\tilde{x}_K + \tilde{\varepsilon}_i , \qquad i = 1, \ldots, N , \tag{5}$$

where μ_i is the i^{th} element of $\underline{\mu}_N$ and λ_{ij} is the (i, j) element of Λ_N. The $\tilde{x}_j$'s have zero means and unit variances, and they are uncorrelated across j; the $\tilde{\varepsilon}_i$'s have zero means and variances d_{ii}, and they are uncorrelated across i. An investor can possess information, however, implying that the $\tilde{\varepsilon}_i$'s are not uncorrelated when conditioned on that information. Then, as far as that investor is concerned, there is not a K-factor model generating returns.

Let Y denote the information set of a given investor. Define the mean and covariance matrix of returns conditional on information:

$$\underline{\mu}_N(Y) \equiv E\{\underline{r}_N | Y\} , \tag{6}$$

$$\Sigma_N(Y) \equiv E\{[\underline{r}_N - \underline{\mu}_N(Y)][\underline{r}_N - \underline{\mu}_N(Y)]' | Y\} . \tag{7}$$

Assumption 2. The conditional covariance matrix can be expressed as

$$\Sigma_N(Y) = \Lambda_N G(Y) \Lambda_N' + F_N(Y) - A_N(Y) , \tag{8}$$

where G(Y) and $A_N(Y)$ are non-negative definite, $F_N(Y)$ is diagonal with i^{th} diagonal element $f_{ii}(Y)$, and Λ_N is the same as in (3). There is an $f(Y) < \infty$ such that

$$0 \leq f_{ii}(Y) \leq f(Y) \text{ for all } i . \tag{9}$$

The restriction in (8) permits the conditional variance of a portfolio's return to exceed the unconditional variance, but only through sources of variation (i) independent across assets or (ii) linearly related to the K factors in (5). To see this, note that if $A_N(Y)$ is absent from (8), then asset returns are generated (conditionally) by a factor model with the same λ_{ij}'s as in (5):

$$\tilde{r}_i = \mu_i(Y) + \lambda_{i1}\tilde{z}_1 + \ldots + \lambda_{iK}\tilde{z}_K + \tilde{\eta}_i , \quad i = 1, \ldots, N , \tag{10}$$

where $\mu_i(Y)$ is the i^{th} element of $\underline{\mu}_N(Y)$. The $\tilde{\eta}_i$'s have zero means and variances $f_{ii}(Y)$, and they are independent across i; $[\tilde{z}_1, \ldots, \tilde{z}_K]$ is a linear transformation of $[\tilde{x}_1, \ldots, \tilde{x}_K]$, but the $\tilde{z}_j$'s need not be mutually uncorrelated.[1] The subtraction of $A_N(Y)$ cannot increase the variance of any portfolio, but a full $A_N(Y)$ generally destroys the uncorrelatedness of the $\tilde{\eta}_i$'s and, thereby, the factor structure in (10).

Note that (8) admits restrictions of the form,

$$\Sigma_N(Y) = h(Y)V_N - A_N(Y) , \tag{11}$$

where h(Y) is a scalar and constant for all N. Let Y be the realization of a random vector, $\underline{\tilde{y}}$. As shown in the appendix, (11) obtains if $\underline{\tilde{r}}_N$ and $\underline{\tilde{y}}$ are jointly Normal or Student t. Thus, the algebraic restriction in (8) can be replaced with a (stronger) distributional assumption:

Assumption 2'. The joint distribution for returns, $\underline{\tilde{r}}_N$, and information, $\underline{\tilde{y}}$, is (i) multivariate Normal or (ii) multivariate Student t.[2]

Assumption 3. The elements of $\underline{\mu}_N$ and Λ_N are finite.

The APT considers an infinite sequence of economies indexed by N. The above assumptions pertain to any set of N assets, but there is a kind of "stationarity" imposed on the parameters:

Assumption 4. The parameters in (1) through (11) relevant to any specific asset are constant for all values of N.[3]

The last assumption requires that the number of assets does not supply information about future returns (i.e., Y does not vary with N).

Arbitrage is defined with respect to the investor's information set, Y. Let $\underline{c}_N$ be a vector of amounts invested in each asset. Arbitrage is defined as the existence of a subsequence (indexed $\hat{N}$) of portfolios such that

$$\underline{c}'_{\hat{N}} \underline{\iota}_{\hat{N}} = 0 \tag{12}$$

$$\lim_{\hat{N} \to \infty} E\{\underline{c}'_{\hat{N}} \underline{r}_{\hat{N}} | Y\} = \infty , \tag{13}$$

and

$$\lim_{\hat{N} \to \infty} \text{var}\{\underline{c}'_{\hat{N}} \underline{r}_{\hat{N}} | Y\} = 0 . \tag{14}$$

An investor arbitrages by forming a subsequence of zero-investment portfolios where, given his information, the variances of the portfolios' returns approach zero while the expected portfolio returns become infinite.[4] Such an opportunity is described as "Y permits arbitrage."

The absence of arbitrage implies restrictions on the investor's expectations.

Lemma 1. If Y does not permit arbitrage, then there exist $m(Y) < \infty$ and $\{\rho_N(Y), \gamma_{1N}(Y), \ldots, \gamma_{KN}(Y)\}$ such that

$$\sum_{i=1}^{N} [E(r_i | Y) - \rho_N(Y) - \sum_{j=1}^{K} \lambda_{ij} \gamma_{jN}(Y)]^2 \leq m(Y) \text{ for } N = 1, 2, \ldots \tag{15}$$

Proof. The proof follows that of Huberman (1981, theorem 1). Let $[\rho_N(Y), \gamma_{1N}(Y), \ldots, \gamma_{KN}(Y)]' \equiv \underline{\theta}_N(Y) = (B'_N B_N)^{-1} B'_N \underline{\mu}_N(Y)$ and let $\underline{b}_N(Y) = \underline{\mu}_N(Y) - B_N \underline{\theta}_N(Y)$.[5] If the lemma is false, then $\|\underline{b}_{\hat{N}}(Y)\|$ goes to infinity on some subsequence, $\hat{N}$.[6] Define the subsequence of portfolios,

$\underline{c}_{\hat{N}}(Y) = \alpha_{\hat{N}}(Y)\underline{b}_{\hat{N}}(Y)$, where $\alpha_{\hat{N}}(Y) = \|\underline{b}_{\hat{N}}(Y)\|^{2p}$ for $p \in (-1, -\frac{1}{2})$. [Note $\underline{c}'_{\hat{N}}(Y)\underline{\imath}_{\hat{N}} = 0$ since $B'_{\hat{N}}\underline{b}_{\hat{N}}(Y) = \underline{0}$.] The conditional mean is $E[\underline{c}'_{\hat{N}}(Y)\underline{r}_{\hat{N}}|Y] = \underline{c}'_{\hat{N}}(Y)\underline{\mu}_{\hat{N}}(Y) = \alpha_{\hat{N}}(Y)\|\underline{b}_{\hat{N}}(Y)\|^2 = \|\underline{b}_{\hat{N}}(Y)\|^{2p+2}$, which becomes infinite as $\hat{N} \to \infty$. The conditional variance is $\text{var}[\underline{c}'_{\hat{N}}(Y)r_{\hat{N}}|Y] = \underline{c}'_{\hat{N}}(Y)\ \Sigma_{\hat{N}}(Y)\underline{c}_{\hat{N}}(Y) = \alpha^2_{\hat{N}}(Y)\underline{b}'_{\hat{N}}(Y)[F_N(Y) - A_N(Y)]\underline{b}_{\hat{N}}(Y) \leq \alpha^2_{\hat{N}}(y)\|\underline{b}_{\hat{N}}(Y)\|^2 f(Y) = \|\underline{b}_{\hat{N}}(Y)\|^{4p+2} f(Y)$, which goes to zero as $\hat{N} \to \infty$. [The inequality follows from (9) and the non-negative definiteness of $A_N(Y)$.] Q.E.D.

It can also be shown there exists a fixed set of constants (given Y) satisfying (15) for all N.

Theorem 1. If Y does not permit arbitrage, then there exist $m(Y) < \infty$ and $\rho(Y), \gamma_1(Y), \ldots, \gamma_K(Y)$ such that

$$\sum_{i=1}^{\infty} \left[E(r_i|Y) - \rho(Y) - \sum_{j=1}^{K} \lambda_{ij}\gamma_j(Y)\right]^2 \leq m(Y) . \tag{16}$$

The proof of theorem 1 is identical to that given by Huberman (1981, theorem 2) and is omitted here.

The key to (15) and (16) is that the factor loadings (λ_{ij}'s) are identified from the unconditional covariance matrix and are, therefore, constant for all information sets. This property is useful in aggregating these restrictions to "coarser" information sets. A preliminary step is to recognize the probabilistic nature of the upper bound, m(Y). For any Y that does not permit arbitrage, lemma 1 guarantees an $m(Y) < \infty$, but m(Y) need not be constant for alternative information sets. Let Y be the realization of a random object, $\tilde{Y}$.

Assumption 5. If no realization of $\tilde{Y}$ permits arbitrage, there exists a function, m(Y), in (15) such that

$$E\{m(\tilde{Y})\} < \infty . \tag{17}$$

Saying that "no realization of $\tilde{Y}$ permits arbitrage" is basically ruling out arbitrage *ex ante*. That is, an investor knows he will not receive information allowing him to arbitrage. In such a case, any less-informed investor faces a constraint on his expectations identical to (15). The "less-informed" investor observes only Y*, a subset of Y.[7]

Lemma 2. If no realization of $\tilde{Y}$ permits arbitrage, and if Y* is a subset of Y, then there exist $m(Y^*) < \infty$ and $\{\rho_N(Y^*), \gamma_{1N}(Y^*), \ldots, \gamma_{KN}(Y^*)\}$ such that

$$\sum_{i=1}^{N} [E(r_i|Y^*) - \rho_N(Y^*) - \sum_{j=1}^{K} \lambda_{ij}\gamma_{jN}(Y^*)]^2 \leq m(Y^*) \text{ for } N = 1, 2, \ldots \tag{18}$$

Proof. Note (15) holds for $\underline{\theta}_N(Y)$ defined in the proof of lemma 1. Let $[\rho_N(Y^*), \gamma_{1N}(Y^*), \ldots, \gamma_{KN}(Y^*)] \equiv \underline{\theta}_N(Y^*) = E[\underline{\theta}_N(Y)|Y^*]$. (Specifically, $\underline{\theta}_N(Y^*) = (B_N'B_N)^{-1}B_N'E[\underline{\mu}_N(Y)|Y^*]$.) Take the expectation of both sides of (15) conditional on Y*. By Jensen's inequality, the ith term on the left side is

$$E\{[E(r_i|Y) - \rho_N(Y) - \sum_{j=1}^{K} \lambda_{ij}\gamma_{jN}(Y)]^2|Y^*\} \geq \{E[E(r_i|Y)|Y^*] - E[\rho_N(Y)|Y^*] - \sum_{j=1}^{K} \lambda_{ij}E[\gamma_{jN}(Y)|Y^*]\}^2 = \{E(r_i|Y^*) - \rho_N(Y^*) - \sum_{j=1}^{K} \lambda_{ij}\gamma_{jN}(Y^*)\}^2.$$

Thus, the lemma holds for $\underline{\theta}_N(Y^*)$ and $m(Y^*) \equiv E[m(Y)|Y^*]$. Q.E.D.

The analog of theorem 1 also holds in this case [and the proof is again identical to that of Huberman (1981, theorem 2)]:

Theorem 2. Under the conditions of lemma 2, there exist $m(Y^*) < \infty$ and $\rho(Y^*)$, $\gamma_1(Y^*)$, . . ., $\gamma_K(Y^*)$ such that

$$\sum_{i=1}^{\infty} \left[E(r_i | Y^*) - \rho(Y^*) - \sum_{j=1}^{K} \lambda_{ij}\gamma_j(Y^*)\right]^2 \leq m(Y^*). \tag{19}$$

A special case for Y^* is the null set.[8] Then (19) becomes a restriction on unconditional means that closely resembles the original APT [Ross (1976, 1977)]:

Corollary. If no realization of $\tilde{Y}$ permits arbitrage, then there exist $m < \infty$ and ρ, γ_1, . . ., γ_K such that

$$\sum_{i=1}^{\infty} \left[E(r_i) - \rho - \sum_{j=1}^{K} \lambda_{ij}\gamma_j\right]^2 \leq m . \tag{20}$$

Of course, (20) can also be obtained by ruling out arbitrage unconditionally (i.e., for an uninformed investor). [See Huberman (1981).]

Several characteristics of the restrictions deserve mention. The sum-of-squares bound in (16) implies an approximate linear relation,

$$E[\underline{r}_N | Y] \approx \rho(Y)\underline{\iota}_N + \Lambda_N \underline{\gamma}(Y) , \tag{21}$$

where $\underline{\gamma}'(Y) \equiv [\gamma_1(Y), \ldots, \gamma_K(Y)]$. The relation is "approximate" in that the mean squared error in (21) vanishes as $N \to \infty$. [A similar statement holds for (19) and (20).] If there is a risk-free asset with return r_F, then it is straightforward to show that $\rho(Y) = \rho = r_F$.[9] The interpretation of the γ's, or "factor premiums," is less straightforward. There are conditions, however, under which γ_j in (20) is the unconditional expected return on a "fully diversified" portfolio with a loading of unity on the j^{th} factor and loadings of zero on the other factors. [See Ingersoll (1982).] Similarly, $\gamma_j(Y)$ in (16) can be interpreted as the conditional expected return on the same portfolio.[10]

3. IMPLICATIONS

3.1 Empirical Work

Fundamental to empirical investigations of the APT is the hypothesis that asset returns obey a factor model as in (5). [E.g., Roll and Ross (1980), Reinganum (1981), and Chen (1982).] Moreover, the factor model is estimated unconditionally, using only time series of returns. This study finds that an unconditional factor model is relevant to the APT even when arbitrage is conditioned on information about future returns.

In addition to investigating a factor model, researchers typically test for a linear relation between the vector of unconditional expected returns, $E(\underline{r})$, and the columns of $(\underline{\iota}\ \Lambda)$. This remains an appropriate test of the APT with information, but the tests can also include conditional expected returns.[11] In fact, the relation can hold for a given information set (or unconditionally) but fail for a more inclusive (finer) information set. The value of theorem 2 to the econometrician is that tests using conditional means need not include the entire set of investors' information. Subsets of this information, such as publicly reported data, can also be used to test the model.

All tests, whether conditional or unconditional, that treat the APT as an exact linear relation face a problem. The bounded sum of squares [as in (20)] implies that linearity holds arbitrarily well for an infinite number of assets, but the linear approximation can be poor for a finite set of assets—

possibly the set chosen by the researcher.[12] It seems necessary for the researcher to argue that his set of assets is a random sample from an arbitrarily large (infinite?) universe of assets. Within a fixed-size random sample, the number of assets violating linearity (to an arbitrarily close approximation) approaches zero in probability as the underlying universe grows large.[13]

3.2 Homogeneous versus Heterogeneous Information

The model requires that an investor receives information consistent with assumption 2 (or 2'). The rational expectations literature makes an important distinction between (i) initial information possessed by investors before the market opens and (ii) revised information that includes inferences from subsequent prices. Normality is commonly assumed as the joint distribution for returns and initial information [e.g., Grossman (1978)]. If information is homogeneous across investors, then the same distribution holds, of course, for returns and revised information. Thus, it is straightforward to apply the no-arbitrage results in section 2 to a sequence of economies with joint Normality (as in assumption 2') and homogeneously informed investors.

If initial information is heterogeneous across investors, then an assumption about initial information is not necessarily preserved in the revision to include prices. To correctly apply the model under rational expectations, assumption 2 (or 2') must describe the revised information set. Nevertheless, it appears that this assumption is valid for some heterogeneous-information economies. For example, Admati (1982) assumes initial information is Normal and obtains a "noisy" rational expectations equilibrium price that is also Normal.[14]

If the model holds with heterogeneous information, then (21) characterizes beliefs across all investors who perceive no arbitrage opportunities. The vectors of such investors' expected returns are approximately (as described earlier) spanned by the columns of $(\underline{\iota}\ \Lambda)$. Aside from the approximation in (21), no-arbitrage investors can disagree only about ρ and $\underline{\gamma}$. For example, if $\rho(Y) = r_F$ and $K = 1$, then investors disagree only about the expected return on a fully-diversified "market-factor" portfolio (e.g., bulls vs. bears). This occurs even if no investor perceives a factor model given his information.[15] Essentially, the same factors that appear in time series of ex post returns also appear cross-sectionally in investors' ex ante returns (or in the time series of ex ante returns for a given investor).

APPENDIX

This appendix demonstrates that (11) is implied by assumption 2'. Let $\underline{r}_N$ be N × 1 and $\underline{y}$ be $\ell \times 1$. We first consider the Student t and then obtain the Normal as a limiting case.

(i) $[\underline{r}_N\ \underline{y}]$ obeys a multivariate Student t distribution with ν degress of freedom. Define

$$\mathrm{cov}\begin{bmatrix}\underline{r}_N \\ \underline{y}\end{bmatrix} \equiv \frac{\nu}{\nu - 2}\begin{bmatrix}\Sigma_{rr} & \Sigma_{ry} \\ \Sigma'_{ry} & \Sigma_{yy}\end{bmatrix} . \tag{A1}$$

Then

$$V_N \equiv \mathrm{cov}\left[\underline{r}_N\right] = \frac{\nu}{\nu - 2}\,\Sigma_{rr} \tag{A2}$$

and

$$\Sigma_N(\underline{y}) \equiv \mathrm{cov}\left[\underline{r}_N | \underline{y}\right] = \frac{\nu}{\ell + \nu - 2}(1 + q)\left[\Sigma_{rr} - \Sigma_{ry}\Sigma_{yy}^{-1}\Sigma'_{ry}\right] , \tag{A3}$$

where

$$q = \frac{1}{\nu}\left[\underline{y} - E(\underline{y})\right]'\Sigma_{yy}^{-1}\left[\underline{y} - E(\underline{y})\right] . \tag{A4}$$

[See Zellner (1971, appendix B.2).] Thus, (A3) satisfies (11) with

$$h(\underline{y}) = \frac{(\nu - 2)(1 + q)}{\ell + \nu - 2} \tag{A5}$$

and

$$A_N(\underline{y}) = \frac{\nu(1 + q)}{\ell + \nu - 2}\,\Sigma_{ry}\Sigma_{yy}^{-1}\Sigma'_{ry} . \tag{A6}$$

(ii) $[\underline{r}_N\ \underline{y}]$ obeys a multivariate Normal distribution. This case is obtained from the above by letting $\nu \to \infty$. Then $h(\underline{y}) = 1$ and $A_N(\underline{y}) = \Sigma_{ry}\Sigma_{yy}^{-1}\Sigma'_{ry}$. [See Zellner (1971, appendix B.1).]

FOOTNOTES

[1]Let $\underline{z} = P(Y)\underline{x}$, where $G(Y)$ is factored as $G(Y) = P(Y)P'(Y)$. Note $E[\underline{z}\underline{z}'] = \bar{G}(Y)$. I am grateful to an anonymous referee for suggesting the formulation in (8) and its interpretation via (10).

[2]An important difference between the two distributions is the effect of information on variances. Since the Student t allows $\underline{y}$ to affect $\Sigma(\underline{y})$, it is possible that some $\underline{y}$'s cause conditional variances of portfolio returns to exceed unconditional variances. Such a scenario is impossible with Normality since $\Sigma(\underline{y})$ is then constant. (See appendix.)

A Student t distribution could also be more satisfactory for empirical reasons. Blattberg and Gonedes (1974) conclude that daily common stock returns closely obey a Student t distribution, and such returns have been used in tests of the APT. [E.g., Roll and Ross (1980).]

[3]Since Λ_N is unique only up to an orthogonal rotation, it is implicitly assumed that a given rotation is preserved as the $(N + 1)^{th}$ row is added.

[4]Arbitrage is "asymptotic" in the sense originally described by Ross (1976, 1977), but the precise definition follows that of Huberman (1981). As Huberman notes, further conditions are needed to guarantee that (12)-(14) imply an unbounded subsequence of (conditional) expected utility.

[5]It is implicitly assumed that subsequences begin with N large enough to give B_N full column rank (cf. assumption 2).

[6]For an N-vector $\underline{v}$ with i^{th} element v_i, $\|\underline{v}\|^2 \equiv \sum_{i=1}^{N} v_i^2$.

[7]It may be useful to describe this setting more formally. Let $\underline{r}_N$ be a random vector on the probability space, (Ω, S, P). Let Y be a random object, $Y:(\Omega, S) \to (\Omega', S')$. Assumption 5 states that $m(Y)$ is Borel-measurable on (Ω', S') and P_y-integrable, where $P_y(A) = P\{\omega: Y(\omega) \in A\}$ for $A \in S'$. (This implies m is finite almost everywhere on P_y). Also note, for example, that $E(\underline{r}_N|Y)$ can be represented as $E(\underline{r}_N|G)$, where G is the σ-field induced by Y $[G \equiv Y^{-1}(S')]$. Saying "Y* is a subset of Y" is shorthand for conditioning on a sub σ-field of G, say G*. If $Y^* = g(Y)$, $g:(\Omega', S') \to (\Omega'', S'')$, then take $G^* = [g(Y)]^{-1}(S'')$. [See Ash (1972, section 6.4).]

[8]That is, condition on the coarsest σ-field, $\{\phi, \Omega'\}$. (Cf. footnote 7.).

[9]See Huberman's (1981) development of this corollary to his theorem 1.

[10]See, in particular, Ingersoll's (1982) theorems 3, 4, and 6. A "fully diversified" portfolio is the limit of a sequence of positive net investment portfolios with weights $\underline{\alpha}_N$ such that $N\|\underline{\alpha}_N\|^2$ is uniformly bounded. The key assumption in this interpretation of the γ's is that $\Omega_N = B_N'B_N/N$ is

nonsingular in the limit. The extension of Ingersoll's theorems to conditional expected returns is straightforward, since O_N is invariant with respect to Y.

[11]The approach of Gibbons and Ferson (1981) could be useful here.

[12]This point is stressed by Shanken (1982). Connor (1981) develops a stronger version of the APT in which the sum of squares is asymptotically zero. Connor makes additional restrictions on asset supplies and imposes a competitive equilibrium instead of the weaker no-arbitrage condition.

[13]Consider a sample of size n from a population of N assets, and hold n fixed as N goes to infinity. For a given $\delta > 0$, let L be the number of assets in the population such that

$$|E(r_i) - \rho - \sum_j \lambda_{ij}\gamma_j| > \delta , \qquad (*)$$

and note $L < m/\delta^2$ if (20) holds. For an asset randomly selected from the population of size N, define $p_N \equiv \text{prob}\{(*)\}$, and note $p_N = L/N < m/N\delta^2$. If ℓ is the number of assets in the sample that satisfy (*), then $\text{prob}\{\ell > 0\} = 1 - (1 - p_N)^n < 1 - (1 - m/N\delta^2)^n$ for $N > m/\delta^2$. Thus $\underset{N\to\infty}{\text{plim}}\ \ell = 0$.

[14]Admati's equilibrium price is actually the limit on a sequence of economies in which the number of agents becomes infinite but the number of assets is fixed. An application of her results to this setting would also require the number of assets to become infinite, and this possibility remains to be explored. [Equilibrium in her model precludes exact arbitrage but not necessarily the asymptotic arbitrage as in (12)-(14).]

[15]There is a special case in which a factor structure is commonly perceived. Assume (i) $\underline{r}$ and $\underline{y}$ are jointly Normal and (ii) the distribution of returns is compact [e.g., Samuelson (1970)] in that, if h is the length of the shortest trading interval, then mean and variance are both of order h. Conditional on $\underline{y}$,

$$\underline{r} = \underline{\mu}(\underline{y})h + \Gamma(\underline{y})\underline{\varepsilon}\sqrt{h}$$

where $\Gamma(\underline{y})\Gamma'(\underline{y}) = \Sigma(\underline{y})$ and $\underline{\varepsilon}$ is a vector of independent standard Normal variates. Note $E\{[\underline{r} - \underline{\mu}(\underline{y})][\cdot]'|\underline{y}\} = h\Sigma(\underline{y})$, and unconditionally, $E\{[\underline{r} - \underline{\mu}][\cdot]'\} = h\Sigma(\underline{y}) + h^2\text{cov}[\underline{\mu}(\underline{y})]$ since $\Sigma(\underline{y})$ is constant under Normality. Thus, differences between the conditional and unconditional covariance matrices are of order h^2. As $h \to dt$, an unconditional factor structure (for V) must also represent $\Sigma(\underline{y})$.

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