Testing Rational Expectations
By The Use Of
Overidentifying Restrictions

by

Richard Startz

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RODNEY L. WHITE CENTER
FOR FINANCIAL RESEARCH
University of Pennsylvania
The Wharton School
Philadelphia, PA 19104

1. Introduction

Introduction of the rational expectations hypothesis has produced a minor boom in new econometric techniques. Because rational expectations forces restrictions on the stochastic interaction of observable and unobservable variables, estimation and hypothesis testing conditional on the existence of rational expectations frequently becomes possible. In particular, economic hypotheses that depend on the unobservable expectations of market participants can be considered if these unobservable expectations are known to be rational. I begin the econometric discussion in this paper with the examination of a well-known method of single equation estimation conditional on rational expectations. I then present a new statistic that tests the rational expectations hypothesis per se.

Rational expectations based tests may be subdivided roughly into three categories. In the first category an entire economic model is known. Behavior depends on the expectations held by market participants of variables whose realizations are determined endogenously within the model. If expectations are rational then the predictions of the model ought to be the same as these expectations. Strong tests within this category have recently been suggested by Wallis [11] and by Revankar [7].

In the second category, we observe a variable that ought under rational expectations to contain all available information about the occurence of another variable. Well-known examples include the idea that stock prices ought to be a random walk, that a forward rate (on foreign exchange for example) ought to be an unbiased predictor of the future spot rate, and that under the permanent income hypothesis today's consumption is the best predictor of tomorrow's consumption. These sorts of tests can usually be executed using ordinary least squares, because under the null hypothesis the error term must be uncorrelated with the observable rational expectation. Extended versions of this notion are considered in Abel and Mishkin [1].

The attention of this paper is focused on the third category. We consider the problem of testing when the realization of a variable is observable, but its market expectation is not. While the market expectation is unobservable we do see some portion of the information that market participants used in forming their expectations. Such information can be used to form instrumental variables. The notion of using outside information to deal with unobservables appeared at least as early as Goldberger [5], in which a limited-information maximum-likelihood estimator was developed, though rational expectation was not considered explicitly. McCallum [6] developed an instrumental variable estimator that yields consistent structural estimates and allows for efficient hypothesis testing. The techniques discussed here follow directly from McCallum's work.

The next section of the paper sets up the problem and then discusses estimation and hypothesis testing. McCallum's estimator produces consistent and asymptotically efficient estimates of structural coefficients, if the rational expectations hypothesis is true. A test of the rational expectations hypothesis itself is then derived, first by an intuitive argument about the properties of conditional expectations and then by formal proof. The test statistic is shown to be a reinterpretation of Basmann's [2,3] test for overidentification. Finally, it is shown how to jointly test structural hypotheses and rational expectations. All the required test statistics can be generated from the results of several auxiliary linear regressions.

The third section of the paper applies the tests to the hypothesis that (rationally) expected inflation is independent of the real interest rate. The hypothesis is rejected at a high level of significance.

2. Hypothesis Testing Under Rational Expectations

We begin with a classical linear regression equation, as in (1). The dependent variable, y, is a linear function of explanatory variables X_1^e and X_2, and of a structural error term u. The structural errors are assumed to be i.i.d. normal and to be independent of both X_1^e and X_2.

$$(1) \qquad y = X_1^e\beta_1 + X_2\beta_2 + u$$

If the investigator knew the values of both sets of explanatory variables, estimation and hypothesis testing could proceed by ordinary least squares. We assume that X_2 is observable, but that X_1^e is not. We do observe X_1, the realization of. X_1^e. X_1^e is a set of expectations existing in the minds of market participants. Market participants base these expectations on the information set Φ. X_1 and the elements of Φ are jointly normally distributed. Assuming for the moment that X_2 is known to market participants at the time expectations are formed, we let $X^e = [X_1^e \; X_2]$, $X = [X_1 \; X_2]$, and include X_2 in Φ. Equation (1) is rewritten in a more convenient form in (2). The null hypothesis, H_0, proposes a set of r linear restrictions, (3). The rational expectations hypothesis, RE, that expectations are the mathematical expected value of the realized variable conditional on available information, is given in (4)

$$(2) \qquad y = X^e\beta + u$$

$$(3) \qquad H_0\text{: } R\beta = b$$

$$(4) \qquad \text{RE: } X^e = E(X|\Phi)$$

X can be decomposed into an expectational component and an expectational error e. X^e and e are independent due to the properties of the normal distribution. There are no restrictions on the correlation of u and e.

$$(5) \qquad X = X^e + e$$

While we do not observe all the information in Φ, we do know a subset Z. The expectation of X^e conditional on Z is a linear function of Z. Let Γ be a q×k matrix of unknown coefficients, where the number of columns in Z and X^e are q and k respectively, we can write (6).

$$(6) \qquad Z\Gamma = E(X^e | Z)$$

Once again applying the properties of the normal distribution, we decompose X^e into its conditional expectation and independent error term v. Independence of v and e follows from the independence of information and errors in (4). Independence of v and u and of Z and u follow from the initial Gauss-Markov assumptions on (2).

$$(7) \qquad X^e = Z\Gamma + v$$

Appropriate substitutions produce the reduced form equations (8) and (9)

$$(8) \qquad X = Z\Gamma + v + e$$

$$(9) \qquad y = Z\Gamma\beta + v\beta + u$$

Alternatively, we write the reduced form more generally as (10) and recognize that rational expectations implies the nonlinear restrictions $\Theta = \Gamma\beta$.

$$(10) \qquad y = Z\Theta + v\beta + u$$

Equations (8) and (9) are seemingly unrelated regression equations with a nonlinear cross-equation constraint. If $q = k$, the system is just identified. Equations (8) and (10) can be estimated individually by ordinary least squares, yielding coefficient estimates $\hat{\Gamma}$ and $\hat{\Theta}$. β may then be estimated by $\hat{\beta} = \hat{\Gamma}^{-1}\hat{\Theta}$. This is the method of indirect least squares. The rational expectations restrictions are not binding so while $\hat{\beta}$ is consistent and asymptotically efficient, no tests of the rational expectations hypothesis itself are possible. In other words, H_0 is testable conditioned on RE, but RE is not testable.

When $q > k$, the system is overidentified. The system may be estimated either by instrumental variable (two-stage least squares, generalized classical linear) methods or by maximum-likelihood methods. These methods are asymptotically efficient and asymptotically equivalent. Since our interest is in IV methods, which are also frequently more convenient, the discussion of maximum-likelihood methods is confined to the next paragraph. Linear estimation occupies the remainder of the paper.

Equations (8) and (9) may be estimated jointly by nonlinear multivariate least squares, taking into account the cross-equation constraint and the contemporaneous correlation of the error terms. The resulting estimator of β is the limited-information maximum-likelihood estimator. The overidentifying restrictions implied by rational expectations may be tested by comparison of the likelihood function from the LIML estimate to the likelihood function from the unconstrained joint estimation of (8) and (10). See [1] for further discussion.

The instrumental variable estimator is most easily explained as if it were literally done by two steps of least squares. In the first stage, equation (8) is estimated by ordinary least squares. In the second stage, y is regressed against the "fitted" values, $Z\hat{\Gamma}$, and a residual $\dot{e}_2$, as in (11).

$$(11) \qquad y = (Z\hat{\Gamma})\beta + e_2$$

The OLS estimator of (11), $\hat{\beta}$, is the two-stage least squares estimator.[1] McCallum proposed use of this estimator in his 1974 article and showed its consistency. If we use all the information available to us, the estimator is also asymptotically efficient. The usual methods may be used for making hypothesis tests about β. These tests are conditional on the validity of the rational expectations hypothesis.

The rational expectations restrictions, $\Theta = \Gamma\beta$, have the following intuitive interpretation. According to equation (10), $Z\Theta$ is the expectation of y conditional on Z. According to (2), $X^e\beta$ is the expectation of y conditional on X^e. Rational expectations, (6), tells us that $Z\Gamma$ is the expectation of X^e conditional on Z. Since the expectation of y conditional on

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[1] Suppose that X_2 is not included in the information set. How is estimation affected? McCallum suggests using $P_Z X_1$ as instruments for X_1 and using X_2 as its own instrument. However, X_2 may be correlated with e, so this suggestion leads to inconsistent estimates. The usual 2SLS estimation, simply excluding X_2 from Z, is consistent.

Z equals the expectation of y conditional on the expectation of X^e conditional on Z, we have (12).

$$(12) \qquad Z\Theta = Z\Gamma\beta$$

In essence, we test rational expectations by testing (12). If we regress $Z\hat{\Theta}$ on $Z\hat{\Gamma}$, we ought to see a perfect fit as the number of observations grows large. A familiar statistic emerges from a little matrix algebra. For convenience, let $P_Z = Z(Z'Z)^{-1}Z'$.

$$(13) \qquad \hat{\Theta} = (Z'Z)^{-1}Z'y, \quad \hat{\Gamma} = (Z'Z)^{-1}Z'X, \ \hat{\beta} = (X'P_ZX)^{-1}X'P_Zy$$

$$(14) \qquad Z\hat{\Theta} = P_Zy, \quad Z\hat{\Gamma}\hat{\beta} = P_ZX\hat{\beta}, \quad \hat{y} = P_Zy, \quad \hat{X} = P_ZX$$

$$(15) \qquad Z\hat{\Theta} - Z\hat{\Gamma}\hat{\beta} = \hat{y} - \hat{X}\hat{\beta}$$

Equation (15) demonstrates that the differences between the two conditional estimates are the residuals from the regression of the fitted y on the fitted X. Let $\mu = v - e\beta$, μ_i are i.i.d. $N(0, \sigma^2)$. The asymptotic distribution of the sum of squares of (15), $S\hat{S}R$, follows.[2]

$$(16) \qquad \hat{\Theta} - \hat{\Gamma}\hat{\beta} = (Z'Z)^{-1}Z'[I - X(X'P_ZX)^{-1}X'P_Z]\mu$$

The asymptotic distribution of $\hat{\Theta} - \hat{\Gamma}\hat{\beta}$ is singular normal.

[2] With some renaming of variables, this proof is taken from Basmann [2].

(17) $\hat{\Theta} - \hat{\Gamma}\hat{\beta} \overset{A}{\sim} SN(0, \sigma^2 \text{plim } [(Z'Z)^{-1} - (Z'Z)^{-1}Z'X(X'P_Z X)^{-1}X'Z(Z'Z)^{-1}])$

(18) $S\hat{S}R = (\hat{y} - \hat{X}\hat{\beta})'(\hat{y} - \hat{X}\hat{\beta}) = (\hat{\Theta} - \hat{\Gamma}\hat{\beta})'(Z'Z)(\hat{\Theta} - \hat{\Gamma}\hat{\beta})$

The sum of squares (15), divided by σ^2, is asymptotically $\chi^2(q-k)$.[3]

It remains to produce a consistent estimator of σ^2. The usual estimator is the sum squared residuals divided by n or by n-k. However, Basmann [1960] recommends use of the sum squared residuals minus the fitted sum of square residuals. Basmann also suggests that the test statistic in small samples is more closely approximated as being F(q,n-k). Either of the estimators in (20) gives a consistent estimate of σ^2.

(19) $SSR = (y - X\hat{\beta})'(y - X\hat{\beta})$

(20) $s^2 = SSR/(n-k)$ or $s^2 = [SSR - S\hat{S}R]/(n-k)$

The test statistics $\lambda = S\hat{S}R/s^2$ and $\lambda/(q-k)$ are distributed asymptotically $\chi^2(q-k)$ and $F(q-k,n-k)$ respectively. These are Basmann's statistics for testing overidentifying restrictions. They test the rational expectations hypothesis independently of any structural tests.

Structural hypothesis testing conditional on rational expectations makes use of the usual asymptotic χ^2 or F tests. To test the r linear restrictions $R\beta = b$, we look at the difference of ("second-stage") residuals using $\hat{\beta}$ and the restricted 2SLS estimator $\hat{b}$.

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[3]See [Searle] p. 69.

(21) $\hat{SSR} = (y - \hat{X}\hat{\beta})'(y - \hat{X}\hat{\beta}), \quad \hat{SSR}^* = (y - \hat{X}\hat{b})'(y - \hat{X}\hat{b})$

(22) $(\hat{SSR}^* - \hat{SSR})/s^2 \overset{A}{\approx} \chi^2(r)$

The numerator of (22) is written out in (23).

(23) $\hat{SSR}^* - \hat{SSR} = \mu' P_Z X(X'P_Z X)^{-1} R'[R(X'P_Z X)^{-1} R']^{-1} R(X'P_Z X)^{-1} X'P_Z \mu$

Both (23) and $S\hat{\hat{S}}R$ can be written as quadratic forms in μ. Since the two quadratic forms are independent, the sum is also distributed χ^2. (24) follows immediately.

(24) $[\hat{SSR}^* - \hat{SSR} + S\hat{\hat{S}}R]/s^2 \overset{A}{\approx} \chi^2(r+q-k)$

The critical value for a $\chi^2(r+q-k)$ is less than the sum of the critical values for a $\chi^2(r)$ and a $\chi^2(q-k)$ at any of the usual levels of significance. Thus if both separate test statistics indicate rejection, the joint hypothesis will also be rejected. In addition, the joint hypothesis may be rejected even though neither separate test fails.

3. Testing the Independence of the Real Rate of Interest and Expected Inflation

Since early discussion by Irving Fisher, there has been an ongoing dispute as to whether the expected real rate of interest is independent of expected inflation. The Fisher equation states that the nominal interest rate equals the expected real interest rate plus the expected inflation rate.

(25) $\quad i = r^e + \pi^e$

Fama [4] found evidence confirming the joint hypothesis that r^e and π^e are independent and that r^e has been constant during much of the post-War period. Recently, Summers [10] applied McCallum's test and rejected independence, though not on the same sample period as Fama. Startz [9] rejected the hypothesis, jointly with other hypotheses, on the same sample period as Fama. Here, we test the Fisher hypothesis by McCallum's method, test separately for the rationality of inflationary expectations, and then test jointly. Therefore, we are testing the same hypothesis as proposed by Fama, without the restriction of a constant expected real interest rate. The data used is described in [9].

Expected inflation is unobservable. However, under rational expectations, the actual inflation rate equals expected inflation plus an uncorrelated prediction error. The expected real interest rate is also unobservable. We can write it as the sum of its mean and deviations around that mean; $r^e = \alpha + (r^e - \alpha)$. Equation (25) can be rewritten in terms of observable variables and a two-part structural error term.

$$(26) \qquad i = \alpha + \beta\pi + (r^e - \alpha) + \beta(\pi^e - \pi)$$

We wish to test the structural hypothesis $\beta = 1$ subject to the rational expectations hypothesis $\pi^e = E(\pi|\Phi)$. Estimation of (26) by ordinary least squares yields a classic example of errors-in-variables, since π is certainly correlated with $\pi^e - \pi$.

Obvious candidates for information useful to market participants in predicting current inflation include recent lagged values of inflation. Under rational expectations, these are uncorrelated with the inflation prediction errors. Under the null hypothesis (which presumably extends to noncontemporaneous as well as contemporaneous correlation) lagged inflation is uncorrelated with the expected real interest rate. We can use lagged inflation rates as instruments and use McCallum's technique, as Summers did. We can also use Basmann's statistic to test for rational expectations. Notice that if $\beta \neq 0$, then the lagged inflation rates are not valid instruments. Therefore the two hypotheses cannot really be tested separately and the joint test statistic derived in the previous section is particularly appropriate.

The tests are run on monthly data from 1/53 through 7/71, for a total of 223 observations. One month treasury bill rates and rates of change of the CPI provide the data. The results of ordinary least squares estimation are:

$$(27) \qquad \underset{}{i} = \underset{(0.112)}{2.476} + \underset{(0.031)}{.300\pi}$$

$$R^2 = .30 \qquad \text{ser} = 1.29 \qquad \text{D.W.} = 0.0573$$

The results of running Fama's test would appear to confirm independence.

$$(28) \qquad \pi = \underset{(0.357)}{-0.847} + \underset{(.102)}{.984i}$$

$$R^2 = .30 \qquad \text{ser} = 2.34 \qquad \text{D.W.} = 1.77$$

The results of instrumental variable estimation using a constant, and three lagged inflation rates are:

$$(29) \qquad i = \underset{(0.249)}{1.403} + \underset{(0.096)}{.776\pi}$$

$$ser = 1.86 \qquad D.W. = 1.71$$

The t-statistic on the hypothesis $\beta = 1$ is 2.345, leading to a rejection with confidence level between .98 and .99. $S\hat{\hat{S}}R$ was 14.5; the regular SSR was 761.7. $\hat{SSR}^*$ and $\hat{SSR}$ were 319.1 and 300.2 for the constrained and unconstrained versions respectively. Thus the F-statistic for $\beta = 1$ is 5.49 (not surprisingly, the square of the t-statistic). The test of rational expectations by means of overidentifying restrictions has a test statistic of 4.21. This statistic, distributed $\chi^2(2)$, is a bit lower than required for rejection at the 90 percent confidence level. The test statistic for the joint test, 9.7, is distributed $\chi^2(3)$. The joint hypothesis can be rejected with approximately the same level of confidence as with the structural test.

4. Summary

Instrumental variables provide a consistent, efficient (and simple) method for coefficient estimation and for testing structural hypotheses. In many economic models, it is difficult to find variables that come with a guarantee of being valid instruments. Overidentifying restrictions are rarely tested in practice. Perhaps this is because it is so difficult to know what economic statement might be associated with rejection. However, the rational expectations hypotheses requires that information known to market participants form valid instruments. Thus, rejection of the overidentifying restrictions leads to a rejection of rational expectations.

In principle, structural hypotheses and the hypothesis of rational expectations are separate issues. It is possible to accept one hypothesis and reject the other at a given level of significance. Of course, if the rational expectations hypothesis is rejected, the structural coefficients are not consistently estimated and structural hypothesis tests are invalid.

In practice, structural hypotheses and the hypothesis of rational expectations are often inextricably intertwined. The independence of the right-hand side variables and the structural error terms frequently only holds under the null hypothesis; this being all that is required for valid hypothesis testing. The Fisher effect example explored in section 3 is a case in hand. If the right-hand side variables are not independent from the structural errors, then even under rational expectations available information need not form valid instruments. In practice, therefore, it is frequently impossible to know whether rejection is properly attributed to failure of the structural hypothesis or failure of rational expectations.

The last element of section 2 is the presentation of a statistic for the joint test of H_0 and RE. It is possible in a given sample that one individual test or the other might fail without rejection of the joint test statistic. If both indicate rejection then the joint test will also cause rejection at the same level of significance. Even if neither individual test leads to rejection, the joint test may be strong enough to reject the joint hypothesis.

Independence of expected inflation and expected real returns has been the subject of statistical investigation for some half century. The evidence in section 3 rejects independence over a sample period in which previous work had appeared to confirm independence. Even though the information set is limited to three lagged values of inflation, the rejection is very strong.

There are a number of economic hypotheses that suggest the use of instrumental variable tests. The question of unbiasedness of forward rates implied in the term structure and observable in foreign exchange markets are perhaps the most obvious. The direct test of rational expectations proposed here is an easily executed addition to the econometric repertoire.

References

[1] Abel, A.B. and F.S. Mishkin, "A Unified Framework for Testing Rationality, Market Efficiency and the Short-Run Neutrality of Monetary Policy", mimeo., University of Chicago, 1980.

[2] Basmann, R.L., "On the Asymptotic Distribution of Generalized Linear Estimators", Econometrica, 28, pp. 97-107, January 1960.

[3] Basmann, R.L., "On Finite Sample Distributions of Generalized Classical Linear Identifiability Test Statistics", Journal of the American Statistical Association, 55, pp. 650-659, December 1960.

[4] Fama, E.F., "Short-Term Interest Rates as Predictors of Inflation", American Economic Review, 65, pp. 269-282, June 1975.

[5] Goldberger, A.S., "Maximum-Likelihood Estimation of Regressions Containing Unobservable Independent Variables", International Economic Review, 13, pp. 1-15, February 1972.

[6] McCallum, B.T., "Rational Expectations and the Natural Rate Hypothesis: Some Consistent Estimates", Econometrica, 44, pp. 43-52, January 1976.

[7] Revankar, N.S., "Testing of the Rational Expectations Hypothesis", Econometrica, 48, pp. 1347-1363, September 1980.

[8] Searle, S.R., Linear Models, John Wiley and Sons, 1971.

[9] Startz, R., "Unemployment and Real Interest Rates: Econometric Testing of Inflation Neutrality", American Economic Review, forthcoming.

[10] Summers, L.H., "The Non-Adjustment of Nominal Interest Rates: A Study of the Fisher Effect", mimeo, M.I.T., August 1980.

[11] Wallis, K.F., "Econometric Implications of the Rational Expectations Hypothesis," Econometrica, 48, pp. 49-74, January 1980.