The Interpretation of One-Parameter Performance Measures

by

Marshall E. Blume

Working Paper #21-80

RODNEY L. WHITE CENTER
FOR FINANCIAL RESEARCH

University of Pennsylvania

The Wharton School

Philadelphia, PA 19104

The Interpretation of One-Parameter Performance Measures

Marshall E. Blume

I. Introduction

The usual interpretation of the traditional one-parameter investment performance measures, first used by Jensen (1968) and now known as "alpha" coefficients by professional investors, rests upon the Sharpe-Lintner capital asset pricing model. It has long been recognized that the market portfolio of risky assets in this model should consist of all risky assets.[1] Since it would be impossible to purchase such a portfolio or even to measure its return, the usual practice has been to substitute a subset of risky assets, whose returns, it is hoped, are highly correlated with the returns of the portfolio of all risky assets. In professional evaluation services, such as provided by Merrill Lynch, the returns on the portfolio of all risky assets is usually approximated by the returns implied by the Standard and Poor's Composite Index of 500 stocks.

Recently, Roll (1977) has devised a numerical example which shows that high correlation between the return used to approximate the market return and the market return itself is not sufficient to preserve, even as a rough approximation, the validity of the Sharpe-Lintner model. Indeed, from his writings, it would appear that he feels that the inability to measure the market portfolio correctly virtually destroys the empirical usefulness of the Sharpe-Lintner model and, by extension, the investment tools associated with these models, such as betas or investment performance measures.[2]

The purpose of this paper is to show that one-parameter performance measures do provide useful information to an investor who wishes to follow the

---

[1]Cf. Markowitz (1959), pp. 297-303.

[2]Cf. Roll (1978, 1980).

mean-variance precepts of Markowitz. This demonstration does not require that the capital asset pricing model describe equilibrium prices.

In the presence of incomplete information, it will be shown that the correct portfolio to use as the reference portfolio in calculating these one-parameter measures is the portfolio which an investor currently holds, not the market portfolio of all risky assets. Nonetheless, most professionally calculated one-parameter performance measures are based upon the Standard and Poor's Composite Index. With this fact in mind, the paper will then show that these professionally calculated measures, better known as alpha coefficients, do provide in some situations relevant and useful information.

II. The Investor's Decision

The usual development of the capital asset pricing model makes the assumption that all investors know the true joint probability distribution of the future returns of all assets--in short, have homogeneous expectations. In such a world, performance measurement would be a meaningless exercise in that every investor would be holding the best possible portfolio of risky assets, namely the market portfolio.[3]

In reality, no investor would know the true joint probability distribution of the future returns of all assets. Thus, he would never be able to answer the question of whether he really did hold the best portfolio of all possible portfolios. A more relevant question to such an investor is the very pragmatic question of whether he could alter his current portfolio in some way so as to obtain a better overall position.

To be more precise, assume that the investor currently holds portfolio p, has assessed a predictive distribution of the returns on this portfolio with mean μ_p and standard deviation σ_p, and finally is willing to utilize a mean-variance one-period world to evaluate his alternatives. If the investor could borrow or lend at the risk-free rate r_f, he could obtain any position along the ray eminating from the risk-free rate through portfolio p, as shown in Figure 1. The investor's assumed optimal position has been designated with the superscript '*'. As drawn, the investor's final portfolio would involve a positive position in the risk-free asset; but more importantly, portfolio p need not be, and most likely is not, on the true efficient set.

---

[3]This statement assumes perfectly fluid capital markets in which there are no transaction costs and that all assets are infinitely divisible.

An investor who actually held p^* would, of course, be desirous of making any changes which would improve his position by, in some sense, moving his portfolio of risky assets towards the efficient set. Such changes could involve increasing or decreasing the levels of some of his current holdings and possibly adding additional assets. His position would obviously be improved if the change allowed him a greater expected return at the same level of risk as before. As an example, if he could modify his risky portfolio p so as to obtain portfolio q as in Figure 2, he could, with appropriate adjustments in his investment in the risk-free portfolio, obtain portfolio q^* for a gain in expected return of p^*q^* with no change in risk. Of course, with the new opportunities given by q, he would probably want, in addition, to alter his risk level.

In this paper, we shall say that portfolio q is <u>relatively more efficient</u> than portfolio p in that q allows a greater expected return than p at any level of risk as measured by standard deviation. It should be noted that this concept of relative efficiency does not require that either portfolio be efficient in an absolute sense.

To summarize, the usual question faced by most is how they might change their existing portfolio so as to obtain a relatively more efficient portfolio. It is precisely this question which the traditional one-parameter performance measures of investment performance answer. The next sections will demonstrate how.

III. Performance Evaluation

The underlying rationale for evaluating investment performance hinges upon the presence of incomplete information. Any investor with complete information would already be holding the best possible portfolio and would have no need for performance evaluation. Therefore, the first task of this section will be to detail exactly what information the investor is assumed to know and what he is assumed not to know.

To characterize the investor's knowledge, the following assumptions will be made:

A1. The investor knows the expected return of his risky portfolio.

A2. The investor has no explicit knowledge about the expected returns of each individual asset which he holds or is considering adding to his portfolio. The only knowledge he has is that the average of these expected returns, weighted by each asset's proportion, is the known expected return for the portfolio.

A3. The investor knows or can learn at no costs the variance-covariance matrix of the returns of the set of assets which he holds or is considering adding to his portfolio.[4]

Several points should be made. First, A3 implies that the investor would know the standard deviation of the returns of his risky portfolio. Thus, he would be able to determine the optimal allocation of his wealth as between his risky portfolio and his risk-free investment.

Second, a comment should be made about the realism of a situation in which the investor knows the expected return for his entire portfolio without knowing the expected return of the components. Such a situation could well occur as follows: An investor who has studied the historical returns on

---

[4] We shall also require that the variance-covariance matrix of any set of risky assets be positive definite. This means that no portfolio of risky assets can be formed which is riskfree.

the S & P Composite Index may well be able to estimate the future expected return on the index, even though he has no explicit information about any individual issue in the index.

Third, the possibility that an investor would know the variance-covariance matrix of the returns of individual assets without knowing their expected values is consistent with a theorem recently proved by Merton (1980). In a continuous time process, Merton showed that the accuracy of the estimator of the expected return depends upon the length of the overall observation period and not upon the number of independent observations within the period. In contrast, the accuracy of the estimator of the variance depends only upon the number of independent observations, which can be made arbitrarily large for a given overall observation period.

Finally, the assumption that the investor lacks any explicit knowledge about the expected returns of individual assets needs to be formalized in some way so as to be consistent with a mean-variance world in which an investor is trying to optimize his position. Consider an investor who is holding portfolio p and has no idea of how to modify it so as to improve his position. For such an investor, there will exist a unique vector of expected returns[5] for which an efficient set algorithm would yield as the efficient portfolio of risky assets portfolio p. We shall say that this vector of expected returns captures the essence of the lack of explicit knowledge about the expected returns of the individual assets. If the investor placed any other values on these expected returns, he could immediately move to a relatively more efficient portfolio.

---

[5]The uniqueness is guaranteed by the assumption that the variance-covariance matrix for the risky assets is positive definite. (Cf. footnote 4).

To develop more formally this vector of expected returns, let us define the following variables:

N	the total number of risky assets including those with zero weight in portfolio p.
x_{ip}	the proportion of the risky portfolio p invested in asset i with zero indicating no investment.
σ_{ij}	the true covariance between asset i and j, or variance if i=j.
σ_p	the standard deviation of portfolio p.
r_f	the risk-free borrowing and lending rate.
μ_p	the true expected return on portfolio p, which is known to the investor.
μ_i	the true expected return on asset i, but which is not know to the investor.
μ_{ip}	the expected return on asset i which would be consistent with the lack of any explicit knowledge of the true μ_i's for an investor who currently hold portfolio p as his risky portfolio.

The values of the μ_{ip}'s will be such that the standard portfolio problem yields as its solution, the x_{ip}'s. The standard portfolio problem is to minimize the portfolio variance subject to an expected return constraint and a wealth constraint. The wealth constraint can be eliminated by measureing expected returns from the risk-free rate. Thus, we have

$$\min_{x_i\text{'s}} \sigma_p^2 = \sum_i^N \sum_j^N x_i x_j \sigma_{ij} \tag{1}$$

$$\text{s.t.} \sum_i^N x_i(\mu_{ip} - r_f) = \mu_p - r_f$$

The goal is to define the μ_{ip}'s so that the above problem yields the x_{ip}'s as the solution.

In Lagrangian form, (1) can be rewritten as

$$h = \sum_{i}^{N} \sum_{j}^{N} x_i x_j \sigma_{ij} - \frac{2}{\lambda}\left[\Sigma x_i\left(\mu_{ip} - r_f\right) - \left(\mu_p - r_f\right)\right] \tag{2}$$

The first N equations of the first order conditions are

$$\sum_j x_j \sigma_{ij} = \frac{1}{\lambda}\left(\hat{\mu}_{ip} - r_f\right), \quad i = 1,\ldots,N \tag{3}$$

If the above is to hold for $x_j = x_{jp}$, we have

$$\mu_{ip} - r_f = \lambda \sum_j x_{jp} \sigma_{ij}, \quad i = 1,\ldots,N \tag{4}$$

$$= \lambda \sigma_{ip}$$

Multiplying (4) by x_{ip} and summing the N equations over i yields

$$\mu_p - r_f = \lambda \sigma_p^2 \tag{5}$$

or

$$\lambda = \frac{\mu_p - r_f}{\sigma_p^2} \tag{6}$$

Substituting (6) into (4) finally gives the desired equation

$$\hat{\mu}_{ip} - r_f = \frac{\sigma_{ip}}{\sigma_p^2}\left(\mu_p - r_f\right)$$

$$= \beta_{ip}(\mu_p - r_f), \quad i = 1,\ldots,N \tag{7}$$

where β_{ip} is the beta coefficient of asset i defined with respect to portfolio p.

Equation (7) defines the μ_{ip}'s which will make an investor content with holding portfolio p. It should be noted that (7) is formally similar to the standard capital asset pricing model. If portfolio p happened to be the true market portfolio, the μ_{ip}'s would be identical to the true μ_i's.

If an investor somehow learned that the true expected return for, say, security 1 was different from the implied μ_{ip}, it would seem that, with this additional information, the investor should be able to improve his position through an increase in the relative efficiency of his risky portfolio. Indeed, if μ_i were greater than $\hat{\mu}_{ip}$, it will turn out that the investor could improve his position by weighting his portfolio more heavily towards this asset. If μ_i were less than $\hat{\mu}_{ip}$, it will turn out that the investor could improve his position by reducing his commitment to this asset. The next section will present a numerical example to illustrate these assertions, and the appendix presents a formal proof.

Thus, the difference between μ_i and $\hat{u}_{ip}$ represents a measure of investment performance. If portfolio p is the portfolio underlying the S & P Composite Index, this difference would be what the financial community now terms "the alpha coefficient."

IV. A Numerical Example

A numerical example will illustrate how the performance measure $\mu_i - \mu_{ip}$ can be used to identify desirable changes in an existing portfolio. Assume that there are four risky securities in the universe with population statistics as given in Table 1.[6] The risk-free rate is assumed to be 3.0 percent.

The hypothetical risky portfolios to be analyzed are shown in Table 2. Portfolio a is on the efficient frontier of risky assets and maximizes the ratio of the expected risk premium of the portfolio to its standard deviation or, in symbols, the ratio of $(\mu_p - r_f)$ to σ_p. It is thus the market portfolio of the Sharpe-Lintner theory. Portfolio b is a portfolio of an investor who plunges into the asset with the greatest expected return. Portfolio c is an equally-weighted portfolio. Portfolio d is on the efficient frontier of risky assets, but does not provide as great a value of the ratio of $(\mu_p - r_f)$ to σ_p as portfolio a. The relative positions of these four portfolios in the expected return-standard deviation space are illustrated in Figure 3.

In Table 3, there are detailed statistics for each of these four portfolios. In addition to the proportions invested in each asset, these detailed statistics include the beta coefficients calculated with respect to each portfolio, the β_{ip}'s, and the implicit expected return consistent with an investor who holds portfolio p with no explicit knowledge of the true expected returns of each asset, the μ_{ip}'s. Finally, the one-parameter performance measures, given by $(\mu_i - \mu_{ip})$, are shown. Again, if p were represented by the

[6]This example is the same one which Roll (1978) used and has been repeated here to facilitate a comparison of the analysis in this section with that of Roll if the reader wishes.

S & P Composite Index, these performance measures would be the alpha coefficients as calculated in the financial community.

The one-parameter performance measures are all zero for portfolio a as they should be since portfolio a is indeed the market portfolio. An investor could not move to a relatively more efficient portfolio of risky assets by altering the relative proportions of his money invested in these four assets.

For the plunging portfolio b, the one-parameter performance measures are positive for each of the first three assets and zero for the fourth. As the appendix shows, an investor could improve his position by increasing his investment in any of the first three assets. Any such increase would, of course, require a reduction in the investment in the fourth asset.

More generally, an investor could move to a relatively more efficient portfolio by the following procedure. First, construct any portfolio of the four assets which of itself has a positive performance measure relative to the existing portfolio. As an example, an equally-weighted portfolio of the first three assets would have a performance measure of 3.47.[7] Second, sell proportionally a portion of the current risky portfolio and use the proceeds to purchase the new portfolio.

To pursue this example in more detail, let n represent the equally-weighted portfolio of the first three assets. On the basis of the postulated characteristics of the securities in Table 1, portfolio n will have an expected return of 7 percent, a standard deviation of 3.16 percent, and a covariance with portfolio b of 5.33. Figure 4 contains a plot of portfolio n and b and the locus of convex combinations of these two portfolios. As can be

[7]The performance measure of a portfolio is in general an average of the performance measures of the individual assets, weighted in proportion to the portfolio weights.

seen, any tilting of b towards n with even 100 percent in n will lead to a relatively more efficient portfolio.

Further analysis discloses that the optimal combination is to place 97.7 percent in n and 2.3 percent in b. In this example, as in general, it is important to note that the alpha coefficient does not provide sufficient information by itself to determine these optimal percentages. All the alpha does is indicate the direction of the change.

For the equally-weighted portfolio c, the performance measures are positive for the first two assets and negative for the second two assets. By increasing the investment in either of the first two assets or by decreasing the investment in either of the second two assets, an investor could move to a relatively more efficient portfolio. Also, any portfolio of the four assets with a positive performance measure when combined with portfolio c in appropriate weights will result in a relatively more efficient portfolio. What the appropriate portfolio weights are would require further analysis.

Portfolio d is on the efficient frontier of risky assets, but it is not the best portfolio possible when there are borrowing and lending opportunities at 3 percent. As with portfolios not on the frontier, the one-parameter performance measures again point to possible changes in the existing portfolio which would lead to a relatively more efficient portfolio.

V. The S & P Composite Index

The correct portfolio to use in calculating beta coefficients and the associated one-parameter performance measures is the portfolio which the investor currently holds. Nonetheless, most professional evaluation services calculate these statistics with respect to the S & P Composite Index of 500 stocks. The obvious question is what information, if any, do these statistics convey to an investor who holds some portfolio other than the index.

Before answering this question, let us consider possible reasons why the S & P Index has received so much prominence in the investment community. There are at least three reasons: First, the returns on the S & P Index roughly measure the returns which all investors in the aggregate received on their investments in these 500 stocks.[8] These 500 stocks do indeed represent a substantial proportion of the market value of all equities. Due to trading and other types of costs, the net aggregate returns would be somewhat less.

Second, performance measurement services, such as that of Merrill Lynch, consistently show that the returns of a large majority of managed equity funds are less than the returns implied by the index. Exhibiting greater returns does not necessarily mean better performance in view of possible differences in risk; nonetheless, these often greater returns on the S & P Index have whetted the appetite of many investors.

Third, it has become possible in the last several years for an investor to buy a portfolio which will replicate quite closely the returns on the

---

[8] The S & P Composite Index weights each stock in proportion to the market of the total common stock authorized and issued less Treasury stock whether held by US investors or by foreign investors. In some few cases, the number of shares which S & P uses in determining the market value of the company's stock differs slightly from the number reported in the company's annual report.

S & P Index at relatively low costs in comparison to the costs of more actively managing an account. An investor of modest means could invest in the Vanguard First Index Trust,[9] while an investor of greater means could, at even lower costs, place his money with an organization like Wells Fargo. Thus, purchasing the index is a viable alternative to active portfolio management.

Now, consider an investor who had his portfolio, say a, evaluated relative to the S & P Index, say s. After examining the evaluation, he would conclude that the professionally-calculated alpha was less than zero, greater than zero, or neither less than nor greater than zero. The question facing the investor is how to use such a conclusion to ascertain whether some combination of his existing portfolio with the index would lead to a relatively more efficient portfolio.

If the professionally-calculated alpha were negative, we have, assuming that the investor knows the expected return on both a and s, the following

$$\mu_a - r_f < \beta_{as}(\mu_s - r_f) \tag{8}$$

The beta in (8) is calculated with respect to portfolio s, while the last two sections showed that the investor should be concerned wth the beta calculated with respect to his existing portfolio--in short, β_{sa} rather than β_{as}. If β_{as} is positive, (8) can be rewritten in terms of the relevant beta, β_{sa}, as

$$(\mu_s - r_f) > \frac{1}{\rho^2_{as}} \beta_{sa}(\mu_a - r_f), \tag{9}$$

---

[9]The annual total costs associated with the First Index Trust are currently around 30 basis points.

where ρ_{as} is the correlation between the returns on portfolios a and s. Whatever the value of ρ_{as}, it can be concluded that if portfolio a has a negative alpha with respect to s, s will have a positive performance measure with respect to a. Thus, an investor could move to a relatively more efficient portfolio by tilting his existing portfolio by an appropriate amount towards the index.

If the professionally-calculated alpha were zero, (8) would hold as an equality as would (9). Nonetheless, if the correlation between a and s were less than 1.0, the performance measure of portfolio s calculated with respect to a would be positive. This result is at variance as to how alphas are normally interpreted. An alpha of zero is often interpreted as being consistent with neither superior nor inferior performance. In fact, an alpha of zero as calculated in the financial community with respect to the S & P Index implies that an investor could move to a relatively more efficient portfolio of risky assets by tilting his existing portfolio by an appropriate amount towards the index. The prescription is thus the same whether the calculated alpha is zero or negative.

Unlike the prior two cases, the interpretation of a positive professionally-calculated alpha is ambiguous. Depending upon the correlation ρ_{as}, the one-parameter performance measure of the index s with respect to the existing portfolio a could be positive, negative, or zero. If this correlation coefficient were known, it would be a simple matter to derive the appropriate one-parameter performance measure from the professionally-calculated alpha.

VI. A Reconciliation With Roll

The differing conclusions of this paper from those of Roll hinge upon differences in interpretation of the use of the one-parameter performance measures. To illustrate, Figure 5 has been reproduced from one of Roll's papers, but with the notation of this paper. He shows in this configuration that the performance measure of c calculated with respect to portfolio m will be positive. Roll concludes that, "if the portfolio manager wants to appear to have ability, he would definitely choose a portfolio such as ... c." He then goes on to assert that such a positive performance measure "is decidedly not an indication of true ability on the part of the portfolio managers," whereas in this paper, such a positive performance measure would be interpreted favorably.

A visual examination of Figure 5 shows that by decreasing the proportion of the portfolio of risky assets placed in m and by increasing the proportion in c the investor would be able to move to a relatively more efficient portfolio. The optimal allocation between the two portfolios would be that which maximized the slope of the line connecting r_f and some combination of the joint portfolio consisting of m and c.

In contrast, Roll appears to be using the performance measure to make an all-or-nothing decision. If an investor had to choose between portfolio m or c, he would clearly choose m. If the performance measure is to be used for this purpose, it will sometimes yield the wrong decisions as Roll has shown. However, it has been known for some time that these one-parameter performance measures do not take into account diversification effects.[10] In an all-or-nothing decision, diversification effects are important, and these performance

---

[10] Cf. Friend and Blume (1970).

measures do not contain sufficient information to make such a decision. In short, Roll appears to be using these one-parameter performance measures to answer the wrong question.

VII. Conclusion

This paper used the concept of relative efficiency to show that the traditional one-parameter performance measuers do provide useful information. The appropriate portfolio to use as the reference portfolio in calculating these measures is the portfolio which the investor currently holds. If a potential portfolio has a positive performance measure with respect to the existing porfolio, the investor can improve the relative efficiency of his existing portfolio by spreading his investment in risky assets over both portfolios.

If the performance measure is negative, the investor should sell short the potential portfolio and use the proceeds to increase the amount invested in his existing portfolio. To the extent that there are some restrictions on short sales, this recommended action may not be feasible. If zero, the investor's existing portfolio provides the greatest level of relative efficiency possible and no investment should be placed in the potential portfolio.

Although the one-parameter performance measures should be calculated with respect to the investor's existing portfolio, most professional evaluation services calculate these measures with respect to the Standard & Poor's Composite Index. These measures are usually termed alpha coefficients. If these coefficients are zero or negative, the paper showed that they had unambiguous interpretations as to how to improve the relative efficiency of an existing portfolio. It should be noted that the interpretation of a zero coefficient differs from the usual interpretation of professional investors. If positive, the implied action was ambiguous and further analysis would be required.

Finally, the paper reconciled the conclusions of this paper with the substantially different conclusions of Roll. The apparent difference seems to be that Roll is concerned with the question of whether an investor should have invested in one portfolio rather than another. The one-parameter performance measures are not capable of answering this all-or-nothing question. They can only answer the question of whether an existing portfolio should be combined in some proportion with a second portfolio, or tilted towards a second portfolio. The one-parameter measures indicate the direction of the tilt, but do not contain sufficient information to indicate the magnitude of the tilt.

Appendix

Let p be the portfolio of risky assets which an investor currently holds and a be some other portfolio not currently held. Portfolio a may contain assets in common with p. Consider a combination of the two portfolios with x_a proportion of total wealth invested in a and x_p in p. There is no requirement that the sum of x_a and x_p be 1.0.

Let us now consider combinations of a and p in a mean-variance world, and we shall initially consider only combinations which maintain the expected return constant at $\mu_p - r_f$. Then, the investor will select the x_a and x_p so as to

$$\min \quad x_a^2 \sigma_{aa} + x_p^2 \sigma_{pp} + 2 x_a x_p \sigma_{ap} \tag{A1}$$

$$\text{s.t.} \quad x_a \mu_a' + x_p \mu_p' = \mu_p' \, ,$$

where the prime indicates that the expected return is measured from the risk-free rate. It should be recalled that the variance-covariance matrix of all risky assets was assumed positive definite.

Rewriting (A1) in Lagrangian form, we have

$$h = x_a^2 \sigma_{aa} + x_p^2 \sigma_{pp} + 2 x_a x_p \sigma_{ap} - \frac{2}{\lambda} \left[x_a \mu_a' + x_p \mu_p' - \mu_p' \right] \tag{A2}$$

The first order conditions are

$$\sigma_{aa} x_a + \sigma_{ap} x_p - \frac{1}{\lambda} \mu_a' = 0 \tag{A3}$$

$$\sigma_{ap} x_a + \sigma_{pp} x_p - \frac{1}{\lambda} \mu_b' = 0 \tag{A4}$$

$$\mu'_a x_a + \mu'_p x_p = \mu'_p \tag{A5}$$

If the investor were content to hold p with no investment in a, x_p would be 1.0 and x_a would be zero. Also, in this case. λ would be μ'_p/σ_{pp} which is positive.[11] By substituting these values into (A3), one can show that

$$\mu'_a = \beta_{ap}\mu'_p, \tag{A6}$$

which is the same expression as (7) in the text and defines the implicit expected returns which would make an investor content with holding just portfolio p as his portfolio of risky assets.

To prepare the way for the effect of increasing or decreasing μ'_a from the implicit value $\beta_{ap}\mu'_p$, note that (A3) and (A4) can be solved for x_a and x_p as follows:

$$x_a = \frac{1}{\lambda}\left[\sigma_{pp}\mu'_a - \sigma_{ap}\mu'_p\right]/\left(\sigma_{aa}\sigma_{pp} - \sigma_{ap}^2\right) \tag{A7}$$

$$x_p = \frac{1}{\lambda}\left[-\sigma_{ap}\mu'_a + \sigma_{aa}\mu'_p\right]/\left(\sigma_{aa}\sigma_{pp} - \sigma_{ap}^2\right) \tag{A8}$$

or, more briefly,

$$x_a \propto \sigma_{pp}\mu'_a - \sigma_{ap}\mu'_p \tag{A9}$$

$$x_p \propto -\alpha_{ap}\mu'_a + \sigma_{aa}\mu'_p, \tag{A10}$$

---

[11] Cf. equation (6) in the text.

where the constant of proportionality is positive and the same for both (A9) and (A10). Of course, in this case x_a is zero and thus the quantity on the right of the proportionality sign would be zero, and the ratio of x_a to x_p zero.

Now, consider the situation in which the investor learns that the true expected return on portfolio a, as measured from the risk-free rate, is greater than $\mu'_a = \beta_{ap}\mu'_p$ by the amount $\Delta\mu$. We shall first show that the investor would hold a positive proportion of portfolio a in his portfolio of risky assets and then show that the resulting portfolio is relatively more efficient then portfolio p.

From (A9), the new proportion to hold in a, $\tilde{x}_a$, and in p, $\tilde{x}_p$, would be

$$\tilde{x}_a \propto \sigma_{pp}(\mu'_a + \Delta\mu) - \sigma_{ap}\mu'_p \qquad \text{(A11)}$$

$$\tilde{x}_p \propto -\sigma_p(\mu'_a + \Delta\mu) - \sigma_{aa}\mu'_p \qquad \text{(A12)}$$

or

$$\tilde{x}_a \propto \left[\sigma_{pp}\mu'_a - \sigma_{ap}\mu'_p\right] + \sigma_{pp}\Delta\mu = \sigma_{pp}\Delta\mu \qquad \text{(A13)}$$

$$\tilde{x}_p \propto \left[-\sigma_{ap}\mu'_a - \delta_{aa}\mu'_p\right] - \sigma_{ap}\Delta\mu \qquad \text{(A14)}$$

In order to have a positive investment in a, the ratio of $\tilde{x}_a$ to $\tilde{x}_p$ must be greater than zero. The variable $\tilde{x}_a$ is clearly greater than zero. In (A14), the term in the brackets is greater than zero. If σ_{ap} is negative, $\tilde{x}_p$ will clearly be positive. Thus, the ratio of $\tilde{x}_a$ to $\tilde{x}_b$ will be positive implying a long position in both a and p. If σ_{ap} is positive, it is possible for a sufficiently large $\Delta\mu$ that the proportion $\tilde{x}_p$ could be negative with the proceeds of the short sale used to increase $\tilde{x}_a$ above 1.0.

Finally, since the variance-covariance matrix was assumed positive definite, the most efficient portfolio of risky assets is unique. In addition, since the prior portfolio with x_a equal to zero is still feasible, the new portfolio consisting of a and p must be relatively more efficient than p. A similar proof shows that if $\Delta\mu$ is negative, the investor will attempt to short portfolio a.[12]

---

[12]It should be noted that if σ_{ap} is negative and $\Delta\mu$ sufficiently negative, the optimal proportions in both a and p could be negative with the proceeds of the short sales used to purchase the risk-free asset.

Bibliography

Friend, Irwin, and Marshall E. Blume. " Measurement of Portfolio Performance Under Uncertainty," American Economic Review, 60 (September 1970), pp. 561-575.

Jensen, Michael. " The Performance of Mutual Funds in the Period 1945-1964," Journal of Finance, 23 (May 1968), pp. 386-416.

Markowitz, Harry M. Portfolio Selection: Efficient Diversification of Investments (New York: John Wiley & Sons, Inc., 1959).

Merton, Robert C. "On Estimating the Expected Return on the Market: An Exploratory Investigation" (NBER Working Paper 444, February 1980).

Roll, Richard. "A Critique of the Asset Pricing Theory's Tests," Journal of Financial Economics, 4 (March 1977), pp. 126-176.

____________. "Ambiguity When Performance is Measured by the Securities Market Line," Journal of Finance,33 (September 1978), pp. 1051-1070.

____________. "The Expected Return Benchmark and Portfolio Performance Evaluation," Journal of Portfolio Management (forthcoming 1980).

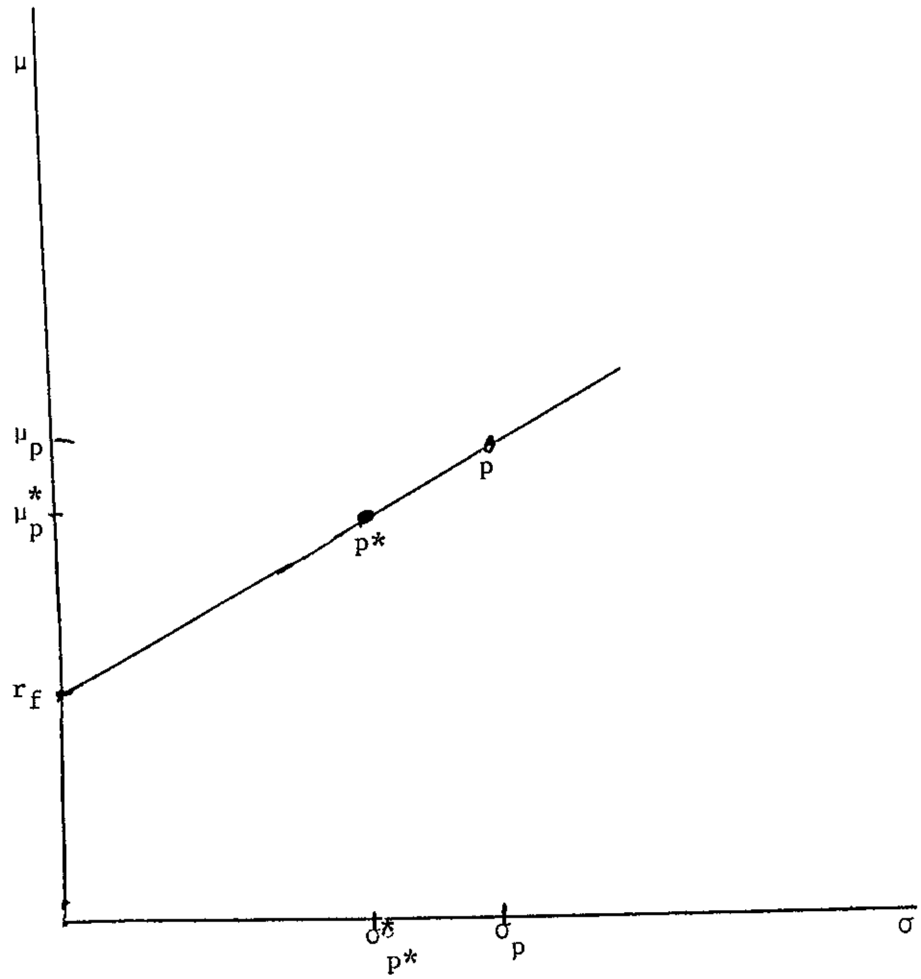

Figure 1 The Investor's Opportunities with Risky Portfolio p.

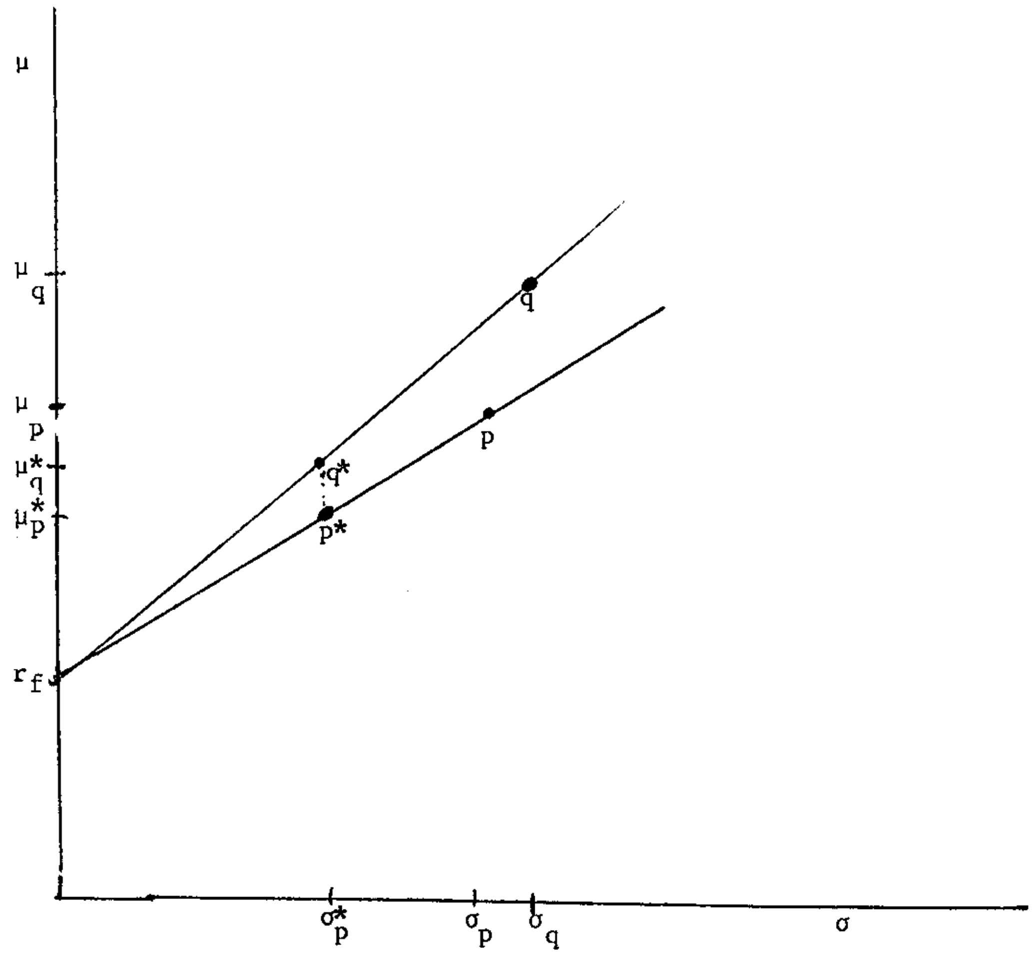

Figure 2 The gain in Relative Efficiency From Moving From Portfolio p to Portfolio q.

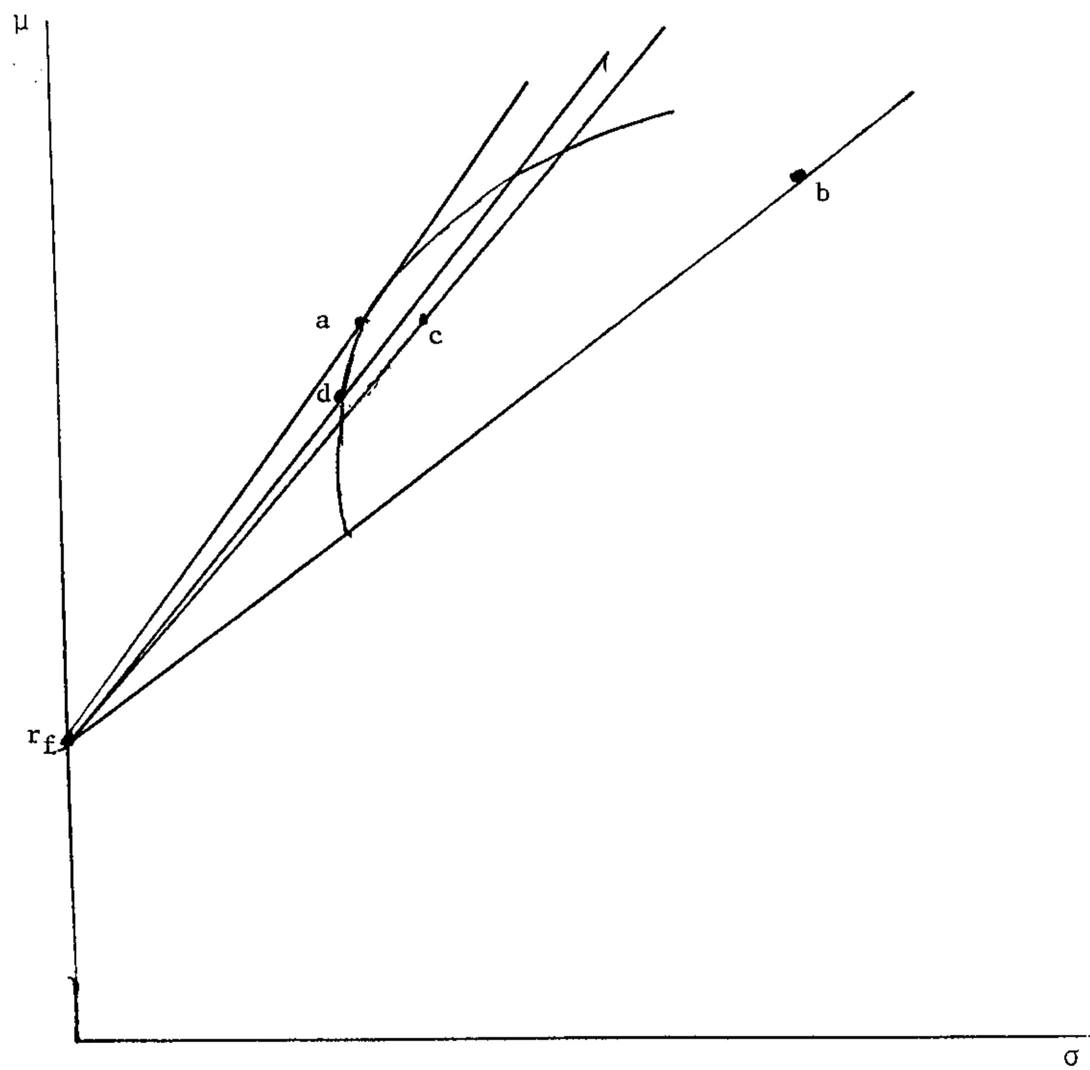

Figure 3 Relative Positions of Hypothetical Portfolios

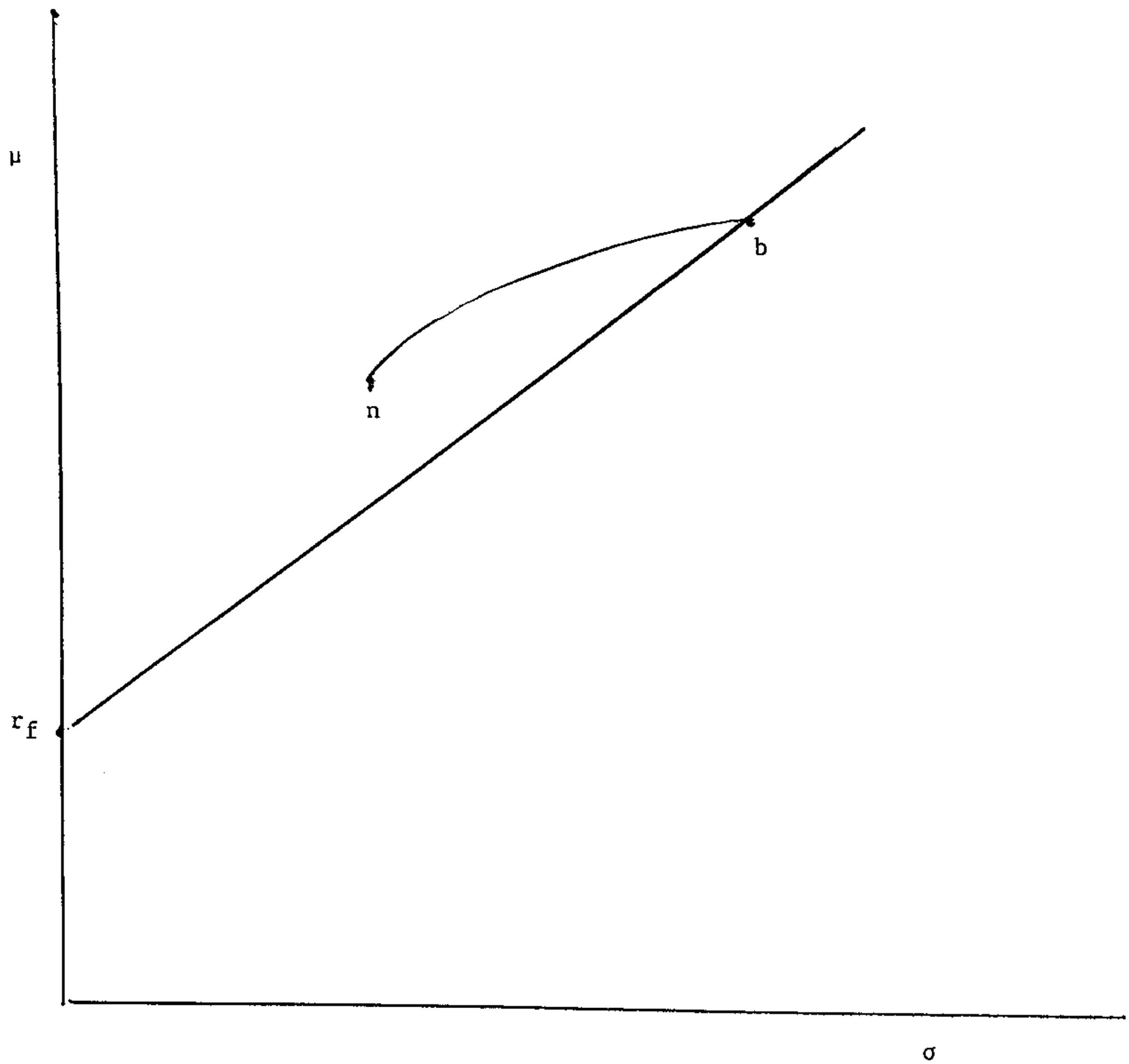

Figure 4 Combining Portfolio b with the New Portfolio n.

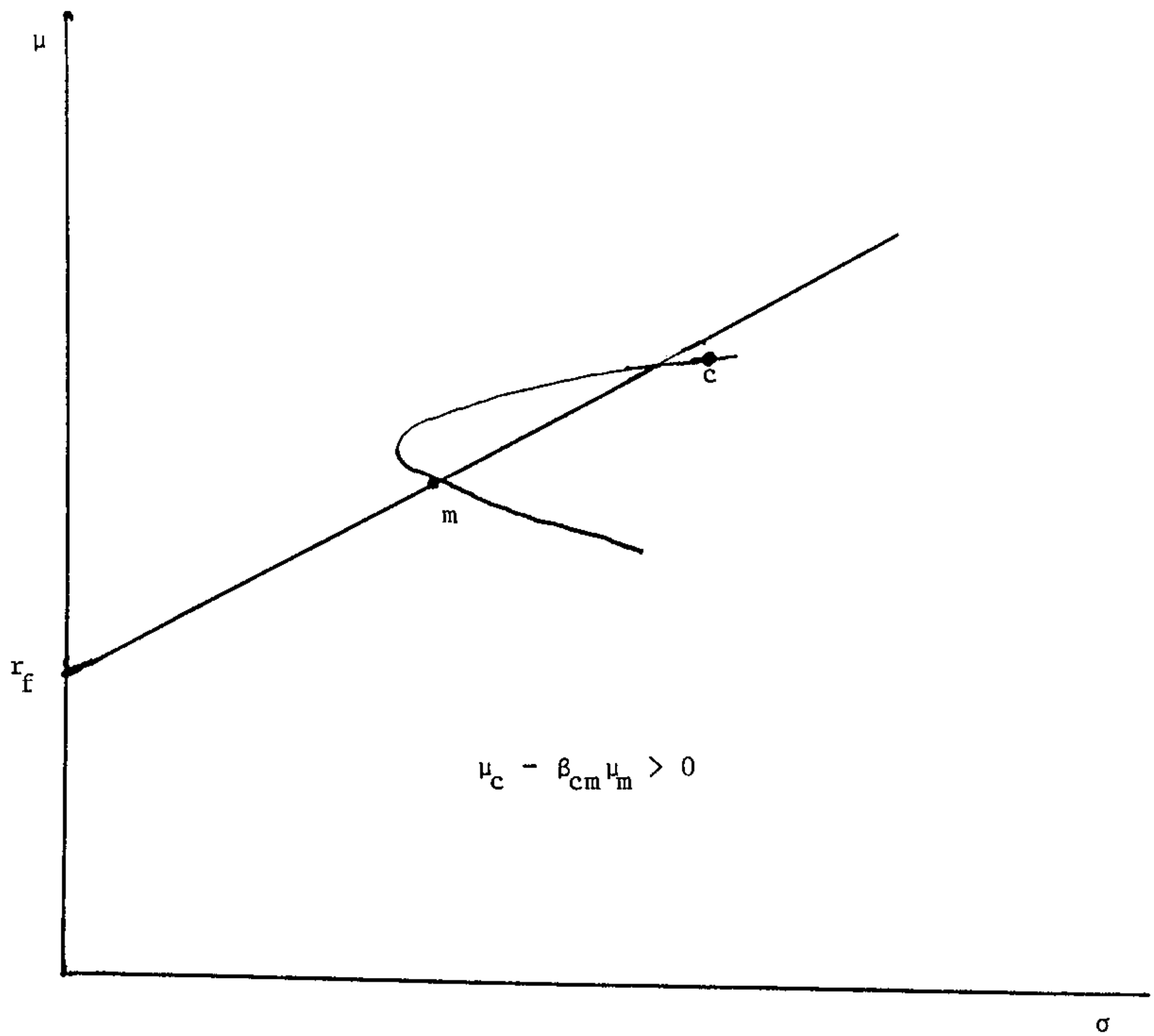

Figure 5 Example from Roll [7] With the One-Parameter Performance Measure of c With Respect to m Positive.

Table 1

Population Statistics for Individual Risky Assets

Asset	Expected Return (%/period)	Variance-Covariance Matrix Asset 1	2	3	4
1	6	10	2	4	5
2	7	2	20	4	1
3	8	4	4	40	10
4	9	5	1	10	60

Table 2

Hypothetical Risky Portfolios

Portfolio	Percentage Invested in Individual Asset 1	2	3	4	Expected Return	Standard Deviation
a	40.5%	32.0%	14.1%	13.5%	7.01%	2.81%
b	0	0	0	100.0	9.00	7.75
c	25.0	25.0	25.0	25.0	7.50	3.37
d	49.2	30.0	11.2	9.7	6.81	2.71

Table 3

Detailed Statistics on the Potental Risky Portfolios

Portfolio	Item	Asset 1	2	3	4
a	Proportion (%)	40.50	32.00	14.10	13.50
	β_{ia}	.75	1.00	1.25	1.50
	$\hat{\mu}_{ia}$	6.00	7.00	8.00	9.00
	$\mu_i - \hat{\mu}_{ia}$	0	0	0	0
b	Proportion (%)	0	0	0	100.00
	β_{ib}	0.08	0.02	0.17	1.00
	μ_{ib}	3.50	3.10	4.00	9.00
	$\mu_i - \mu_{ib}$	2.50	3.90	4.00	0
c	Proportion (%)	25.00	25.00	25.00	25.00
	β_{ic}	0.46	0.59	1.27	1.67
	$\hat{\mu}_{ic}$	5.08	5.67	8.74	10.52
	$\mu_i - \hat{\mu}_{ic}$	.92	1.33	-0.74	-1.52
d	Proportion (%)	49.20	30.00	11.20	9.70
	β_{id}	0.88	1.02	1.17	1.32
	$\hat{\mu}_{id}$	6.36	6.99	7.46	8.02
	$\mu_i - \hat{\mu}_{id}$	-0.36	0.01	0.54	0.98