The CAPM and Mean-Variance Efficient Portfolios

Ex Ante and *Ex Post* Data

by

Irwin Friend, Randolph Westerfield
and
Joao Ferreira

Working Paper No. 10-80 REV.

RODNEY L. WHITE CENTER
FOR FINANCIAL RESEARCH

University of Pennsylvania

The Wharton School

Philadelphia, PA 19104

The CAPM and Mean-Variance Efficient Portfolios

Ex Ante and *Ex Post* Data

Irwin Friend, Randolph Westerfield and Joao Ferreira

Several types of evidence have recently been presented raising fundamental questions about the validity and descriptive usefulness of the mean-variance capital asset pricing model (CAPM), including versions advanced by Sharpe, Lintner, Black, Merton and Kraus-Litzenberger. This evidence has included new regression tests of the CAPM based on market portfolios encompassing a broader range of risky assets than common stocks alone and utilizing *ex ante* (expected) as well as *ex post* (realized) rates of return; analyses of actual stock portfolios and the assets and liabilities held by large samples of individual investors to test the reasonableness of the customary joint assumptions of the relevance of covariance as distinguished from variance measures of risk and of the homogeneity of expectations made in most theoretical versions and applications of the CAPM; and surveys of individual investors to determine the risk measures relevant to their investment behavior, with particular reference to covariance and variance measures.[1]

This paper adopts a different approach to the testing of the Sharpe-Lintner and Black versions of the CAPM and their basic premise that a market portfolio is *ex ante* mean-variance efficient. Our approach is to determine the exact composition of *ex ante* and *ex post* efficient portfolios derived at given levels of risk-free returns, and to compare these efficient portfolios with equal-and value-weighted portfolios constructed from the same sample of common stocks. Our basic purpose is to examine the correspondence between the efficient portfolios derived and the equal- and value-weighted portfolios as the size of the sample of common stocks used in this analysis increases and more closely approximates the market portfolio.

For the first time to our knowledge, we derive efficient common stock portfolios from *ex ante* data covering the expected returns on each of a number of individual stocks provided by a large sample of institutional investors. We compare the properties of these efficient portfolios with those that would be predicted on the assumption that the Sharpe-Lintner or Black version of the CAPM is valid.

Two types of efficient portfolios are derived--one with no constraints on short sales as assumed by Sharpe-Lintner and Black, the other with no short sales permitted. The composition of the unconstrained efficient portfolios corresponding to given risk-free rates of return are calculated using both full rank and diagonal estimation procedures for the variance-covariance matrix, while the composition of the constrained efficient portfolios is calculated using only diagonal estimation procedures.[2] The sensitivity of the efficient portfolios to plausible changes in several key assumptions are examined, and these portfolios are then compared both with equal- and value-weighted common stock portfolios.

The main result of this analysis is that with no short sale restrictions, close to half of all stocks in efficient portfolios are held in short positions, whereas with no short sales permitted, only a small proportion of stocks enter the efficient portfolios. Furthermore, there is no tendency for the relative frequency of short positions to decrease as the size of the sample of stocks increases. These findings are inconsistent with CAPM theory that requires the market portfolio to be ex ante efficient. They are, on the other hand, generally consistent with the results obtained by other investigators, including Haim Levy, from a similar analysis of ex post data.[3] However, while departures from expectations might be used as an argument to explain the discrepancies between the observed ex post results and those predicted by theory, this argument is not as applicable to ex ante data. On the other hand, because of data unavailability, the variance-covariance matrix will be estimated using ex post data, but as explained subsequently this is probably not so serious as the use of ex post mean returns as a proxy for expected returns. In any case the sensitivity of our results to errors in the ex post variance-covariance matrix used is examined with no changes in the conclusions reached. This error analysis further supports our conclusion that it is unlikely the true optimal portfolio could be a portfolio with all positive investment proportions. The market price of risk implied by these data (i.e., $\frac{E(R_p)-R_f}{Var(R_p)}$ where $E(R_p)$ is the expected return on the optimal portfolio of common stocks, R_f is return on the risk-free asset, and Var (R_p) is the variance of return on the optimal portfolio) is very much larger than that implied by historical ex post returns of the market portfolios used in earlier studies.

Data on expected returns

It has been frequently demonstrated that developing reliable estimates of expected return for individual stocks is very difficult if one must rely on ex post realizations.[4] To partially overcome this problem, the study utilizes two survey sources of data on the rates of return on individual common stocks expected by investors. The first consists of monthly estimates of the long-run annual rates of return on each of over 300 stocks expected by the Institutional Counsel Service of Wells Fargo Bank.[5] The second consists of a number of Institutional Investor Surveys, covering rates of return expected by many more institutional investors but for a much smaller sample of companies, conducted by one of the authors in 1972, 1974, 1976, 1977 and 1978 under the auspices of the American Telephone and Telegraph Co.[6] Both of these sets of estimates of expected return are based on a discounted cash flow approach implicit in adding the dividend yields (the ratios of next year's dividends to current prices) to the estimated long-run annual growth rates in earnings, dividends and stock prices which are assumed to be the same. We believe that both set of estimates, but particularly the second, provide more useful insights into meaningful investor expectations than the short-run (ordinarily annual) forecasts made by a number of brokerage firms which are rarely the bases for investment decisions by the forecaster and and which are frequently tied to the firm's selling efforts.

The expected returns collected from the Institutional Investor Surveys are especially useful because they represent data regularly compiled by the reporting firms exclusively for their own investment decisions and because they reflect the expectations of a substantial fraction of the market rather than those of a single investor. We therefore rely mainly on these data in this paper and use the Wells Fargo Bank data primarily for comparison purposes.

In these Institutional Investor Surveys, expected annual rates of return were collected on 66 common stocks from 33 financial institutions in August 1974; the corresponding numbers were 49 and 33 in March 1976, 56 and 29 in February 1977 and 49 and 47 in September 1978.[7] The institutions covered, including commercial banks, insurance companies and investment counselling firms, were those with the largest equity portfolios and the response rates averaged over 80%.[8] The stocks covered, all listed on the New York Stock Exchange (NYSE) and over $100 million in size, were as a whole of lower than average risk, with ordinary least squares beta coefficients and standard deviations (based on 60 monthly rates of return) varying in the 1974 survey from .57 to 1.16 and .035 to .080, respectively, and with a comparable range for other years.[9]

The annual rate of return on a particular stock expected by a specific institution was obtained for each of these years by adding a spot dividend yield (adjusted for the expected growth rate)[10] to the expected annual growth rate in per share earnings over a five year (or, if the data were available, longer) time period used by these institutions to estimate expected returns for purposes of their investment decisions. Though the expected annual rate of return derived in this manner is frequently treated as an internal rate of return with an infinite (or very long-term) horizon, it can also be regarded as a one period return with a five year horizon. In each of the years covered, mean expected returns were computed for every stock for which at least five institutions regularly estimated the long-run, i.e., five years or longer, expected growth rate.[11] In computing the mean expected return for each stock, equal weight was given to the estimate of each institution.

There are four important assumptions made in using the mean expected return for each stock derived in this manner as a measure of the rate of return required by the market for that stock. First, it is assumed that a five year growth estimate is an adequate approximation of expected long-run growth, so that earnings, dividends and prices can all be assumed to grow at the same rate. (In the Wells Fargo data, this problem does not arise since the growth estimates are explicitly long-run and effectively have an infinite horizon.) To determine whether this assumption affects our results significantly, part of the analysis is redone on the assumption that after five years, the price-dividend ratios of these stocks revert to the beginning of period price-dividend ratios for New York Stock Exchange stocks as a whole (as against the beginning of period price-dividend ratios for individual stocks which is implicit in our procedures).[12]

A second assumption made is that for large institutional investors expected returns can be taken as a proxy for market required returns. This in turn assumes that markets are generally efficient, that institutional investors buy or sell stocks with prices changing until expected rates are approximately equal to required rates, and that any discrepancies between these rates are relatively small and short-lived. Obviously, the required rates of all investors would be relevant under the efficient market assumption if short sales are permitted without any restrictions, but only the required rates of holders of a particular stock if short sales are not permitted. For the 1974, 1976 and 1977 surveys, no reliable information was available on whether the reporting institutions did or did not hold a particular stock, but this information was available in the 1978 survey so that separate analyses could be carried out for institutions holding a stock as well as for all reporting institutions.

Third, it is assumed in our analysis of the survey data that the mean of the expected returns reported by institutional investors is an adequate representation of the return required by this sector of the market. This would of course be true under the assumption of homogeneous expectations, an assumption made in the development of the Sharpe-Lintner and Black CAPM, but it is not necessarily true under heterogeneous expectations. However, value-weighted means of expected returns (i.e., weighted by the total equity holdings of each institutional investor) generally provided results quite close to the equally-weighted means. Moreover, while heterogeneity of expectations might be considered to reduce the market's required rate of return, the regression of the mean expected rate of return for each stock on the heterogeneity of expectations (estimated by the standard deviation of the returns for that stock expected by the different institutions) as well as on beta and residual standard deviation measures of risk provides no such evidence.[13] As another test of the possible implications of heterogeneous expectations for our results on optimal portfolios, part of the analysis will be repeated subtracting one standard deviation of the returns for each stock expected by the different institutions from the mean expected return for that stock to re-estimate the rates of return on individual stocks required by the market. One other, and perhaps the most satisfactory, test carried out on the sensitivity of our results to heterogeneous expectations is the derivation of optimal portfolios for individual institutions based on their own expectations.

Finally, the assumption is made that institutional investors are representative of the market for NYSE stocks as a whole. Such investors in recent years have accounted for about 50% of the ownership and 75% of the public trading in NYSE stocks.[14] The corresponding percentages for our sample of 50 institutions in 1978 are estimated to be roughly half the total for all institutions

<u>Determination of optimal portfolios</u>: <u>Institutional survey data</u>

To determine the composition of optimum portfolios of common stocks from the Institutional Investor Survey data on expected rates of returns, in each year covered we first selected a group of 10 stocks at random from the entire available sample and repeated this process nine times so that we finally had 10 groups of stocks with 10 issues each; we similarly formed 10 groups of stocks with 15 issues each; and continued this process until all issues were included. For each group of stocks, we estimated such characteristics of optimal portfolios as the investment proportions vector, the mean return, variance, implied market price of risk, percentage of number of stocks held short to total number held, and value of stocks held short to value held long.[15] We then derived the averages of these portfolio characteristics over the 10 groups of stocks selected in each portfolio size class (i.e., 10, 15, etc.)[16] The main reason for this size classification is to determine the changes in the relative importance of short positions in the optimal portfolios as the number of available stocks increases so that a conclusion may be drawn about the implied proportion of short positions in an optimal portfolio of all NYSE stocks.[17] In addition, changes in the expected excess return-risk (measured by variance) trade-off of these portfolios as the number of stocks increases will be examined to estimate the implicit market price of risk for NYSE stocks as a whole. This implied market price of risk will be compared to independent estimates derived from other data. In this first stage of analysis we rely upon point estimates, whereas in the second part we use interval estimates incorporating sampling error.

The derivation of optimal portfolios is based primarily on a full rank historical model where the variance-covariance matrix for each stock portfolio covered was estimated from a sample of 60 months of ex post returns for the period preceding the month in which the ex ante or expected future returns were reported.[18] Since we are using monthly estimates of the variance-covariance matrix in conjunction with estimates of expected annual returns over a five year (or longer) period, there is an implicit assumption that the term structure of expected returns is relatively flat which we believe is generally a reasonable assumption for institutional investors and is consistent with the absence of any clear relationship between the estimated growth rate and the length of the horizon.[19] The ex ante returns and the ex post variances and covariances provide the basis for calculating the investment proportions vector of efficient portfolios. Thus a significant limitation is that we rely on historical data for the estimation of ex ante variances and covariances. However, this is probably not so serious as the use of ex post mean returns as a proxy for expected returns, since it has been shown under plausible assumptions that variances and covariances of returns can be estimated far more accurately from the available time series of realized returns than can the expected returns.[20] The procedures we use to estimate optimal portfolios for each combination of stocks summarized in Appendix B are similar to those followed in previous literature and require only the estimation of the composition of a single efficient stock portfolio represented by the point in the expected return-standard deviation plane where the line drawn from the risk-free rate is tangent to the efficient set. The risk-free rate primarily used was the one-year Treasury bill rate at the beginning of the period for which expected future returns were estimated but several other risk-free rates were also tested.

It should be noted that the approach followed here does not require the assumption that a stock market index is a proxy for the entire market portfolio of risky assets.[21] The assumption rather is that randomly increasing the number of assets in a portfolio should give an investment proportion vector closer to what would be obtained from the market portfolio than results indicated by a smaller sample of assets.

Composition of optimal portfolios: Institutional survey data

Table 1 presents the characteristics of the optimal portfolios computed from the initial analysis of the Institutional Investor Survey data on expected returns which breaks down the entire sample for each of the years 1974, 1976, 1977 and 1978 into smaller sub-samples. The characteristics presented for the optimum portfolios include the number of securities in the portfolio; the annual expected mean return; the annualized variance of return obtained from the financial data files of the Rodney L. White Center; the market price of risk computed from the annualized data; the ratio of the number of stocks in the optimum portfolio held in short positions; and the ratio of the value of short positions to the value of long positions. For comparative purposes, similar characteristics where applicable are presented for equally-weighted and value-weighted portfolios of the same stocks utilized for computation of the optimal portfolios, with the mean expected returns again based on the *ex ante* data. In this analysis, it is assumed that there are no short-sale restrictions, the algebraic sum of the portfolio weights of the individual stocks is assumed to be one, no distinction is made between investors who do and do not own a stock, the expected five year (or, where available, longer-term) growth rates are assumed to be the expected long-run growth rates, and the risk-free rate for each period covered is taken to be the one-year Treasury bill rate at the beginning of the period.

There are a number of striking differences between the optimal portfolios and the portfolios with either equal or value weights. Short positions occur with high frequency in the optimal portfolios. While the expected returns and variance of returns are not too different among the different types of portfolios with a relatively small number of stocks, this in no longer true as the portfolios increase in size. For the optimal portfolios, either expected returns increase rapidly or the variance of returns declines rapidly as the number of stocks rises, while for the other portfolios there are only modest variations in the expected value and variance of returns with changes in portfolio size. As a result, the estimated market price of risk increases rapidly with portfolio size in the optimal but not in the other portfolios.

In each year the largest optimal portfolios summarized in Table 1 show over 40% of the number and value of stocks held in short positions, with the relative importance of short positions increasing moderately with the number of stocks in the portfolio. For the entire sample of 49 stocks in 1978, the largest number available in this analysis, 49% of the stocks and almost as high a proportion of their value were held in short positions. While it does not appear possible from theory alone to determine how the percentage of number or value of short positions in the optimal portfolio varies with the number of assets in the portfolio, the empirical evidence strongly suggests that as the number of stocks in the portfolio increases the percentage of short positions probably increases and in any case does not decrease. The slope coefficients in the regressions of both the ratios of the number and value of short positions to total positions in the optimal portfolios on the log of the number of stocks are statistically significant (more than twice the standard error) and positive in virtually all cases and are never negative.[22] Consequently, for ex ante data, the Sharpe-Lintner version of CAPM seems to point to short positions by investors as a whole in a high proportion of individual stocks which is inconsistent with the equilibrium assumptions of that theory as well as with the actual description of institutional (or other) investor behavior.

In addition to the large numbers of short positions in the optimal portfolios at a given risk-free rate, Table 1 also indicates that the optimal portfolios have a market price of risk very much higher than any plausible estimate that has been made based on the historical realized rates of return in the stock market. Thus, one such estimate based on annual returns of a value-weighted portfolio of New York Stock Exchange (NYSE) stocks since the turn of the century estimates the market price of risk in the neighborhood of two[23] contrasted with estimates ranging from 11 to 30 for larger portfolios summarized in Table 1, with even higher market prices of risk implied for larger optimal portfolios in view of the significantly positive correlation in all years between the market price of risk and the log of the number of stocks in the portfolio. The results for equally-weighted and value-weighted portfolios of the same stocks covered by the optimal portfolios (and using the same *ex ante* returns and variance of returns) point to a market price of risk ranging from two to four, much closer to the estimates obtained from the long-term historical data for NYSE stocks as a whole than to those derived from the optimal portfolios.

Possible sources of bias in estimating the investment proportion of optimal portfolios

There are a number of possible reasons to explain these results which contradict the prediction of the Sharpe-Lintner theory that the value-weighted investment proportions vector is *ex ante* mean-variance efficient. The most obvious are that investors do not as a rule sell short or are inhibited by institutional constraints from so doing, that their assessment of an asset's risk is not closely related to its beta coefficient whether retrospective or prospective, or that their expectations are not homogeneous. Though these are also assumptions made in the customary development of the Sharpe-Lintner theory, there is plenty of external evidence to raise serious questions about their validity.[24] If investors effectively do not or cannot sell short, it is not surprising that optimal portfolios imply more short positions than are reflected in actual portfolios and that the actual excess of return on risky assets as a whole over the risk-free rate per unit of risk (i.e., the market price of risk) is below the level theoretically available with unconstrained short-selling. Heterogeneous expectations and the concentration of investors on a subset of available stocks (or other risky assets) because of information and transaction costs would tend to yield the same disparities between optimal and observed portfolios. These disparities might conceivably not be statistically significant or might simply reflect econometric or data deficiencies in our analysis and it is necessary to the extent possible to correct for these deficiencies as well as to determine whether changing the Sharple-Lintner assumptions results in a closer approximation between optimal and observed portfolio behavior.

At least two possible data deficiencies in our analysis should be noted. First, the risk-free rate used may not be appropriate. Second, even if homogeneous expectations are a tenable assumption, the estimates of the expected rates of return on individual stocks may be subject to substantial error for several reasons but in particular because the extension of a five year growth rate to an infinite horizon may be unwarranted. Both of these potential problems have been explored, with representative results summarized in Table 2. Neither adding 2% to or subtracting 2% from the one-year Treasury bill rate changed significantly any of the conclusions drawn from Table 1, indicating that the analysis is not very sensitive to moderate differences in the estimation of the appropriate risk-free rate.[25] Thus for 1977, adding 1.9% to the Treasury bill rate to approximate the average difference between the zero-beta rate of return and the risk-free rate over the period 1956-68[26] does not change the relative importance of short positions though it does lower somewhat the apparent overstatement of the market price of risk (see Table 2). As a consequence, our analysis casts doubt on the validity of the Black as well as the Sharpe-Lintner version of CAPM.[27] Similarly, the qualitative results reported in Table 1 are not greatly affected by re-estimating for 1978 the expected rates of return on individual stocks under the assumption that after the first five years into the future, growth in earnings and dividends are assumed to revert to the average for all NYSE stocks (Table 2).[28] The relative importance of short positions is not changed and the overstatement of the market price of risk is even more pronounced.

To test whether changes in some of the more important assumptions in the development of Sharpe-Lintner theory, i.e., homogeneous excectations and no constraints on short selling, would narrow the gap between optimal portfolios and those acutally held, the analyses summarized in Table 3, 4, and 5 were carried out. In Table 3, the analysis of Table 1 is repeated for

1978 except that first the assumption of the identity of mean expected return and the market's required rate of return implicit in homogeneous expectations is no longer made, and second only the ex ante data reported by institutions holding the stock are reflected in the estimates of expected returns on individual stocks. The effect of heterogeneous expectations on the required rate of return for an individual stock is initially estimated by lowering the mean expected return of that stock by one standard deviation of the expected return reported by the responding institutions. The result is to decrease somewhat the relative importance of short positions in optimal portfolios and the estimates of the market price of risk, but not to change substantially the qualitative conclusions of the earlier analysis.

A more satisfactory way of adjusting for the heterogeneity of expectations (at least in expected returns) is to compute the characteristics of optimal portfolios for each institution.[29] Some of the results of this analysis are summarized in the right hand panel of Table 3 which presents the average minimum and maximum values of the specified portfolio characteristics (i.e., expected return, variance of return, market price of risk, ratio of number of short positions and ratio of value of short to long positions). These results imply that the earlier findings on short positions and market price of risk are not mainly attributable to heterogeneous expectations. Thus they raise questions about other assumptions of the CAPM, including the basic proposition that investors effectively measure risk by co-variance of return with that in the market portfolio.

Confining the analysis of ex ante data only to institutions holding the stock, which is more appropriate if short sales are effectively precluded, again decreases somewhat the relative importance of short positions and estimates of the market price risk, but does not change the qualitative results appreciably.[30] However if short sales are precluded, this constraint should be built directly into the construction of optimal portfolios. Partly to avoid computational problems and partly as a check against our earlier use of a full-rank historical model, the actual derivation of the optimal portfolios for this part of the analysis is based on a standard single index (or diagonal) model which makes the simplifying assumption that for any two stocks the disturbance terms in the market model are uncorrelated.[31] It has been shown that for historical data this model does a better job of forecasting the future correlation matrices (for one-year and five-year predictions) than a full-rank historical model which assumes that the correlation matrix for the period being predicted is identical with that in the past ten years.[32] In this part of the analysis summarized in Tables 4 and 5, we shall construct optimal portfolios not only with short selling constraints but also without such constraints for direct comparison with the earlier results.

In deriving these optimal portfolios from the single index model, there are five necessary inputs: the expected return on the i'th stock; the beta coefficient of return on the i'th stock; the residual standard deviation of return on the i'th stock; the standard deviation of return on the market portfolio;and the risk-free rate. As explained earlier, the expected return on the i'th stock is obtained from ex ante data reported in the Institutional Investor Survey while the ordinary least squares beta coefficient and residual standard deviation of return on the i'th stock are derived from ex post data on monthly market rates of return for the preceeding 60 month period computed from the financial data files of the Rodney L. White Center and from monthly returns for this period implied by the New York Stock Exchange Composite index. In the present analysis, the market proxy is used mainly as a simple tool for approximating the covariance characteristics of stocks in the sample.

Three different estimates of the standard deviation of return on the market portfolio were used in the application of the index model (.13 .19, and .28) based not only on recent data but also on historical data back to the turn of the century,[33] and three different estimates of the risk-free rate for each period based on the one-year Treasury bill rate at the beginning of the period and 200 basis points higher and lower. While the midpoints of the range of values assumed both for the standard deviation of return on the market portfolio and for the risk-free rate seem to be the most plausible values and were used in deriving the results shown in Tables 4 and 5, the other estimates were used to test the sensitivity of our results to the assumptions made.

Table 4 presents the characteristics of optimal portfolios for the years 1974, 1976, 1977 and 1978 based on the single index model which may be compared directly with the comparable results in Table 1 based on a full rank model. The relative frequency and value of short positions is not as great as in the earlier analysis though the ratio of the value of short to long positions remains quite high with no tendency to decline as the number of stocks in the portfolio increases. However, the single index model implies even higher values of the market price of risk than the full rank model.[34]

The corresponding results of the analysis of optimal portfolios in Table 5 assuming no possibility of short sales are also contradictory to the capital asset pricing model. The ratio of the number of stocks held long in the optimal portfolios to the total number of stocks available for portfolio acquisition ranges form 10% to 20% for the largest portfolios considered in each of the years and shows a fairly consistent tendency to decline with the total number of stocks in the sample. Thus, the imposition of the constraint of no short selling on Sharpe-Lintner theory seems to imply that relatively few stocks would actually be held in investor portfolios. The implication that the great bulk of stock are not held by individual investors is obviously inconsistent with the capital asset pricing model, but is consistent with actual portfolios held.[35] It should be noted that the short sales constraint helps to reduce the market price of risk implied by the optimal portfolios, but it remains higher than that obtained from ex-post data. Without short sales, the expected value and variance of returns in optimal portfolios is no longer very sensitive to portfolio size.

Optimal portfolios: Other data

The results reported earlier, based on the *ex ante* Institutional Investor Survey data, have been checked against two other sources of information, -- the realized returns estimated from *ex post* data and the expected returns discussed earlier obtained from Wells Fargo.

Table 6 summarizes an analysis of optimal portfolios for 1977 and 1978 from a full rank historical model based exclusively on the arithmetic average return from ex post data for estimating mean returns and analagous sample values for estimation of the variance-covariance matrix. The procedures followed and the sample of stocks covered are identical with those described earlier in the discussion of Table 1 except that mean ex post returns for the previous 60 months are substituted for mean ex ante returns in estimating expected returns for the future. Both the 1977 and 1978 results based on ex post data substantiate the dubious qualitative results relating to short positions and the market price of risk obtained earlier from the ex ante data derived from the periodic surveys of a sizable sample of institutional investors.

The final analysis of optimal portfolios makes use of the Wells Fargo ex ante data discussed earlier, covering 335 stocks as of February 1, 1978, and is based on the single index model used in Table 4. Now only one institutional investor is used to estimate expected returns, which for our purposes is a major deficiency, but many more stocks are covered and the expected returns which are estimated as internal rates of return have essentially an infinite horizon. The beta coefficients and residual standard deviations used for individual stocks are those estimated by Wells Fargo and apparently reflect accounting data for these stocks as well as their past covariation with general market movements. As in the analysis presented in Table 4, several different estimates of the risk-free rate and of the standard deviation of returns on the market portfolio were tested, but all of them reinforce the conclusion that with either *ex post* or *ex ante* measures there is a large frequency of short positions in optimal portfolios.[36] With an annualized risk-free rate of 7% and standard deviation of 19%, which are based upon the historical data, short positions constituted slightly over 50% of the total number in the optimal portfolios. Nor does the substitution of the other estimates made of the risk-free rates and standard deviation of the market change this result appreciably, with the estimated short ratio ranging from 49% to 53%.[37] The estimated market price of risk is much more sensitive to different values of the standard deviation of the market and especially of the risk-free rate.

Direct test of errors in estimates of ex ante optimal portfolios

While we have tested the sensitivity of our estimates of the composition of *ex ante* optimal portfolios (based on *ex ante* expected returns and an *ex post* variance-covariance matrix) to several different assumptions about the value of the standard deviation of returns on the market portfolio and the nature of the inter-correlations between the disturbance terms in the market model, a more direct test of the size of the potential error in using historical data for estimation of *ex ante* variances and covariances is highly desirable. Since the results obtained earlier in this paper characterized the level of the individual institutional investor as well as the aggregate level, our estimates of the *ex ante* expected returns could be considered free from error for optimization purposes.

However, it can also be argued that investors consider their expectations subject to error so that allowance should be made for the sampling error in the mean return as well as in the variance-covariance matrix. Consequently, in the analysis described below, we test whether our results would be changed qualitatively if we allow for sampling errors both in the mean as well as variances and covariances of returns. We still retain the assumption that expected returns reported by institutional investors provide better estimates of the relevant ex ante expected returns than mean returns obtained from historical data.

In the earlier analysis of this paper, for a given set of N assets we have estimated a set of weights X_i, i=1, 2,...., N, for a given risk-free rate. The resulting tangency portfolio p is mean-variance efficient in the Sharpe-Lintner sense and appears to contain a high frequency of short positions using a mixture of ex ante and ex post data. The question to be considered here is whether from our sample data we can rule out the possibility that ex ante (i.e. population) efficient portfolios have all positive investments in all assets. For this, we need an estimate of the variance of the vector $\hat{X}_p$.

Our estimation procedure was to replace the vector $X_p = Y_p / b^*$ where $Y_p = V^{-1}\mu$ (and b^*, V^{-1} and μ are, respectively, a constant, the inverse of the variance-covariance matrix and the universe mean risk premium) with the sample analogue $\hat{X}_p = \hat{Y}_p / \hat{b}^*$(see appendix B for details). It has been demonstrated by Jobson and Korkie that $\hat{X}_p$ computed in this manner is unbiased.[38] Jobson and Korkie also derived an expression for the sample variance of $\hat{X}_p$. The results of a Monte Carlo simulation experiment suggest that $\hat{X}_p$ is normally distributed for sample sizes of 300 or more if stationarity and multivariate normality in common stock returns can be assumed. Unfortunately, the variance of X_p is not obtainable in a closed form but for large samples ($N \geq 300$) the numerical approximation procedures suggested by Jobson and Korkie do converge. So far we have been concerned with estimating the vector X_p with $\hat{X}_p$. Now we wish to estimate X_p when it satisfies $X_p < f$ where f is set at the .95 confidence limit value and is obtained from sample observations using the statistical procedures set out by Jobson and Korkie. In essence, letting $\hat{X}_p + 1.65\,\hat{\delta} = f$, we have Prob $(X_p < f) = .95$, where $\hat{\delta}$ is an estimate of the standard error of X_p. This may be interpreted as follows: there is a 95% probability that X_p satisfies $X_p < f$.

Using the basic procedures set forth we have computed the f confidence limit value for X_p at .95 probability, assuming normality for various sample sizes N.[39] The results are presented in Table 7. As can be seen at $\hat{X}_p + 1.65\,\hat{\delta}$ the frequency of negative investments is much less than for $\hat{X}_p$ but in no case is there a total absence of negative values when the maximum number of common stocks are admitted. The frequency of negative investments does not appear to be correlated with N the sample size and exceeds 5% when all stocks are admitted 2 times out of 4.

Thus it appears our tangency portfolios do not generally satisfy one of the essential qualities of a market portfolio, which must have positive investments in all assets. It might be argued that this exercise is not a completely satisfactory test of the Sharpe-Lintner theory because of the many assets omitted from our sample. The omission of even a single asset can in principle cause the estimated X_p to alter its composition. However, this does not seem to be a plausible position since there does not appear to be any tendency for the frequency of negative proportions to decrease as N gets larger (at $\hat{X}_p$ or f).

Other implications of ex ante data

There are many other potentially interesting analyses of the *ex ante* data and the derived optimal portfolios examined in this paper, including an analysis of the predictive ability of beta coefficients recomputed with respect to the optimal portfolios as compared with the market portfolio or with an appropriately weighted portfolio of the sample stocks, and more fundamentally an analysis of the risk characteristics for each stock which together with the reported expected returns for that stock would better explain the weightings of these stocks in the institutional portfolios. Most of this analysis has had to be postponed to a subsequent paper but some preliminary results are available. These suggest that beta coefficients computed with respect to the optimal portfolios whose characteristics were summarized in Table 1 can be superior in predictive performance to the other beta measures tested.

Thus, beta coefficients computed with respect to the optimal portfolio (β_i^*) derived from our February 1977 sample of 48 stocks does a significantly better job of predicting actual market returns (R_i) over the next year than either the beta coefficients computed with respect to the market portfolio as measured by the S&P Composite Stock Index (the customary beta measure) or with respect to the value-weighted and equally weighted returns of the 48 sample stocks.[40] The regression between R_i and β_i^* computed from the optimal portfolio is

$$R_i = .096 \underset{(-1.80)}{-.022\beta_i^*} \qquad \bar{R}^2 = .0675$$

(where $\bar{R}^2$ is the coefficient of determination adjusted for degrees of freedom and the figure in parentheses represents the t-statistic.)

The beta coefficient in this regression is both statistically significant and has the expected sign since in the predictive period (February 1977 to February 1978) the realized rate of return as measured by the Standard & Poors 500 index ($R_m = -.08$) was lower than the risk-free rate measured by the U.S. Treasury bill rate ($R_f = .06$). The corresponding results based on the other beta measures (when they were derived from the S&P index or from the value-weighted or equally weighted returns of the 48 sample stocks.) were completely insignificant. It might be noted that an earlier paper[41] found that while the power of the customary beta coefficients in explaining either *ex ante* or *ex post* returns was not very high, their performance was better for the *ex ante* returns. Our preliminary results cited above suggest that it may be possible to use the *ex ante* data to derive more meaningful beta coefficients, but a more fundamental use of these data would be an attempt to obtain new insights into how investors assess risk whether this take the form of covariance, variance, or other measures. We also plan to use these data to assess how return expectations are formed and how well they predict subsequent realized returns.

Concluding comments

The preceding analysis casts new doubt on the validity of the Sharpe-Lintner and Black versions of the capital asset pricing model and the central prediction that the value-weighted market portfolio is *ex ante* efficient. While the evidence in this paper indicates that these versions of the CAPM imply optimal portfolios which have very little relationship to the portfolios actually held, and while other types of evidence in earlier papers raise further doubts about the performance of this model, it is not clear what deficiencies in the theoretical assumptions made account for most of the difference between the model and investor behavior. There is some evidence in this paper that heterogeneous expectations are not adequate to explain the empirical discrepancies observed. Another plausible explanation of the observed discrepancies between theory and behavior, which is consistent both with the analysis in this paper and a wide range of different types of evidence presented in earlier papers,[42] is that an investor's assessment of an asset's risk is not closely related to its beta coefficient. The validity of this explanation would of course be extremely damaging to all current versions of capital asset pricing theory.

APPENDIX A

List of Institutional Investors and Stocks Covered in 1978 Survey

The banks responding were Bank of New York, Chase, Chemical, Cleveland Trust, First National Bank of Boston, Harris Bank, Manufacturers Hanover, Mellon Bank, Morgan Guaranty, St. Louis Union Trust, Security Pacific, Wells Fargo, Hartford National Bank & Trust, Detroit Bank & Trust, Girard Bank, Crocker National, Bankers Trust, Brown Brothers, National Bank of Detroit, Wilmington Trust, Marine Midland Bank, Provident National, Lincoln First Bank of Rochester, Bank of America, and five (5) banks which asked not to be identified. The investment firms responding were IDS Advisory, Lionel D. Edie, T. Rowe Price, Putnam Advisory, Scudder, Stevens and Clark, Stein Roe and Farnham, Boston Company, and two (2) others which asked not to be identified. The insurance companies were Aetna, Equitable, Metropolitan, Prudential, Travelers, John Hancock, Connecticut General, Continental, and one (1) other that asked not to be identified.

The stocks covered were American Can, American Electric Power, Atlantic Richfield, Baltimore Gas & Electric, Borden Inc., Central & Southwest, Colgate Palmolive, Commonwealth Edison, Consolidated Edison, Continental Group, Detroit Edison, Duke Power, Exxon, Florida Power & Light, Freeport Minerals, International Business Machines, Johnson & Johnson, Lucky Stores, Maytag Co., Middle South Utilities, Mobil, National Steel, New England Electric System, Niagara Mohawk Power, Northern States Power, Ohio Edison, Pacific Gas & Electric, Philip Morris, Philadelphia Electric, Procter & Gamble, Public Service Co.-Indiana, Ralston Purina, Revlon Inc., Reynolds R.J., Rockwell International, Safeway Stores, Sears Roebuck, Shell Oil, Southern, Southern California Edison, Standard Brands, Standard Oil of California, Standard Oil of Indiana, Sun Inc. Texas Utilities, Union Oil of California, Virginia Electric & Power, Warner Lambert, Wisconsin Electric Power.

APPENDIX B

EFFICIENT SET ALGORITHMS

Full Rank Procedures

To derive the composition of the portfolio of risky assets, when there is a risk-free asset, we made use of the following results on efficient portfolios (see Merton, Robert "Analytical Derivation of Efficient Portfolio Frontier," Journal of Financial and Quantitative Analysis, September 1972, pp. 1858-1871 and Roll, Richard, *op. cit.*, Appendix, for similar analysis):

I) Let us represent the risk-free rate by R_F, the vector of returns of the risky assets by R, the return of the portfolio of all assets by R_P and assume: a) the covariance matrix V of returns of the risky assets is non-singular; b) at least two risky assets have different returns.

The proportions (X_p) invested in risky assets along the efficient frontier are given by

$$\text{Min} \quad \tfrac{1}{2} \; X'VX$$
$$R_P = X'R + X_{m+1} \; R_F$$
$$X'\ell + X_{m+1} = 1$$

ℓ denoting the unit vector and X an Nx1 vextor representing investment proportions..
The solution to this problem is

(1) $$X_p = \frac{(R_P - R_F)}{a - 2bR_F + cR_F^2} \quad V^{-1} \quad (R - R_F \; \ell)$$

with $a = R'V^{-1}R$, $b = RV^{-1}\ell$ and $c = \ell'V^{-1}\ell$

II) The proportions invested in each asset along the efficient frontier are a linear combination of two mutual funds; one mutual fund containing only risky assets and the other mutual fund containing only the risk-free asset.

It is easy to prove that the proportions invested in each asset of the mutual fund of risky assets are exactly the same as the proportions of a portfolio orthogonal to the portfolio of risky assets whose return is equal to R_F.

In fact, the composition of the portfolio of risky assets having return r_p is given by (2)

(2) $$X_p = V^{-1} \; (R \; \ell) \; A^{-1} \; \binom{r_p}{1}$$

With $A = \begin{Vmatrix} a & b \\ b & c \end{Vmatrix}$. On the other hand, the returns of two minimum variance orthogonal portfolios of risky assets, whose covariance matrix is non-singular and whose returns we represented by r_p and R_f, are related by the expression (2)

$$r_p = \frac{b\,R_F - a}{c\,R_F - b}$$

By substitution of r_p in (I)

$$X_p = V^{-1} \; (R \; \ell) \; \frac{\begin{pmatrix} c & -b \\ -b & a \end{pmatrix}}{ac - b^2} \cdot \begin{pmatrix} \dfrac{bR_F - a}{cR_F - b} \\ 1 \end{pmatrix}$$

$$= \frac{1}{ac - b^2} \; V^{-1} \; (R\ell) \cdot \begin{pmatrix} \dfrac{ac - b^2}{b - cR_F} \\ - \dfrac{ac - b^2}{b - cR_F} R_F \end{pmatrix}$$

$$= \frac{1}{b - cR_F} \; V^{-1} \; (R - R_F\ell) = \frac{Y_p}{(R - R_f\ell) V^{-1}\ell} = \frac{Y_p}{b^*}$$

From the previous results we can conclude that to determine the composition of the tangent portfolio of risky assets corresponding to a risk-free rate R_F, we can use the following procedure:

a) Determine r_p using the relationship

$$r_p = \frac{bR_F - a}{cR_F - b}$$

b) Calculate the proportions invested in each risky asset using the result

$$X_p = V^{-1} \ (R \ell) \quad A^{-1} \ \left(\begin{matrix} r_p \\ 1 \end{matrix} \right)$$

Diagonal Procedures

The calculations needed to determine the proportions invested in each risky asset of the tangent portfolio can be substantially simplified if we assume that the single index model describes reality adequately.[1]

Assuming that a risk-free asset exists whose return R_F is smaller than the return of any efficient portfolio (otherwise, it wouldn't exist), the solution is

$$(3) \quad X_i = \frac{\left((\bar{R}_i - R_f) - \beta_i \, \sigma_m^2 \left[\dfrac{\sum_i \dfrac{R_i - R_f}{\sigma_{\varepsilon}^2} \beta_i}{1 + \sum_i \dfrac{\beta_i^2 \, \sigma_m^2}{\sigma_{\varepsilon_i}^2}} \right] \right) \Big/ \sigma_{\varepsilon_i}^2}{\sum_{i=1}^{N} \left\{ \left(\bar{R}_i - R_f - \beta_i \, \sigma_m^2 \dfrac{\sum_i \dfrac{\bar{R}_i - R_f}{\sigma_{\varepsilon_i}^2} \beta_i}{1 + \sum_i \dfrac{\beta_i^2 \, \sigma_m^2}{\sigma_{\varepsilon_i}^2}} \right) \Big/ \sigma_{\varepsilon_i}^2 \right\}}$$

---

1 $R_i = \alpha_i + \beta_i I + \varepsilon_i$,

$E \ (\varepsilon_{m+1} \ \varepsilon_i) = 0 \qquad i = 1, 2, \ldots, N,$

$E \ (\varepsilon_i \ \varepsilon_j) = 0 \qquad i = 1, 2, \ldots, N, \quad j = 1, 2, \ldots, N, \ i \neq j$

R_i = the return of security i,

I = a market index

ε_i = random variable with mean zero and variance, $\sigma^2_{\varepsilon i}$, and

σ_m^2 = variance of the market index

This result is almost the same as that derived by Elton, Gruber and Padberg _op. cit._, p. 1346. The difference reflects the fact that we have assumed the algebraic rather than the absolute sum of the weights is equal to one. For the case when short sales are not allowed see Elton, Gruber and Padberg _op. cit._, pp. 1347-1350.

Table 1
Characteristics of Optimal Portfolios Derived from Full Rank Model
Using Institutional Survey *Ex Ante* Data,
and Comparison with Other Portfolios Using Same Data,
for 1974, 1976, 1977 and 1978

Part A: Optimal Portfolios

1974

Portfolio Characteristics								
Number of stocks in portfolio	10	15	20	25	30	35	40	48
Expected return (annual)	.169	.194	.197	.216	.232	.238	.281	.364
Variance (annual)	.022	.024	.021	.023	.020	.019	.022	.025
Market price of risk	3.07	3.86	4.62	4.93	6.67	7.13	8.05	10.12
Ratio short positions/total no.	.370	.433	.475	.460	.447	.449	.468	.417
Ratio value short/long positions	.347	.553	.637	.701	.767	.793	.860	.942

1977

Number of stocks in portfolio	10	15	20	25	30	35	40	47
Expected return (annual)	.147	.147	.157	.159	.154	.167	.160	.159
Variance (annual)	.020	.016	.015	.011	.009	.009	.006	.004
Market price of risk	4.81	6.14	7.11	9.77	11.81	12.88	18.59	30.36
Ratio short positions/total no.	.270	.340	.400	.400	.463	.426	.450	.447
Ratio value short/long positions	.251	.405	.533	.577	.640	.680	.733	.765

1976

Number of stocks in portfolio	10	15	20	25	34
Expected return (annual)	.140	.141	.155	.163	.167
Variance (annual)	.020	.017	.013	.012	.006
Market price of risk	[illegible].35	4.51	7.10	8.46	16.13
Ratio short positions/total no.	.220	.393	.415	.468	.471
Ratio value short/long positions	.187	.404	.543	.633	.751

1978

Number of stocks in portfolio	10	15	20	25	30	35	40	49
Expected return (annual)	.170	.180	.179	.203	.192	.199	.225	.243
Variance (annual)	.026	.022	.015	.017	.012	.010	.010	.007
Market price of risk	3.40	4.42	6.59	6.88	9.13	11.15	14.93	24.06
Ratio short positions/total no.	.340	.440	.450	.476	.463	.480	.458	.490
Ratio value short/long positions	.392	.521	.620	.690	.709	.762	.812	.896

Part B: Other Portfolios

I. Portfolios with Equal Weights

1974

Portfolio Characteristics								
Number of stocks in portfolio	10	15	20	25	30	35	40	48
Expected return (annual)	.141	.144	.142	.144	.145	.145	.144	.145
Variance (annual)	.017	.016	.016	.015	.015	.015	.015	.015
Market price of risk	2.41	2.66	2.65	2.81	2.88	2.95	2.92	2.96

1977

Number of stocks in portfolio	10	15	20	25	30	35	40	47
Expected return (annual)	.136	.135	.135	.136	.135	.136	.135	.135
Variance (annual)	.025	.024	.023	.023	.023	.023	.022	.022
Market price of risk	3.48	3.65	3.69	3.86	3.74	3.84	3.67	3.88

1976

Number of stocks in portfolio	10	15	20	25	34
Expected return (annual)	.132	.131	.133	.132	.132
Variance (annual)	.023	.024	.022	.021	.021
Market price of risk	2.93	2.72	3.11	3.15	3.17

1978

Number of stocks in portfolio	10	15	20	25	30	35	40	49
Expected return (annual)	.143	.143	.142	.143	.142	.142	.142	.142
Variance (annual)	.026	.030	.027	.027	.026	.026	.026	.026
Market price of risk	2.26	2.00	2.21	2.21	2.22	2.25	2.24	2.31

II. Portfolios with Value Weights

1974

Portfolio Characteristics								
Number of stocks in portfolio	10	15	20	25	30	35	40	48
Expected return (annual)	.139	.139	.140	.144	.141	.143	.142	.142
Variance (annual)	.019	.016	.017	.016	.016	.015	.015	.015
Market price of risk	2.00	2.37	2.27	2.64	2.46	2.79	2.70	2.85

1977

Number of stocks in portfolio	10	15	20	25	30	35	40	47
Expected return (annual)	.141	.142	.142	.144	.143	.143	.143	.143
Variance (annual)	.032	.028	.028	.026	.027	.026	.025	.025
Market price of risk	2.92	3.29	3.34	3.57	3.48	3.64	3.68	3.73

1976

Number of stocks in portfolio	10	15	20	25	34
Expected return (annual)	.136	.136	.138	.137	.137
Variance (annual)	.040	.036	.037	.033	.032
Market price of risk	1.75	1.97	1.99	2.15	2.25

1978

Number of stocks in portfolio	10	15	20	25	30	35	40	47
Expected return (annual)	.151	.149	.149	.148	.147	.148	.148	.148
Variance (annual)	.028	.029	.028	.026	.027	.026	.029	.025
Market price of risk	2.42	2.26	2.36	2.50	2.38	2.51	2.51	2.54

Table 2
Characteristics of Optimal Portfolios Derived From Full Rank Model
Using Institutional Survey *Ex Ante* Data,
Substituting Different Risk-Free Rate and Different Long-Term Growth Estimates,
and Comparison With Other Portfolios Using Same Data,
for 1977 and 1978

Part A: Optimal Portfolios

Portfolio Characteristics	Estimated Zero-Beta Return Used as Risk-Free Rate: 1977 Sample of Stocks								Growth Rates Assumed to Revert After 5 years to Market Average: 1978 Sample of Stocks							
Number of stocks in portfolio	10	15	20	25	30	35	40	47	10	15	20	25	30	35	40	49
Expected return (annual)	.159	.159	.178	.169	.186	.198	.191	.201	.339	.367	.380	.444	.491	.518	.540	.611
Variance (annual)	.030	.021	.024	.015	.016	.016	.010	.008	.046	.041	.029	.032	.025	.019	.017	.008
Market price of risk	2.379	3.323	3.652	5.443	5.911	6.996	10.273	14.022	5.579	6.900	10.352	11.271	16.156	23.062	27.555	67.698
Ratio short positions/total no.	.400	.413	.455	.448	.430	.454	.425	.404	.450	.427	.455	.488	.483	.474	.467	.469
Ratio value short/long positions	.348	.486	.666	.641	.722	.789	.799	.843	.472	.644	.707	.757	.825	.836	.864	.915

Part B: Other Portfolios

I. Portfolios with Equal Weights

Portfolio Characteristics																
Number of stocks in portfolio	10	15	20	25	30	35	40	47	10	15	20	25	30	35	40	49
Expected return (annual)	.136	.135	.135	.134	.134	.135	.135	.135	.217	.221	.216	.215	.214	.215	.216	.216
Variance (annual)	.025	.023	.024	.022	.022	.022	.022	.022	.027	.029	.027	.026	.026	.025	.026	.025
Market price of risk	1.884	1.973	1.951	2.083	3.024	2.066	2.081	2.081	4.993	4.666	4.931	5.004	5.109	5.192	5.131	5.194

II. Portfolios with Value Weights

Portfolio Characteristics																
Number of stocks in portfolio	10	15	20	25	30	35	40	47	10	15	20	25	30	35	40	49
Expected return (annual)	.142	.143	.141	.145	.142	.143	.142	.143	.218	.222	.217	.219	.215	.217	.217	.217
Variance (annual)	.031	.028	.028	.026	.026	.026	.025	.025	.028	.028	.027	.027	.026	.026	.026	.025
Market price of risk	1.688	1.888	1.887	2.127	2.015	2.109	2.138	2.147	4.712	4.846	4.924	5.030	5.025	5.114	5.149	5.258

Table 3

Characteristics of Optimal Portfolios Derived from Full Rank Model
Using Institutional Survey Ex Ante Data,
Adjusting for Heterogeneous Expectations,
and Comparison with other Portfolios Using Same Data, 1978

Part A: Optimal Portfolios

Portfolio Characteristics	Mean Expected Returns of Each Stock Lowered by One Standard Deviation								Minimum Value of Portfolio Characteristics of Optimal Portfolio Averaged over Sample of Individual Institutions				Maximum Value of Portfolio Characteristics of Optimal Portfolio Averaged Over Sample of Individual Institutions			
Number of stocks in portfolio	10	15	20	25	30	35	40	49	10	15	20	25	10	15	20	25
Expected return (annual)	.156	.156	.164	.165	.171	.177	.177	.187	.158	.163	.172	.171	.216	.262	.261	.280
Variance (annual)	.023	.018	.013	.012	.011	.010	.009	.006	.020	.017	.015	.011	.038	.044	.028	.031
Market price of risk	3.22	4.10	6.02	6.79	8.18	9.85	10.72	17.99	2.15	3.03	4.06	4.32	4.74	6.64	6.94	7.88
Ratio short positions/total no.	.330	.420	.460	.470	.480	.480	.458	.408	.350	.406	.385	.404	.520	.500	.495	.512
Ratio value short/long positions	.317	.473	.595	.653	.708	.731	.772	.856	.370	.527	.607	.661	.494	.648	.743	.768

Part B: Other Portfolios

I Portfolios with Equal Weights

Portfolio Characteristics								
Number of stocks in portfolio	10	15	20	25	30	35	40	49
Expected return (anual)	.140	.138	.138	.138	.139	.139	.138	.139
Variance (annual)	.027	.029	.027	.026	.025	.025	.026	.025
Market price of risk	2.10	1.86	2.05	2.08	2.17	2.19	2.13	2.17

II Portfolios with Value Weights

Portfolio Characteristics								
Number of Stocks in portfolio	10	15	20	25	30	35	40	49
Expected return (annual)	.146	.144	.145	.144	.144	.145	.144	.144
Variance (annual)	.029	.029	.027	.027	.026	.026	.026	.025
Market price of risk	2.21	2.14	2.26	2.23	2.31	2.35	2.35	2.40

Table 4

Characteristics of Optimal Portfolios Derived from Diagonal Model,
Using Institutional Survey *Ex Ante* Data,
and Comparison With Other Portfolios Using Same Data,
For 1974, 1976, 1977 And 1978

Part A: Optimal Portfolios

1974

Portfolio Characteristics								
Number of stocks in portfolio	10	15	20	25	30	35	40	48
Expected return (annual)	.163	.207	.221	.262	.272	.253	.250	.293
Variance (annual)	.011	.018	.019	.029	.030	.020	.012	.014
Market price of risk	5.83	6.01	6.42	5.49	5.76	7.75	13.06	14.13
Ratio short positions/total no.	.390	.400	.350	.400	.367	.343	.400	.417
Ratio value short/long positions	.500	.700	.740	.800	.830	.810	.760	.790

1977

Number of stocks in portfolio	10	15	20	25	30	35	40	47
Expected return (annual)	.155	.212	.200	.193	.17[illegible]	.182	.183	.183
Variance (annual)	.015	.012	.010	.008	.004	.003	.003	.003
Market price of risk	7.19	13.07	15.7	17.26	30.70	38.59	44.76	46.99
Ratio short positions/total no.	.400	.400	.300	.280	.233	.229	.225	.213
Ratio value short/long positions	.630	.690	.670	.670	.670	.650	.650	.660

1976

Number of stocks in portfolio	10	15	20	25	34
Expected return (annual)	.157	.166	.163	.162	.159
Variance (annual)	.012	.012	.009	.007	.006
Market price risk	7.57	8.53	11.2[illegible]	13.62	15.[illegible]
Ratio short position/total no.	.300	.267	.200	.240	.294
Ratio value short/long positions	.440	.550	.560	.580	.600

1978

Number of stocks in portfolio	10	15	20	25	30	35	40	49
Expected return (annual)	.443	.416	.375	.318	.304	.294	.281	.277
Variance (annual)	.015	.012	.008	.003	.003	.002	.002	.002
Market price risk	23.8	26.7	35.1	69.8	76.2	89.2	1[illegible]1	129
Ratio short position/total no.	.300	.333	.450	.160	.200	.229	.200	.122
Ratio value short/long positions	.890	.880	.870	.840	.830	.820	.820	.850

Part B: Other Portfolios

I. Portfolios with Equal Weights

1974

Portfolio Characteristics								
Number of stocks in portfolio	10	15	20	25	30	35	40	48
Expected return (annual)	.139	.138	.138	.140	.140	.139	.142	.145
Variance (annual)	.012	.013	.013	.013	.013	.014	0.13	0.13
Market price of risk	3.19	2.89	2.98	3.06	2.98	2.79	3.15	3.31

1977

Number of stocks in portfolio	10	15	20	25	30	35	40	47
Expected return (annual)	.127	.130	.131	.130	.130	.132	.134	.135
Variance (annual)	.021	.020	.018	.017	.018	.017	.018	.018
Market price of risk	3.65	4.10	4.60	4.70	4.53	4.85	4.86	4.78

1976

Number of stocks in portfolio	10	15	20	25	34
Expected return (annual)	.137	.134	.133	.134	.132
Variance (annual)	.014	.014	.016	.016	.015
Market price of risk	5.08	4.81	4.39	4.37	4.35

1978

Number of stocks in portfolio	10	15	20	25	30	35	40	49
Expected return (annual)	.144	.144	.142	.142	.142	.144	.144	.142
Variance (annual)	.019	.019	.020	.020	.022	.021	.020	.020
Market price of risk	3.27	3.18	2.88	2.87	2.72	2.91	2.96	2.94

II. Portfolios with Value Weights

1974

Portfolio Characteristics								
Number of stocks in portfolio	10	15	20	25	30	35	40	48
Expected return (annual)	.144	.141	.142	.142	.141	.141	.142	.142
Variance (annual)	.013	.013	.013	.013	.013	.013	.013	.012
Market price of risk	3.31	3.03	3.20	3.21	3.11	3.07	3.24	3.52

1977

Number of stocks in portfolio	10	15	20	25	30	35	40	47
Expected return (annual)	.129	.129	.133	.133	.132	.135	.140	.143
Variance (annual)	.021	.021	.018	.018	.019	.019	.018	.019
Market price of risk	3.91	3.77	4.67	4.68	4.27	4.63	5.04	4.85

1976

Number of stocks in portfolio	10	15	20	25	34
Expected return (annual)	.140	.139	.138	.139	.137
Variance (annual)	.015	.019	.019	.018	.018
Market price of risk	5.00	3.93	3.93	4.17	4.14

1978

Number of stocks in portfolio	10	15	20	25	30	35	40	49
Expected return (annual)	.157	.155	.152	.152	.152	.152	.151	.150
Variance (annual)	.026	.025	.026	.024	.025	.024	.024	.023
Market price of risk	2.89	2.83	2.64	2.81	2.74	2.81	2.84	2.87

Table 5

Characteristics of Optimal Portfolios Derived From Diagonal Model Using Institutional Survey *Ex Ante* Data, Introducing the Constraint of No Short Selling, For 1974, 1976, 1977 and 1978[1]

1974

Portfolio Characteristics								
Number of stocks in portfolio	10	15	20	25	30	35	40	48
Expected return (annual)	.148	.153	.155	.161	.161	.161	.179	.183
Variance (annual)	.010	.011	.012	.013	.013	.013	.012	.013
Market price of risk	4.54	4.60	4.63	4.77	4.77	4.77	6.40	6.56
Ratio long positions/total no.	.500	.333	.300	.240	.200	.171	.175	.125

1977

Portfolio Characteristics								
Number of stocks in portfolio	10	15	20	25	30	35	40	47
Expected return (annual)	.132	.168	.161	.167	.159	.164	.164	.164
Variance (annual)	.017	.019	.016	.016	.016	.013	.013	.013
Market price of risk	4.95	6.22	6.98	6.98	7.10	8.94	9.06	9.06
Ratio long positions/total no.	.400	.067	.100	.080	.100	.114	.125	.106

1976

Portfolio Characteristics					
Number of stocks in portfolio	10	15	20	25	34
Expected return (annual)	.146	.146	.146	.149	.147
Variance (annual)	.012	.012	.012	.012	.012
Market price of risk	6.72	6.72	6.72	6.87	7.04
Ratio long positions/total no.	.500	.333	.250	.200	.206

1978

Portfolio Characteristics								
Number of stocks in portfolio	10	15	20	25	30	35	40	49
Expected return (annual)	.167	.165	.165	.165	.165	.159	.159	.159
Variance (annual)	.017	.016	.016	.016	.016	.013	.013	.013
Market price of risk	5.02	5.23	5.23	5.23	5.23	5.88	5.88	5.88
Ratio long positions/total no.	.300	.267	.200	.160	.133	.143	.125	.102

[1]A comparison with other portfolios using same data, i.e. equal-weighted and value-weighted portfolios, is presented in Table 4.

Table 6

Characteristics of Optimal Portfolios Derived From Full Rank Model
Based on Ex Post Data For Same Sample of Stocks Used in Ex Ante Analysis,[1]
and Comparison With Other Portfolios Using Same Data,
for 1977 and 1978

Part A: Optimal Portfolios

Portfolio Characteristics	1977								1978							
Number of stocks in portfolio	10	15	20	25	30	35	40	47	10	15	20	25	30	35	40	49
Expected return (annual)	.251	.237	.312	.300	.313	.391	.341	.302	.250	.271	.264	.349	.449	.554	.800	.726
Variance (annual)	.095	.046	.067	.061	.040	.067	.032	.016	.071	.082	.053	.061	.084	.094	.139	.033
Market price of risk	2.04	3.97	3.81	3.99	6.45	4.98	8.96	15.61	2.75	2.50	3.83	4.82	4.70	5.32	5.37	20.56
Ratio short positions/total no.	.420	.480	.490	.464	.487	.491	.48	.489	.400	.427	.470	.460	.477	.477	.463	.490
Ratio value short/long positions	.592	.640	.746	.745	.814	.842	.854	.882	.61	.711	.765	.819	.875	.913	.944	.959

Part B: Other Portfolios

I. Portfolios with Equal Weights

Portfolio Characteristics	1977								1978							
Number of stocks in portfolio	10	15	20	25	30	35	40	47	10	15	20	25	30	35	40	49
Expected return (annual)	.121	.119	.115	.116	.114	.115	.114	.115	.108	.101	.101	.104	.102	.100	.102	.103
Variance (annual)	.026	.025	.022	.024	.022	.023	.023	.022	.026	.030	.027	.027	.026	.026	.026	.026
Market price of risk	2.47	2.48	2.63	2.52	2.61	2.62	2.57	2.67	2.03	1.60	1.77	1.86	1.82	1.78	1.85	1.91

II. Portfolios with Value Weights

Portfolio Characteristics	1977								1978							
Number of stocks in portfolio	10	15	20	25	30	35	40	47	10	15	20	25	30	35	40	49
Expected return (annual)	.125	.128	.125	.125	.126	.125	.125	.126	.115	.108	.107	.110	.106	.107	.108	.109
Variance (annual)	.032	.030	.027	.027	.026	.026	.026	.025	.028	.029	.028	.026	.027	.026	.026	.025
Market price of risk	2.17	2.41	2.51	2.55	2.67	2.68	2.69	2.77	2.19	1.87	1.92	2.16	1.96	2.07	2.10	2.16

[1] The stocks are the same as those used in Table 1 (and subsequent tables) for 1977 and 1978 with the mean *ex post* returns for the preceding 60 months (annualized) substituted for mean *ex ante* returns.

References

Blume, Marshall E. and Friend, Irwin, 1978, The Changing Role of the Industrial Investor, John Wiley & Sons, Inc., New York, A Twentieth Century Fund Report.

Common Stock Measures, 1974, Rodney L. White Center for Financial Research, University of Pennsylvania, Philadelphia, PA.

Elton, Edwin J. and Gruber, Martin J., 1973, "Estimating the Dependance Structure of Share Prices-Implication for Portfolio Selection," Journal of Finance, Vol. XXVII, No. 5, pp. 1203-1239, December.

Elton, Edwin J. Gruber, Martin J. and Padberg, Manfred W., 1976, "Simple Criteria for Optimal Portfolio Selection," Journal of Finance, Vol XXXI, No. 5, pp. 1341-1357, December.

Fama, Eugene F. and MacBeth, J.D., 1973, "Risk, Return and Equilibrium: Empirical Tests," Journal of Political Economy, Vol. 71, pp. 607-636, May.

Friend, Irwin and Westerfield, Randolph, 1981, "Risk and Capital Asset Pricing," forthcoming in Journal of Banking and Finance, September.

Friend, Irwin and Wester, Randolph, 1980, "Co-Skewness and Capital Asset Pricing," Jounral of Finance, Vol. 35, No. 4, pp. 897-913, September.

Friend, Irwin, Westerfield, Randolph and Granito, Michael, 1978, "New Evidence on the Capital Asset Pricing Model," Journal of Finance, Vol. XXXIII, No. 3, pp. 903-917, June.

Friend, Irwin and Blume, Marshall E., 1975, "The Demand for Risky Assets," American Economic Review, Vol. LXV, No. 5, pp. 900-922, December.

Jobson, J.D. and Korkie, B., 1980, "Estimation for Markowitz Efficient Portfolios," Journal of American Statistical Association, Vol. 75, No. 371, pp. 544-554, September.

Levy, Haim, 1979, "The Capital Asset Pricing Model: Theory and Empiricism," Mimeo, March.

Merton, Robert C., 1980, On Estimating the Expected Return on the Market: An Exploratory Investigation, Working Paper No. 444, National Bureau of Economic Research, Cambridge, MA., JEL No. 520.

Roll, Richard, 1977, "A Critique of Asset Pricing Theory's Tests," Journal of Financial Economics, Vol. 4, pp. 129-176, March.

Security Market Line, Wells Fargo Bank, February 1, 1978.

The New York Stock Exchange Fact Book 1978, New York Stock Exchange, pp. 53 and 55, New York.

Footnotes

[1]For a discussion of this evidence, see Irwin Friend and Randolph Westerfield, "Risk and Capital Asset Pricing," Journal of Banking and Finance, forthcoming; Irwin Friend and Randolph Westerfield, "Co-Skewness and Capital Asset Pricing," Journal of Finance, September 1980; and Irwin Friend, Randolph Westerfield and Michael Granito, "New Evidence on the Capital Asset Pricing Model," Journal of Finance, June 1978. In addition, Richard Roll, "A Critique of Asset Pricing Theory's Tests," Journal of Financial Economics (March 1977) has recently questioned the testability of the CAPM.

[2]Full rank estimation for computing the composition of the constrained efficient portfolios was not feasible from the available algorithms known to the authors.

[3]Haim Levy, "The Capital Asset Pricing Model: Theory and Empiricism," Mimeo, March 1979.

In another paper which has received widespread attention ("A Critique of the Asset Pricing Theory's Tests," op. cit.), Richard Roll has shown that it is possible to construct a market proxy that supports the Sharpe-Lintner model even though this proxy has a .895 correlation with the market proxy used in one well-known test which resulted in a rejection of that model. We suspect that the computed tangent porfolio used by Roll, representing some unknown combination of assets, is replete with short positions which would be as inconsistent with Sharpe-Lintner theory as the test rejected by Roll.

[4]Robert C. Merton, On Estimating the Expected Return on the Market: An Exploratory Investigation, Working Paper No. 444, National Bureau of Economic Research, February 1980.

[5]The estimates of long-run rates of return and measures of risk used appear in Security Market Line, Wells Fargo Bank, February 1, 1978. The data for this period covered 335 stocks.

[6]See Appendix A for a list of the institutions and stocks covered in 1978. A similar survey was conducted in late 1979, but the results were not available in time for inclusion in this article.

[7] Two other Surveys conducted in June 1972 and February 1978 were not included in the analysis of these data, the first because of the much smaller size of the institutional sample (7) than in the subsequent years, the second because of the availability of more recent data in the same year.

[8] A number of the responses could not be used because the institutions did not have available the information required (in particular, 5 year or longer expected growth rates).

[9] These estimates were computed using the Standard & Poor 500 Composite Index and return relatives taken from a data tape containing monthly returns on all NYSE firms compiled by the Rodney L. White Center for Financial Research at the Wharton School, University of Pennsylvania.

[10] The adjustment used was to multiply the spot yield $(\frac{D_0}{P_0})$ by one plus one-half the expected growth rate $(1+\frac{g}{2})$ so that the expected rate of return $i = \frac{D_0}{P_0}(1+\frac{g}{2})+g$, where D_0 and P_0 represent the initial annualized quarterly dividend rate and stock price per share.

[11] There were 47 such stocks in 1974, 34 in 1976, 48 in 1977 and 49 in 1978. The long-run expected growth rates reported for these stocks generally referred to five year periods. However, several institutions reported 10 year or 20 year growth figures and a few (3 in the most recent period) also reported internal rates of return with an infinite horizon as well as five year growth rates. There did not appear to be any significant relationship between the explicit or implicit estimated growth rate and the length of the horizon.

[12] In other words, the stock price of the i'th stock at time 0 is set equal to the discounted sum of dividends over the next five years plus a discounted stock price at the end of the fifth year estimated as $\frac{P_{0m}}{D_{0m}} \cdot D_{0i}\frac{(1+g)^5}{(1+i_e)^5}$ instead of $\frac{P_{0i}}{D_{0i}} \cdot D_{0i}\frac{(1+g)^5}{(1+i_e)^5}$, where $\frac{P_{0m}}{D_{0m}}$ represents the initial price-dividend ratio for the stock market as a whole.

[13] Friend, Westerfield and Granito, <u>op</u>. <u>cit</u>., pp. 906-907.

[14] <u>The New York Stock Exchange 1978 Fact Book</u>, pp. 53 and 55.

[15] The investment proportions vector, X_p, is estimated as $\hat{X}_p$, the sample point estimate. $\hat{X}_p=\hat{Y}_p/\hat{b}^*$ and $\hat{Y}_p = \hat{V}^{-1}\hat{\mu}$ ($\hat{b}^*$ is a constant, V^{-1} is the inverse of the variance-covariance matrix and μ is the universe mean risk premium) see appendix B for details.

[16] There is, of course, only one group of issues included in the last portfolio size class, which covers all stocks sampled in that year.

[17] There is no obvious analytical solution to the relationship between the relative importance of short positions in optimal stock portfolios and the number of stocks in the portfolio. However, empirical analysis based on <u>ex post</u> data indicates that the relative importance of short positions increases with the portfolio size class. (Levy, <u>op</u>. <u>cit</u>.)

[18] These data were obtained from the financial data files of the Rodney L. White Center. The computational procedures are set out in Appendix B. The returns are dividend adjusted time-weighted rates of return.

[19] A number of institutions prepared reported shorter or longer horizons than five years and others provided estimates for more than one horizon. With the flat term structure assumption, the fact that monthly betas are used in conjunction with annual returns does not affect our results so long as the customary additional assumption is made that the relevant investment horizon for capital asset pricing theory is very short and can be approximated by a one month period.

[20] Robert C. Merton, op. cit. Moreover, in a supplementary analysis, we have checked the sensitivity of our results to the full rank model used by estimating the variance-covariance matrix with a diagonal model. This supplementary analysis consists of a diagonal model discussed later in this paper where three different estimates of the standard deviation of returns on the market portfolio were used. In addition, instead of using the historical data for the estimation of the correlation between returns on any two stocks which is done in the full rank model, or using the assumption in the diagonal or standard single index model presented subsequently that the disturbance terms in the market model are uncorrelated, we also tested an overall mean model which sets every correlation coefficient equal to the average of all historical correlation coefficients for the past 60 months of ex post returns, with no substantive change in any of the results presented in this paper. According to empirical analysis by Edwin J. Elton and Martin J. Gruber ("Estimating the Dependance Structure of Share Prices-Implication for Portfolio Selection, Journal of Finance, December 1973), this overall mean model outperformed the standard single index models in its ability to forecast the correlation matrix over the next five years, and both outperformed the full historical model.

[21] This assumption has been questioned though in other tests of the CAPM we have found that our results did not appear particularly sensitive to different constructions of the market portfolio. See Friend and Westerfield, "Risk and Capital Asset Pricing", and Friend, Westerfield and Granito, op. cit.

22 Haim Levy has shown that the optimal proportion in the i^{th} asset, x_i, with expected return $\bar{R}_i$ and residual standard deviation $\sigma_{\varepsilon i}$, assuming the diagonal model (see appendix B), is given by

$$\frac{a}{\sigma_{\varepsilon i}}\left[\frac{\bar{R}_i-R_f}{\sigma_{\varepsilon i}} - \sum_{\substack{j=1\\ j\neq i}}^{n} \frac{\bar{R}_j-R_f}{\sigma_{\varepsilon j}} \quad C_{ij}\right]$$

where a >0 and C_{ij} is the partial correlation coefficient between asset i and j. The ex ante values for $\bar{R}_i$ always satisfy $\bar{R} - R_F > 0$. In this case short selling (i.e. $x_i < 0$) will take place only if the C_{ij} coefficients are positive (which they virtually always are) and high enough to offset the value of $(\bar{R}_i-R_f)/\sigma_{\varepsilon i}$. When the full rank estimation procedures (described in appendix B) are used, our empirical evidence suggests that as n (the number of securities) is increased the C_{ij} effect tends to become increasingly more important and offsets the $\bar{R}_i > R_f$ effect.

In contrast, when diagonal estimation procedures are used, the related frequency of cases where $x_i < 0$ shows no systematic tendency to change as n increases (although the dollar value of the $x_i < 0$ does show some tendency to increase with larger n). It may be that diagonal estimation of the variance-covariance structure of asset returns leads to understated values when compared to full rank procedures. For another discussion of these points see Haim Levy, op. cit.

23 Irwin Friend and Marshall Blume, "The Demand for Risky Assets," American Economic Review, December 1975.

24 For example, see NYSE Fact Book for data on short sales and Marshall Blume and Irwin Friend, The Changing Role of the Industrial Investor, John Wiley, 1978, for information on risk assessment and homogeneity of expectations.

25 Similar results were obtained when a five year treasury obligation was used as the risk-free rate.

26 Eugene Fama and J.D. MacBeth, "Risk, Return and Equilibrium: Empirical Tests," Journal of Political Economy' May 1973.

27 The results of the stylized test suggested by Roll op.cit. are equally damaging to the validity of Black's version of the CAPM, For the year 1978, for instance, the minimum return stock has a return that is smaller than the return of the global minimum variance portfolio. Hence, the test was performed by calculating the weights of the securities in the global minimum variance portfolio and in the efficient portfolio corresponding with the same return as the return of the maximum return stock. The results show that there are fourteen stocks (out of 48) which have negative weights in both portfolios. Therefore, an all positive investment proportions vector is not likely to be located on the positively sloped part of the minimum variance frontier, which contradicts both the Black and Sharpe theory.

[28] This assumption is implicitly made by predicating that the price/dividend ratio for the individual stocks at the end of the five year period will be equal to the price/dividend ratio for all NYSE stocks at the beginning of the period.

[29] There were 15 institutions that were used in this analysis. Each of these institutions held at least 10 stocks; 14 held at least 20 stocks and 12 at least 25 stocks but this number went down drastically thereafter.

[30] For 1978, the only year for which such data were available, the market price of risk implied by the largest optimal portfolio, covering 49 stocks, was 18.3, the ratio of short positions to the total number of stocks in the portfolio was 41% and the ratio of the value of short to all positions was 47% (or a short/long value ratio of 87%).

[31] Edwin J. Elton, Martin J. Gruber and Manfred W. Padberg, "Simple Criteria for Optimal Portfolio Selection," Journal of Finance, December 1976.

[32] Edwin J. Elton and Martin J. Gruber, op cit, Journal of Finance, December 1973.

[33] Friend and Blume, "The Demand for Risky Assets," op. cit., p. 916.

[34] Similar results to those contained in Table 4 are obtained when the risk-free rates are raised or lowered by 200 basis points except that the relative number of short positions is increased somewhat with the higher risk-free rates and decreased somewhat with the lower risk-free rates. Results similar to Table 4 are also obtained when the assumed standard deviation of the market portfolio is either doubled or halved, except again that the relative number of short positions is increased somewhat with the higher standard deviation and decreased somewhat with the lower standard deviation.

[35] See Blume and Friend, 1978, op cit.

[36] The total number of stocks admitted into the Wells Fargo population is 335. The optimum portfolio had the following characteristics: relative frequency of negative investment proportions = 51%; variance of optimum portfolio = .00564; and return of optimum portfolio = .182

[37] The other risk-free rates tested were 5% and 13% while the other standard deviations were 10% and 22%.

[38] See Jobson and Korkie, "Estimation for Markowitz Efficient Portfolios," Journal of American Statistical Association, September 1980, pp. 17-21 for the precise computational methods.

[39]The confidence limit values are computed using return data from January 1950 to August 1978 following the requirements of convergence (i.e. $N \geq 300$).

[40]The beta coefficients were estimated from monthly data over the preceding 60 month period.

[41]Friend, Westerfield and Granito, _op. cit._

[42]E.g., see Blume and Friend, 1978, _op. cit._, and Friend and Westerfield, _op. cit._