STOCHASTIC DOMINANCE WITH A RISKLESS ASSET: THE CONTINUOUS CASE

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Introduction

In recent articles [9] [10], the authors developed First, Second and Third degree stochastic dominance rules under the assumption that investors can borrow or lend money at some riskless interest rate, r. These rules are denoted by the acronym SDR, which stands for Stochastic Dominance with a Riskless Asset, to distinguish from Stochastic Dominance (SD) rules which were developed in the late sixties.

SD rules are simple, in the sense that in establishing a preference all one has to do is to check whether a certain condition holds with regard to the two well-defined risky options F and G. SDR rules are much more complex, since we have to determine if a preference exists between $\{F_\alpha\}$ and $\{G_\beta\}$ according to a certain rule; $\{F_\alpha\}$ and $\{G_\beta\}$ include all the infinite combinations of the risky options F and G respectively, and the riskless asset. Thus, the SDR rules, in principle, involve an infinite number of comparisons. However, Levy and Kroll [10] established various criteria which make it possible to circumvent the infinite number of comparisons and hence determine if a preference exists. In any empirical research, the number of comparisons involved according to the criteria developed in [10] is finite and is a function of the number of observations at our disposal. To be more precise, in empirical studies, the cumulative distributions of the risky options are given as step functions with a finite number of steps which in turn implies a finite number of comparisons. Levy and Kroll [11]

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developed an algorithm for a step function and use it in order to establish First, Second and Third degree stochastic dominance with riskless assets. The main finding of the empirical study is that the efficient sets shrink dramatically once borrowing and lending is allowed and, in some cases, the efficient set includes only one or two risky portfolios.

The purpose of this paper is to find SDR rules for continuous density functions; a case where the cumulative distribution is not given as a step function. We develop and investigate the theoretical properties of SDR rules and do not deal with an algorithm applicable to empirical studies. Nevertheless, recall that the rules developed in [10] are general and hold for discrete as well as continous distributions. Thus, deriving SDR criteria for continous distributions we obtain rules which are stated in terms of the parameters of the distributions under consideration rather than in a general form. Though these rules differ in their structure from the rules given in [10], the parametric approach yields the same efficient set as the non-parametric rules developed earlier, since the two sets of rules provide necessary and sufficient conditions for dominance.

The advantage of the parametric approach developed in this paper is three-fold: (a) For some continuous distributions the rule is very simple and stated in terms of the expected return, variance etc. Thus, there is no need for any computer calculation or for many comparisons in order to determine if preference exists; (b) Parametric approach yields some theoretical results, e.g. cases where a "separation" exists; (c) In case of some continuous distributions (e.g. log normal), one cannot apply directly the rule developed by Levy and Kroll [10] without an infinite number of comparisons.[1] However, using the parametric approach, one can use the knowledge of the parameters of the continuous distribution under consideration in order to state if a preference exists by SDR rules, which are solely functions of the distribution parameters.

Thus, in a sense, the parametric approach is a further investigation of the rules developed in [10] for continuous distributions. For some continuous distributions, we obtain simple rules in terms of the parameters, but for some others, the preference criterion may remain a complex one even in the parametric approach.

In the second section, we present some notations and state the well known SD and SDR criteria. In the third section, we develop SDR criteria for continuous distributions, and in the fourth section, we apply these decision rules to three important distributions: uniform, normal and log-normal. In section V, we employ the rule developed for lognormal distribution, and derive the mean-variance efficient frontier. Concluding remarks are given in Section VI.

II. SD and SDR rules

Let X and Y be the returns of two risky ventures with cumulative distributions F and G respectively. Also denote by $Q_F(P)$ and $Q_G(P)$ the P order quantile of F and G. Let u be a Von Neuman-Morgenstern utility function. U_1 stands for the set of all u with $u' \geq 0$. U_2 denotes

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1. Note that the algorithms of SDR rules applies only to step functions. Since the lognormal distribution is continuous, it is impossible to apply the SDR rules given in [10], since it involved infinite number of comparisons.

the set such that for all $u \in U_2$, $u' \geq 0$ and $u'' \leq 0$. U_3 denotes the set of all u with $u' \geq 0$, $u'' \leq 0$, $u''' \geq 0$. The Stochastic Dominance rules without a riskless asset is given in theorems 1a, 1b and 1c. In the theorems, we apply the quantile terminology.[2]

Theorem 1. A necessary and sufficient condition for a dominance of F over G by all $u \in U_i$, $(i = 1, 2, 3)$ is given by the following conditions:

a. First degree Stochastic Dominance (FSD):

For all $u \in U_1$: $Q_F(P) \geq Q_G(P)$ for all P in $[0,1]$ and a strict inequality for at least some P.

b. Second degree Stochastic Dominance (SSD):

For all $u \in U_2$; $\int_0^P Q_F(t)dt \geq \int_0^P Q_G(t)dt$ for all P in $[0,1]$ and a strict inequality for at least some P.

c. Third degree Stochastic Dominance (TSD):

For all $u \in U_3$; $\int_0^P \int_0^t Q_F(z)dzdt \geq \int_0^P \int_0^t Q_G(z)dzdt$ for all p in $[0,1]$

and $\int_0^1 Q_F(t)dt \geq \int_0^1 Q_G(t)dt$, with at least one strict inequality.

For proofs of these theorems, see Quirk & Saposnik [15], Fishburn [5], Hadar & Russell [6], Hanoch & Levy [7], Rothschild & Stiglitz [16], and Whitmore [18]. Further developments of SD rules for decreasing absolute risk aversion utility functions were developed recently by Vickson [17] and Mayer [13].

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2. $Q_F(p)$ is given by $Pr_F(x \leq Q_F(p)) = P$. For a proper definition of Q for the discrete case, see Levy & Kroll [10], p. 554. The quantile approach was first used in [10]. This framework was a necessary one for developing the SDR rules. Levy and Kroll proved that the rules given here and the rules defined in terms of the cumulative distributions are equivalent.

Denote the mixture of the random variable X with a riskless asset by X_α and the cumulative distribution of this variable by F_α, i.e., $X_\alpha = (1-\alpha)r + \alpha X$ where $0 < \alpha < \infty$ and r stands for the riskless interest rate. Similarly, the mixture of Y and the riskless asset will be denoted by Y_β and its cumulative distribution is denoted by G_β. The set $\{X_\alpha\}$ dominates the set $\{Y_\beta\}$ (or the set $\{F_\alpha\}$ dominates the set $\{G_\beta\}$) if and only if each Y_β which belongs to $\{Y_\beta\}$ is dominated by at least one X_α taken from the set $\{X_\alpha\}$. Necessary and sufficient conditions for a dominance of $\{F_\alpha\}$ over $\{G_\beta\}$ are given in theorems 2a, 2b and 2c.

Theorem 2

A necessary and sufficient condition for a dominance of $\{F_\alpha\}$ over $\{G_\beta\}$ by all $u \in U_i$, $(i = 1, 2, 3)$ is given by the following conditions:

a. First degree Stochastic Dominance with Riskless Asset (FSDR) for $i = 1$:

$$\sup_{F(r) < P \leq 1} \frac{Q_G(P) - r}{Q_F(P) - r} \leq \inf_{0 \leq P < F(r)} \frac{Q_G(P) - r}{Q_F(P) - r} \tag{1}$$

b. Second degree Stochastic Dominance with a Riskless Asset (SSDR) for $i = 2$:

$$\sup_{P_0 < P \leq 1} \frac{\int_0^P (Q_G(t) - r))dt}{\int_0^P (Q_F(t) - r))dt} \leq \inf_{0 \leq P < P_0} \frac{\int_0^P (Q_G(t) - r))dt}{\int_0^P (Q_F(t) - r))dt} \tag{2}$$

where P_0 is the solution of the following equation:

$$\int_0^{P_0} (Q_F(t) - r)dt = 0 \tag{3}$$

c. Third degree Stochastic Dominance with a Riskless Asset (TSDR), for $i = 3$

If there is no P_1 in the range $[0,1]$ which solves the equation

$$\int_0^{P_1}\int_0^{t} (Q_F(t) - r)dzdt = 0 \tag{4}$$

then the condition is

$$\mathrm{Max}\left[\sup_{P_1 < P \leq 1} \frac{\int_0^{P}\int_0^{t} (Q_G(z) - r)dzdt}{\int_0^{P}\int_0^{t} (Q_F(z) - r)dzdt},\ \frac{\int_0^{1} (Q_G(t) - r)dt}{\int_0^{1} (Q_F(t) - r)dt}\right] \leq$$

$$\leq \inf_{0 < P \leq P_1} \frac{\int_0^{P}\int_0^{t} (Q_G(z) - r)dzdt}{\int_0^{P}\int_0^{t} (Q_F(t) - r)dzdt} \tag{5}$$

if there is no P_1 in the range $[0,1]$ that solve the eq. (4), then the condition for dominance reduces to,

$$\frac{\int_0^{1} (Q_G(t) - r)dt}{\int_0^{1} (Q_F(t) - r)dt} \leq \inf_{0 \leq P \leq 1} \frac{\int_0^{P}\int_0^{t} (Q_G(z) - r)dzdt}{\int_0^{P}\int_0^{t} (Q_F(z) - r)dzdt} \tag{6}$$

For a proof of these theorems, see Levy & Kroll [10].

Before we turn to the investigation of these SDR rules in the specific case of continuous distributions, we would like to mention that, though the SDR rules are mathematically correct, it <u>appears</u> (even if each cumulative distribution is a step-function as obtained in empirical studies) that they are impractical. For example take the SSDR case; in order to find the SUP in the range $P_0 < P \leq 1$, the rule tells us that we have to calculate the left hand side of (2) for an infinite number of points since there are an infinite number of values of P in this range. However, as is shown in the appendix of this paper, the rules can be applied easily to distributionswith a step function, and the critical values (i.e. the SUP and INF) should be calculated only at the boundaries of the steps, which is of course finite. If the distribution functions are not given as step functions, infinite calculation of SUP and INF is involved, and a parametric approach is called for.

III. SDR Criteria for Continuous Distributions

Theorems 2a, 2b and 2c hold for step-functions as well as continuous distributions. However, since in the continuous case one cannot calculate the value of INF and SUP for <u>every</u> value P, we will try to find the conditions for <u>internal</u> extremum over the appropriate ranges of P's for which the SUP and INF are defined, and then, by comparison of the appropriate values at the extremum points, one can establish a parametric condition for a dominance requiring only one comparison, i.e. that of the values of the relevant function at extremum points.

To simplify the mathematical terms, let's adopt the following notation:

$$\delta(P) = \frac{Q_G(P) - r}{Q_F(P) - r} \tag{7}$$

$$\gamma(P) = \frac{\int_0^P (Q_G(t) - r)dt}{\int_0^P (Q_F(t) - r)dt} \tag{8}$$

$$\mu(P) = \frac{\int_0^P \int_0^t (Q_G(z) - r)dzdt}{\int_0^P \int_0^t (Q_F(t) - r)dzdt} \tag{9}$$

Let us start developing the FSDR criterion for continuous distributions. In order to do so we have to develop the first and second order conditions for internal extremum of $\delta(p)$. A suspected point P^* for extremum of $\delta(P)$ is where the following equality holds:

$$\frac{\partial\delta(P)}{\partial P} = \frac{Q_G'(P^*)[Q_F(P^*) - r] - Q_F'(P^*)[Q_G(P^*) - r]}{[Q_F(P^*) - r]^2} = 0 \qquad (10)$$

This first order condition can be rewritten as

$$\delta(P^*) = Q_G'(P^*)/Q_F'(P^*) \qquad (10')$$

The second order conditions can be obtained as follows:

$$\frac{\partial^2\delta(P)}{\partial P^2} = \frac{[Q_G''(P)(Q_F(P) - r) + Q_G'(P)Q_F'(P) - Q_F''(P)(Q_G(P) - r) - Q_F'(P)Q_G'(P)](Q_F(P) - r)^2}{(Q_F(P) - r)^4}$$

$$- \frac{2[Q_G'(P)(Q_F(P) - r) - Q_F'(P)(Q_G(P) - r)](Q_F(P) - r)Q_F'(P)}{(Q_F(P) - r)^4}.$$

After some reductions and a substitution of first order condition in this expression, we obtain:

$$\frac{\partial^2\delta(P)}{\partial P^2}\bigg|_{P=P^*} = \frac{Q_G''(P^*) - Q_F''(P^*)\dfrac{Q_G'(P^*)}{Q_F'(P^*)}}{[Q_F(P^*) - r]} \qquad (11)$$

The extremum obtained at P^* is a maximum if (11) is negative and a minimum if (11) is positive. We are interested in minimum for points below $F(r)$ and maximum for points above $F(r)$ (see eq. (1)). Since $Q_F(P) \gtrless r$ if and only if $P \gtrless F(r)$, it is easy to show from (11) that the sole condition for a minimum at a suspected point below $F(r)$ and for a maximum at a suspected point above $F(r)$ is:

$$Q_G''(P^*) < Q_F''(P^*)\ \delta(P^*) \qquad (12)$$

Therefore, in order to reveal a dominance by FSDR in the case of continuous distributions, we do not have to compute $\delta(P)$ for all P in the range $[0,1]$. It is sufficient to calculate $\delta(P)$ for suspected points P^* of internal extremum.[3] Thus, for a specific continuous distribution, we have to solve (11) and find the values of P^* for which (11) holds. For these P^*'s we must check inequality (12). Suppose that two values of P^* solve (11). If for these two values (12) holds, we know that the extremum point of $\delta(P^*)$ for $P^* < F(r)$ is a minimum, and the extremum point $\delta(P^*)$ for $P^* > F(r)$ is indeed a maximum. Thus, if the minimum is greater than the maximum, then inequality (1) holds, and dominance by FSDR can be established for the specific continuous distribution under consideration. Nevertheless, it may be that (12) does not hold, which implies that either there is no internal extremum, or that (11) holds for <u>all</u> values of P. This would in turn imply that one cannot utilize (11) and (12) directly. In these cases, we have to analyze more carefully the behavior of the function $\delta(P)$, in order to examine whether the condition given in eq. (1) indeed holds. In the next section, we will demonstrate cases where (11) and (12) hold, as well as cases where internal extremum point do not exist and hence (11) and (12) do not hold. In the last case a further investigation is called for. However, before we turn to sepecific continuous distributions, let us summarize the conditions for FSDR, and provide similar conditions for SSDR and TSDR.

<u>Theorem 3a</u>

In order to check if the FSDR condition holds (see eq. (1)), it is necessary

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3. In general we have the density function rather than the quantiles. However, one can use the relationship $Q'(P) = \frac{1}{f(x_p)}$ where $f(x_p)$ is the density function at point $x_p = Q_p$. Using this relationship one can calculate (11) and (12).

and sufficient to check if inequality (1) holds at border points, or suspected points P^*, where the suspected points fulfill the following two conditions:

$$\delta(P^*) = Q'_G(P^*)/Q'_F(P^*)$$

and

$$Q''_G(P^*) < Q''_F(P)\delta(P^*)$$

Note that if the above conditions do not hold, $\delta(P)$ must be monotonic, and it is enough to calculate its values at the border points. Since there is no internal extremum, the highest values of $\delta(P)$ must be located within the borders of the appropriate ranges (see eq. (11). The proof of this theorem has been already discussed above.

Theorem 3b

In order to determine the existence of inequality (2) of SSDR it is necessary and sufficient to examine $\gamma(P)$ at border points or suspected internal extremum points, P^*, which fulfill the conditions:

$$\gamma(P^*) = \delta(P^*)$$

and (13)

$$Q'_G(P^*) < Q'_F(P^*)\ \gamma(P^*)$$

Theorem 3c

In order to determine the existence of inequalities (5) or (6) of TSDR, it is necessary and sufficient to examine $\mu(P)$ at border points or suspected internal extremum points, P^*, which fulfill the following two conditions:

$$\mu(P^*) = \gamma(P^*)$$

and (14)

$$Q_G(P^*) < Q_F(P^*)\mu(P^*)$$

Proof of Theorems 3b and 3c

The proof of theorems 3b and 3c is based on the proof of theorem 3a. Let us prove that (13) are the first and second conditions of extremum of $\gamma(P)$.

Define $S_G(P)$ and $S_F(P)$ by the equations:

$$S_G(P) - r = \int_0^P [Q_G(t) - r]dt \quad \text{and} \quad S_F(P) - r = \int_0^P [Q_F(t) - r]dt$$

It follows by definition that,

$$\gamma(P) = \frac{S_G(P) - r}{S_F(P) - r}$$

Thus, $\gamma(P)$ has the same form as $\delta(P)$ and by analogy to the formation of first and second order conditions of $\delta(P)$, we can conclude that the proper conditions for extremum $\gamma(P)$ are:

$$\frac{S'_G(P)}{S'_F(P)} = \frac{S_G(P) - r}{S_G(P) - r} = \gamma(P) \tag{15}$$

and $\qquad S''_G(P) < S''_F(P)\ \gamma(P).$

However, since $S'_G(P) = Q_G(P) - r$, $S''_G(P) = Q'_G(P)$, $S'_F(P) = Q_F(P) - r$ and $S''_F(P) = Q'_F(P)$, we obtain that the first order condition and the proper second order condition for extremum of $\gamma(P)$ given by (15) are identical to the conditions given by (13). A similar line of reasoning holds for the case of TSDR: all one has to do is to define $S_G(P) - r = \int_0^P \int_0^t [Q_G(z) - r]dzdt$ and $S_F(P) - r = \int_0^P \int_0^t [Q_F(z) - r]dzdt$, and to follow the same arguments used in proving 3a.

IV. SDR Criteria for Specific Continuous Distributions

In this section we apply the criteria developed in the previous section to some classes of continuous distributions of economic interest. We begin by illustrating the case of the location-scale family of distributions where $\delta(P)$ does not have an internal extremum and hence is checked only on the borders of the appropriate ranges. Then we proceed to illustrate the case of the log normal distribution, a case where internal extremum points exist.

a) Location-scale family of distribution

Define $X_{\alpha,\beta}$ as $X_{\alpha,\beta} = \alpha + \beta Z$ where $-\infty < \alpha < \infty$, $\beta > 0$ and Z is a random variable. The set $\{X_{\alpha,\beta}\}$ is defined as a location-scale family with respect to variable Z. Example of such families are: normal distributions, symmetric stable distributions and uniform distributions.[4] Obviously, also the set $\{F_\alpha\}$ which consists of combinations between F and a riskless asset is a location-scale family with respect to the generating distribution function F. Let X_F and Y_G belong to the same location-scale family with respect to Z, i.e. $X = a + bZ$ and $Y = c + dZ$ (we delete the subscripts F and G where no confusion can arise). By definition, the P order quantile of X and Y is also a linear function of the P order quantile of Z, namely, $X(p) = a + bZ(P)$ and $Y(P) = c + dZ(P)$. In this case $\delta(P) = \dfrac{c + dZ(P) - r}{a + bZ(P) - r}$. According to (10'), if there is an internal extremum for $\delta(P)$, then at this point we have

$$\frac{c + dZ(P) - r}{a + bZ(P) - r} = \frac{(c + dZ(P) - r)'}{(a + bZ(P) - r)'} \tag{16}$$

which is reduced to

$$\frac{c + dZ(P) - r}{a + bZ(P) - r} = \frac{dZ'(P)}{bZ'(P)} = \frac{d}{b} \tag{17}$$

Equation (17) can be rewritten as

$$b = \frac{d(a-r)}{c-r} \tag{18}$$

Thus, if equation (17) holds, it must hold for every P (since $Z(P)$ disappears from (18)). However, (17) holds only if the parameters a, b, c, d and r fulfill equation (18). However, if (18) holds, $\delta(P)$ is constant for all P and

4. Discussion of the location-scale family of distributions can be found in Ali [2] and Bawa [4].

is equal to d/b for every p. Thus, if (18) holds, we can say that in this specific case that X and Y are identical in FSDR sense. In other words, for each combination of Y and r there is some combination of X and r which creates identical distribution, and vice versa.

Let us turn to analyze the more interesting case where (18) does not hold. In this case there is no internal extremum and hence $\delta(p)$ is either increasing or monotonic decreasing function with one discontinuity point as the denominator of $\delta(p)$ approaches zero. Namely, the discontinuity point is at the value P for which $a + bZ(P) - r = 0$. Thus, in order to check if the FSDR condition holds, we have to check $\delta(P)$ only at the borders of the proper intervals, $[0, F(r))$ and $(F(r), 1]$ (see eq. 1). It is clear that if the numerator of $\delta(P)$ becomes positive before the denominator becomes positive, then $\delta(P)$ approaches $-\infty$ below $F(r)$ and approaches $+\infty$ above $F(r)$ and there is no dominance by FSDR. That is, a necessary condition for the dominance of F over G by FSDR is that the denominator of $\delta(P)$ becomes positive before the numerator, which is identical to the condition $F(r) < G(r)$. If this condition holds in the case where F and G belong to the same location-scale distribution, then we can safely claim that $\delta(P)$ is increasing monotonically up to infinity as p increases from zero to $F(r)$, and increases monotonically from minus infinity as p increases from $F(r)$ to 1. Therefore, in this case in order to determine if FSDR exists we have to check whether the necessary condition $F(r) < G(r)$ holds and in addition if $\delta(0) \geq \delta(1)$. Note that since $\delta(P)$ is monotonic, it suffices to calculate only $\delta(0)$ and $\delta(1)$. These two conditions are stated formally in the following theorem:

<u>Theorem 4</u>

Let F and G belong to the same location-scale family of distribution. Then F dominates G by FSDR if and only if:[5]

$$F(r) < G(r)$$

and

$$\delta(0) \geq \delta(1) \tag{19}$$

We have already shwon that $F(r) < G(r)$ is a necessary condition for dominance. Also, if $F(r) < G(r)$ and $\delta(0) \geq \delta(1)$, eq. (1) holds which implies dominance by FSDR. If $\delta(0) < \delta(1)$, eq. (1) does not hold and F does not dominate G by FSDR.

Condition $F(r) < G(r)$ given in (19) can be rewritten as $P_{r_F}(X \leq r) \leq P_{r_G}(Y \leq r)$. However since $X = a + bZ$ and $Y = c + dZ$, the last inequality implies,

$$P_{r_F}(a + bZ \leq r) \leq P_{r_G}(c + dZ \leq r) \quad \text{or}$$

$$P_{r_F}(Z \leq \frac{r-a}{b}) \leq P_{r_G}(Z \leq \frac{r-c}{d})$$

But the last inequality holds if and only if $\frac{r-a}{b} < \frac{r-c}{d}$ or $\frac{a-r}{b} > \frac{c-r}{d}$. Thus, condition $F(r) < G(r)$ given in (19) is equivalent to condition (20)

$$\frac{a-r}{b} > \frac{c-r}{d} \tag{20}$$

Having the parameters a, b, c, d and the riskless interest rate r, one can easily check if (20) holds, <u>regardless</u> of the shape of the distribution of Z. However, in order to check if $\delta(0) \geq \delta(1)$ we should have information on the distribution of Z. Let's illustrate condition (19) for two important distributions which belong to a location scale family.

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5. Theorem (4) holds for a broader class than the location-scale family. This theorem holds also if $X = a + bH(Z(P)$ and $Y = c + dH(Z(P))$ where H is some monotonic increasing function.

Normal distribution

First recall that $Z \sim N(0,1)$. Thus,

$$E_{X_F} = E(a + bZ) = a \quad \text{and} \quad \sigma(X_F) = b$$

$$E_{Y_G} = E(c + dZ) = c \quad \text{and} \quad \sigma(Y_G) = d$$

The necessary condition $F(r) < G(r)$ for dominance of F over G given by eq. (20) can be rewritten simply as,

$$\frac{E_{X_F} - r}{\sigma_F} \geq \frac{E_{X_G} - r}{\sigma_G} \tag{20'}$$

Let us examine $\delta(P)$ in the normal case. $\delta(P)$ can be rewritten as

$$\delta(P) = \frac{d + \frac{(c-r)}{Z(P)}}{b + \frac{a-r}{Z(P)}} \tag{21}$$

As indicated by (19) we have to examine $\delta(P)$ only at $P = 0$ and $P = 1$. However, at these two values $Z(P)$ is equal to $-\infty$ and $+\infty$, respectively, and hence $\delta(0) = \delta(1) = d/b$. Since $\delta(0) = \delta(1)$, the sole condition for dominance of F over G by FSDR in the normal case is given by (20'). However, this condition is well-known, and simply assert that the market line of F is above the market line of G. It is easy to verify that in the normal case condition (20') (or the condition $F(r) < G(r)$) is also a necessary and sufficient condition for dominance by SSDR.[6]

Uniform distribution

Let $Z \sim U(0,1)$. Thus, $Z(0) = 0$ and $Z(1) = 1$. The necessary condition $F(r) \leq G(r)$ or its formulation given by (20) must also hold in this case, since

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6. The condition $F(r) \leq G(r)$ and its equivalency to (20') for normal distributions has been discussed in Levy and Kroll [9].

the uniform distributions belong to the location-scale family of distributions. Let us examine the second condition for dominance of F over G. Recalling that $Z(0) = 0$ and $Z(1) = 1$ we have, $\delta(0) = \frac{c-r}{a-r}$ and $\delta(1) = \frac{c+d-r}{a+b-r}$. Thus F dominates G if in addition to condition (20) (or (20')) $\delta(0) \geq \delta(1)$. We preclude cases where the distribution of the random variable is either completely to the right of r (a case where no one will mix the risky asset with r) or completely left to r (a case where no one would invest in the risky asset), and hence each distribution should start to the left of r and ends to the right of r.[7] The requirement $\delta(0) \geq \delta(1)$ is then equal to,

$$\frac{c-r}{a-r} > \frac{c+d-r}{a+b-r} \qquad (22)$$

Since a is the point when distribution F starts ($F(a) = 0$) and $a+b$ is the point where F ends ($F(a+b) = 1$), we know from the above explanation that $a-r < 0$ and $a+b-r > 0$. Thus (22) can be rewritten as

$$(c-r)(a+b-r) < (c+d-r)(a-r) \qquad (22')$$

It can be easily shown that (22') holds if and only if (20) holds. Namely, if $F(r) \leq G(r)$, then also $\delta(0) \geq \delta(1)$, and hence $F(r) \leq G(r)$ is a necessary and sufficient condition for dominance of F over G by FSDR (as well as by SSDR).

To sum up, the condition $F(r) \leq G(r)$ is a necessary condition for dominance for location-scale family of distributions, and it is necessary and sufficient condition for the normal and uniform distributions. For these distributions, dominance by FSDR or SSDR transforms into the well known rule which asserts that the market line of the dominated portfolio is lower. Thus, in this specific case we have a separation by FSDR and SSDR, since only one risky portfolio with the highest market line is included in the efficient set.

7. For the elimination of these trivial cases, see Levy & Kroll [10].

b. Lognormal distributions

We have shown above that for the location scale family of distributions, there is no internal extremum of $\delta(P)$. Now we turn to the lognormal distribution where internal extremum may exist. The lognormal distribution has a specific importance in economics. There are theoretical as well as empirical studies which claim that the distribution of rate of return on securities is well fitted by the lognormal distribution.[8] We first demonstrate how one can use the rules developed in Section III in order to establish FSDR in the lognormal case and then proceed in developing SSDR for lognormal distributions. Before we turn to the theorems we would like to mention that Levy & Kroll [9] dealt with the FSDR and lognormal distribution in a different, and perhaps more complicated, framework.

Theorem 4

Let F and G be two-parameter lognormal distributions of the risky options and r the riskless interest rate. Then F dominates G by FSDR if and only if

$$\begin{aligned} &\text{(a)} \quad F(r) < G(r) \\ &\text{(b)} \quad \sigma_F \geq \sigma_G \end{aligned} \tag{23}$$

when σ^2 stands for the variance of $\log X$, where X is the rate of return.

Proof

Let μ_F, σ_F and μ_G, σ_G be the mean and variance of $\log X$ under distribution F and G respectively, X is the rate of return and r the riskless interest rate. The P quantile of the lognormal distribution is given by $Q_\Lambda(P)$ where

$$Q_\Lambda(P) = e^{\mu + Z_N(P)\sigma} \tag{24}$$

8. See Lintner [12] and Merton [14].

$Z_N(P)$ being the P quantile of the standardized normal distribution (see Aitchison and Brown [1] and Levy [8]). Thus,

$$Q'_{\Lambda}(P) = e^{\mu + Z_N(P)} Z'_N(P)\sigma = Z'_N(P)\sigma Q_{\Lambda}(P) \tag{25}$$

Using Theorem 3a, at extremum points P^*, we have,

$$\frac{\sigma_G Q_{\Lambda_G}(P^*)}{\sigma_F Q_{\Lambda_F}(P^*)} = \delta(P^*) \tag{26}$$

In general the left hand side of (26) which we denote by $S(P)$ can be rewritten as

$$S(P) = \frac{\sigma_G Q_{\Lambda_G}(P)}{\sigma_F Q_{\Lambda_F}(P)} = \frac{\sigma_G}{\sigma_F} e^{\mu_F - \mu_G + Z_N(P)(\sigma_G - \sigma_F)} \tag{27}$$

Taking the derivative of (27) we obtain,

$$\frac{\partial S(P)}{\partial P} = Z'_N(P)[\sigma_G - \sigma_F]e^{\mu_F - \mu_G + Z_N(P)(\sigma_G - \sigma_F)} \tag{28}$$

Thus, $\frac{\partial S(P)}{\partial P}$ is negative (which implies that $S(P)$ decreases as P increases) if and only if $\sigma_F > \sigma_G$.[9]

Thus, if indeed there are internal extremum points, from the fact that $S(P)$ is decreasing with P, we can conclude from (26) that $\delta(P^*)$ in the range $0 \leq P < F(r)$ must be greater than $\delta(P^{**})$ in the range $F(r) < P \leq 1$, when P^* and P^{**} are the values appropriate to the two extremum points. We shall see by means Figure 1 that if we add the condition $F(r) < G(r)$ (as required by the theorem) then indeed there is an internal minimum in the range $0 < P < F(r)$ and internal maximum in the range $F(r) < P \leq 1$. Thus the condition

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9. In the case $\sigma_F = \sigma_G$, $\delta(P^*)$ is constant for all P's (see (27)), and in this case it is trivial that F dominates G by FSDR if $F(r) < G(r)$. (See Figure 2 and the appropriate discussion, below).

$\sigma_F < \sigma_G$ guarantess that the minimum in the first range is greater than the maximum in the second range which implies further that F dominates G by FSDR.

Insert Figure 1

Figure 1 demonstrates a case where $F(r) < G(r)$ and $\sigma_F > \sigma_G$. Recall that two lognormal distributions intersect at most once and that if $\sigma_F > \sigma_G$, then G must cross F from below as shown in this figure.[10] From the definition of $\delta(P)$, it is obvious that $\delta(0) = \delta(\tilde{P}) = 1$, and that $0 < \delta(P) < 1$ for every P in the range $0 < P < \tilde{P}$. For $\tilde{P} < P < F(r)$, $\delta(P)$ increases with P and approaches infinity as P approaches $F(r)$. Thus, $\delta(P)$ must have a global internal minimum below $F(r)$. Similar arguments lead to the conclusion that $\delta(P)$ has an internal maximum above $F(r)$. Thus, from these properties, and the fact that $S(P)$ is decreasing with P, (see (26)), we can conclude that the minimum is greater than the maximum and hence F dominates G by FSDR. It is simple to show by way of the same arguments that if $F(r) > G(r)$, and $\sigma_F \geq \sigma_G$, or if $F(r) < G(r)$ and $\sigma_F < \sigma_G$, there is no FSDR.

Insert Figure 2

Figure 2 summarizes the situation where $F(r) < G(r)$ and $\sigma_F \geq \sigma_G$. In this case F dominates G by FSDR. The function $S(P)$ decreases with P, and crosses $\delta(P)$ at the minimum and maximum points A and B respectively, where A is above B. Note that if $\sigma_F = \sigma_G$, then $S(P)$ is horizontal with a tangency point at A and B, i.e. $\delta(P_1^*) = \delta(P_2^*)$. However, this condition is also sufficient for dominance of F over G by FSDR (see (1)).

Let us turn to SSDR rules in case of lognormal distributions.

Theorem 5

Let F and G and r be as in Theorem 4. Then F dominates G by SSDR, if and only if:

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10. For further details of the properties of lognormal distributions, see Aitchison and Brown [1].

(a) F dominates G by SSD

or

(b) $\sigma_F \geq \sigma_G$ and $P_0 < P_1$ where P_0 and P_1 are the values which respectively solve equations (28) and (29):

$$\int_0^{P_0} \mathrm{Exp}[\mu_F + Z_N(P)\sigma_F]dP = rP_0 \tag{28}$$

$$\int_0^{P_1} \mathrm{Exp}[\mu_G + Z_N(P)\sigma_G]dP = rP_1 \tag{29}$$

Proof

Sufficiency. Since in general SSD implies SSDR (see Levy & Kroll [10]) it is obvious that condition (a) is sufficient for SSDR. Thus, we have to prove that if there is no SSD then there is SSDR of F over G if and only if $\sigma_F \geq \sigma_G$ and $P_0 < P_1$. If $\sigma_F \geq \sigma_G$ and $F(r) < G(r)$, F dominates G by FSDR (see Theorem 4). Since FSDR implies SSDR, what is left to prove is that F dominates G by SSDR in the case $\sigma_F \geq \sigma_G$, $P_0 < P_1$ and $F(r) > G(r)$. This specific situation is depicted in Figure 3. Once again recall that

Insert Figure 3

$\sigma_F > \sigma_G$ implies that G crosses F from below and that there is only one intersection point. We use this figure to show that $\gamma(P)$ has an internal global minimum below P_0 and a global internal maximum above P_0, and that the global minimum is above the global maximum, which implies by (2) that F dominates G by SSDR.

Let us first examine the behavior of $\gamma(P)$ in the range $0 \leq P < P_0$. First note that $\gamma(0) = \frac{-r}{-r} = 1.$[11] It is obvious from the definition of $\gamma(P)$ and from figure 3 that as P increases $\gamma(P)$ starts to decline since it is below 1. However, when P approaches P_0, then the denominator of $\gamma(P)$

11. Use L'hoptal rule to obtain this result.

qpproaches zero from below, but since $P_0 < P_1$ (see Figure 3), the numerator of $\gamma(P)$ is still negative and hence $\gamma(P)$ approaches $+\infty$. Thus, in the range $0 \leq P < P_0$, $\gamma(P)$ starts from 1, falls below 1, and at P_0 approaches $+\infty$, which implies that in this range $\gamma(P)$ has an internal minimum. Similar arguments lead to the conclusion that in the range $P_0 < P < 1$, $\gamma(P)$ has an internal maximum. Thus, in order to prove the sufficiency, what is left to show is that the minimum is greater than the maximum. Denote by P^* and P^{**} the points corresponding to the minimum and maximum of $\delta(P)$, respectively, where $P^* < P^{**}$. According to the first order conditions given in theorem 3b, at these extremum points we have $\gamma(P) = \delta(P)$. Therefore, instead of proving that $\gamma(P^{**}) < \gamma(P^*)$, it is enough to prove that $\delta(P^{**}) < \delta(P^*)$.

We shall show that both P^* and P^{**} are in the range where $\delta(P)$ is continuous and decreasing. If we succeed in demonstrating this, we complete the proof since $P^* < P^{**}$, and at the extremum points $\gamma(P) = \delta(P)$ (see Theorem 3b). Thus, the minimum is greater than the maximum (since $\delta(P)$ is declining) and F dominates G by SSDR (see eq. (2)). Let us begin by showing that indeed P^* and P^{**} are in the range where $\delta(P)$ is decreasing.

We proved that $\gamma(P)$ has a global minimum in the range $[0, P_0)$. However, at the point $P = F(r)$, $\gamma(P)$ is still declining, since its denominator which is negative has zero change in the neighborhood of $F(r)$ (since $Q_F(P) - r = 0$ at the point $P = F(r)$, but at this point $Q_G(P) - r > 0$ see Figure 3). Hence the numerator of $\gamma(P)$ becomes less negative. Thus, $\gamma(P)$ must decline in the neighborhood of $F(r)$ and the value corresponding to the minimum value of $\delta(P)$ must fulfill $P^* > F(r)$.

Since $P^{**} > P^*$, and since $P = F(r)$ is the only discontinuity point of $\delta(P)$, both P^* and P^* are located in the range $P > F(r)$ when $\delta(P)$ is continuous. However, it is easy to verify from Figure 3 that in the range $P > F(r)$, $\delta(P)$ is decreasing with P. This completes the proof.[12]

Insert Figure 4

Figure 4, which is plotted by a computer summarizes the case when F dominates G by SSDR with two lognormal distributions. As can be seen, $\delta(P)$ is decreasing and continuous in the range $F(r) < P \leq 1$, and $\gamma(P^*) > \gamma(P^{**})$; this implies SSDR.

Necessity. So far we have completed the sufficiency part of the theorem. The necessity part is straightforward: $P_0 < P_1$ is a necessary condition for dominance by SSDR for all distributions (see Levy & Kroll [10]) and it is also necessary condition in the specific case of lognormal distributions. If $\sigma_F < \sigma_G$ and there is no SSD of F over G, then $E_G(X) > E_F(X)$ must hold (because $\sigma_F < \sigma_G$, and $E_F(X) > E_G(X)$ is equivalent to SSD with lognormal distributions, see Levy [8]). However, SSDR of F over G, implies that there is some α such that F_α dominates G by SSD. But if $E_G(X) > E_F(X)$, for all $\alpha \leq 1$ we always have $E_G > E_{F_\alpha}$ since $E_G(X) > E_F(X) > r$. Thus, F_α cannot dominate G by SSD, since a necessary condition for dominance by SSD is that $E_{F_\alpha}(X) \geq E_G(X)$. For $\alpha > 1$ we also cannot find F_α which dominates G by SSD, since both F and G start from zero, and hence the left tail of F is above G (see Figure 1).

V. The Lognormal efficient set frontier

Levy [8] and Levy & Kroll [9] established the SSD and FSDR efficient frontiers in the μ-σ space for the case of lognormal distributions. Here we extend their work by identifying also the SSDR efficient frontier. The dashed area in Figure 5 presents the feasbile set in terms of the mean and

12. We proved by cumbersome algebra that $\delta(P)$ is decreasing in this range. However, since $\sigma_F > \sigma_G$ we must have the lognormal distributions as drawn in Figure 3, we decided to omit the mathematical proof, and to rely on the simple graphical devise.

variance of log-returns (see Levy & Kroll [9]). Levy [8] prove that the segment BD is the SSD efficient frontier, and Levy & Kroll [9] proved that CD is the FSDR efficient frontier. It is obvious that the SSDR efficient frontier must lie within segment CD, since the SSDR efficient set is a subset of FSDR efficient set. Below we shall analyze which part of CD is SSDR efficient.

Let P_0 be the solution of (30)

$$\int_0^{P_0} [e^{\mu(\sigma)+Z_N(t)\sigma} - r]dt = 0 \tag{30}$$

Note that the quantile of a lognormal distribution $Q_\Lambda(P)$ is given by

$$Q_\Lambda(P) = e^{\mu+Z_N(P)\sigma}$$

Since on the efficient set a change in σ causes a change in μ we write $\mu(\sigma)$ rather than μ in (30), emphasizing that μ is an implicit function of the portfolio's σ.[13]

Equation (30) can be rewritten as

$$\int_0^{P_0} e^{\mu(\sigma) + Z_N(t)\sigma} dt = rP_0 \equiv T \tag{31}$$

According to Theorem 5, on the efficient set curve as we increase σ, P_0 which solves (30) or (31) must increase as well (otherwise we have dominance by SSDR).

However, since r is constant, P which solves (30) increases as one increases σ if and only if $\frac{\partial T}{\partial \sigma}$ is negative (see (31)). In other words if $\frac{\partial T}{\partial \sigma} < 0$, one must increases P_0 in order to satisfy (30), which means that σ and P increase together, a property which identifies the efficient set. (see Theorem 5). Thus, on the efficient set, we have

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13. We use a very similar technique to the one employed by Baumol [3] in identifying the efficient frontier.

$$\frac{\partial\left(\int_0^{P_0} e^{\mu(\sigma) + z(t)\sigma} dt\right)}{\partial\sigma} \leq 0 \qquad (32)$$

which can be rewritten as

$$\frac{\partial \int_0^{P_0} e^{\mu(\sigma)+z(t)\sigma} dt}{\partial\sigma} = \int_0^{P_0} e^{\mu(\sigma)+z(t)\sigma}\left[\frac{\partial\mu(\sigma)}{\partial\sigma} + z(t)\right]dt \leq 0$$

Therefore, on the SSDR efficient segment of the frontier the following holds,

$$\frac{-\int_0^{P} z(t)\, e^{\mu+z(t)\sigma} dt}{\int_0^{P_0} e^{\mu+z(t)\sigma} dt} \geq \frac{\partial\mu(\sigma)}{\partial\sigma} \qquad (33)$$

or

$$\frac{\partial\mu(\sigma)}{\partial\sigma} \leq \frac{-\int_0^{P_0} z(t)e^{\mu+z(t)\sigma} dt}{rP_0} \quad \text{(see eq. 30))} \qquad (34)$$

Inequality (34) provides a condition to examine whether we are on the efficient or inefficient SSDR set. One should calculate the right hand side of (34), and eliminate the segment of the frontier where $\frac{\partial\mu(\sigma)}{\partial\sigma}$ is greater than this value. Thus, we relegate to the inefficient SSDR set, a segment like CE (see Figure 5) when the slope is higher than a given value obtained from the right hand side of (34).

V. Concluding Remarks

Stochastic dominance decision rules with the allowance for borrowing and lending at a riskless interest rate have been developed only in recent years. These rules hold for empirical distributions (with a cumulative step-function) as well as for any theoretical distributions with a continuous cumulative distributions. However, though these rules always hold, one can not apply them for distribution with continuous cumulative distributions, since it involved an infinite number of comparisons.

In this paper we developed the stochastic dominance rules with a riskless asset in such a way that, in principle, one can apply these rules to any theoretical distribution. We demonstrated the application of these new rules to uniform, normal and lognormal distributions. In addition to the possibility of such applications, which is impossible to apply without the rules developed in this paper, the stochastic dominance rules in the continuous framework, yields, in some cases, stochastic dominance criteria which are stated simply in terms of the distributions parameters (e.g., mean, variance, etc.). However, for some distributions the decision rules may be still complicated, even if one uses the continuous framework.

Finally, we used the rules developed in this paper to investigate the mean-variance risk-averters efficient frontier when the distributions are assumed to be lognormal.

Appendix. SDR rules for step-function cumulative distributions

In this appendix, we show that in the case of discrete distributions, the dominance conditions FSDR, SSDR, and TSDR can be greatly simplified since we do not have to trace the quantiles at every P, but only the quantiles at the upper or lower end of each step. A step is defined as a range of P where the quantiles of F and G are both constants. Thus, according to this definition the expression $(Q_G(P) - r)/(Q_F(P) - r)$ is a constant in each step. Therefore, in employing FSDR criterion (see equation (1)) only one computation for each step is needed.

SSDR - Since we are looking for either SUP or INF, we have to prove that

$$\gamma(P) \equiv \int_0^P [Q_G(t) - r]dt / \int_0^P [Q_F(t) - r]dt \tag{A-1}$$

is either monotonic non-increasing or monotonic non-decreasing within each step. Given that the above claim is correct, and considering the fact that $\gamma(P)$ is continuous at all points,[1] we are justified in examining the quantiles only at one end of each step. Let us look at a specific step i. Denote with $\underline{P}_i$ and $\overline{P}_i$ the lower and upper borders of this step. The quantiles in the i step will be denoted by $Q_F(i)$ and $Q_G(i)$. Let us further denote:

$$\int_{-\infty}^{\underline{P}_i} [Q_F(P) - r]dp = a \tag{A-2}$$

$$\int_{-\infty}^{\underline{P}_i} [Q_G(p) - r]dp = b \tag{A-3}$$

$$\text{then } \gamma(\underline{P}_i) = b/a \tag{A-4}$$

1. There is, of course, one discontinuity point $P = P_0$ where the denominator of (A-1) is zero. A discussion on this issue will be done at the end of this Appendix.

For every P in the range $(\underline{P}_i, \overline{P}_i)$ we have:

$$\frac{b + \int_{\underline{P}_i}^{P} [Q_G(i) - r]dt}{a + \int_{\underline{P}_i}^{P} [Q_F(i) - r]dt} = \frac{b + (P-\underline{P}_i)(Q_G(i) - r)}{a + (P - \underline{P}_i)(Q_F(i) - r)} \qquad \text{(A-5)}$$

differentiating with respect to P:

$$\frac{d\gamma(P)}{dP} = \frac{(Q_G(i)-r)[a + (P-\underline{P}_i)(Q_G(i)-r)] - [b + (P-\underline{P}_i)(Q_G(i)-r)](Q_F(i)-r)}{[a + (P-\underline{P}_i)(Q_F(i) - r)]^2}$$

After certain reductions we have:

$$\frac{d\gamma(P)}{dP} = \frac{a(Q_G(i)-r) - b(Q_F(i)-r)}{[a + (P-\underline{P}_i)(Q_F(i)-r)]^2} \qquad \text{(A-6)}$$

The numerator of (A-6) is independent of P, and hence does not change signs within a step. Thus, the whole expression is either non-decreasing or non-increasing within the step, and hence it is enough to examine $\gamma(P)$ at the ends of each quantile, and there is no need to calculate $\gamma(P)$ for each P within the quantile.

<u>TSDR</u>: We have to prove that in the case of discrete distribution functions the expression:

$$\mu(P) \equiv \int_0^P \int_0^t [Q_G(z)-r]dzdt / \int_0^P \int_0^t [Q_F(z)-r]dzdt \qquad \text{(A-7)}$$

is either monotonic non-increasing or monotonic non-decreasing within each step i. Denote as before by $\underline{P}_i$ and $\overline{P}_i$, the lower and upper bounds of the i-th step. Let,

$$\int_0^{\underline{P}_i} \int_0^t [Q_F(z)-r]dzdt = a \qquad \text{(A-8)}$$

and

$$\int_0^{\underline{P}_i} \int_0^t [Q_G(z)-r]dzdt = b \qquad \text{(A-9)}$$

thus $\mu(\underline{P}_i) = b/a$ and for every $\underline{P}_i < P < \overline{P}_i$. By definition,

$$\mu(P) \equiv b + \int_{\underline{P}_i}^{P} \int_{\underline{P}_i}^{t} [Q_G(i)-r]dzdt]/[a + \int_{\underline{P}_i}^{P}\int_{\underline{P}_i}^{t} [Q_F(i)-r]dzdt \qquad (A-10)$$

Because $Q_G(i)$ and $Q_F(i)$ are both constant within the step, we obtain,

$$\mu(P) \equiv b^* + [Q_G(i)-r](\frac{1}{2}P^2 - \underline{P}_iP)]/[a^* + [Q_F(i)-r](\frac{1}{2}P^2 - \underline{P}_iP)] \qquad (A-11)$$

where a^* and b^* are independent of P and defined by the following expression:

$$b^* = b + \frac{1}{2}\underline{P}_i^2(Q_G(i) - r) \qquad (A-12)$$

$$a^* = a + \frac{1}{2}\underline{P}_i^2(Q_F(i) - r) \qquad (A-13)$$

Differentiating $\mu(P)$

$$\frac{d\mu(P)}{dP} = \frac{(Q_G(i)-r)(P-\underline{P}_i)[a^* + (Q_F(i)-r)(\frac{1}{2}P^2 - \underline{P}_iP) -}{(a^* + [Q_F(i)-r](\frac{1}{2}P^2 - \underline{P}_iP))^2}$$

$$\frac{-[b^* + (Q_F(i)-r)(\frac{1}{2}P^2 - \underline{P}_iP)](Q_F(i)-r)(P-\underline{P}_i)}{(a^* + [Q_F(i) - r](\frac{1}{2}P^2 - \underline{P}_iP))^2}$$

After appropriate reductions in the numerator and denoting the denominator by B^2 we have:

$$\frac{d\mu(P)}{d(P)} = \frac{(P-\underline{P}_i)[a^*(Q_G(i)-r) - b^*(Q_F(i)-r)]}{B^2} \qquad (A-14)$$

and since $P > \underline{P}_i$ the sign of (A-14) is independent of P within the step i.

Finally, note that $\gamma(P)$ and $\mu(P)$ are not defined at the points P_0 and P_1, where P_0 and P_1 are the solutions to the equations:

$$\int_0^{P_0} [Q_F(z)-r]dt = 0 \qquad (A-15)$$

$$\int_0^{P_1}\int_0^{t} [Q_F(z)-r]dzdt = 0 \qquad (A-16)$$

Denote by P_0' and P_1' the solutions to the equations:

$$\int_0^{P_0'} [Q_G(t)-r]dt = 0 \tag{A-17}$$

$$\int_0^{P_1'} \int_0^{t} [Q_G(z)-r]dzdt = 0 \tag{A-18}$$

If $P_0' < P_0$, then F cannot dominate G by SSD, since in such a case when P approaches P_0 from below, $\gamma(P)$ approaches $-\infty$ and condition (2) of Theorem 2 cannot hold. Similarly, $P_1' > P_1$ is a necessary condition for TSDR. Thus, $F(r) \leq G(r)$, $P_0 \leq P_0'$ and $P_1 \leq P_1'$ are necessary conditions for FSDR, SSDR and TSDR respectively. The implication of these results in the case of discrete distributions is that in addition to the necessary computation at the end of each step we first must confirm that these necessary conditions hold. It must be further mentioned that if P_0 and P_0' are in the same step (say i^*) then if one fails to check this necessary condition, he can reach a misleading conclusion. Such a situation is illustrated in Figure A-1, in which P_0' and P_0 are in the same step. The lower bound of this step is $\underline{P}$ and the upper bound is $\overline{P}$. $P_0' < P_0$ and hence $\gamma(P) \to -\infty$ as P becomes closer to P_0 from the left. So condition (2) of Theorem 4 does not hold and there is no SSDR of F over G. However, if one checks $\gamma(P)$ only at the end points $\underline{P}$ and $\overline{P}$, he can mistakenly conclude that there is SSDR, because $\gamma(\underline{P}) > \gamma(\overline{P})$.

Figure A-1

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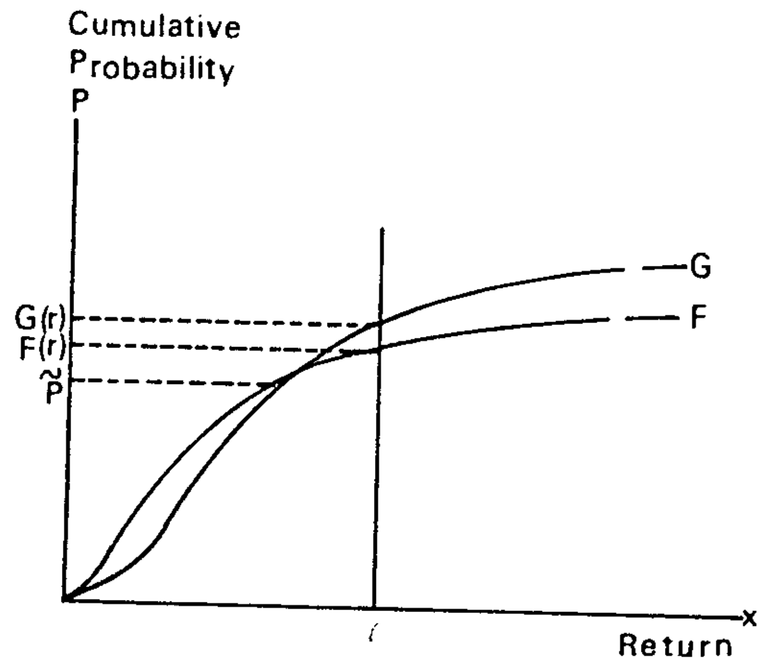

Figure 1

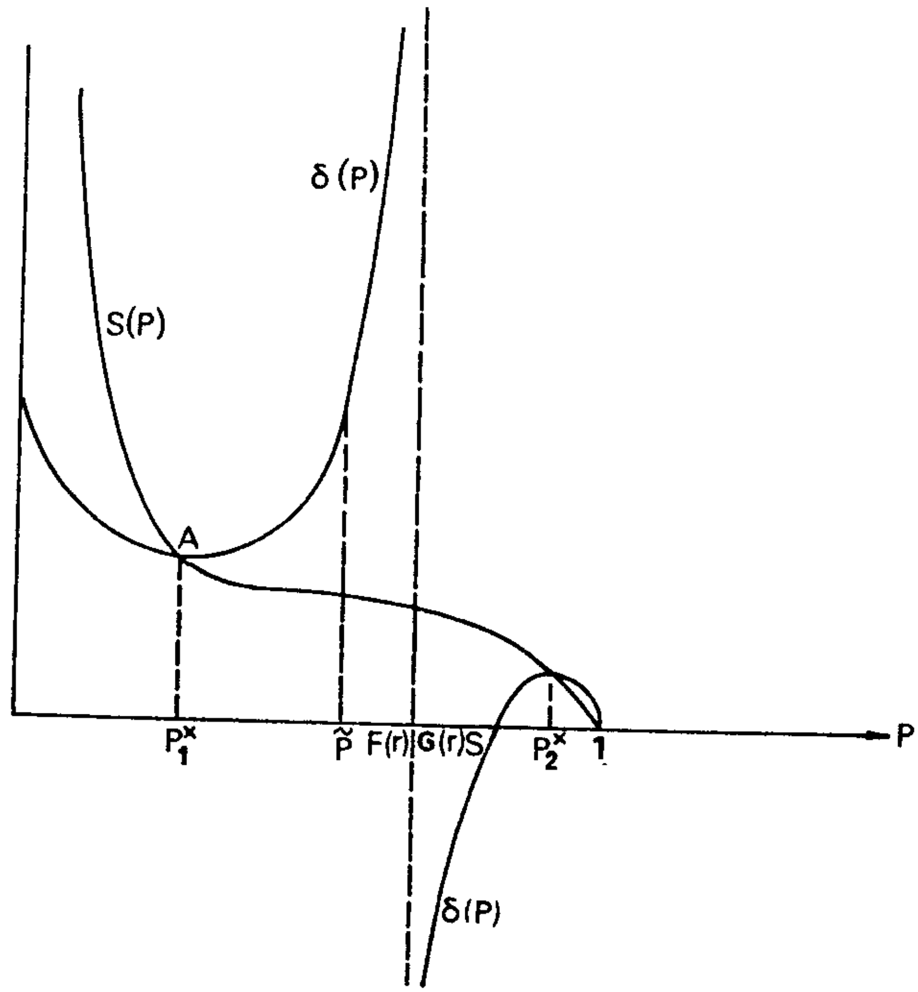

Figure 2

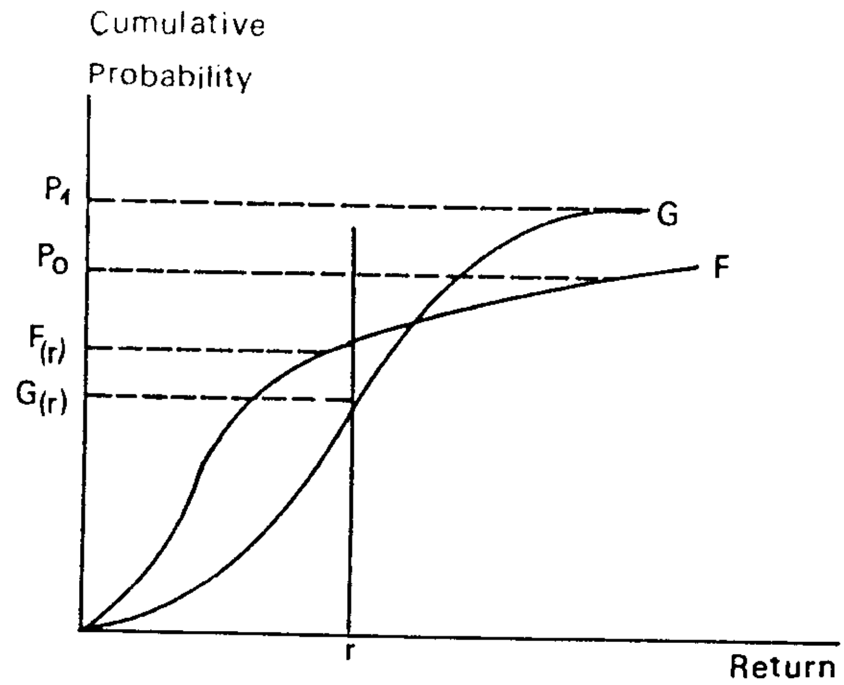

Figure 3

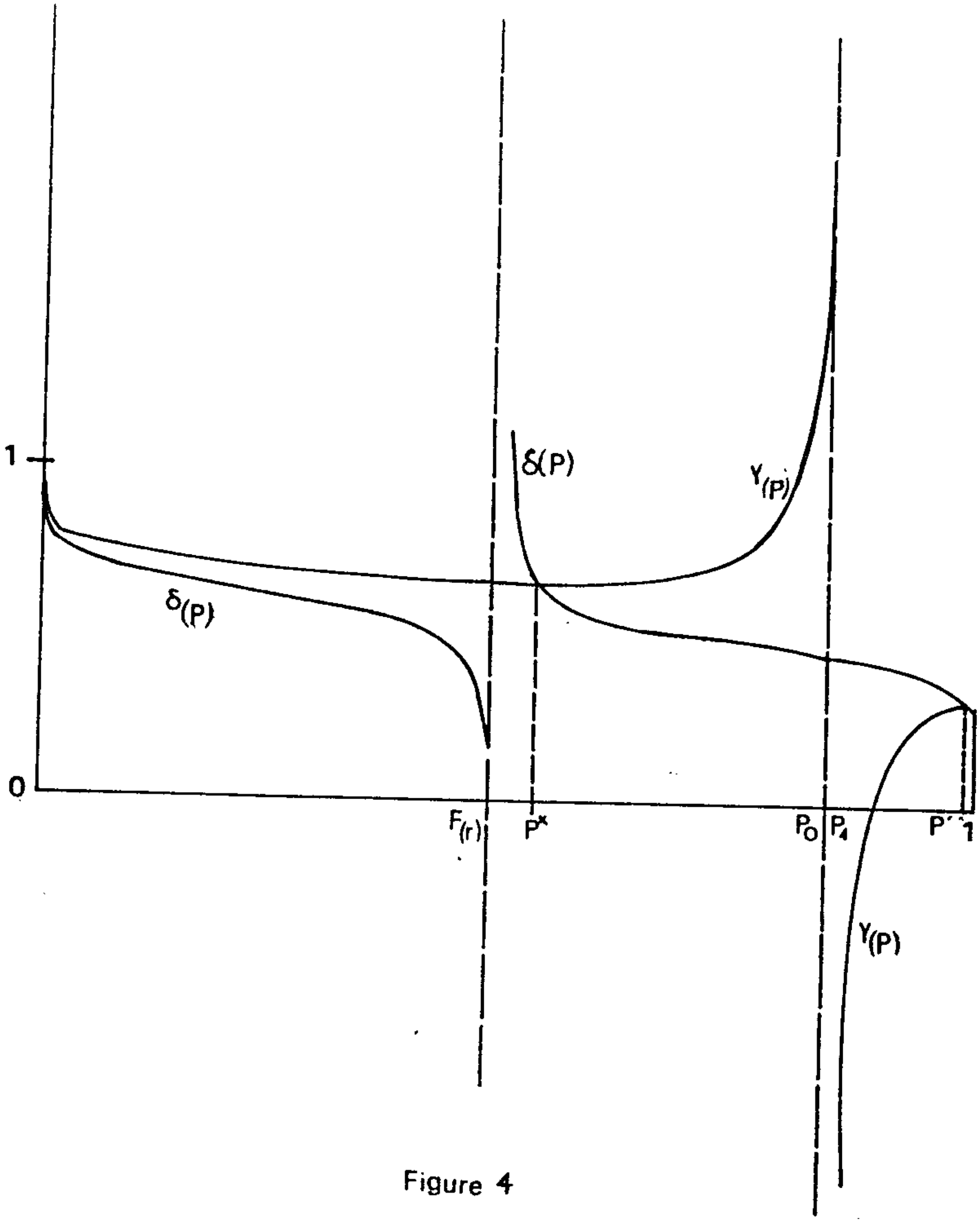

Figure 4

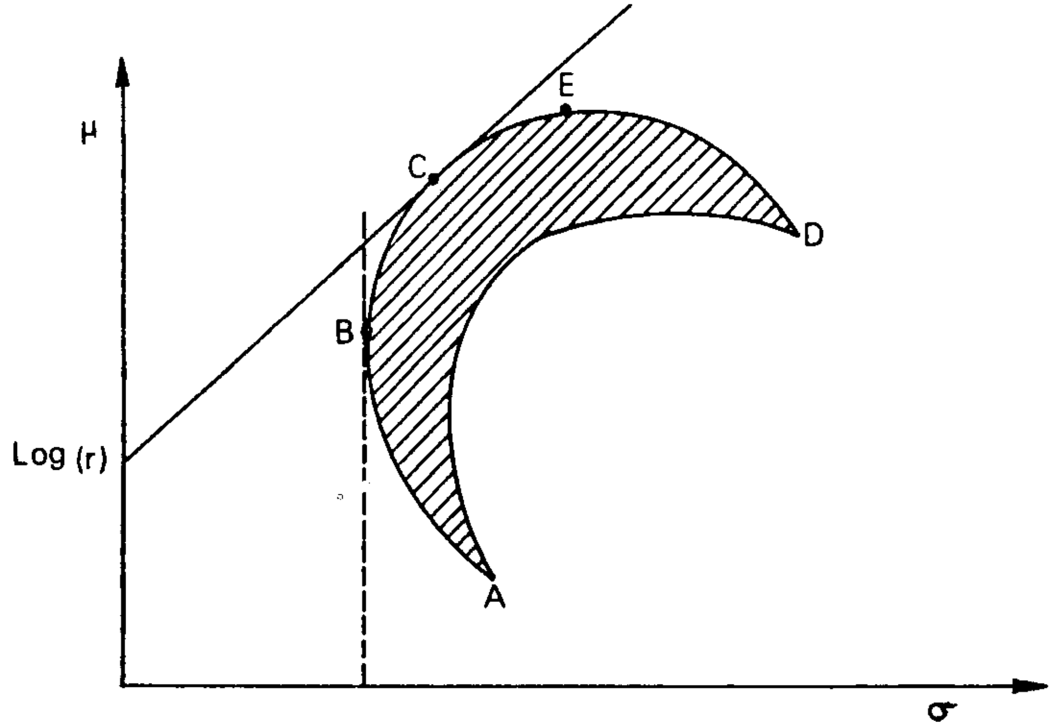

Figure 5

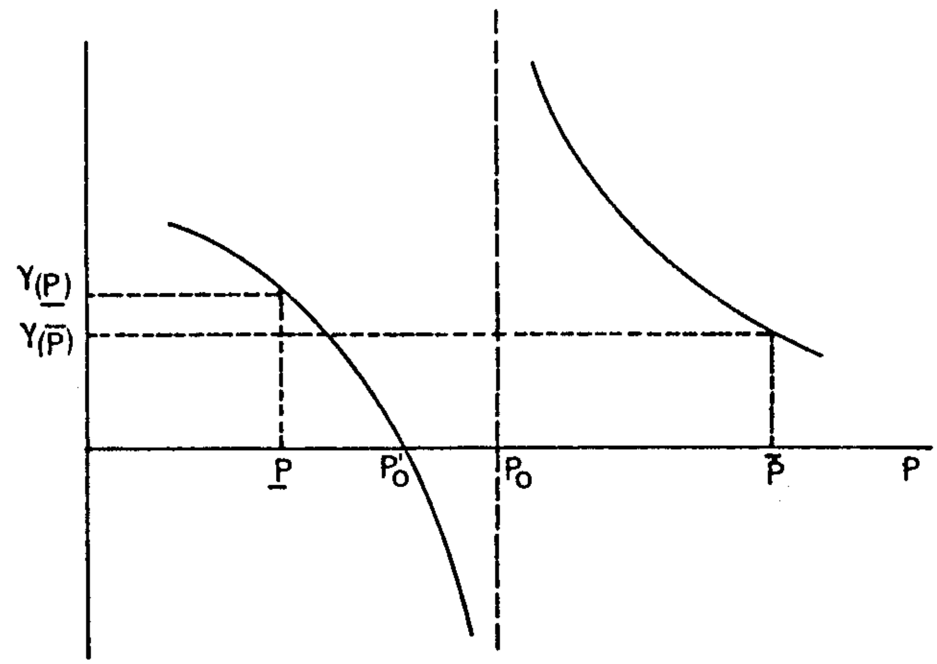

Figure A-1