Notes On Taxation and Risk Taking

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The modern literature on taxation and risk taking begins with the work of Domar and Musgrave (1944). A number of subsequent papers have continued these investigations. With the notable exceptions of Stiglitz (1972) and Kanbur (1981), these studies have been exercises in partial equilibrium analysis. An earlier Stiglitz paper (1970), for example, studies the effect of taxation on the portfolio selection decisions of expected utility-maximizing investors.

The present paper is intended as an initial step toward a general equilibrium analysis of the effect of taxation on risky capital formation. The two relatively simple general equilibrium models used for this purpose incorporate unlimited entry by expected utility-maximizing entrepreneurs. The first model is a special case of the Kihlstrom-Laffont (1979) model in which entrepreneurs bear all risks. In the second model, entrepreneurs are able to transfer risks to non-entrepreneurs by selling output shares in a stock market. This model is described in detail in Kihlstrom-Laffont (1980).

We consider proportional income taxation as well as proportional taxation of the income from capital. We are not explicitly concerned with how the tax proceeds are spent. In particular, they are not assumed to be redistributed.

The paper is presented in four sections. In the first, we describe two partial equilibrium results, one of which is equivalent to a result in Stiglitz (1970). Our restatement of Stiglitz's result facilitates its application in a general equilibrium framework. The second section describes the Kihlstrom-Laffont (1979) model and uses the partial equilibrium analysis of Section 1 to study the effect of capital and income taxation on risky capital formation. This analysis is very closely related to the work of Kanbur (1981). The results are ambiguous, but the sources of the ambiguity are described. Section 3 applies the same analysis to the study of income taxation and capital taxation in the Kihlstrom-Laffont (1980) model. The ambiguities arising in the model with risk sharing are eliminated in this model because of the possibilities for risk sharing.

For the most part, the paper considers the case in which losses are subject to the same tax rate as income. Thus losses are, in effect, subsidized. This case is referred to as the case in which loss offsets are provided. The case in which loss offsets are not provided is discussed briefly at the end of Section 3.

The fourth section considers some extensions of the Kihlstrom-Laffont (1980) model. In these extensions, Stiglitz's (1970) results are directly applicable to the general equilibrium analysis.

1. A PARTIAL EQUILIBRIUM RESULT

We will consider a model in which individuals receive non-capital as well as capital income. It is first assumed that capital income is subject to proportional taxation, but that non-capital income is untaxed. Non-capital income is denoted by A and is non-random. Before tax capital income is denoted by I and may or may not be random depending on the choices made by the investor. If the capital tax rate is t, then after tax capital income is $I(1-t)$ and income from all sources is $W = A+I(1-t)$. It should be emphasized that these expressions for after tax capital income and total income are assumed to hold even if I is negative, i.e. even if investors suffer losses. Thus losses are assumed to be subsidized by the existence of "loss offset" provisions.

All of the investors considered are assumed to maximize the expected value of a von Neumann-Morgenstern utility function u, whose sole argument is total income. We will be concerned with the influence of taxation on risk taking in two very simple general equilibrium models. Before moving on to the analysis of these models, we state a partial equilibrium result that is closely related to a result obtained by Stiglitz (1970). Stiglitz shows that under two widely-accepted hypotheses about attitudes toward risk, increases in the marginal tax rate cause investors to raise their demand for risky assets. Stiglitz's hypotheses are that as wealth increases, investors become less risk averse in the absolute Arrow-Pratt sense but more risk averse in the relative Arrow-Pratt sense. Formally, he assumes that

$$R_a(W,u) \equiv -\frac{u''(W)}{u'(W)}$$

is a non-increasing function of wealth and that

$$R_r(W,u) = WR_a(W,u)$$

is a nondecreasing function of wealth. The theoretical and empirical support for these hypotheses are discussed by Stiglitz and by the papers he refers to.

We will show that Stiglitz's result can be viewed as a consequence of a more fundamental proposition which asserts that, under the Stiglitz hypotheses, increases in the marginal tax rate cause investors to be less risk averse. For the purpose of stating this result formally, we define

$$\Psi(I,t) \equiv u(A + I(1 - t)) \quad ,$$

which is the investor's utility function of before-tax capital income I. Our result asserts that if u is such that $R_a'(W,u) \leq 0$ and $R_r'(W,u) \geq 0$, then, for each I, the Arrow-Pratt absolute risk aversion measure

$$R^a(I,t,\Psi) = -\frac{\Psi''(I,t)}{\Psi'(I,t)}$$

is a non-increasing function of t.

PROPOSITION 1: <u>If</u> $R_a(W,u)$ <u>is a non-increasing (decreasing) function of</u> W <u>while</u> $R_r(W,u)$ <u>is a nondecreasing (increasing) function of</u> W, <u>then</u> $1 > t_1 \geq t_2 > 0$ <u>implies</u>

$$(1.1) \qquad R^a(I,t_2,\Psi) \geq (<) \; R^a(I,t_1,\Psi)$$

<u>for all</u> I <u>at which</u> $A+I(1-t_1) > 0$.

PROOF: Note first that

$$R^a(I,t,\Psi) = -(1 - t)\,\frac{u''(A + I(1 - t))}{u'(A + I(1 - t))} \quad .$$

Since $1 > t_1 \geq t_2$, $\left[\frac{A}{1-t_2} + I\right] > 0$ and $R_a(W,u)$ is non-increasing, we obtain

$$-\left[\frac{A}{(1-t_2)} + I\right](1-t_1)\,\frac{u''(A + I(1 - t_1))}{u'(A + I(1 - t_1))}$$

$$\leq -\left[\frac{A}{(1-t_2)} + I\right](1-t_1)\,\frac{u''\left(\left[\frac{A}{(1 - t_2)} + I\right](1 - t_1)\right)}{u'\left(\left[\frac{A}{(1 - t_2)} + I\right](1 - t_1)\right)}$$

Furthermore, since $R_r(W,u)$ is nondecreasing, $t_1 \geq t_2$ implies

$$- \left[\frac{A}{(1-t_2)} + I\right](1-t_1) \frac{u''\left(\left[\frac{A}{(1-t_2)} + I\right](1-t_1)\right)}{u'\left(\left[\frac{A}{(1-t_2)} + I\right](1-t_1)\right)}$$

$$\leq - \left[\frac{A}{(1-t_2)} + I\right](1-t_2) \frac{u''\left(\left[\frac{A}{(1-t_2)} + I\right](1-t_2)\right)}{u'\left(\left[\frac{A}{(1-t_2)} + I\right](1-t_2)\right)}$$

The theorem follows immediately from these inequalities when $\left[(A/(1-t_2)) + I\right] > 0$. ||

While we have already noted that Stiglitz's result is an immediate corollary of Proposition 1, it should also be pointed out that when combined with Pratt's Theorem 7, Stiglitz's result implies Proposition 1. Thus Proposition 1 can be viewed as an equivalent restatement of Stiglitz's result. By restating the result in this form, we are able to apply it directly to answer a rather general class of questions about the effect of taxation on risk taking. One such question is, of course, the one posed by Stiglitz; viz., how does taxation affect the portfolio decisions made by an investor faced with a choice between a safe and a risky asset? We are also able to ask, for example, how taxation influences the equilibrium level of risk taking in the economy as a whole. In general, Proposition 1 implies that if all individual utility functions exhibit decreasing absolute and increasing relative risk aversion, then an increase in the capital tax rate has the same effect on the economic equilibrium as a decrease in the level of risk aversion of all investors. In the sections that follow, we apply Proposition 1 to an analysis of the effects of capital income taxation in two specific simple general equilibrium models.

It is also possible to use the approach just described to investigate the effect on risk taking of income taxation. For that purpose, we let I represent total before-tax income and define

$$\Phi(I,t) \equiv u((1-t)I) \quad .$$

The following proposition is easily established using well-known arguments.

PROPOSITION 2: *If* $R_r(W,u)$ *is an increasing (nondecreasing) function of* W, *then the Arrow-Pratt absolute risk aversion measure*

$$R^a(I,t,\Phi) = -\frac{\Phi''(I,t)}{\Phi'(I,t)}$$

is a decreasing (non-increasing) function of t.

In its strong form, Proposition 2 implies that, when relative risk aversion is increasing, an increase in the income tax rate has the same effect on equilibrium as a decrease in risk aversion.

2. THE FREE ENTRY MODEL WITH NO STOCK MARKET

In this section, we study the effect of taxation in the model of Kihlstrom-Laffont (1979). The formal analysis will explicitly deal with the case of capital taxation. We will, however, indicate how the analysis changes if all income is taxed. Briefly, the model has the following features. There is a large number of individuals. In general, individuals differ in their attitudes toward risk, but we will consider only the case in which all individuals are the same. Each individual has a choice between supplying his capital for debt which pays a fixed (non-random) return or using his capital to become an entrepreneur. If an individual chooses to be an entrepreneur, he receives the random profits of the firm he creates. The only market is the market for debt capital. Equilibrium occurs when supplies equal demands in this market.

Formally, the set of individuals is identified with the interval $[0,1]$. There are two commodities: a production good that is interpreted as capital (in the Kihlstrom-Laffont paper this good is interpreted as labor) and a consumption good that we call income. All individuals begin with the same initial allocation of each good. Specifically, each individual owns one unit of capital and A units of income. Since capital is purely a production good, individuals have no desire to consume it. As a result, the sole argument of individual utility is income. The utility function of which the expected value is maximized by each of the identical individuals is denoted by u. We assume that u has a continuous second derivative and that $u' > 0$ while $u'' < 0$.

Any individual can become an entrepreneur and create a firm that produces income from capital. The process of creating a firm requires the use of one capital unit. The same technology is available to all individuals. It is described by the production function

$$g(K,x)$$

where K = the amount of capital employed in addition to the one unit used to create the firm, and

x = the value taken by a random variable $\tilde{x}$.

In this model, $\tilde{x}$ is assumed to have the same distribution for all firms. For our current purposes, no specific assumptions need be made about the correlation between the $\tilde{x}$'s that affect the output of different firms. The set X in which $\tilde{x}$ takes its values is, for simplicity, assumed to be a finite set of real numbers. If $Y \subset X$, we denote the objective probability of Y by $m(Y)$. We assume that, for all $(K,x) \in [0,\infty) \times X$,

$$g(K,x) \geq 0 \quad ,$$

and that $g(0,x) = 0$ for all $x \in X$. In addition, we assume that the second derivative of g with respect to K is continuous and that $g_K > 0$ while $g_{KK} < 0$.

Finally, there is assumed to exist a K^* at which

$$Eg_K(K^*,\tilde{x}) = Eg(K^*,\tilde{x})/(1 + K^*)$$

as illustrated in Figure 1.

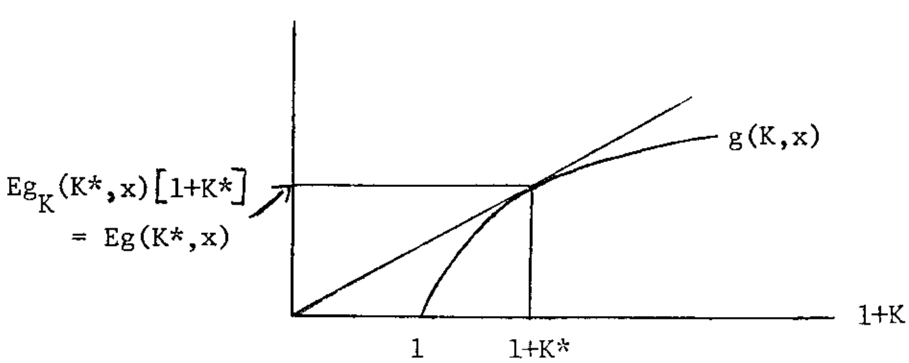

Figure 1

There is assumed to be a competitive capital market in which exchanges take place before $\tilde{x}$ is known. The equilibrium capital price is denoted by r. Capitalists, i.e. nonentrepreneurs, sell their labor for r. If the returns to capital are taxed at rate t, entrepreneurs choose K to maximize

$$\text{(2.1)} \qquad Eu(A + (1 - t)[g(K,\tilde{x}) - rK]) \quad ,$$

subject to the solvency constraint

$$m(A + (1 - t)[g(K,\tilde{x}) - rK] \geq 0) = 1 \quad .$$

In (2.1),

$$g(K,x) - rK$$

is the entrepreneur's before-tax profit and t is the marginal tax rate on income earned from capital. The r units of before-tax capital income earned by capitalists are also taxed at rate t.

With capital taxation, all individuals will be indifferent when faced with the worker-entrepreneur choice if

$$\text{(2.2)} \qquad F^1(r,K;t) \equiv u(A+(1-t)r) - Eu(A + (1-t)[g(K,\tilde{x}) - rK]) = 0 \quad .$$

When this equality is acheived and α individuals decide to be entrepreneurs, the demand for capital will be αK. If α is such that this demand equals $1-\alpha$, the supply of capital from nonentrepreneurs, then equilibrium is acheived. Thus, the equilibrium number of entrepreneurs is determined by K and the equation

$$\alpha K = 1 - \alpha \quad .$$

Specifically,

$$\alpha = \frac{1}{1 + K} \quad .$$

The equilibrium can be identified with the (r,K) pair that satisfies (2.2) and the first-order condition for maximization of (2.1) (assuming that an interior maximum is achieved). This first-order condition is

$$(2.3) \qquad F^2(r,K;t) = Eu'(A + (1-t)[g(K,\tilde{x}) - rK])[g_K(K,\tilde{x}) - r] = 0 \; .$$

One special case of some interest arises when u is linear; i.e., when all individuals are risk neutral. In this case, (2.3) reduces to

$$Eg_K(K,\tilde{x}) = r$$

and (2.2) becomes

$$r = Eg(K,\tilde{x}) - rK \quad .$$

Combining these equalities, one obtains

$$(2.4) \qquad r = Eg_K(K,\tilde{x}) = \frac{Eg(K,\tilde{x})}{[1 + K]} \quad .$$

Thus, if all individuals are risk neutral, the equilibrium K is the K^* of Figure 1. Kihlstrom-Laffont (1979) showed that in this case the equilibrium is efficient in the "first-best" sense of Arrow (1964) and Debreu (1959). They also showed that, when u is risk averse, the equilibrium K can exceed or fall below K^*. The Kihlstrom-Laffont analysis of the effect on K of an increase in the risk averseness of u concluded that the direction of this effect is anbiguous even if the absolute risk aversion of u is nonincreasing, and $g(K,x)$ and $g_K(K,x)$ always change in the same direction as x changes.

Under the hypotheses just mentioned, the equilibrium r was shown to fall when risk aversion increases. In view of Proposition 1, the Kihlstrom-Laffont analysis permits us to conclude that, when $R_a'(W,u) \leq 0$ and $R_r(W,u) \geq 0$ and when x changes $g(K,x)$ and $g_K(K,x)$ in the same direction, an increase in the capital tax rate, t, will raise the equilibrium r. The effect of t on the equilibrium K will, however, be ambiguous. Thus the increase in t may cause K to move closer to K^* or to move farther away from K^*.

The Kihlstrom-Laffont (1979) analysis can also be applied to study the effect of income tax rate changes on the equilibrium. In this case, Proposition 2 provides the basis for the application of the analysis. The results are again ambiguous. Specifically, even if we assume that $R_a'(W,u) \leq 0$ and $R_r'(W,u) \geq 0$ and $g(K,x)$ and $g_K(K,x)$ move in the same direction when x changes, the effect on K^* of an increase of the income tax rate, t, is ambiguous.

For the case of capital taxation, the nature of the ambiguity can be described easily with the aid of Figures 2a and 2b, which are slight reinterpretations of Figure 4 in Kihlstrom-Laffont (1979).

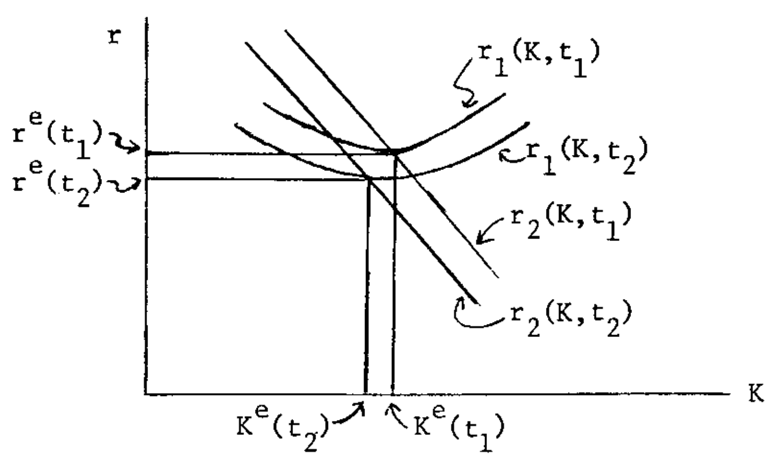

Figure 2a

$t_1 > t_2$

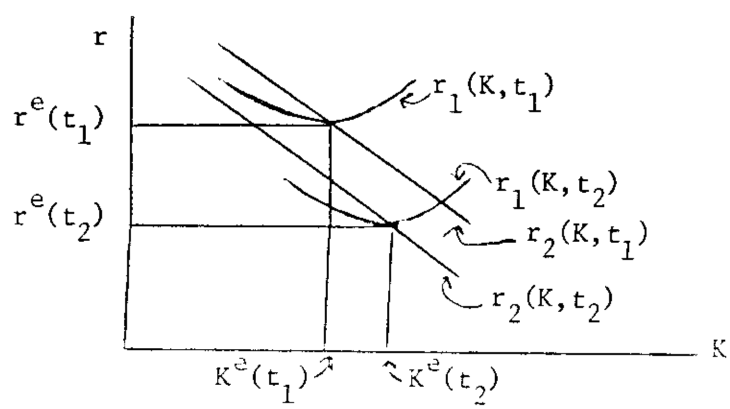

Figure 2b

$t_1 > t_2$

In these figures, $r_i(K,t)$ is the function defined implicitly by

$$F^i(r_i(K,t),K;t) = 0 \quad ,$$

for $i = 1$ and $i = 2$.

The equilibrium r and K values arising when the marginal tax rate is t are denoted by $r^e(t)$ and $K^e(t)$ respectively. Either Figure 2a or 2b may describe the situation when u exhibits nonincreasing absolute and nondecreasing relative risk aversion and when $g(K,x)$ and $g_K(K,x)$ always change in the same direction as x changes. Note that, in both figures, an increase in t from t_2 to t_1 causes both $r_1(K,t)$ and $r_2(K,t)$ to shift upward. The fact that

$$\partial r_i/\partial t > 0$$

for $i = 1$ and $i = 2$ follows from

$$\partial r_i/\partial t = - F^i_t/F^i_r \quad ,$$

$$F^1_r > 0 ,$$
$$F^1_t < 0 ,$$
$$F^2_r < 0 ,$$

and

$$F^2_t > 0 \quad .$$

The positivity of F^1_r is an immediate consequence of $u' > 0$. The negativity of F^2_r is established in Kihlstrom-Laffont (1979) and depends crucially on the hypotheses that u exhibits non-increasing absolute risk aversion and that $g(K,x)$ and $g_K(K,x)$ move in the same direction as x changes. The negativity of F_t^1 and positivity of F_t^2 follow from the analysis of Kihlstrom-Laffont (1979) and from Proposition 1 of this paper.

Because $\partial r_i/\partial t > 0$ for $i = 1$ and $i = 2$, $r^e(t_1)$ unambiguously exceeds $r^e(t_2)$. In Figure 2a, $K^e(t_1)$ also exceeds $K^e(t_2)$; but in Figure 2b, $K^e(t_1)$ is lower than $K^e(t_2)$. Hence $K^e(t)$ can either rise or fall with t.

We now argue that the sign of $\partial K^e/\partial t$ is ambiguous because the income of both capitalists and entrepreneurs is taxed at the same rate. The argument begins with a demonstration that if only capitalists are taxed, an increase in t lowers $K^e(t)$. We then show that if only entrepreneurs are taxed, an increase in t raises $K^e(t)$. In the analysis of both of these cases, we continue to assume that u exhibits nonincreasing absolute and nondecreasing relative risk aversion and that $g(K,x)$ and $g_K(K,x)$ change in the same direction when x changes.

Consider then the case in which only capitalists are taxed. In this case, (2.2) becomes

$$G^1(r,K;t) = u(A + (1-t)r) - Eu(A + g(K,\tilde{x}) - rK) = 0 ,$$

while (2.3) becomes

$$G^2(r,K;t) = Eu'(A + g(K,\tilde{x}) - rK)\left[g_K(K,\tilde{x}) - r\right] = 0 .$$

Note that a change in t has no influence on G^2. When $r_1(K,t)$ is defined by $G^1(r_1(K,t),K;t) = 0$,

$$\partial r_1/\partial t = - G_t^1/G_r^1 .$$

Since

$$G_t^1 = - u'(A + (1 - t)r) < 0$$

and

$$G^1_r \;=\; (1-t)u'(A + (1-t)r) + KEu'(A + g(K,\tilde{x}) - rK) \;>\; 0 \;,$$

we conclude that

$$\partial r_1/\partial t \;>\; 0 \;.$$

As a consequence of these remarks, we can replace Figures 2a and 2b by Figure 3, in which $K^e(t)$ is seen to fall unambiguously when t rises.

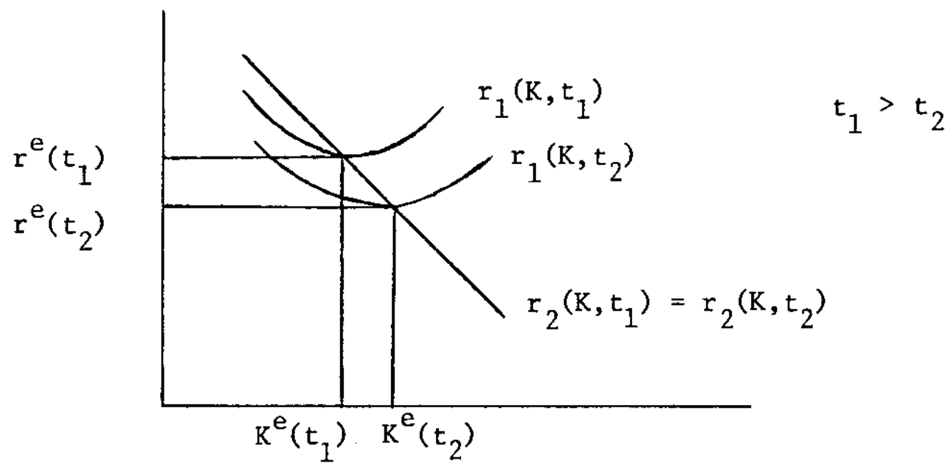

Figure 3

Note that in Figure 3, r_2 is a decreasing function of K. This feature of r_2 is a consequence of the hypotheses that u exhibits increasing absolute risk aversion and that $g(K,x)$ and $g_K(K,x)$ change in the same direction when x changes.

Now consider the case in which entrepreneurial profits are taxed at rate t but capitalists are untaxed. In this case, (2.2) becomes

(2.5) $$H^1(r,K;t) = u(A + r) - Eu(A + (1-t)[g(K,\tilde{x}) - rK]) \;=\; 0 \;,$$

while (2.3) continues to hold. Thus, the equilibrium values for r and K are those which simultaneously satisfy (2.5) and (2.3). As before, we denote these values by $r^e(t)$ and $K^e(t)$. We have already studied the effect of a t increase on the function $r_2(K,t)$ defined implicitly by

$$F^2(r_2(K,t),K;t) \;=\; 0 \quad .$$

We specifically noted that

$$\partial r_2/\partial t \;>\; 0 \quad .$$

If we now define $r_1(K,t)$ by

$$H^1(r_1(K,t),K;t) \;=\; 0 \quad ,$$

then

$$\partial r_1/\partial t \;=\; -\, H^1_t/H^1_r$$

where

$$H^1_r = u'(A+r) + (1-t)KEu'(A + (1-t)[g(K,\tilde{x}) - rK]) \;>\; 0$$

and

$$H^1_t = Eu'(A + (1-t)[g(K,\tilde{x}) - rK])[g(K,\tilde{x}) - rK] \quad .$$

At the equilibrium, the sign of H'_t can be determined by the fact that (2.3) holds. In particular, (2.3) implies that

$$H^1_t = Eu'(A + (1-t)[g(K,\tilde{x}) - rK])[g(K,\tilde{x}) - g_K(K,\tilde{x})K] \quad . \tag{2.6}$$

Since $g(K,x)$ is a strictly concave function of K for every x, (2.7) $g(K,x) - g_K(K,x)K > 0$

for all x. Combining (2.6), (2.7), and the positivity of u', we obtain

$$H_t^1 > 0 .$$

Thus,

$$\partial r_1/\partial t < 0$$

at the equilibrium, and for small changes in t, say from t_2 to $t_1 > t_2$, the equilibrium values of $r^e(t)$ and $K^e(t)$ will shift as shown in Figure 4.

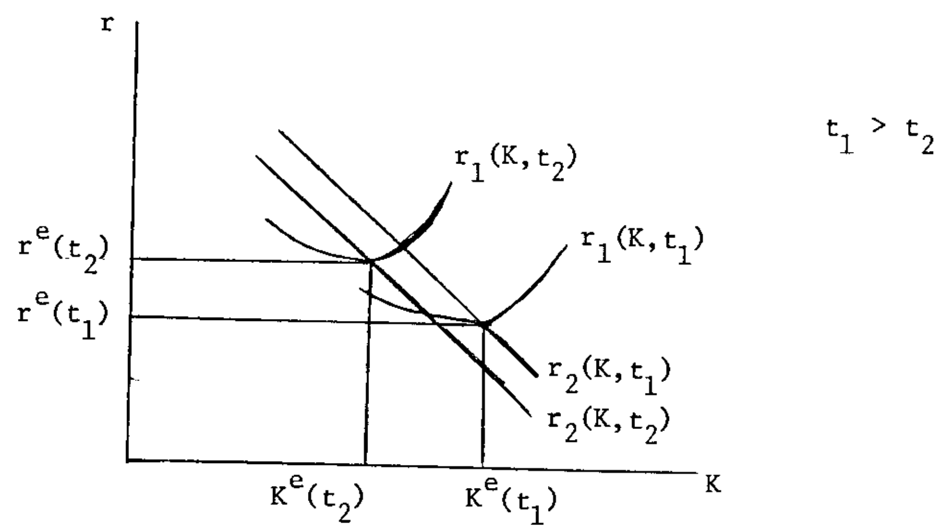

Figure 4

Since $\partial r_1/\partial t < 0$ and $\partial r_2/\partial t > 0$, the effect of the t change on $r^e(t)$ is ambiguous. The equilibrium K value, $K^e(t)$, unambiguously rises, however. A more formal demonstration that

$$\partial K^e/\partial t > 0$$

can be accomplished by implicitly differentiating (2.3) and (2.5). Specifically, since $K^e(t)$ and $r^e(t)$ are solutions to

$$H^1(K^e(t), r^e(t), t) = 0$$
$$F^2(K^e(t), r^e(t); t) = 0 ,$$

we obtain

$$\partial K^e/\partial t = -1/\Delta \begin{vmatrix} H^1_t & H^1_r \\ F^2_t & F^2_r \end{vmatrix} ,$$

where

$$\Delta = \begin{vmatrix} H^1_K & H^1_r \\ F^2_K & F^2_r \end{vmatrix} .$$

We have already observed that

$$F^2_t < 0 ,$$
$$F^2_r > 0 ,$$

and

$$H^1_r > 0 ,$$

and that at the equilibrium

$$H^1_t > 0 .$$

It is also easily verified using the arguments in Kihlstrom-Laffont (1979) that $u'' < 0$ and $g_{KK} < 0$ imply

$$F^2_K < 0$$

and

$$H^1_K = 0 .$$

Thus

$$\Delta = H_K^1 F_r^2 - H_r^1 F_K^2 > 0 \quad ,$$

and

$$\begin{vmatrix} H_t^1 & H_r^1 \\ F_t^2 & F_r^2 \end{vmatrix} = H_t^1 F_r^2 - F_t^2 H_r^1 < 0 \quad ,$$

so that

$$\partial K^e / \partial t > 0 \quad .$$

3. THE GENERAL EQUILIBRIUM MODEL WITH A STOCK MARKET

In the model of this section, we introduce a stock market to the model of Section 2. The resulting stock market model is described in slightly more generality than the model of Section 2. Specifically, we drop the assumption that all individuals are alike. This is done primarily as an aid to interpretation. The propositions of this section, however, apply only to the case in which all individuals are alike. The fourth section does briefly discuss the more general case, however.

Again we describe the model for the case of capital taxation and indicate briefly how the case of income taxation differs. The set of individuals is still identified with the interval $[0,1]$. These individuals are now divided into n types. The Lebesgue measure of the set of type i individuals is μ_i. We assume throughout that $\mu_i > 0$ for all i. As before, there are two commodities: capital and income. All individuals again begin with one unit of capital and A units of income, and the sole argument of individual utility functions is income. The utility function of all type i individuals is denoted by u_i. We assume that u_i has a continuous second derivative, and that $u_i' > 0$ while $u_i'' \leq 0$.

Any individual can create a firm or firms that produce income from capital. The process of creating each firm requires the use of one capital unit. Capital employed for this purpose is referred to as set-up capital or entrepreneurial capital. The technology employed by every firm is the same and is decribed by the production function

$$g(K,x)$$

where K = the amount of nonentrepreneurial or "operating" capital employed, and

x = the value taken by a random variable $\tilde{x}$.

The random variable $\tilde{x}$ is now assumed to be the same for all firms. This is a stronger hypothesis than that made in Section 2. It means that the output of all firms is determined by the same random influences. The set X in which $\tilde{x}$ takes its values is, for simplicity, assumed to be a finite set of real numbers. If $Y \subset X$, we continue to denote the objective probability of Y by $m(Y)$. We assume that, for all (K,x),

$$g(K,x) \geq 0$$

and that $g(0,x) = 0$ for all x. In addition, we assume that the second derivative of g with respect to K is continuous and that $g_K > 0$ while $g_{KK} < 0$. Finally, for all $x \in X$, there is assumed to exist a $K(x)$ at which

$$g_K(K(x),x)[1 + K(x)] = g(K(x),x)$$

as illustrated in Figure 5.

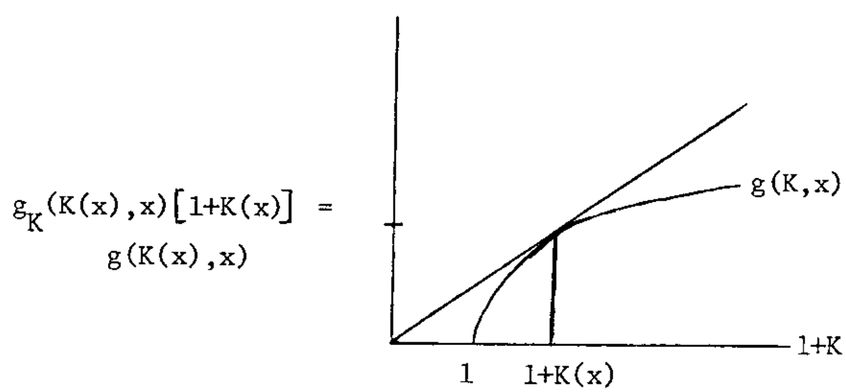

Figure 5

Again this is a stronger assumption than that made in Section 2.

A competitive debt market is assumed to exist in which capital can be exchanged for income. The income value of a capital unit in this market is denoted by r. Entrepreneurs who create firms can now raise operating capital as well as entrepreneurial capital by issuing debt or by selling equity. A firm that employs K units of operating capital and obtain B capital units in the debt market generates random profits which equal

$$\tilde{\pi}(K,B) = g(K,x) - rB \quad . \tag{3.1}$$

The entrepreneur who creates this firm will be obliged to obtain $K-B$ capital units by selling shares. The market for firm shares is assumed to be competitive. There is specifically a function N, that specifies for each possible (K,B) the capital value of shares to firms employing K capital units and raising B of these units in the debt market. All income received as a return to capital is taxed at rate t.

Suppose now that an entrepreneur creates a firm that employs K capital units and buys B units of capital in the debt market. If this entrepreneur retains $\gamma\times 100\%$ of the firm's shares, his after-tax income will equal

$$\tilde{W}_E(K,B,\gamma) = A + \{r[N(K,B)(1-\gamma) - (K-B)] + \gamma[g(K,\tilde{x}) - rB]\}(1-t) \quad . \tag{3.2}$$

Consider next a nonentrepreneur, a capitalist in our terminology, who buys $\gamma\times 100\%$ of the shares of a firm that uses K units of operating capital and that has raised B capital units by selling debt. This capitalist's random after-tax income is

$$\tilde{W}_C(K,B,\gamma) = A + \{r[1 - \gamma N(K,B)] + \gamma[g(K,\tilde{x}) - rB]\}(1 - t) \quad . \tag{3.3}$$

An entrepreneur of type i chooses (K,B,γ) to maximize

$$Eu_i(\tilde{W}_E(K,B,\gamma)) \tag{3.4}$$

subject to the non-negativity constraint

$$(3.5) \qquad m(\tilde{W}_E(K,B,\gamma) \geq 0) = 1 \quad .$$

We let $(\hat{K}_i,\hat{B}_i,\hat{\gamma}_i)$ denote the solution to this problem.[1/]

It is important to observe at this point that even if entrepreneurs of type i could invest in firms for which $(K,B) \neq (\hat{K}_i,\hat{B}_i)$, they would not find it advantageous to do so. The proof of this result, which is given in Kihlstrom-Laffont (1980), makes crucial use of the assumption that the same random variable $\tilde{x}$ enters the production function of all firms. The Kihlstrom-Laffont arguments also imply that capitalists of type i find it optimal to invest only in firms for which $(K,B) = (\hat{K}_i,\hat{B}_i)$. They choose γ to maximize

$$(3.6) \qquad Eu_i(\tilde{W}_C(\hat{K}_i,\hat{B}_i,\gamma))$$

subject to

$$(3.7) \qquad m(\tilde{W}_C(\hat{K}_i,\hat{B}_i,\gamma) \geq 0) = 1 \quad .$$

In equilibrium, we must have

$$\max_{\{(K,B,\gamma) \text{ satisfying } (3.5)\}} Eu_i(\tilde{W}_E(K,B,\gamma)) = \max_{\{\gamma \text{ satisfying } (3.7)\}} Eu_i(\tilde{W}_C(\hat{K}_i,\hat{B}_i,\gamma))$$

for all i. As shown in the earlier Kihlstrom-Laffont paper, this will only be true if

$$N(K,B) \leq 1 + K - B$$

1/ Of course, $(\hat{K}_i,\hat{B}_i,\hat{\gamma}_i)$ depends on N and r. Our notation suppresses that dependence.

for all (K,B) and

$$N(\hat{K}_i,\hat{B}_i) = 1 + \hat{K}_i - \hat{B}_i$$

for all i. Following Kihlstrom-Laffont, we, in fact, assume that

(3.8) $$N(K,B) = 1 + K - B$$

for all (K,B). Under this assumption, the Modigliani-Miller theorem holds, so that $W_C(K,B,\gamma)$ and $W_E(K,B,\gamma)$ are equal and independent of B. In fact,

$$\tilde{W}_C(K,B,\gamma) = \tilde{W}_E(K,B,\gamma) = A + \{r[1-\gamma(1+K)] + \gamma g(K,\tilde{x})\}(1 - t)) \quad .$$

Thus, all individuals of type i hold $\hat{\gamma}_i \times 100\%$ of the shares in a firm for which $K = \hat{K}_i$ where $(\hat{K}_i,\hat{\gamma}_i)$ maximizes

(3.9) $$Eu_i(A + \{r[1 - \gamma(1 + K)] + \gamma g(K,\tilde{x})\}(1 - t)$$

subject to

(3.10) $$m(A + \{r[1 - \gamma(1 + K)] + \gamma g(K,\tilde{x})\}(1 - t) \geq 0) = 1 \quad .$$

Supply equals demand in the market for shares to firms with $K = \hat{K}_i$ when there are

(3.11) $$\hat{\nu}_i = \mu_i\hat{\gamma}_i$$

of these firms. When r has adjusted to the point at which

$$(3.12) \qquad \sum_{i=1}^{n} \hat{\nu}_i(1 + \hat{K}_i) = 1 \quad ,$$

the market for capital is also in equilibrium.

In summary, an equilibrium of the economy just described is a vector $\langle(\hat{\gamma}_i,\hat{K}_i)_{i=1}^n, r\rangle$ such that, for each i, $(\hat{\gamma}_i,\hat{K}_i)$ maximizes (3.9) subject to (3.10) and such that (3.12) holds when $\hat{\nu}_i$ is defined by (3.11). If we define C by

$$(3.13) \qquad C \equiv \gamma(1 + K) \quad ,$$

an equilibrium $\langle(\hat{\gamma}_i,\hat{K}_i)_{i=1}^n, r\rangle$ can be identified with a vector

$$(\hat{C}_i,\hat{K}_i)_{i=1}^n, r$$

such that $(\hat{C}_i,\hat{K}_i)$ maximizes

$$(3.14) \qquad Eu_i(A + \left[r(1 - C) + C\,\frac{g(K,\tilde{x})}{1 + K}\right](1 - t))$$

subject to

$$(3.15) \qquad m(A + \left[r(1 - C) + C\,\frac{g(K,\tilde{x})}{1 + K}\right](1 - t) \geq 0) = 1$$

and such that

$$(3.16) \qquad \sum_{i=1}^{n} \mu_i \hat{C}_i = 1 \quad .$$

If all income is taxed, the model just described can still be applied if, in (3.2), (3.3), (3.9), (3.10), (3.14), (3.15) and all other expressions involving $\tilde{W}_E$ and $\tilde{W}_C$, the factor $(1-t)$ is applied to A as well as the income from capital.

We can now apply the results of Section 1 to the model just described for the purpose of studying the effect of taxation on risk taking. As an immediate corollary of Proposition 1, we observe that, if t is the tax on capital income and if all u_i exhibit decreasing absolute and increasing relative risk aversion, then an increase in the capital tax rate has the same effect on $\langle(\hat{Y}_i,\hat{K}_i)_{i=1}^n,r\rangle$ as an increase in the Arrow-Pratt absolute risk aversion measure of all u_i. Similarly, when t is the rate of tax on all income, Proposition 2 implies that, if $R_r'(W,u_i) \geq 0$ for all i, then an increase in the income tax rate has the same effect on the equilibrium as an increase in $R_a(W,u_i)$ for all i.

We would like to be more specific about the form of the effect which a t increase has on the equilibrium. For this purpose, we return to the special case in which all individuals have the same utility function u. When t is a tax on capital income, the analysis of Kihlstrom-Laffont (1980) implies that, for this special case, the unique equilibrium $\langle\hat{Y},\hat{K},r\rangle$ can be identified with the $\hat{K}$ that maximizes

$$(3.17) \qquad Eu\left(A + \left[\frac{g(K,\tilde{x})}{1 + K}\right](1 - t)\right) \quad ,$$

if we let $\hat{Y}$ and r be defined by

$$(3.18) \qquad \hat{Y} = 1/(1 + \hat{K})$$

and

$$(3.19) \qquad r = \frac{Eu'\left(A + \left[\frac{g(\hat{K},\tilde{x})}{1 + \hat{K}}\right](1 - t)\right)\frac{g(\hat{K},\tilde{x})}{1 + \hat{K}}}{Eu'\left(A + \left[\frac{g(\hat{K},\tilde{x})}{1 + \hat{K}}\right](1 - t)\right)}$$

In this equilibrium, $1/(1+\hat{K})$ capital units are used to create firms while the remaining $1-(1/(1+\hat{K})) = \hat{K}/(1+\hat{K})$ capital units are employed as operating capital.

When t is reinterpreted as a tax on all income, the equilibrium $\hat{K}$ is that which maximizes

$$(3.17') \qquad Eu\left(\left[A + \frac{g(K,\tilde{x})}{1 + K}\right](1 - t)\right) \quad .$$

With $\hat{K}$ thus redefined, (3.18) continues to hold. For this case, the expression (3.19) for the equilibrium r must also be modified by multiplying A by $(1-t)$.

Note that, in these equilibria, there is no demand for riskless debt, since

$$1 - \hat{\gamma}(1 + \hat{K}) = 0 \quad .$$

Thus, if we were to adopt the standard approach which uses demand for the safe asset to measure the risk averseness of behavior, we would be led to conclude that taxation has no effect on risk taking. This is a questionable conclusion since changes in t do, in general, change $\hat{K}$, the K level that characterizes the equilibrium.[2] In fact, when t is the tax on capital income, Proposition 1 implies that, when $R_a' \leq 0$ and $R_r' \geq 0$, the effect on $\hat{K}$ of a t change is the same as that of an increase in risk aversion. Similarly, Proposition 2 implies that, when t is a tax on all income, $R_r' \leq 0$ implies that a t change has the same effect on $\hat{K}$ as an increase in the level of risk aversion. Our third proposition gives conditions on g that make it possible to determine

---

[2] One important special case in which $\hat{K}$ is unaffected by a t change occurs when g exhibits stochastic constant returns to scale; i.e., when

$$g(K,x) = h(K)x$$

for all K and x.

the effect of an increase in risk aversion on $\hat{K}$. For the purpose of stating this proposition, we modify our notation to make the dependence of K on u explicit. Thus, $\hat{K}(u)$ will be used to denote the K value that maximizes (3.17). Once again the formal analysis proceeds on the assumption that t is a tax on capital income. As before, the only modification required to apply the same analysis to the case of income taxation is the replacement of A by A(1-t).

We also use $\gtrsim$ to denote the partial ordering "more risky than." This ordering was introduced by Rothschild and Stiglitz (1970). Thus, we will use the notation:

$$\tilde{Z}_1 \gtrsim \tilde{Z}_2$$

to mean that $\tilde{Z}_1$ is more risky than $\tilde{Z}_2$ in the Rothschild-Stiglitz sense.

We also note finally that when u is linear; i.e., when all individuals are risk neutral; $\hat{K}(u) = K^*$, where K^* maximizes

$$\frac{Eg(K,\tilde{x})}{(1 + K)} \quad .$$

In this case, $\hat{K}(u)$ is independent of t.

PROPOSITION 3: <u>Suppose that</u> u <u>and</u> v <u>are two utility functions such that, for all</u> W,

$$(3.20) \qquad -\frac{u''(W)}{u'(W)} > -\frac{v''(W)}{v'(W)} \quad .$$

<u>If</u> $g(\hat{K}(v),x)$ <u>and</u> $g(K^*,x)$ <u>are both increasing functions of</u> x <u>and if</u> $g(\hat{K}(v),x) - g_K(\hat{K}(v),x)(1+\hat{K}(v))$ <u>and</u> $g(K^*,x) - g_K(K^*,x)(1+K^*)$ <u>are both increasing (decreasing) functions of</u> x, <u>then</u>

$$\hat{K}(u) > \hat{K}(v) > K^* \quad .$$
$$(K(u) < \hat{K}(v) < K^*)$$

Now assume that $g(K,x)$ is an increasing function of x and $g(K,x)-g_K(K,x)(1+K)$ is an increasing or a decreasing functin of x for all K between $\hat{K}(u)$ and K^*. Then the following results obtain:

$$(3.21) \qquad \frac{g(K^*,\tilde{x})-Eg(K^*,\tilde{x})}{1+K^*} \gtrsim \frac{g(\hat{K}(v),\tilde{x})-Eg(\hat{K}(v),\tilde{x})}{1+\hat{K}(v)} \gtrsim \frac{g(\hat{K}(u),\tilde{x})-Eg(\hat{K}(u),\tilde{x})}{1+\hat{K}(u)}$$

and

$$(3.22) \qquad \frac{Eg(K^*,x)}{1+K^*} > \frac{Eg(\hat{K}(v),\tilde{x})}{1+\hat{K}(v)} > \frac{Eg(\hat{K}(u),\tilde{x})}{1+\hat{K}(u)} \quad .$$

Furthermore, if we suppose that ξ and ζ are any two increasing concave functions such that

$$(3.23) \qquad -\frac{\zeta''(W)}{\zeta'(W)} \geq -\frac{\xi''(W)}{\xi'(W)}$$

for all W, and that

$$(3.24) \qquad E\xi\left(\frac{g(\hat{K}(u),\tilde{x})}{1+\hat{K}(u)}\right) \geq (>) \quad E\xi\left(\frac{g(\hat{K}(v),\tilde{x})}{1+\hat{K}(v)}\right) ,$$

then we can conclude that

$$(3.25) \qquad E\zeta\left(\frac{g(\hat{K}(u),\tilde{x})}{1+\hat{K}(u)}\right) \geq (>) \quad E\xi\left(\frac{g(\hat{\hat{K}}(v),\tilde{x})}{1+\hat{K}(v)}\right) .$$ [3]

As a special case, we obtain that

$$E\xi\left(\frac{g(\hat{K}(v),\tilde{x})}{1+\hat{K}(v)}\right) \geq E\xi\left(\frac{g(K^*,x)}{1+K^*}\right)$$

---

[3] The fact that (3.24) implies (3.25) when ξ and ζ are related by (3.23) was pointed out to us by Oliver Hart. The proof given below of this fact is Hart's. We would also like to thank Sandy Grossman for his contributions to our discussion of this result and its proof.

implies

$$E\zeta\left(\frac{g(\hat{K}(v),\tilde{x})}{1+\hat{K}(v)}\right) \geq E\zeta\left(\frac{g(K^*,x)}{1+K^*}\right) .$$

REMARKS: Proposition 2 demonstrates that a decrease in risk aversion results in an equilibrium K choice that involves more risks. Specifically, (3.21) asserts that the deviations of equilibrium individual incomes from their means are more risky. This, of course, implies that the variance of equilibrium individual income is larger when risk aversion is lower. Because of (3.22), the increase in the riskiness of income is compensated for by an increase in average income. The final conclusion of the proposition asserts that if some individual prefers the less risky income distribution $g(\hat{K}(u),\tilde{x})/(1+\hat{K}(u))$, then every individual ξ who is at least as risk averse will also prefer the less risky distribution. If, in other words, the increase in average income from $Eg(\hat{K}(u),\tilde{x})/(1+\hat{K}(u))$ to $Eg(\hat{K}(v),\tilde{x})/(1+\hat{K}(v))$ fails to compensate ξ for the increase in riskiness of $g(K,\tilde{x})/(1+K)$ when K changes from $\hat{K}(u)$ to $\hat{K}(v)$, then the same increase in mean income will also fail to compensate more risk averse individuals for the same increase in income risks.

PROOF: We consider only the case in which $g(\hat{K}(v),\tilde{x})-g_K(\hat{K}(v),\tilde{x})(1+\hat{K}(v))$ is increasing in x. The first-order condition satisfied by $\hat{K}(u)$ is

$$(3.26) \qquad F(\hat{K}(u),u) = 0$$

where

$$F(K,u) = E\{u'(W(K,\tilde{x}))[g(K,\tilde{x}) - g_K(K,\tilde{x})(1+K)]\}$$

and

$$W(K,x) = A + \left[\frac{g(K,x)}{1+K}\right](1 - t) \quad .$$

Similarly, $\hat{K}(v)$ solves

$$(3.27) \qquad F(\hat{K}(v),v) = 0 \quad .$$

Note that

$$(3.28) \qquad F_K(K,u) = - E\{u'(W(K,\tilde{x}))g_{KK}(K,\tilde{x})(1+K)\}$$
$$- E\{u''(W(K,\tilde{x}))\ g(K,\tilde{x}) - g_K(K,\tilde{x})(1+K)\ ^2(1+K)^2\} \quad .$$

Since $g_{KK} < 0$, (3.28) implies that $F_K(K,u) > 0$ for all $K \geq 0$ and all u for which $u' > 0$ and $u'' \leq 0$.

Pratt has shown that (3.20) holds if and only if

$$u(I) = \phi(v(I)) \quad ,$$

where $\phi' > 0$ and $\phi'' < 0$. As a result, $F(K,u)$ can be rewritten as

$$(3.29) \qquad F(K,u) = E\{\phi'(v(W(K,\tilde{x})))v'(W(K,\tilde{x}))\Big[g(K,\tilde{x})-g_K(K,\tilde{x})(1+K)\Big]\}$$
$$= E\{\phi'(v(W(K,\tilde{x})))v'(W(K,\tilde{x}))\Big[g(K,\tilde{x})-g_K(K,\tilde{x})(1+K)\Big]\,\Big|\,x\varepsilon X^-\}m(X^-)$$
$$+ E\{\phi'(v(W(K,\tilde{x})))v'(W(K,\tilde{x}))\Big[g(K,\tilde{x})-g_K(K,\tilde{x})(1+K)\Big]\,\Big|\,x\varepsilon X^+\}m(X^+) \quad .$$

where

$$X^+ = \{x \varepsilon X\colon g(K,x) - g_K(K,x)(1+K) > 0\}$$

and

$$X^- = \{x \varepsilon X\colon g(K,x) - g_K(K,x)(1+K) < 0\} \quad .$$

If $g(\hat{K}(v),x)$ and $g(\hat{K}(v),x)-g_K(\hat{K}(v),x)(1+\hat{K}(v))$ both increase with x, there exists a number λ such that

$$v(W(\hat{K}(v),x)) < \lambda \qquad \text{if} \quad x \in X^- \tag{3.30}$$

and

$$v(W(\hat{K}(v),x)) > \lambda \qquad \text{if} \quad x \in X^+ \quad . \tag{3.31}$$

Since $\phi'' < 0$, (3.30) and (3.31) imply that

$$\phi'(v(W(K,x)))\left[g(K,x)-g_K(K,x)(1+K)\right] < \phi'(\lambda)\left[g(K,x)-g_K(K,x)(1+K)\right] \tag{3.32}$$

for all $x \in X^+ \cup X^-$, when $K = \hat{K}(v)$. Combining (3.27), (3.29) and (3.32), we observe that

$$F(\hat{K}(v),u) < \phi'(\lambda)F(\hat{K}(v),v) = 0 \quad . \tag{3.33}$$

Inequality (3.33), the first-order condition (3.26), and the fact that $F_K < 0$ together imply that

$$\hat{K}(u) < \hat{K}(v) \quad .$$

The situation is depicted in Figure 5.

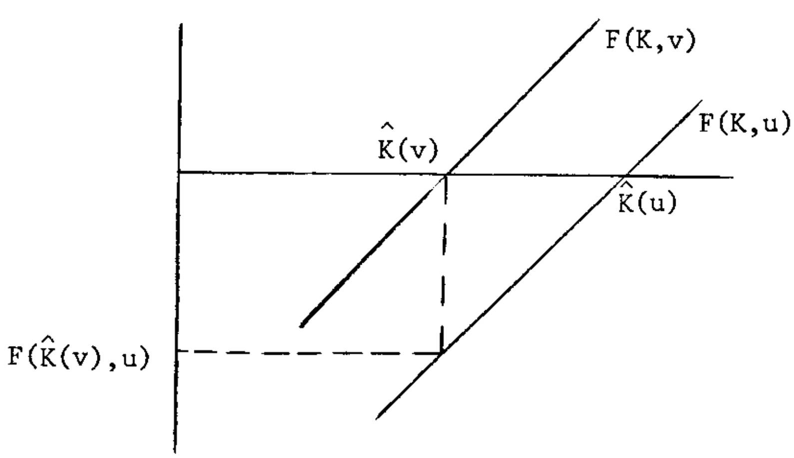

Figure 5

The inequalities involving K^* are, of course, obtained as a special case.

To prove that

$$\frac{g(\hat{K}(v),\tilde{x}) - Eg(\hat{K}(v),\tilde{x})}{1 + \hat{K}(v)} \gtrless \frac{g(\hat{K}(u),\tilde{x}) - Eg(\hat{K}(u),\tilde{x})}{1 + \hat{K}(u)} \quad ,$$

we let ξ be any strictly concave function and prove that

$$E\xi\left(\frac{g(K,\tilde{x}) - Eg(K,\tilde{x})}{1 + K}\right)$$

is an increasing (decreasing) function of K when $g(K,x)$ increases with x and $g(K,x)-g_K(K,x)(1+K)$ increases (decreases) with x for all K between $\hat{K}(u)$ and K^*. Thus $E\xi\left(\frac{g(K,\tilde{x})-Eg(K,\tilde{x})}{1 + K}\right)$ is an increasing (decreasing) function of K, when $\hat{K}(u)$ >(<) $\hat{K}(v)$. As a result,

$$E\xi\left(\frac{g(\hat{K}(u),\tilde{x}) - Eg(\hat{K}(u),\tilde{x})}{1 + \hat{K}(u)}\right) > E\xi\left(\frac{g(\hat{K}(v),\tilde{x}) - Eg(\hat{K}(v),\tilde{x})}{1 + \hat{K}(v)}\right)$$

for all concave functions.

The proof that

$$E\xi\left(\frac{g(K,\tilde{x}) - Eg(K,\tilde{x})}{1 + K}\right)$$

is increasing in K is accomplished by simply differentiating $E\xi$ with respect to K to obtain

$$(\partial/\partial K)E\xi\left(\frac{g(K,\tilde{x}) - Eg(K,\tilde{x})}{1 + K}\right)$$

$$= \frac{1}{(1+K)^2} \text{ COV } \left(\xi'\left(\frac{g(K,\tilde{x}) - Eg(K,\tilde{x})}{1 + K}\right), g_K(K,\tilde{x})(1+K)-g(K,\tilde{x})\right) \quad .$$

This covariance is positive when $g(K,x)$ and $g_K(K,x)(1+K)-g(K,x)$ increase together. It is negative if $g(K,x)$ increases when $g_K(K,x)-g(K,x)$

decreases.

The conclusion that $Eg(K^*,x)/(1+K)$ exceeds both $Eg(\hat{K}(v),\tilde{x})/(1+\hat{K}(v))$ and $Eg(\hat{K}(u),\tilde{x})/(1+\hat{K}(u))$ is an immediate consequence of the fact that K^* maximizes $Eg(K,\tilde{x})/(1+K)$. In order to prove that

$$\frac{Eg(\hat{K}(v),\tilde{x})}{1+\hat{K}(v)} > \frac{Eg(\hat{K}(u),\tilde{x})}{1+\hat{K}(u)} \quad ,$$

we show that the failure of this inequality leads to a contradiction. We specifically show that

$$\text{(3.34)} \qquad \frac{Eg(\hat{K}(u),\tilde{x})}{1+\hat{K}(u)} \geq \frac{Eg(\hat{K}(v),\tilde{x})}{1+\hat{K}(v)}$$

implies

$$\text{(3.35)} \qquad Ev(A+\left[\frac{g(\hat{K}(u),\tilde{x})}{1+\hat{K}(u)}\right](1-t)) > Ev(A+\left[\frac{g(\hat{K}(v),\tilde{x})}{1+\hat{K}(v)}\right](1-t)) \quad ,$$

a contradiction to the fact that $\hat{K}(v)$ maximizes $Ev(A+ g(K,\tilde{x})/(1+K)\ (1-t))$. We begin by noting that (3.34) and the monotonicity of v imply that

$$\text{(3.36)} \qquad Ev(A+\left[\frac{g(\hat{K}(u),\tilde{x})}{1+\hat{K}(u)}\right](1-t)) = Ev(A+\left[\frac{Eg(\hat{K}(u),\tilde{x})}{1+\hat{K}(u)} + \frac{g(\hat{K}(u),\tilde{x})-Eg(\hat{K}(u),\tilde{x})}{1+\hat{K}(u)}\right](1-t))$$

$$> Ev(A+\left[\frac{Eg(\hat{K}(v),\tilde{x})}{1+\hat{K}(v)} + \frac{g(\hat{K}(u),\tilde{x})-Eg(\hat{K}(u),\tilde{x})}{1+\hat{K}(u)}\right](1-t)) \ .$$

Since v is strictly concave, (3.21) implies that

$$\text{(3.37)} \qquad Ev(A+\left[\frac{Eg(\hat{K}(v),\tilde{x})}{1+\hat{K}(v)} + \frac{g(\hat{K}(u),\tilde{x})-Eg(\hat{K}(u),\tilde{x})}{1+\hat{K}(u)}\right](1-t))$$

$$> Ev(A+\left[\frac{Eg(\hat{K}(v),\tilde{x})}{1+\hat{K}(v)} + \frac{g(\hat{K}(v),\tilde{x})-Eg(\hat{K}(v),\tilde{x})}{1+\hat{K}(v)}\right](1-t))$$

$$= Ev(A+\left[\frac{g(\hat{K}(v),\tilde{x})}{1+\hat{K}(v)}\right](1-t)) \quad .$$

Combining (3.36) and (3.37), we obtain (3.35), the desired inequality.

We can now conclude the proof by showing that (3.24) implies (3.25) when ζ is at least as risk averse as ξ in the sense that (3.23) holds uniformly.[4/] First, note that, by the mean value theorem, there exists a $\bar{K}$ between $\hat{K}(v)$ and $\hat{K}(u)$ at which

$$(3.38) \qquad \frac{g(\hat{K}(v),\tilde{x})}{1+\hat{K}(v)} - \frac{g(\hat{K}(u),x)}{1+\hat{K}(u)} = \frac{g_K(\bar{K},x)(1+\bar{K}) - g(\bar{K},x)}{(1+\bar{K})^2}(\hat{K}(v) - \hat{K}(u)) \quad .$$

If $g(K,x)$ and $g(K,x)-g_K(K,x)(1+K)$ are both increasing in x for all K between $\hat{K}(v)$ and $\hat{K}(u)$, then, as shown above, $\hat{K}(v)-\hat{K}(u)$ is negative. When $g(K,x)$ increases and $g(K,x)-g_K(K,x)(1+K)$ decreases when x changes for all K between $\hat{K}(v)$ and $\hat{K}(u)$, then $\hat{K}(v)-\hat{K}(u)$ is positive. In either case,

$$\frac{g_K(\bar{K},x)(1+\bar{K}) - g(\bar{K},x)}{(1+\bar{K})^2}(\hat{K}(v) - \hat{K}(u))$$

is an increasing function of x. Because of (3.38),

$$\frac{g(\hat{K}(v),x)}{1+\hat{K}(v)} - \frac{g(\hat{K}(u),x)}{1+\hat{K}(u)}$$

is therefore an increasing function of x when $g(K,x)-g_K(K,x)(1+K)$ increases with x for all K between $\hat{K}(u)$ and $\hat{K}(v)$. If $g(K,x)$ also increases with x for all K between $\hat{K}(u)$ and $\hat{K}(v)$, then

$$\frac{g(\hat{K}(v),x)}{1+\hat{K}(v)} - \frac{g(\hat{K}(u),x)}{1+\hat{K}(u)}$$

can be treated as an increasing function, say r, of $g(\hat{K}(u),x)/(1+\hat{K}(u))$. If we now let

$$\tilde{y} = g(\hat{K}(u),\tilde{x})/(1+\hat{K}(u)) \qquad ,$$

4/ As noted above, this part of the proof is due to Oliver Hart.

and define

$$h(y) \;=\; y + r(y) \quad ,$$

then

$$h(\tilde{y}) \;=\; g(\hat{K}(v),\tilde{x})/(1 + \hat{K}(v)) \quad .$$

Since

$$Eu(A + \left[\frac{g(\hat{K}(u),\tilde{x})}{1 + \hat{K}(u)}\right](1 - t)) \;\geq\; Eu(A + \left[\frac{g(\hat{K}(v),\tilde{x})}{1 + \hat{K}(v)}\right](1 - t))$$

and

$$Ev(A + \left[\frac{g(\hat{K}(v),\tilde{x})}{1 + \hat{K}(v)}\right](1 - t)) \;\geq\; Ev(A + \left[\frac{g(\hat{K}(u),\tilde{x})}{1 + \hat{K}(u)}\right](1 - t)) \quad ,$$

there must be some y, call it $\bar{y}$, in the interval

$$\left[\min_{x \varepsilon X} \frac{g(\hat{K}(u),x)}{1 + \hat{K}(u)} \;,\; \min_{x \varepsilon X} \frac{g(\hat{K}(u),x)}{(1 + \hat{K}(u))}\right]$$

at which

$$r(\bar{y}) \;=\; 0 \quad .$$

Thus,

$$\tilde{y} \;\leq\; (\geq) \;\; \bar{y}$$

implies

$$h(\tilde{y}) \;=\; \tilde{y} + r(\tilde{y}) \;\;\leq (\geq)\;\; \bar{y} \quad .$$

If we now denote the cumulative distribution functions of $\tilde{y}$ and $h(\tilde{y}) = \tilde{y}+r(\tilde{y})$ by F and G respectively, then it is easily shown, from what has just been asserted, that

$$F(y) \leq (\geq) G(y)$$

if $y \leq (\geq) \bar{y}$.

Thus, F and G are related by the "single crossing property" of Diamond-Stiglitz (1974). Suppose now that (3.24) holds, i.e. suppose that

$$E\xi(\tilde{y}) \geq E\xi(\tilde{y} + r(\tilde{y}))$$

and that ζ is at least as risk averse as ξ in the sense that (3.23) holds. Integrating by parts, we obtain

$$(3.39) \qquad E\xi(\tilde{y}) - E\xi(\tilde{y} + r(\tilde{y})) = \int \xi(\tilde{y})d(F(\tilde{y}) - G(\tilde{y})) = \int (G(\tilde{y}) - F(\tilde{y}))\xi'(\tilde{y})d\tilde{y}$$

and

$$(3.40) \qquad E\zeta(\tilde{y}) - E\zeta(\tilde{y} + r(\tilde{y})) = \int \zeta(\tilde{y})d(F(\tilde{y}) - G(\tilde{y})) = \int (G(\tilde{y}) - F(\tilde{y}))\zeta'(\tilde{y})d\tilde{y} \quad .$$ [5/]

Since ζ is at least as risk averse as ξ, we can write

$$\zeta(y) = k(\xi(y)) \quad ,$$

where k is an increasing and concave function. Thus,

$$(3.41) \qquad \int (G(\tilde{y}) - F(\tilde{y}))\zeta'(\tilde{y})d\tilde{y} = \int (G(\tilde{y}) - F(\tilde{y}))k'(\xi(\tilde{y}))\xi'(\tilde{y})d\tilde{y}$$

Since $G(y)-F(y)$ is nonnegative (nonpositive) when $y \leq (\geq) \tilde{y}$, since ξ is an increasing function and since k is concave,

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[5/] Since X is a finite set, the integrals must be interpreted as Stiglitz's integrals.

$$(G(y) - F(y))k'(\xi(y)) \geq (G(y) - F(y))k'(\xi(\bar{y}))$$

for all y. Thus

$$(3.42) \qquad \int (G(\tilde{y}) - F(\tilde{y}))k'(\xi(y))\xi'(y)dy > k'(\xi(\bar{y}))\int (G(\tilde{y}) - F(\tilde{y}))\xi'(\tilde{y})d\tilde{y} \ .$$

Combining (3.39), (3.40), (3.41), and (3.42) we obtain

$$(3.43) \qquad E\zeta(\tilde{y}) - E\zeta(\tilde{y} + r(\tilde{y})) > k'(\xi(\bar{y}))(E\xi(\tilde{y}) - E\xi(\tilde{y} + r(\tilde{y}))) \ ,$$

where $k'(\xi(\bar{y})) > 0$. Thus (3.24) implies (3.25) when ζ is at least as risk averse as ξ. ||

Using Propositions 1, 2 and 3, we can describe the effect of an increase in capital and income taxation on $\hat{K}$. For this purpose, we let $\hat{K}(t)$ denote the equilibrium $\hat{K}$ when t is the marginal (income or capital) tax rate.

COROLLARY: Assume that either

i) $R_r(W,u)$ is increasing and all income is taxed at rate t,

or

ii) $R_r(W,u)$ is increasing, $R_a(W,u)$ is decreasing and capital income is taxed at rate t.

Suppose that $t_1 > t_2$. If $g(\hat{K}(t_1),x)$ and $g(K^*,x)$ are both increasing functions of x and $g(\hat{K}(t_1),x)-g_K(\hat{K}(t_1),x)(1+\hat{K}(t_1))$ and $g(K^*,x)-g_K(K^*,x)(1+K^*)$ are both increasing (decreasing) functions of x, then

$$K(t_2) > K(t_1) > K^*$$
$$(K(t_2) < K(t_1) < K^*) \ .$$

Now assume that $g(K,x)$ is an increasing function of x and $g(K,x)-g_K(K,x)(1+K)$

is an increasing or a decreasing function of x for all K between $\hat{K}(t_2)$ and K^*, then the following results obtain:

$$\frac{g(K^*,\tilde{x})-Eg(K^*,\tilde{x})}{1+K^*} \gtrless \frac{g(\hat{K}(t_1),\tilde{x})-Eg(\hat{K}(t_1),\tilde{x})}{1+\hat{K}(t_1)} \gtrless \frac{g(\hat{K}(t_2),\tilde{x})-Eg(\hat{K}(t_2),\tilde{x})}{1+\hat{K}(t_2)}$$

and

$$\frac{Eg(K^*,\tilde{x})}{1+K^*} \geq \frac{Eg(\hat{K}(t_1),\tilde{x})}{1+\hat{K}(t_1)} \geq \frac{Eg(\hat{K}(t_2),\tilde{x})}{1+\hat{K}(t_2)} .$$

Furthermore, if we suppose that ξ and ζ are any two increasing concave functions such that

$$-\frac{\zeta''(W)}{\zeta'(W)} \geq -\frac{\xi''(W)}{\xi'(W)}$$

for all W, and that

$$E\xi\left(\frac{g(\hat{K}(t_2),\tilde{x})}{1+\hat{K}(t_2)}\right) \geq E\xi\left(\frac{g(\hat{K}(t_1),\tilde{x})}{1+\hat{K}(t_2)}\right)$$

then we can conclude that

$$E\zeta\left(\frac{g(\hat{K}(t_2),\tilde{x})}{1+\hat{K}(t_2)}\right) \geq E\zeta\left(\frac{g(\hat{K}(t_1),\tilde{x})}{1+\hat{K}(t_1)}\right)$$

As a special case, we obtain that

$$E\xi\left(\frac{g(\hat{K}(t_1),\tilde{x})}{1+\hat{K}(t_1)}\right) \geq E\xi\left(\frac{g(K^*,\tilde{x})}{1+K^*}\right)$$

implies

$$E\zeta\left(\frac{g(\hat{K}(t_1),\tilde{x})}{1+\hat{K}(t_1)}\right) \geq E\zeta\left(\frac{g(K^*,\tilde{x})}{1+K^*}\right) \qquad ||$$

Before concluding, we consider the effects on the analysis of the elimination of the loss offset. In the model as described, the elimination of the loss offset has no effect on the results. This is true for the case of income tax as well as for the case of a capital income tax.

The first point at which the elimination of the loss offset could cause problems is in the determination of the equilibrium N function. It is still true, however, that, even without the loss offset, $N(K,B) = 1+K-B$ is an equilibrium N function.

When N is assumed to equal $1+K-B$ and when all individuals are the same the equilibrium K remains that at which

$$Eu(A + \frac{g(K,\tilde{x})}{1 + K}(1 - t))$$

is maximized. This observation is a consequence of the assumption that $g(K,x) \geq 0$ for all K and x. Because of this assumption, individuals never suffer losses in equilibrium. As a result, the existence or non-existence of a loss offset can have no effect on the analysis.

It should be observed, however, that when individuals differ,

$$r\left[1 - \gamma(1 + K)\right] + \gamma g(K,\tilde{x})$$

will not, in general, equal

$$\frac{g(K,\tilde{x})}{1 + K}$$

in equilibrium. Thus, income from capital may be negative even if $g(K,\tilde{x})$ is always positive. As a consequence, a more general analysis will be forced to deal with the issues raised by the existence or non-existence of a loss offset.

In the present very special form of the model, we can analyze some of these issues by allowing $g(K,x)$ to take negative values. In this case, the equilibrium K will be that at which

$$E(u(A + \left[\frac{g(K,\tilde{x})}{1 + K}\right]) \mid g(K,\tilde{x}) \leq 0)m(g(K,\tilde{x}) \leq 0)$$

$$+ \quad E(u(A + \left[\frac{g(K,\tilde{x})}{1 + K}\right](1 - t) \mid g(K,\tilde{x}) \geq 0)m(g(K,\tilde{x}) \geq 0)$$

is maximized. While we will not explicitly present the calculation, the reader can verify that implicit differentiation results in an ambiguous expression for

$$\partial\hat{K}/\partial t$$

even if $R_a(W,u)$ is non-increasing in W, $R_r(W,u)$ is nondecreasing in W and $g(K,x)-g_K(K,x)(1+K)$ and $g(K,x)$ both increase (for example) with x. Thus, the corollary cannot in general be extended to the case of no loss offset, if g can take negative values.

The intuition behind the ambiguity just observed is easily understood in terms of the analysis underlying Proposition 3. In that proposition, it was shown that a decrease in risk aversion produced an unambiguous effect on K. As Proposition 1 demonstrates, the imposition of a capital income tax with loss offset provisions unambiguously increases risk aversion if $R_a'(W,u) \leq 0$ and $R_r'(W,u) \geq 0$. If, however, the loss offset is eliminated, the imposition of the capital tax no longer unambiguously increases risk aversion. In fact, the elimination of the loss offset can be viewed as the introduction of a risk aversion increasing concave transformation, T, defined by

$$T(I) = \begin{cases} (\frac{1}{1 - t})I & ; \text{ if } I \leq 0 , \\ I & ; \text{ if } I \geq 0 . \end{cases}$$

T is illustrated in Figure 6.

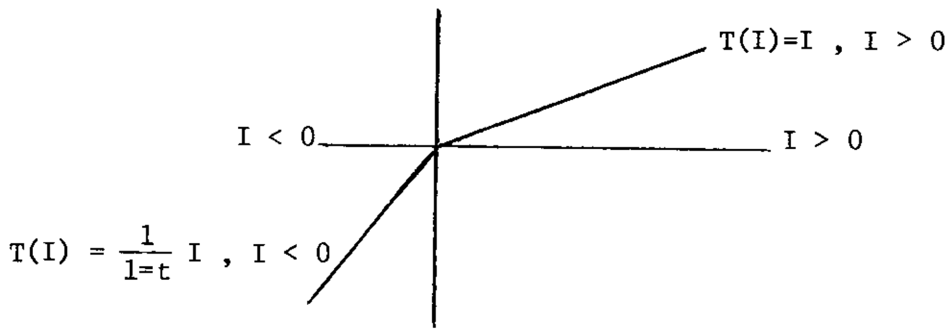

Figure 6

We can imagine before-tax income as being subjected to T before being substituted for I in Ψ. The resulting utility function, $\Psi(T(I),t)$, of I is that which is maximized when there is no loss offset. The utility function maximized with a loss offset is simply $\Psi(I,t)$. Now it is possible to show that $\Psi(T(I),t)$ is a concave tranformation of $\Psi(I,t)$. Thus, considered as a function of I, $\Psi(T(I),t)$ is more risk averse than $\Psi(I,t)$. As a result of this observation and Proposition 3, we can conclude that the elimination of the loss offset reduces risk taking. We cannot, however, say whether there is more or less risk taking with the tax but without the loss offset than there is without the tax. That is, we cannot say whether $\Psi(T(I),t)$ is more or less risk averse than $\Psi(I,0)$.

4. AN EXTENSION OF THE STOCK MARKET MODEL

In this section, we briefly consider a generalization of the model of Section 3. In this generalization, multiple technologies are available to entrepreneurs. Specifically, we assume the existence of m risky technologies with production functions

$$g^j(K,x)$$

in which K = the amount of non-entrepreneurial or operating capital employed, and

x = the value taken by a random variable, $\tilde{x}_j$.

The random variable $\tilde{x}_j$ is the same for all firms of type j created. Again this means that the output of all type j firms is determined by the same random influences. The variables $\tilde{x}_j$ take their values in a finite set X_j of real numbers. No assumptions are made about the correlation structure of $\tilde{x}_j$ and $\tilde{x}_k$, $k \neq j$. The process of setting up a type j firm is assumed to consume s_j units of capital. Each g^j is assumed to satisfy the hypotheses imposed on g in Section 3. In addition to the m random technologies, there is also assumed to be a nonrandom technology for which the production function is

$$g_0(K) \quad .$$

The amount of capital required to set up a type o firm is s_0. The function g_0 also satisfies the hypotheses imposed on g in Section 3.

In this model, we also assume that all individuals of type i begin with A_i units of income. We do not assume that $A_i = A_j$ if $i \neq j$.

If, in this economy, there exist riskless debt markets as well as markets for firm shares and if entrepreneurs can use purchased capital to set up firms, the equilibrium will be achieved when we have found a vector $\langle r,(C^i,K^i)_{i=1}^n\rangle$ such that, for each i,

$$(C^i,K^i) = ((C_0^i,\ldots,C_m^i),(K_0^i,\ldots,K_m^i))$$

maximizes

$$Eu_i(A_i + (1-t)\left[r(1 - \sum_{j=0}^m C_j^i) + \sum_{j=1}^m C_j^i \frac{g^j(K_j^i,\tilde{x}_j)}{(s_j+K_j^i)} + C_0^i \frac{g_0(K_0^i)}{s_0+K_0^i}\right])$$

subject to

$$m(A_i + (1-t)\left[r(1 - \sum_{j=0}^m C_j^i) + \sum_{j=1}^m C_j^i \frac{g^j(K_j^i,\tilde{x}_j)}{(s_j+K_j^i)} + C_0^i \frac{g_0(K_0^i)}{s_0+K_0^i}\right] > 0) = 1$$

where m is the probability measure induced by the random vector $(\tilde{x}_1,\ldots,\tilde{x}_m)$ and such that

$$\sum_{i=1}^n \left[\sum_{j=1}^m C_j^i\right]\mu_i = 1 \quad .$$

Note that, in the formalization just described, the taxes are imposed only on the income from capital.

The analysis will focus on special cases in which the technology or the utility functions satisfy certain special hypotheses. One class of utility functions considered is the class of strictly concave functions exhibiting constant absolute risk aversion. This class is denoted by $\mathcal{U}_{\text{CARA}}$. We also define $\mathcal{U}_{\text{QUAD}}$ to be the class strictly concave quadratic utility functions. Other classes considered are defined for $\gamma < 1$ by letting

$$\mathcal{U}_{\text{HARA}}(\gamma) = \{u \mid \text{for some } B > 0,\ u:(-B,+\infty) \to R \text{ and } u(W) = \gamma(W+B)^\gamma\} \qquad \text{if } \gamma \neq 0$$

and

$$\mathcal{U}_{HARA}(\gamma) = \{u \mid \text{for some } B > 0,\ u: (-B,+\infty) \to R \text{ and } u(W) = \text{Log }(W+B)\} \quad \text{if } \gamma = 0 \quad .$$

While these classes of utility functions are very special, they have come to assume very important roles in the literature of finance in particular and of the economics of uncertainty in general.

Four possible hypotheses are considered:

(H.1) For $i = 1,\ldots,n$, $u_i \in \mathcal{U}_{CARA}$;

(H.2) There exists some $\gamma < 1$ such that, for all $i = 1,\ldots,n$, $u_i \in \mathcal{U}_{HARA}(\gamma)$;

(H.3) For $i = 1,\ldots,n$, $u_i \in \mathcal{U}_{QUAD}$;

and

(H.4) For $j = 1,\ldots,m$, there exists a function h_j such that

$$g^j(K,x) = h_j(K)x \quad .$$

Proposition 4 describes the equilibria of the economies just described under each of these hypotheses.

PROPOSITION 4: <u>In equilibrium</u>, $K_0^i = K_0^*$ <u>where</u>

$$g_0'(K_0^*) = g_0(K_0^*)/(s_0 + K_0^*) \quad .$$

<u>In addition, the equilibrium</u> r <u>is</u>

$$r = g_0(K_0^*)/(s_0 + K_0^*)$$

<u>If either (H.1), (H.2), (H.3) or (H.4) holds, then, for each</u> $j = 1,\ldots,m$,

there exists a K_j^* *such that, for* $i = 1,\ldots,n$,

$$K_j^i = K_j^* \quad .$$

If, in particular, (H.4) holds, then K_j^* *satisfies*

$$h_j'(K_j^*) = h_j(K_j^*)/(s_j + K_j^*) \quad .$$

When (H.1), (H.2), or (H.3) holds, there exists a vector $(C_1^*,\ldots C_m^*)$ *such that, for each* $i = 1,\ldots,n$ *and* $j = 1,\ldots,m$, *the equilibrium* C_j^i *can be written*

$$C_j^i = a^i C_j^* \quad , \tag{4.1}$$

where a^i *is a positive real number. If, in fact, (H.2) holds, then we can also find a* C_0^* *such that (4.1) holds for* $j = 0$ *as well as* $j = 1,\ldots,m$.

The proof of this proposition is relatively straightforward and is omitted.

Proposition 4 implies that, if either (H.1), (H.2) or (H.3) holds, then in equilibrium individuals of each type i simply choose a^i to maximize

$$Eu(A + (1 - t)[r(1 - a^i) + a^i\tilde{y}]) \tag{4.2}$$

subject to

$$m(A + (1 - t)[r(1 - a^i) + a^i\tilde{y}] \geq 0) = 1 \quad , \tag{4.3}$$

where

$$r = g_0(K_0^*)/(s_0 + K_0^*)$$

and

$$\tilde{y} = \sum_{j=1}^{m} C_j^*\left[g_j(K_j^*,\tilde{x}_j)/(s_j + K_j^*)\right] \quad .$$

If (H.4) holds and $m = 1$, individuals of each type i solve a maximization problem of the same type. In this case, however,

$$\tilde{y} = h_1(K_1)\tilde{x}_1/(s_1 + K_1^*) \quad .$$

The preceding remarks imply that, if (H.1), (H.2) or (H.3) hold or if (H.4) holds with $m = 1$, then individuals solve a simple portfolio problem. As a result, Stiglitz's (1970) result can be applied directly to analyze these cases. If either (H.1) or (H.2), then Stiglitz's hypotheses that $R_r' \geq 0$ and $R_a' \leq 0$ are clearly satisfied. If (H.4) holds with $m = 1$, we must add these restrictions on R_a' and R_r' to our list of assumptions. In either case, an increase in the capital tax rate t causes a^i to rise for each i. As a result, fewer riskless firms are created in equilibrium and more capital is allocated to the creation of risky firms. The same conclusion emerges if t is the income tax rate, if (H.1), (H.2) or (H.3) hold or if (H.4) holds with $m=1$ and if $R_r' \geq 0$.

The analysis of this section has shown that under certain special assumptions on either preferences or on technology Stiglitz's partial equilibrium portfolio theory results apply directly to the simple general equilibrium analysis. If preferences are restricted, they must be restricted to a subclass in which portfolio separation is achieved. These subclasses were described by Cass-Stiglitz (1970). If the technology is restricted, it must satisfy stochastic constant returns to scale. Diamond (1967) introduced this hypothesis. While these assumptions about preferences and technology are restrictive, they are also of central importance in the existing literature.

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