TOWARD EFFICIENCY ANALYSIS OF DIVERSIFIABLE ASSETS

by

Yoram Kroll

Working Paper No. 6-81

RODNEY L. WHITE CENTER
FOR FINANCIAL RESEARCH
University of Pennsylvania
The Wharton School
Philadelphia, PA 19104

Toward Efficiency Analysis of Diversifiable Assets

Yoram Kroll*

I. Introduction

Ranking risky projects by anonymous investors can be carried out using Stochastic Dominance efficiency criteria. These criteria are necessary and sufficient conditions for preference by all anonymous investors which maximize expected utility. The Stochastic Dominance criteria can be safely used to compare two investment strategies (portfolios) where no further diversification is considered. However an economically important issue is the meaning of such a comparison between two assets when these two assets can be further diversified with each other or with other assets.

Hadar & Russell (1971) and Levy & Sarnat (1971) proved that if there is a dominance of one asset over another then this dominance is maintained in the case these two assets can be diversified with a third asset whose return is independent of the returns of these two assets. The proofs of Hadar & Russell and Levy & Sarnat were related only to the First and Second Degree Stochastic Dominance rules. Levy & Kroll (1978a) extended this theorem for all optimal efficiency rules. Note that even if one option does not dominate the other by one of the Stochastic Dominance rules it is still possible that there will be a dominance if diversification with a third independent prospect is considered. This point is demonstrated by Levy & Kroll (1978a) who also established a few rules where both dominance and no dominance conditions between two prospects are not altered by diversification with a third independent prospect.

---

*University of Pennsylvania

Recently, Levy & Kroll (1976, 1978a) considered the effect of diversification of risky assets with a riskless one. By assuming borrowing and lending of a risk-free asset, they were able to increase dramatically the effectiveness of the Stochastic Dominance rules (For an empirical investigation of this effect see Levy & Kroll (1979)).

In this study we extend the works of Levy & Kroll and others in two dimensions. First, by using the conditional returns of the prospects rather than the marginal returns, we establish simple, sufficient rules for analyzing dominance relationships between two assets which can be diversified with a third prospect which can be an independent as well as a dependent prospect. Some of the previous theorems of Hadar & Russell (1971), Levy & Sarnat (1971) and Levy & Kroll (1978a) are obtained here as specific cases.

Secondly, we extend Levy & Kroll (1976, 1978b) analysis, in which they considered the effect of borrowing and lending of a risk-free asset, to the case of borrowing and lending of risky asset. I.e., we extend the Levy & Kroll (1976, 1978b) efficiency analysis, in a way that not only diversification of risky assets with a risk free asset can be considered but also diversification of risky assets with another risky asset. This extension has a clear economic relevance, as one may claim that due to fluctuations in the market there is no pure risk free return either in real terms or in nominal terms. Our main theorems will be furnished by a numerical example in which we demonstrate the main features and the techniques to implement them.

II The Theorems of Stochastic Dominance of Diversifiable Assets

Before we present our theorems and extensions we should present necessary notations, assumptions and previous related theorems.

Let X and Y be the returns on two risky prospects. Let F_X and G_Y denote the cumulative distribution functions (CDF) of X and Y respectively. Let U_i

(i = 1,2,3) be classes of Von Nueman & Morgenstern utility functions. Where $u \varepsilon U_1$ if $u' > 0$; $u \varepsilon U_2$ if $u' > 0$ and $u'' < 0$; and $u \varepsilon U_3$ if $u' > 0$, $u'' < 0$; and $u''' > 0$. Assuming the expected utilities always exist we can state the Stochastic Dominance relationships between F_X and G_Y as follows.

Theorem 1 F dominates G for every $u \varepsilon U_i$ (or FD_iG) if and only if:

a. For i=1 $G(z) - F(z) \geq 0$ for every z. (1)

b. For i=2 $\int_{-\infty}^{z} [G(t) - F(t)]dt \geq 0$ for every z. (2)

c. For i=3 $\int_{-\infty}^{z} \int_{-\infty}^{v} [G(t) - F(t)]dtdv \geq 0$ for every z and $E_F \geq E_G$. (3)

and at each case a strict inequality for at least one z must hold.

Parts a, b and c of the theorems are called First, Second and Third Degree Stochastic Dominance rules, or FSD, SSD and TSD rules respectively. The proofs of these rules are given by: Quirk & Suposnik (1962), Fishburn (1964), Hanoch & Levy (1969), Hadar & Russell (1969), Rotchshild & Stiglitz (1971), Whitmore (1970). A review of these rules and the rules for decreasing absolute risk aversion utilities is given in Levy and Kroll (1980).

An alternative formulation of the SD rules, which we will also use in this paper, is based on the quantiles of the distributions. Let the inverse functions of F and G be denoted by $Q_F(P)$ and $Q_G(P)$. If F is a discrete distribution function, then $Q_F(P)$ should be redefined as the minimum value with probability of P to get this minimum value or less than this minimum value. Levy & Kroll (1978b) showed that theorem 1 can be restated in terms of the quantiles as follows:

Theorem 1': F dominates G for every $u \in U_i$, if and only if:

a. For i=1 $$Q_F(P) - Q_G(P) \geq 0 \text{ for every } P. \tag{1'}$$

b. For i=2 $$\int_0^P [Q_F(t) - Q_G(t)]dt \geq 0 \text{ for every } P. \tag{2'}$$

c. For i=3 $$\left.\begin{array}{l} \int_0^P \int_0^v [Q_F(t) - Q_G(t)]dtdv \geq 0 \text{ for every } P \\ \text{and } \int_0^1 [Q_F(P) - Q_G(P)]dp \geq 0 \end{array}\right\} \tag{3'}$$

and in each case for at least one P a strict inequality must hold.

Denote by W the return on a third risky asset. W can be independent as well as dependent of X and Y. Denote by $X|_w$ and $Y|_w$ the conditional returns of X and Y for W = w. The case W is independent of X and Y is simply the case where $X|_w = X$ and $Y|_w = Y$ for each w.

In the following theorem we present a sufficient rule for a dominance of AX + BW over AY + BW for every $A > 0$ and $-\infty < B < \infty$.

Theorem 2: If $X|_w D_i Y|_w$, (i = 1,2,3) for every w then $(AX + BW)D_i(AY + BW)$ for every $A > 0$ and $-\infty < B < \infty$.

Proof: The proof is quite simple and for the sake of brevity we present a proof only for the case where W is a discrete variable that can take values $w_1, w_2, w_3, \ldots$ with probabilities $q_1, q_2, q_3, \ldots$, respectively. With these assumptions and notation we can write Eu(AX + BW) and Eu(AY + BW) as follows:

$$Eu(AX + BW) = \sum_{t=1}^{\infty} q_t Eu(AX|_{w_t} + Bw_t) \tag{4}$$

$$Eu(AY + BW) = \sum_{t=1}^{\infty} q_t Eu(AY|_{w_t} + BW_t) \tag{5}$$

as w_t is constant $X|_{w_t} D_i Y|_{w_t}$ implies $(AX|_{w_t} + BW_t)D_i(AY|_{w_t} + BW_t)$ for every $A>0$ and $-\infty < B < \infty$ (See for example, Levy & Kroll (1978a) theorem 1 and note 3). By the optimality of SD rules $(AX_{w_t} + Bw_t)D_i(AY|_{w_t} + Bw_t)$ if and only if $EU(AX|_{w_t} + Bw_t) > EU(AY|_{w_t} + Bw_t)$ for every $u \varepsilon U_i$.

This result holds for every t by the assumption of the theorem. Therefore, as $q_t \geqslant 0$ for all t (q_t are probabilities) one can conclude from (4) and (5) that also $EU(AX + BW) > EU(AY + BW)$ for every $u \varepsilon U_i$. Thus, $(AX + BW)D_i(AY + BW)$. Q.E.D.

The case of independence between W and X and Y can be observed as a specific case of theorem 2. Thus, we can have the following corollary.

Corollary 1: If XD_iY (i = 1,2,3) then for every W which is independent of X and Y also $(AX + BW)\ D_i\ (AY + BW)$, for all $A > 0$ and $-\infty < B < \infty$.

Corollary 1 is equivalent to theorem 5 of Hadar & Russell (1971), and theorem 1 of Levy & Kroll (1978). The proof of this corollary is immediate from theorem 2 here. In the case of independence between W and X and Y, $X|_w = X$ and $Y|_w = Y$. Thus, one can replace each $X|_w$ and $Y|_w$ in theorem 2 by X and Y respectively, and by that obtain the corollary.

Note that the conditions of theorem 2 are much stronger than the simple requirement of XD_iY. I.e., it is possible that XD_iY but $X|_w \not{D} Y|_w$ for every possible W. On the other hand, the existence of the stronger condition of the theorem enables us to safely conclude that each investor with $u \varepsilon U_i$ prefers to diversify X with W rather than Y with W.

Also note that theorem 2 is only a sufficient rule for dominance between diversifiable assets, it is not a necessary rule. I.e., it is possible that there are w for which $X|_w$ does not dominate $Y|_w$ but every investor with $u \varepsilon U_i$ prefers to diversity X with W rather than Y with W. In the following theorems we develop more effective sufficient rules for dominance of diversifiable assets. But before we present our new theorem we need additional notations, definitions, and previous theorems which were developed by Levy & Kroll (1976, 1978b).

Let X_α denote the combination $\alpha X + (1-\alpha)W$ and by Y_β the combination $\beta y + (1-\beta)Y$. The sets $\{X\}_\alpha$ and $\{Y\}_\beta$ denote all the possible combinations of X_α and Y_β respectively. $\{X\}_\alpha$ dominates $\{Y\}_\beta$ or $\{X\}_\alpha D_i \{Y\}_\beta$ if and only if for every combination of Y with W there is at least one combination of X with W, which dominates it. Levy and Kroll (1978a) proved that if W is a <u>risk free</u> prospect and α and β are positive (i.e., short sales of risky assets are excluded) then in order to find a dominance of $\{X\}_\alpha$ over $\{Y\}_\beta$ one does not have to compare all possible combinations of Y with W against all combinations of X with W. In fact, it is only necessary and sufficient to find one X_α which dominates one Y_β in order to prove dominance of $\{X\}_\alpha$ over $\{Y\}_\beta$. This claim is stated in theorem 3.

<u>Theorem 3:</u> $\{X\}_\alpha D_i \{Y\}_\beta$ for i = 1,2,3, if and only if for some β_0 there is α_0, such that $X_{\alpha_0} D_i Y_{\beta_0}$.

For a proof of this theorem see Levy & Kroll (1978) or Levy & Kroll (1979). The importance of this theorem is that it is sufficient (as well as necessary) to find only one X_α which dominates one Y_β in order to guarantee that for each element of $\{Y\}_\beta$ one can find a dominant element in $\{X\}_\alpha$. However, this theorem provides no help in determining the existence of such an X_α which dominates some Y_β. Levy & Kroll (1978) developed an operative rule for a dominance of $\{X\}_\alpha$ over $\{Y\}_\beta$ when W is constant. Their main findings (which will be extended here to the case of risky W) are presented below.

Define for a constant W the functions $S^i_F(P)$ and $S^i_G(P)$ as follows: $S^i_F(P)$ is equal to $Q_F(P) - w$ for $i = 1$, $\int_0^P [Q_F(t) - w]dt$ for $i = 2$

and $\int_0^P \int_0^v Q_F(t) - w]dtdv$ for $i = 3$.

The definition of $S_G^i(P)$ is similiar. Denote by P^i the maximum P for which $S_F^i(P) < 0$. Note that X_F is a risky return and w is a riskless return. thus one should expect that w should be below the mean of X and above the minimum return on X. These two conditions imply the existence of $0 < P^i < 1$ for $i = 1,2$. With these assumptions and notations let us state Levy & Kroll (1978) theorem.

Theorem 4: Let X and Y be risky returns with CDF of F and G respectively. Let w be a riskless return where $\min X < w < E(X)$. Then $\{X\}_\alpha D_i \{Y\}_\beta$, if and only if:

For $i = 1,2,$

$$\underset{P^i<P\leq 1}{\mathrm{SUP}} \; S_G^i(P)/S_F^i(P) \leq \underset{0\leq P<P^i}{\mathrm{Inf}} \; S_G^i(P)/S_F^i(P) \tag{6}$$

and for $i = 3$; if $P^3 \leq 1$:

$$\mathrm{Max}[S_G^2(1)/S_F^2(1), \; \underset{P^3\leq P\leq 1}{\mathrm{SUP}} \; S_G^3(P)/S_F^3(P)] \leq \underset{0\leq P<P^3}{\mathrm{Inf}} \; S_G^3(P)/S_F^3(P) \tag{7}$$

if there is no $P^3 \leq 1$.

$$S_G^2(1)/S_F^2(1) \leq \underset{0\leq P\leq 1}{\mathrm{Inf}} \; S_G^3(P)/S_F^3(P) \tag{8}$$

Levy & Kroll (1979) implemented this theorem in an efficiency analysis of 204 mutual funds for the periods 1943-1974, 1953-1974, 1965-1974. The most striking result of their study is the dramatic increase in the effectiveness of the efficiency analysis. For example the efficient sets according to TSD have been reduced to 1-2 mutual funds in most cases described there. This result was obtained of course by assuming borrowing and lending at a constant risk free rate-w. However one may claim that the borrowing and lending rate fluctuates. (For example the rates on Treasury Bills). Thus, a more valid analysis should include a variable borrowing and lending rate rather than a constant one. In the following theorem we use our theorem 2 to extend theorem 4 for the case where W is a risky return rather than a riskless one.

Theorem 5: Let X, Y and W be the risky return on three prospects. Denote by $[\underline{\alpha}_w, \bar{\alpha}_w]$ the range of $\alpha, (\underline{\alpha}_w < \bar{\alpha}_w)$ for which $(X_{|w})_\alpha D_i (Y_{|w})$. (Note that there is not necessarily such a range). Then $\{X\}_\alpha D_i \{Y\}_\beta$ if: $\bigcap_W \{[\underline{\alpha}_w, \bar{\alpha}_w]\} \neq \emptyset$. I.e., the intersection of all the ranges $[\underline{\alpha}_w, \bar{\alpha}_w]$ over all possible w is not empty.

Proof: If for every β one can find α such that $X_\alpha D_i Y_\beta$ than $\{X\}_\alpha D_i \{Y\}_\beta$ I.e., $Eu(X_\alpha) > Eu(Y_\beta)$ for every u in U_i. Again we discuss the case where W is a discrete random variable. Thus exactly as before we can express $Eu(X_\alpha)$ and $Eu(Y_\beta)$ as follows:

$$Eu(X_\alpha) = \sum_{t=1}^{\infty} q_t Eu(\alpha X_{|w_t} + (1-\alpha)w_t) \qquad (9)$$

$$EU(Y_\beta) = \sum_{t=1}^{\infty} q_t EU(\beta Y_{|w_t} + (1-\beta)w_t) \qquad (10)$$

where q_t is the probability that $W = w_t$.

As w_t is constant one can use theorem (4) for each w_t in order to verify dominance of $\{X_{|w_t}\}_\alpha$ over $\{Y_{|w_t}\}_\beta$ for i = 1,2,3. However even if there is a dominance of $\{X_{|w_t}\}_\alpha$ over $\{Y_{|w_t}\}_\beta$ for every t, one can't conclude that $\{X\}_\alpha D_i \{Y\}_\beta$. The reason is that maybe for a given β we can't find the same α which leads to a dominance of $(X_{|w_t})_\alpha$ over $(Y_{|w_t})_\beta$ for all t.

This last requirement is necessary for the use of (9) and (10) in establishing a sufficient rule for a dominance of $\{X\}_\alpha$ over $\{Y\}_\beta$. If for all t the intersection of all the ranges $[\underline{\alpha}_{w_t}, \bar{\alpha}_{w_t}]$ is not empty one can find an α such that $(X_{|w_t})_\alpha D_i Y_{|w_t}$ for all t. We have to prove now that if $(X_{|w_t})_\alpha D_i Y_{|w_t}$ for any t, then for each β we can find at least one γ such that $(X_{|w_t})_\gamma D_i (Y_{|w_t})_\beta$. As w_t is <u>constant</u> for each t, then $(X_{|w_t})_\alpha D_i Y_{|w_t}$ implies $((X_{|w_t})_\alpha)_\beta D_i (Y_{|w_t})_\beta$. (You can verify this claim simply by considering it as a specific case of theorem 2).

But, by definition:
$$((X_{|w_t})_\alpha)_\beta \equiv (\alpha X_{|w_t} + (1-\alpha)w_t)\beta + (1-\beta)w_t$$
$$= \alpha\beta X_{|w_t} + (1-\alpha\beta)w_t \equiv (X_{|w_t})_{\alpha\beta}.$$
Thus, $(X_{|w_t})_{\alpha\beta} D_i (Y_{|w_t})_\beta$. Therefore by selecting $\gamma = \alpha\beta$ we can guarantee that $(X_{|w_t})_\gamma D_i (Y_{|w_t})_\beta$ for every t and therefore according to (9) and (10) for each β we can find γ such that for every u in U_i, $Eu(X_\gamma) > Eu(Y_\beta)$.

I.e., $\{X\}_\alpha D_i \{Y\}_\beta$, Q.E.D.

<u>Discussion of the Theorem</u>

The rule given by this theorem is only a sufficient rule for dominance. It is possible that the conditions of the theorem do not hold, but all investors with u in U_i prefer selections from the set $\{X\}_\alpha$ over selections from the set $\{Y\}_\beta$. Also note that in general the conditions of the theorem neither imply nor are implied by dominance of X over Y. It is possible that $\{X\}_\alpha D_i \{Y\}_\beta$ and $X \not{D}_i Y$, and even not for all w there is a dominance of $X_{|w}$ over Y_w. This property of the rule tends to increase the effectiveness of this rule relatively to the rule which is expressed by Theorem 2, where there is a requirement that $X_{|w} D_i Y_{|w}$ for all w. On the other hand, it is possible that XD_iY but yet $\{X\}_\alpha$ does not dominate $\{Y\}_\beta$. Only in the case where W is independent of X and Y can we infer that XD_iY implies $\{X\}_\alpha D_i \{Y\}_\beta$. The main contribution of the theorem is that W can be any random variable. We do not restrict it to be a risk-free return or independent of X and Y as is assumed in previous studies. On the one hand, this theorem is only sufficient for a dominance of $\{X\}_\alpha$ over $\{Y\}_\beta$. However, in order to use this theorem we do not have to know the exact probabilities of the various possible w's. According to Kroll & Levy (1979) we can extend our theorem also to the case where the borrowing rate is higher than lending rate. For the sake of brevity we do not introduce this possible extension. In addition the theorem requires a knowledge of the distibutions of $X_{|w}$ and $Y_{|w}$ for all possible w rather than the marginal distribution of X and Y. However, utilizing this additional information increases the economic meaning of the comparison between assets which can be further diversified with other assets such as w. In the next numerical example we demonstrate the important features of this rule, as well as possible ways to derive $X_{|w}$ and $Y_{|w}$ for various w's.

II. Possible Ways to Derive Conditional Returns and a Numerical Example

As we mentioned before the conditional returns rather than marginal returns should be used. Only in the case of independence between W and X and Y we can use the marginal returns (which are equal to the conditional returns). Basically one should regress X and Y on W in order to obtain the conditional returns $X_{|w}$ and $Y_{|w}$. This regression is not necessarily a linear one. However, in the next example we use linear regression as a first approximation. In order to get a benchmark for the properties of the conditional return one can use the Sharp's Single Index Model of the Capital Asset Pricing Model, where the return X_i of each security i can be simply described as a linear function of the return on the market portfolio-X_m plus an error term. Equation (11) represents this relationship.

$$X_{it} = w_t + \beta_{it}[X_{mt} - w_t] + u_{it} \quad (11)$$

X_{it} is the rate of return on security i at period t, w_t is the risk free rate at period t (this rate might be different in other periods), β_{it} is the risk factor of security i at period t, and u_{it} is a random error term. In general, one <u>cannot</u> deduce from (11) a linear relationship between X_{it} and w_t because as w_t changes, a new equilibrium, is obtained in which both β_{it} and the market portfolios might change. Take the partial of (11) with respect to w_t to obtain:

$$\frac{\partial X_{it}}{\partial w_t} = 1 + \frac{\partial \beta_{it}}{\partial w_t}[X_{mt} - w_t] + \beta_{it}[\frac{\partial X_{mt}}{\partial w} - 1] \qquad (12)$$

From the differential equation (12) we notice that if the partial of β_i is zero and the partial of X_{mt} is close to constant for w_t in the relevant range, then we can approximate $X_{|w}$ by a linear function of w_t as follows:

$$X_{i|w_t} = a_i + b_i w_t + u_{it} \qquad (13)$$

We will use this assumption as a first approximation in the following numerical example.

A. <u>Numerical Example</u>

Suppose that X and Y are the returns on two mutual funds and w is the return on Treasury Bills. The 5-year annual returns of these prospects are given below:

Annual Return on X(%)	-1.0	+28.0	+5.0	+15.0	+25.0
Annual Return on Y(%)	15.0	0.0	40.0	5.0	-10.0
Annual Return on W(%)	5.0	4.0	6.0	2.0	8.0

Assume that these empirical distributions of X, Y and W represent the correct ones. Now look at investors who borrow and lend at the T.B. rates, and buy one of the funds X or Y. These investors compare the set $\{X\}_\alpha$ versus the set $\{Y\}_\beta$.

By regressing X and Y on W we get:

$$X_{|w} = a + 0.50w + u \tag{14}$$

$$Y_{|w} = c - 0.25w + u \tag{15}$$

For our purpose the parameters a and c are not important. Also note that the significance level of the parameters 0.5 and −0.25 might be very low as we have only five observations on each variate. In practice we have to use many more observations to obtain reliable estimates of these parameters. The efficiency analysis will be done here with respect to dominance by FSD. Note that the cumulative distribution functions of X and Y cross each other (See Figure 1). Therefore X does not dominate Y by FSD. In the example we will prove that even though there is no dominance of X over Y by FSD there is a dominance of $\{X\}_\alpha$ over $\{Y\}_\beta$. The necessary calculations are presented in Table 1. The calculations utilize both the conditions of theorem 5 and the algorithm to determine dominance for constant W and discrete X and Y which is presented in Levy & Kroll (1979). The calculations for the case in which W = 2 will be explained in detail. First we calculate $X_{|w}$ and $Y_{|w}$. In the first year X = 1.0 but W in this year is equal to 5, rather than 2. Therefore according to our regression equation (14), $X_{|2} = -1.0 + 0.5(2-5) = -2.5$. By the same technique $Y_{|2} = 15 - 0.25(2-5) = 15.75$. According to this method we replace the annual marginal returns to by the conditional returns. Note, that in the fourth year W = 2. Thus, the marginal return is equal to the conditional return in this year. I.e., $Y_{|2} = 15 - 0.5(2-2) = 15 = Y$ and $X_{|2} = 5 - 0.25(2-2) = 5 = X$. The second step is to reorder $X_{|w}$ and $Y_{|w}$ from the lowest return to the highest return. (See columns (5) and (6)). The ranked returns represent the empirical quantiles of the distribution of $X_{|w}$ and $Y_{|w}$. We assume an equal probability of 1/5 for each result. For example, for W = 2 there is probability P of 1/5 that X will be equal or below -2.50 given that w = 2. There is probability, P = 2/5 of being at or below 3.00, and so on.

The next step is to calculate $S_F^i(P)$ and $S_G^i(P)$, where $S_F^i(P)$ is equal to the ranked $X_{|w}$ minus w = 2 and $S_G^i(P)$ is equal to the ranked $Y_{|w}$ minus w = 2. In column (9) we calculate the ratio $S_G^i(P)/S_F^i(P)$. According to theorem (4) if the INF of this ratio in the segment where $S_F^i(P)$ is negative then there is a dominance of $\{X_{|w}\}_\alpha$ over $\{Y_{|w}\}_\beta$ by FSD. In our case for w = 2 the Inf for negative $S_F^i(P)$ is equal to 2.333 and the SUP for positive $S_F^i(P)$ is only 1.560 (See Table 1). Therefore $\{X_{|2}\}_\alpha D_i \{Y_{|2}\}_\beta$. In addition from column (9) we can conclude that for each β there is γ = αβ such that $(X_{|2})_{\alpha\beta} D_i (Y_{|2})_\beta$ for <u>every</u> α in the range [1.560, 2.333].

We use exactly the same methods to determine the range of $[\underline{\alpha}, \bar{\alpha}]$ for the other cases of w = 4,5,6,8. In our example the range $[\underline{\alpha}, \bar{\alpha}]$ in case of w = 2 turns out to be a subrange of all the other ranges.[1] Namely the intersection of all the ranges is [1.560, 2.333]. As the intersection of all ranges is not empty we conclude that $\{X\}_\alpha D_i \{Y\}_\beta$.

[1]This result might be due to the positive regression coefficient of X on W together with the negative one in the regression of Y on W. However we cannot conclude that in such a case it is enough to check dominance only for the lowest W, because in general this result is not guaranteed.

Conclusion

In this paper we extend the basic Stochastic Dominance rules for a preference between two risky assets X and Y to include the option of diversification with a third risky asset-W, which can be dependent as well as independent of X and Y. In our analysis we use the conditional returns $X|_W$ and $Y|_W$ rather than the marginal returns X and Y. As a result we were able to specify very simple conditions for a dominance of diversification strategy of X with W rather than Y with W. Our analysis is an extension of previous studies of Hadar & Russell (1971), Levy & Sarnat (1971) and Levy & Kroll (1978), where it was assumed that W is independent of X and Y. These previous results are obtained here as a specific case of the general case where W can be dependent on X and Y. Levy & Kroll (1976, 1978b, 1979) developed Stochastic Dominance criteria between two assets which can be diversified with a riskless one. In theorem 5 of this paper we also extend these criteria to the case where diversification is with a risky asset rather than a riskless one. The main properties of our theorem are discussed and demonstrated through a numerical example. In the general case, where W is dependent of X and Y, a dominance of X over Y neither implies nor is implied by a dominance of diversification strategy of X with W over diversification of Y with W. Thus the effectiveness of the rule proposed here should be analyzed empirically. Such an empirical efficiency study concerning the performance of mutual funds is the subject of our forthcoming paper. It is very clear that such an empirical study will have better economic meaning than previous studies where diversification is not considered at all or it is assumed that X and Y can be diversified only with a risk-free asset such as Treasury Bills, where the return on the Treasury Bills was assumed to be constant over the years. According to our approach such a questionable assumption is not needed.

REFERENCES

[1] Fishburn, P.S. Decision and Value Theory, New York, Wiley, 1964.

[2] Hadar, J. and Russell, W.R., "Rules for Ordering Uncertain Prospects," American Economic Review, (1969), Vol. 59, pp. 25-34.

[3] ________ and ____________, "Stochastic Dominance and Diversification", Journal of Economic Theory, (1971), Vol. 3, pp. 288-305.

[4] Hanoch, G. and Levy, H., "The Efficiency Analysis of Choices Involving Risk," (1969), Review of Economic Studies, Vol. 36, pp. 335-346.

[5] Kroll, Y. and Levy H., "Stochastic Dominance with a Riskless Asset: An Imperfect Market", Journal of Financial and Quantitative Analysis, (1979), Vol. 24, pp. 179-204.

[6] ________ and _______, "Stochastic Dominance" A Review and Some New Evidence", Research in Finance, (1980), Vol. 2, pp. 163-227.

[7] Levy, H. and Kroll Y., "Stochastic Dominance With Riskless Assets", Journal of Financial and Quantitative Analysis, (1976), Vol. 11, pp. 743-773.

[8] ________ and ________, "Ordering Uncertain Options With Borrowing and Lending," Journal of Finance, (1978), Vol. 33, pp. 553-573.

[9] ________ and ________, "Investment Decision Rules, Diversification and Investor's Wealth," Econometrica, (1978), Vol. 41, pp. 125-130.

[10] ________, and ________, "Efficiency Analysis With Borrowing and Lending: Criteria, and their Effectiveness" Review of Economics and Statistics, (1979).

[11] Levy, H. and Sarnat, M., "A Note on Portfolio Selection and Investor's Wealth," Journal of Financial and Quantitative Analysis, (1971), Vol. 6, pp. 639-642.

[12] Quirk, J.P. and Saposnick, R., "Admissability and Measurable Utility functions", Review of Economic Studies, (1962), Vol. 29, pp. 140-146.

[13] Rothschild, M., and Stiglitz, J.E., "Increasing Risk: I. A definition", Journal of Economic Theory, (1970), Vol. 2, pp. 225-243.

[14] Whitmore, G.A., "Third Degree Stochastic Dominance" American Economic Review, Vol. 60, pp. 457-459.

<u>Table 1</u>

Numerical Calculations for Determining a Dominance of $\{X\}_\alpha$ Over $\{Y\}_\beta$ by FSD

	(1) Years	(2) [w]	(3) $X\|_w$	(4) $Y\|_w$	(5) Ranked $X\|_w$	(6) Ranked $Y\|_w$	(7) $S_F^i(P)$	(8) $S_G^i(P)$	(9) $S_G^i(P)/S_F^i(P)$
W = 2	1	5.0	-2.50	15.75	-2.50	-8.50	-4.50	-10.50	2.333
	2	4.0	+27.00	0.50	+3.00	0.50	1.00	-1.50	-1.500
	3	6.0	+3.00	41.00	+15.00	5.00	13.00	3.00	0.231
	4	2.0	+15.00	5.00	+22.00	15.75	20.00	13.75	0.688
	5	8.0	+22.00	-8.50	+27.00	41.00	25.00	39.00	1.560
	range of $[\underline{\alpha}, \bar{\alpha}]$ for W = 2 is:				$1.560 < \alpha < 2.333$				
W = 4	1	5.0	-1.50	15.25	-1.50	-9.00	-5.50	-13.00	2.364
	2	4.0	28.00	0.00	4.00	0.00	0.00	-4.00	$-\infty$
	3	6.0	4.00	40.50	16.00	4.50	12.00	0.50	0.042
	4	2.0	16.00	4.50	23.00	15.25	19.00	11.25	0.592
	5	8.0	23.00	-9.00	28.00	40.50	24.00	36.50	1.520
	range of $[\underline{\alpha}, \bar{\alpha}]$ for W = 4 is:				$1.520 < \alpha < 2.364$				
W = 5	1	5.0	-1.00	15.00	-1.00	-9.25	-6.00	-14.25	2.375
	2	4.0	28.50	-0.25	4.50	-0.25	-0.50	-5.25	10.500
	3	6.0	4.50	40.25	16.50	4.25	11.50	-0.75	-0.070
	4	2.0	16.50	4.25	23.50	15.00	18.50	10.00	0.540
	5	8.0	23.50	-9.25	28.50	40.25	23.50	35.25	1.500
	range of $[\underline{\alpha}, \bar{\alpha}]$ for W = 5 is:				$1.500 < \alpha < 2.375$				
W = 6	1	5.0	-0.50	14.75	-0.50	-9.50	-6.50	-15.50	2.385
	2	4.0	29.00	-0.50	5.00	-0.50	-1.00	-6.50	6.500
	3	6.0	5.00	40.00	17.00	4.00	11.00	-2.00	-0.182
	4	2.0	17.00	4.00	24.00	14.75	18.00	8.75	0.486
	5	8.0	24.00	-9.50	29.00	40.00	23.00	34.00	1.478
	range of $[\underline{\alpha}, \bar{\alpha}]$ for W = 6 is:				$1.478 < \alpha < 2.385$				
W = 8	1	5.0	0.50	14.25	0.50	-10.00	-7.50	-18.00	2.400
	2	4.0	30.00	-1.00	6.00	-1.00	-2.00	-9.00	4.500
	3	6.0	6.00	39.50	18.00	3.50	10.00	-4.50	-0.450
	4	2.0	18.00	3.50	25.00	14.25	17.00	6.25	0.368
	5	8.0	25.0	-10.00	30.00	39.50	22.00	31.50	1.432
	range of $[\underline{\alpha}, \bar{\alpha}]$ for W = 8 is:				$1.432 < \alpha < 2.400$				