CORPORATE FINANCIAL POLICY IN MARKETS WITH SHORT SALE RESTRICTIONS

by

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1. Introduction

It is well-known that in complete markets where there is no taxation a change in firms' financial policies does not change the economy's real allocation.[1] The relevance of financial policies of firms must thus depend on the incompleteness of the market's structure. The study of this topic may be simplified if we first concentrate on the classification of market incompleteness.

Sufficient conditions for a complete market are: 1. That the state-dependent vectors of securities' returns span a vector space whose dimension is the number of states, and 2. That there are no restrictions on consumer short-selling of securities. In this paper we shall discuss markets in which the second of the above sufficient conditions does not hold, and we shall show that in such markets definitive statements may be made about firm financial policies. In particular it will be shown that in a market where short sales are restricted, no firm should borrow less than it can safely return; every firm should, on the margin, thus be a risky borrower. This result holds whether or not there is taxation.

In order to motivate the result, we consider the following example:

Consider an economy with two future states and one firm. Let the firm have a state-dependent revenue vector (3,6); i.e., the total revenue made available by the firm in state 1 is 3 and in state 2 is 6. Now suppose that the firm has two kinds of securities on the market. The first security, called "debt," promises a return of 1 independent of state, whereas the second security, called "equity," will pay off whatever is not paid to debtholders.

The payoff vectors will thus be:

equity: (2,5); debt: (1,1).

We shall furthermore suppose that the price of the whole debt vector is .75, and that the price of the whole equity return vector is 2.7.

A consumer who maximizes a state-dependent utility function over state-dependent consumption will typically be faced with the following maximization problem:

$$\max U(x_1, x_2)$$

s.t.

(1) $x_1 = 2e + d$ (state 1 consumption)

(2) $x_2 = 5e + d$ (state 2 consumption)

(3) $2.7e + .75d \leq W$

(4) $e, d \geq 0.$

In the above equations e and d represent the proportion of the firm's debt and equity purchased by the consumer, and equation (3) represents the consumer's wealth constraint. The inequality (4) gives the no-short sale constraint.

Figure 1 gives a graphical representation of the (x_1, x_2) possibilities afforded the consumer by equations (1)-(4) above. The rays extending from the origin are the return vectors of the firm's debt and equity, and the line AB represents the locus of optimal (x_1, x_2) opportunities.

The indifference curves labelled 1, 2, and 3 indicate three typical consumer optimal portfolios. Consumer 1 is a non-corner maximizer, and divides his wealth equally between purchases of the bond and the equity return vectors. Consumer 2 invests all of his wealth in the firm's equity and consumer 3 invests all of his wealth in the firm's debt. For each of the three consumers we may calculate Kuhn-Tucker shadow prices for a unit of

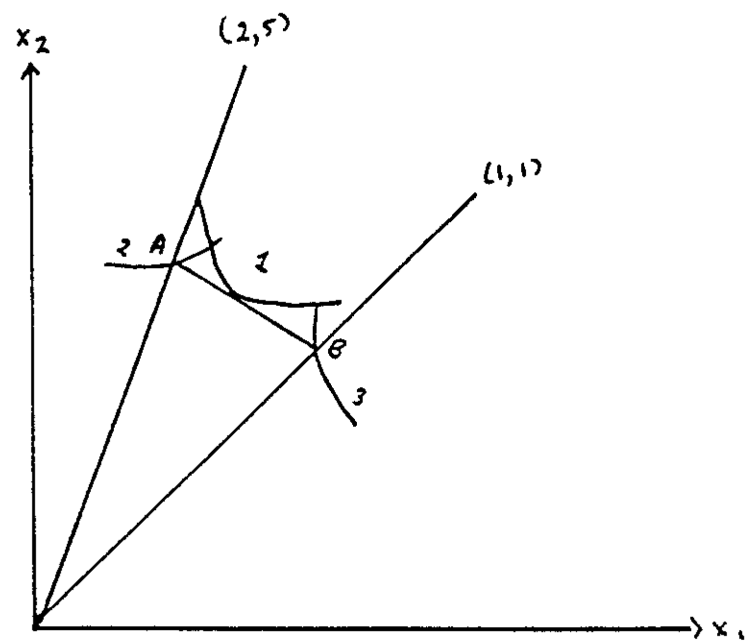

Figure 1

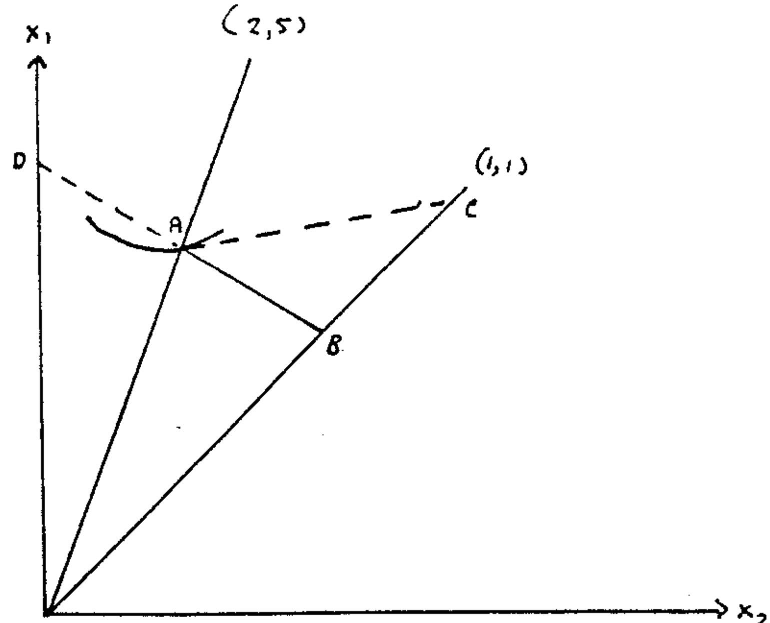

Figure 2

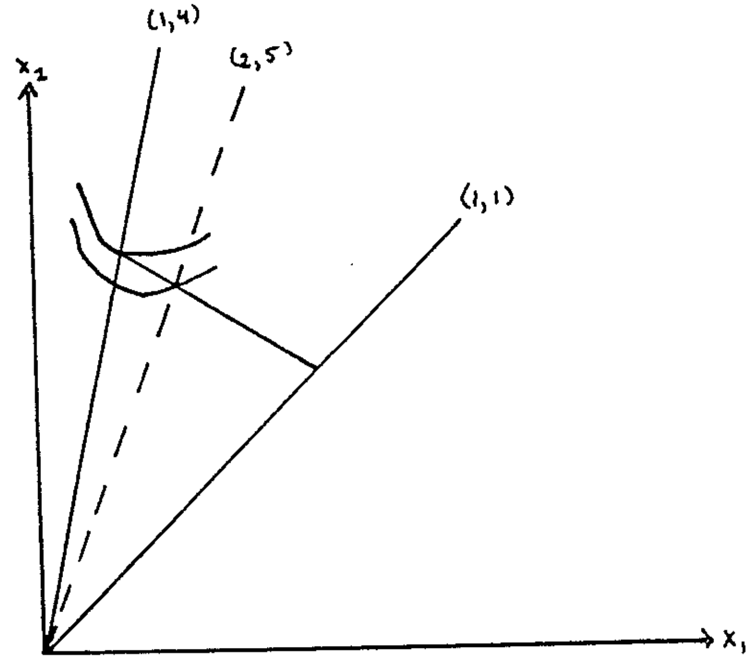

Figure 3

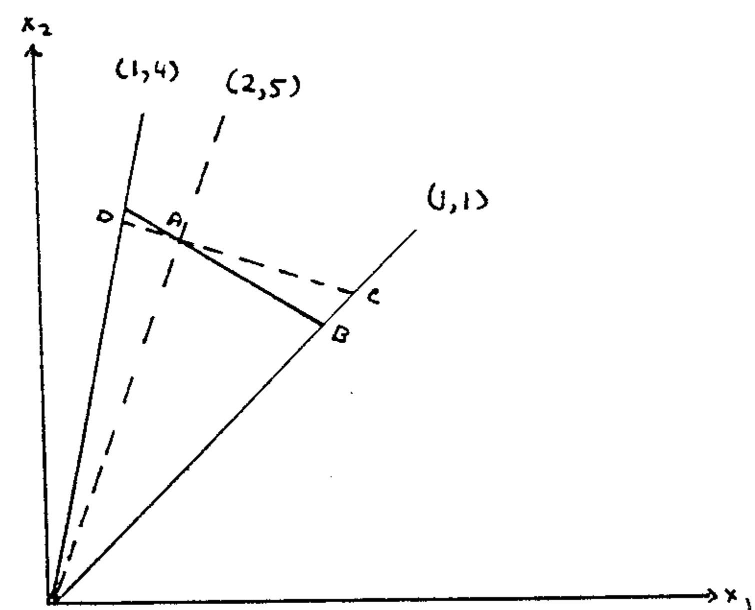

Figure 4

revenue in states 1 and 2. For consumer 1 these prices are derivable from the slope of the line segment AB by solving the equations

(5) $2q_1 + 5q_2 = 2.7$

(6) $q_1 + q_2 = .75.$

The solution to these equations yields

(7) $q_1 = .35, q_2 = .4.$

The implicit (shadow) prices for consumers 2 and 3 are not derivable directly from the line AB but must be derived from their utility functions. It is clear, however, that consumer 2 places a higher implicit value on state 2 consumption than does consumer 1, and a lower implicit value on state 1 consumption. The reverse holds for consumer 3. Thus, for consumer 2 we shall have:

shadow price for state 1 consumption $< .35$

shadow price for state 2 consumption $> .40$

Since consumer 2 purchases the equity of the firm, moreover, it follows from the Kuhn-Tucker theorem that his implicit value for the firm's equity corresponds to the market value. Thus--denoting by r_1 and r_2 consumer 2's shadow prices--we must have

(8) $2r_1 + 5r_2 = 2.7.$

Finally, consumer 2 values the firm's bond at less than its market price. To see this, consider the line AC, whose slope is determined by consumer 2's implicit prices (Figure 2). Since point C is above B, and since this point corresponds to the amount of the firm's debt that the consumer could purchase were he to invest all his wealth in the firm's debt, it follows that at his shadow prices consumer 2 considers the debt to be overpriced. (Another way to see this is to consider what would happen if consumer 2 could short sell the firm's debt. In this case he would be able to choose a portfolio on the line AD in Figure 2, thus increasing his utility.)

Now consider what would happen if the firm increased its debt from 1 to 2. The new return vectors would be

equity: (1,4); debt: (2,2).

In Figure 3 we see that if the new return vectors are priced at the prices implicit in the old budget line AB that consumer 2 is clearly better off than before, whereas the utility achievable by consumers 1 and 3 is unchanged. In fact, we shall show that there exist prices which would make all consumers better off; these are indicated by the line CD in Figure 4.

The argument which we have illustrated above is formalizable and extends to the case where both corporations and individuals are taxed. It follows from this argument that the firm in the illustration should issue at least 3 units of debt, so that the equity return and debt return vectors which it sells are (0,3) and (3,3) respectively. Note that in the two-state case illustrated here it also follows that issuing more than 3 units of debt is not desirable; to see this compare the cone which results from, say, 4 units of debt (i.e., the cone created by the equity return vector of (0,2) and the debt return vector of (3,4)) with the cone which results from 3 units of debt. The latter cone is clearly larger, and it may be shown that consumer 3 would be worse off if the amount of debt is increased beyond 3 units. Unfortunately, this last argument is not fully generalizable: It does <u>not</u> follow that the optimal amount of debt for the firm is that debt level beyond which the firm becomes a risky borrower. It is, however, true that any change in the firm's debt level which enlarges the cone of feasible portfolios will be preferred by all consumers. This may be extremely difficult to calculate, however. As an example, take a firm with a revenue vector of (6,4,3) with a current level of debt of 3. Thus the equity return vector of the firm is currently (3,1,0) and the debt return vector is (3,3,3). If the firm increases its debt to 4, the return vectors will be (assuming, as we shall throughout the paper, costless bankruptcy):

equity: (2,0,0); debt: (4,4,3).

Without knowledge of the current return vectors of all of the other firms in the market, it is not possible to know whether a debt level of 4 will enlarge the cone of feasible consumption vectors or not. If, for example the firm is the only firm operating in the market, an increase in its debt from 3 to 4 will replace the currently available cone with one which intersects it along a plane. In general, any argument which relates to an increase in the size of the feasible cone must rely on the fulfillment of the first of the two sufficient conditions for completeness mentioned in the second paragraph above.

This paper will formalize the above arguments. Its structure is as follows: Sections 2, 3, and 4 set out the basic model. After defining an exchange equilibrium in Section 5, we prove the results for the no-tax case in Section 6. Section 7 deals with the case of corporate and personal taxation.

2. Preliminaries

We consider a two-period model with one physical good which is used both for consumption and for production. We shall assume that the price of the physical good is unity both today and in every state of the world tomorrow. Uncertainty in the model is represented by states of the world: There are assumed to be M of these, one of which will occur tomorrow. We shall not require that agents in the model have homogeneous expectations (i.e., agents may have different subjective probabilities of the occurrence of states), but we shall require that all consumers believe every state of the world to have a positive probability of occurrence, and that all agents agree on the number and description of states of the world.

There will be J firms in the model, each of which purchases inputs today for production tomorrow, and each of which finances this production with a combination of debt and equity. For a given firm j, we shall denote the value of j's equity by p_j^e and the value of j's debt by p_j^d. The value of firm j will thus be $p_j^e + p_j^d$.

3. Firms

Each of the J firms has a stochastic production function. If firm j purchases z_j physical units of input today, we shall denote the physical output of the firm tomorrow in state m by $y_{jm}(z_j)$. The functions y_{jm} will be assumed to have the following properties:

P.1. $y_{jm}(z_j) \geq 0$ for $z_j \geq 0$.

P.2. y_{jm} is increasing and concave.

P.3. at least one of the y_{jm} is strictly concave.

By our assumption on the price of the physical commodity, the value of firm j's production in state m is $y_{jm}(z_j)$.

A firm may choose to finance part (or all) of its production by selling debt. Denote the nominal debt issued by firm j by d_j. We shall assume that d_j represents a promise to the purchasers of the debt to pay them up to d_j in any state of the world tomorrow, the value of the firm's production permitting. Formally, we denote the returns to purchasers of the firm's debt b_j given a decision z_j on the size of the firm's inputs by

$$(9) \qquad r^d_{jm}(z_j,d_j) = \begin{cases} d_j & \text{if } y_{jm}(z_j) \geq d_j \\ y_{jm}(z_j) & \text{otherwise.} \end{cases}$$

The top line of expression (9) denotes the case where the firm does not go bankrupt, whereas the bottom line denotes the case of default on the debt; in this latter case bondholders get all of the firm's production income, though this is still less than the nominal value of the firm's debt.

Given a firm decision to purchase inputs z_j and issue d_j of nominal debt, returns in state m of the world tomorrow to consumers who purchase shares of the firm today will be denoted by

$$(10) \qquad r^e_{jm}(z_j,d_j) = \begin{cases} y_{jm}(z_j) - d_j & \text{if } y_{jm}(z_j) \geq d_j \\ 0 & \text{otherwise.} \end{cases}$$

For simplicity we shall assume that firms have no initial debt and no production in the first period.

4. Consumers

A typical consumer i in our model will be endowed, at the start of the model (today), with a non-negative commodity endowment $\bar{w}_i$ and with a portfolio of shares in each of the model's firms. Denote the initial shareholding of consumer i in firm j by $\bar{e}_{ij}$ where this denotes the proportion of firm j's equity held by i. Note that since we assume that firms initially have no debt, there are no initial bondholders in the model. We shall assume that the initial shareholdings of consumers are both non-negative and that initial shareholders own all of the each of the firms in the market:

$$(11) \qquad \bar{e}_{ij} \geq 0 \text{ for every i and j; } \sum_i \bar{e}_{ij} = 1 \text{ for every j.}$$

Initial shareholders benefit from the sale of both shares and bonds in the firms in which they hold shares, but they must also help finance these firm's inputs. If the value of firm j's equity is p_j^e and the value of its debt is p_j^d, and if firm j has decided to invest z_j in inputs for next period's production, then the returns from their initial shareholdings to consumer i are

$$(12) \qquad \sum_j \bar{e}_{ij}(p_j^e + p_j^d - z_j).$$

The consumer's consumption problem is to choose consumption today x_{io} and a state-dependent consumption vector for tomorrow's consumption $x_{i1},\ldots,x_{iM}$, and a share and bond portfolio, $e_i = (e_{i1},\ldots,e_{iJ})$ and $d_i = (d_{i1},\ldots,d_{ij})$ respectively. The consumer's consumption and portfolio decisions are consistent only when they fulfill the budget constraints:

$$(13) \qquad x_{io} \leq \sum_j \bar{e}_{ij}(p_j^e + p_j^d - z_j) - \sum_j e_{ij}p_j^e - \sum_j d_{ij}p_j^d + \bar{w}_i$$

$$(14) \qquad x_{im} \leq \sum_j d_{ij}r_{jm}^e + \sum_j d_{ij}r_{jm}^d$$

In addition to the budget constraints, we shall impose another constraint on the consumer's decisions: We shall allow no short sales of securities. That is, we shall demand that

$$(15) \qquad e_{ij} \geq 0,\ d_{ij} \geq 0 \text{ for every i and j.}$$

The constraint on short sales is the primary condition which we shall need to establish the results of the model, and it is worthwhile to justify this condition as approximating a "real world" state of affairs. The conventional short sales assumption made in many capital market models allows consumers to sell short unlimited amounts of any firm's stocks or bonds. This is equivalent to the assumption that consumers themselves issue stocks or bonds which are indistinguishable from the equity or debt which is issued by the firms. This condition is said to correspond to the "real world" short sales mechanism whereby investors borrow securities from other investors (usually through the intermediation of a broker), sell these securities, and replace them at the lender's demand by repurchasing them at a later date on the open market (if a dividend has been paid in the meantime, the borrower pays this to the lender also). In fact there are a number of differences between the theoretical short sales mechanism and the "real world" short sales mechanism:

1. The theoretical mechanism allows the short seller the immediate use of the proceeds of his sale, whereas in the "real world" the seller generally deposits the proceeds of the sale for safe keeping with the broker. Thus, while in the theoretical models the short seller can exploit the fact that securities are not paying a rate of return which he considers sufficient given the level of the firm's risk, in the "real world" only expected declines in prices can be exploited.[2]
2. The "real world" mechanism is relatively expensive, whereas the theoretical model involves no expenses whatsoever.
3. In the "real world" it is often difficult to obtain the required quantity of stock to short; this is especially true if the stock which is desired to short is is not widely held. In the theoretical model the short seller "issues" stock which has the same attributes as the marketed stock, so that this difficulty does not exist.

The above points motivate condition (15); we note, furthermore, that all of the results of this paper are obtained if the no-short sales condition is weakened to the following condition:

$$(16) \qquad e_{ij} \geq -a_{ij},\ d_{ij} \geq -b_{ij};\quad a_{ij},\ b_{ij} > 0.$$

Condition (16) weakens the short sales condition to allow <u>limited</u> short sales; although for the sake of notational simplicity we shall not invoke (16) in the proofs of this paper, it may readily be established that all of these proofs go through if (16) holds instead of (15).

We shall assume that consumers choose (x_i, e_i, d_i) so as to maximize a utility function $U_i(x_i)$ defined over the consumption vector. We shall assume that U_i is increasing and concave:

U.1. $U_i(x_i) \geq U_i(\bar{x}_i)$ where $x_i \geq \bar{x}_i$

U.2. U_i is concave.

U.3. $\dfrac{\partial U_i}{\partial x_{im}} \to +\infty$ when $x_{im} \to 0$.

The last condition, (U.3), insures that consumers will carry out all of their obligations even if limited short selling is allowed.

The following Lemma defines the first order conditions for consumer maximization.

<u>Lemma 1</u>: Let (e_i^*, d_i^*, x_i^*) be consumer i's utility-maximizing portfolio-consumption choice given prices $p^e = (p_1^e, \ldots, p_J^e)$, $p^d = (p_1^d, \ldots, p_J^d)$, and given choices of inputs and debt (z_j, d_j) for every firm j. Then there exist <u>implicit state prices</u> $q^i = (q_1^i, \ldots, q_M^i)$ for each consumer i such that:

1.1 $\quad p_j^e \geq \sum_m q_m^i r_{jm}^e(z_j, d_j)$

1.2 $\quad p_j^d \geq \sum_m q_m^i r_{jm}^d(z_j, d_j)$,

where

1.3 $\quad q_m^i = \dfrac{\partial U_i / \partial x_{im}}{\partial U_i / \partial x_{io}}$ and the partial derivatives are evaluated at x_i^*.

Furthermore, if $e_{ij}^* > 0$, then equality holds in (1.1), and if strict inequality holds, it follows that $e_{ij}^* = 0$. A similar statement holds with respect to d_{ij}^*.

Proof:

The proof follows directly from the Kuhn-Tucker Theorem.

The meaning of the theorem is straightforward. Given each consumer's portfolio choices, the quotients of the partial derivatives of the utility function of the consumer (1.3 in Lemma 1) define state prices by which consumer i may be said to price the securities available in the market. If a security has been purchased by the consumer, it will be found that the consumer's implicit valuation of the security agrees with the market price. On the other hand, the consumer will not purchase securities for which the market price exceeds his implicit valuation. Were we to allow short sales, these latter securities are the ones which the consumer would wish to short.

Note that it follows from Lemma 1 that the market price of any security having state-returns $r = (r_1, \ldots, r_M)$ may be defined as

(17) $$p_r = \max_i \sum_m q_m^i r_m.$$

Note furthermore that in general the implicit prices will not be equal across consumers; if the market has such a property, it is said to be complete. Complete markets have highly desirable properties, which were first explored by Arrow (1963-64). As noted in the introduction to this paper, sufficient conditions for the equality of implicit state prices are that the set of available security returns span an M-dimensional vector space and that short sales are allowed. Given our short sale constraint, the markets we shall be discussing will generally not be complete.

5. An exchange equilibrium

Given a choice of inputs and debt levels (z_j, d_j) for each firm j, we shall call $\{(x_i, e_i, d_i), (p_j^e, p_j^d)\}$ an exchange equilibrium if the following conditions are fulfilled:

E.1. (x_i, e_i, d_i) satisfies the budget constraints (13) and (14) and the short-sale restrictions, and x_i maximizes U_i for all combinations of consumption and portfolio vectors which have this property.

E.2. Market supply and demand for all securities are equal; i.e.,

$$\sum_i e_{ij} = \sum_i d_{ij} = 1.$$

We shall denote an exchange equilibrium by writing

$$e\{(z_j, d_j)\} = \{(x_i, e_i, d_i), (p_j^e, p_j^d)\}$$

6. Pareto superior debt

Suppose an exchange equilibrium is established in which some firm j can issue more riskless debt; i.e.

(18) $$d_j < \min_m y_{jm}(z_j).$$

Then it may be shown that an increase in firm j's debt will be unanimously preferred by all consumers. It thus follows that there exists, for each firm, an optimal minimal level of debt: For a given level of outputs z_j, firm j's debt d_j should be chosen such that

(19) $$d_j \geq \min_m y_{jm}(z_j).$$

The theorem which establishes the above statements is the following:

Theorem 1: Let $e\{(z_h, d_h)\} = \{(x_i, e_i, d_i), (p_h^e, p_h^d)\}$ be an exchange equilibrium such that $d_j < \min_m y_{jm}(z_j)$. Then no consumer will be made worse off by a marginal increase in firm j's debt.

Proof:

Suppose that firm j increases its debt from d_j to $d_j + \delta$, where $d_j + \delta < \min_m y_{jm}(z_j)$. We first note that x_i is a feasible allocation with the new debt level of firm j. To see this, we form portfolios $(\hat{e}_i, \hat{d}_i)$ for each individual i and set new debt and equity prices $(\hat{p}_h^e, \hat{p}_h^d)$ for each security as follows:

$$\hat{e}_{ih} = e_{ih} \text{ for all } h=1,\ldots,J$$

$$\hat{d}_{ih} = d_{ih} \text{ for all } h \neq j.$$

$$\hat{d}_{ij} = \frac{e_{ij}\delta + d_{ij}d_j}{d_j + \delta} .$$

prices are set by

$$(\hat{p}_h^e, \hat{p}_h^d) = (p_h^e, p_h^d) \text{ for all } h \neq j.$$

$$\hat{p}_j^e = p_j^e - \frac{\delta}{d_j} p_j^d$$

$$\hat{p}_j^d = \frac{d_j + \delta}{d_j} p_j^d.$$

It is easily verified that $(\hat{e}_i, \hat{d}_i)$ fulfill the budget constraints (13) and (14) at prices $(\hat{p}_h^e, \hat{p}_h^d)$. Furthermore, we note that since (e_i, d_i) fulfilled condition (E.2), that

$$\sum_i \hat{e}_{ij} = \sum_i \hat{d}_{ij} = 1$$

At the new prices $(\hat{p}_h^e, \hat{p}_h^d)$, however, the new portfolios open the possibility of a utility-improving trade for any consumer i who--in the exchange equilibrium $e\{(z_h, d_h)\}$-- had $e_{ij} > 0$ and $p_j^d > \sum_m q_m^i d_j$. Let this consumer trade $\hat{d}_{ij}$ for some additional equity in firm j at the new market prices $\hat{p}_j^e$ and $\hat{p}_j^d$. Such a trade will give him the portfolio

$$\tilde{e}_{ij} = \hat{e}_{ij} + \frac{\hat{d}_{ij}\hat{p}_j^d}{\hat{p}_j^e}$$

$$\tilde{d}_{ij} = 0$$

$$(\tilde{e}_{ih}, \tilde{d}_{ih}) = (e_{ih}, d_{ih}),\ h \neq j.$$

Since the trade is made at prevailing market prices, it will leave the utility of the individual who purchases i's debt unchanged. On the other hand, since i feels that firm j's debt is overvalued at current market prices the trade will leave consumer i better off. To prove the latter fact, we apply Taylor's theorem; denoting consumer i's new consumption vector by x_i, Taylor's theorem gives

$$U_i(\tilde{x}_i) - U_i(x_i) = \sum_m (\tilde{x}_{im} - x_{im}) \frac{\partial U_i}{\partial x_{im}} + (\tilde{x}_{io} - x_{io}) \frac{\partial U_i}{\partial x_{io}} .$$

Dividing through by $\frac{\partial U_i}{\partial x_{io}}$ gives

$$U_i(\tilde{x}_i) - U_i(x_i) = \sum_m (\tilde{x}_{im} - x_{im})q_m^i + (\tilde{x}_{io} - x_{io}) .$$

Now note that

$$\tilde{x}_{io} = x_{io}$$

$$\tilde{x}_{im} = x_{im} + \frac{\hat{d}_{ij}\hat{p}_j^d}{\hat{p}_j^e} (y_{jm} - d_j - \delta) - \hat{d}_{ij}(\hat{d}_j + \delta).$$

Then using Taylor's theorem, it follows that

$$U_i(\tilde{x}_i) - U_i(x_i) = \sum_m \frac{\hat{d}_{ij}\hat{p}_j^d}{\hat{p}_j^e} (y_{jm} - d_j - \delta)\, q_m^i - \sum_m \hat{d}_{ij}(d_j + \delta) q_m^i > 0.$$

This last inequality follows since for small δ, $p_j^d > \sum_m q_m^i d_j$ implies that

$$p_j^d > \sum_m q_m^i (d_j + \delta),$$

and since

$$\hat{p}_j^e = \sum_m q_m^i (y_{jm} - d_j) - \frac{\delta}{d_j} p_j^d$$

$$= \sum_m q_m^i (y_{jm} - d_j) - \delta \max_h \sum_m q_m^h, \qquad \text{by (17)}$$

$$< \sum_m q_m^i (y_{jm} - d_j - \delta).$$

QED

An intuitive explanation of the theorem's proof is that as firm j increases its debt, it is making the cone of feasible allocations larger. This increase in the size of the feasible cone allows trade based on differences in perceived values which previously could not take place because of the short sales constraint. As long as the firm in question does not increase its debt beyond the point of bankruptcy (i.e., as long as $d_j < \min_m y_{jm}(z_j)$), the firm's old return vectors will be spanned by positive linear combinations of the new vectors and an increase in firm j's debt will not reduce the welfare of any individual.

Note furthermore that in the proof of Theorem 1 the shift from portfolio consumption vector $\hat{x}_i$ to $\tilde{x}_i$ takes place by consumer i's trading new debt for new equity at the prices $\hat{p}_j^e$ and $\hat{p}_j^d$. However, even if the trade took place at new prices $\tilde{p}_j^e$ and $\tilde{p}_j^d$, where

(20) $\hat{p}_j^e < \tilde{p}_j^e$ and $\hat{p}_j^d > \tilde{p}_j^d$,

consumer i would be better off as long as $(\tilde{p}_j^e, \tilde{p}_j^d)$ is sufficiently close to $(\hat{p}_j^e, \hat{p}_j^d)$. Trade at these prices would not only make consumer i better off, but would also make all other consumers who hold debt and equity of firm j better off.[3]

While it is clear from the theorem that firms would always be in debt up to the limit of bankruptcy, it is more difficult to make unequivocal statements about the effects of an increase in firm debt beyond this point. In a two-state world it is easily shown that is is an optimal debt level d_j is that which is equal to $\min_m y_{jm}(z_j)$, but this argument does not easily generalize to more than two states. By a method similar to that used to prove Theorem 1, we may establish the following theorem:

<u>Theorem 2</u>: If a marginal change in any firm's debt level resuls in a larger feasible cone, that change will make no consumer worse off.

<u>Proof</u>:

We shall sketch the proof, which is similar to the proof of Theorem 1. If the feasible cone increases in size, then the previous allocation may be generated by a new set of portfolios which are both affordable and which satisfy (E.2). New possibilities for trade will occur if the new set of portfolios fails to satisfy the first order conditions for a consumer maximum given in Lemma 1, and the resulting trade will, at the margin, leave no consumer worse off.

QED

7. Pareto superior debt in the presence of corporate and personal taxation and short sale restrictions

The argument made in the previous section may be extended to the case where there is both personal and corporate taxation. Consider the case where firms are taxed at a rate t_c and where each individual i is taxed at rate t_i.[4] Denote the firm payoffs to shareholders in the absence of bankruptcy and given inputs z_j and debt d_j by

$$(21) \qquad r^e_{jm}(z_j, d_j) = y_{jm}(z_j) - d_j - t_c\left(y_{jm}(z_j) - (d_j - p^d_j) - z_j\right)$$

$$= (1 - t_c)\left(y_{jm}(z_j) - d_j\right) + t_c(p^d_j + z_j).$$

Expression (21) assumes that both "interest" (the difference between the debt repayments in period two and net firm proceeds from sale of debt in the first period) and the first period cost of inputs are tax deductable.

Firm payoffs to bondholders in the absence of bankruptcy will be written

$$(22) \qquad r^d_j = d_j.$$

In the presence of personal taxes, net-of-tax payoffs received by shareholders and bondholders may be written

$$(23) \qquad r^{ei}_{jm}(z_j, d_j) = r^e_{jm} - t_i(r^e_{jm} - p^e_j) = (1 - t_i)r^e_{jm} + t_i p^e_j$$

$$(24) \qquad r^{di}_j(d_j) = r^d_j - t_i(r^d_j - p^d_j) = (1-t_i)r^d_j + t_i p^d_j.$$

From the first-order conditions for consumer maximization it may be established that

$$(25)\qquad p_j^e = \frac{\sum_m (1 - t_i)\, q_m^i r_{jm}^e}{1 - t_i \sum_m q_m^i} \quad \text{if } e_{ij} > 0.$$

$$(26)\qquad p_j^d = \frac{\sum_m (1 - t_i) q_m^i d_j}{1 - \sum_m q_m^i t_i} \quad \text{if } d_{ij} > 0$$

Denote

$$(27)\qquad a = \frac{p_j^d}{d_j}\ .$$

It follows from (26) that in the no-bankruptcy case considered here, a is independent of d_j. We may now write

$$(28)\qquad r_j^{di}(d_j) = d_j \beta_i ,$$

where

$$(29)\qquad \beta_i = (1 - t_i) + t_i a.$$

Now suppose that we are currently at an exchange equilibrium, that firm j can increase its debt without incurring the risk of bankruptcy, i.e., that $r_{jm}^e > 0$ for every m, and suppose that the firm's debt is increased from d_j to $d_j + \delta$. Setting the new debt and equity prices equal to

(30) $$\hat{p}_j^e = p_j^e - a\delta$$

(31) $$\hat{p}_j^d = a(d_j + \delta),$$

we may derive

(32) $$r_{jm}^{ei}(z_j, d_j + \delta) = r_{jm}^{ei} - (1-t_i)[(1-t_c) - t_c a]\delta - t_i a\delta$$

$$= r_{jm}^{ei} - \alpha_i \delta.$$

(33) $$r_j^{di}(d_j + \delta) = r_j^{di} + \beta_i \delta.$$

Since the new r_{jm}^{ei} and r_j^{di} differ from the previous only by a constant, we may define a new portfolio for each consumer i which will give him the same consumption he had before. This new portfolio may be written

(34) $$\hat{e}_{ih} = e_{ih}$$

(35) $$\hat{d}_{ih} = d_{ih},\ h \neq j$$

(36) $$\hat{d}_{ij} = \frac{e_{ij}\alpha_i\delta + d_{ij}r_j^{di}}{r_j^{di} + \beta}$$

We now show that the new portfolio is both affordable and feasible in a market sense (i.e., $\sum_i \hat{d}_{ij} \leq 1$). First note that since

(37) $$\alpha_i = (1 - t_i)[(1 - t_c) - t_c a] - t_i a$$

(38) $\beta_i = (1 - t_i) + t_i a,$

we clearly have

(39) $\alpha_i < \beta_i.$

From the assumption that the market was previously at an exchange equilibrium it follows that

(40) $\sum_m e_{ij} = \sum_m d_{ij} = 1,$

and we thus have

(41) $\hat{d}_{ij} - d_{ij} = \frac{e_{ij}\alpha_i \delta - d_{ij}\beta_i}{\beta_i(d_j + \delta)} < \frac{\beta_i(\delta e_{ij} - d_{ij})}{\beta_i(d_j + \delta)}$

and therefore

(42) $\sum_i (\hat{d}_{ij} - d_{ij}) < 0,$

which proves market feasibility.

To show that the new portfolio is affordable, note that

$$(43) \qquad \hat{e}_{ij}\hat{p}^e_j + \hat{d}_{ij}\hat{p}^d_j = e_{ij}(p^e_j - a\delta) + \frac{e_{ij}\alpha_i\delta + d_{ij}\beta_i d_j}{\beta_i(d_j + \delta)}[a(d_j + \delta)]$$

$$< e_{ij}p^e_j - a\delta e_{ij} + \frac{e_{ij}\beta_i\delta + d_{ij}\beta_i d_j}{\beta_i} a$$

$$= e_{ij}p^e_j - a\delta e_{ij} + ae_{ij}\delta + ad_{ij}d_j$$

$$= e_{ij}p^e_j + d_{ij}p^d_j.$$

We may now repeat the argument made in the proof of Theorem 1: The new portfolio allows every individual--at the new level of firm j's debt--to achieve his previous allocation. Since the new portfolio forces some individuals who previously did not hold the firm's debt to hold it now, these individuals will be motivated to trade their holdings of debt with other individuals. These trades will lead to a new allocation in which the level of utility will rise. We have thus proven:

Theorem 3: In the presence of corporate and individual taxation, no individual will be made worse off by an increase in debt of any firm which can do so without incurring the risk of bankruptcy.[5]

8. Summary and conclusions

When there are short sale restrictions in the market, all consumers will view some debt as being a desirable feature of corporate financial structure; this proposition holds whether or not there is corporate or personal taxation. A minimal desirable level of debt is that beyond which the firm risks bankruptcy. It is difficult to generalize about debt levels beyond this point, except to consider their effects on the whole cone of feasible returns with which they leave consumers. If this cone is enlarged, then an expansion of the firm's debt will be desirable.

Footnotes

1. The literature on the subject is vast and well-known. For an exposition of the complete markets case, see Stiglitz (1969). Studies of cases where financial structure becomes relevant have not treated market incompleteness directly, but rather have focused on specific market imperfections. Market imperfections whose effect on capital structuare has been studies include bankruptcy costs and taxes (Kraus and Litzenberger 1973), agency costs (Jensen and Mecklin 1976), and informational imperfections (Ross 1977).

2. Since in a two-period model there are no second-period prices for securities (since all firms liquidate in the second period), it is impossible to model "real world" short sales in such a model.

3. In the two-state case discussed in the introduction, the final result of an increase in the debt level of the firm would be to establish a new equilibrium with relative prices as illustrated in Figure 4.

4. Section 7 assumes flat-rate taxes on both corporate and personal income. The argument applies also to progressive taxation, but the notation required becomes exceedingly unwieldy.

5. Taggart (1980) has argued that in incomplete markets "all-equity capital structures [may be] perfectly rational for at least some firms." The argument of this section shows that this is not true if the cause of market incompleteness is the restriction of short sales.

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