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Optimal Claims in Automobile Insurance

by

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Abstract

In automobile insurance the premium rate which the insured must pay increases whenever he files a claim. Here we present the optimal claims strategy of the insured. It is shown that the insured will file a claim only if the damage exceeds some critical value which depends on his premium rate and on time. We present a method for obtaining the critical values, and investigate some of their properties. We investigate the effect of increasing the risk of the damages distribution on the claims policy and on the welfare of the insured (we used Rothschild and Stiglitz's [1970], and Diamond and Stiglitz's [1974] definitions of increased risk). Surprisingly, we show that increased risk may improve the welfare of the insured. We also explore the effect of increased current income on the optimal claims policy, and show that such an increase tends to augment the critical values. Further intuitive properties of the optimal claims policy are provided in a numerical example.

Optimal Claims in Automobile Insurance

I. Introduction

In this paper we explore the optimal strategy of claiming damages in automobile insurance. In most countries, the insurance premiums which the insured must pay, increase whenever he files a claim. Thus the insured must weigh the benefits from being paid the damages against the added costs of paying higher insurance rates in the future. As a result the insured will file a claim only when the damage is higher than some critical value. Since the decisions of the insured (to claim or not to claim) affect his state in the future (higher insurance rates or lower insurance rates), one needs to solve a dynamic programming problem to obtain the optimal strategy of claiming insurance. Several authors investigated this problem (see, e.g., Lemaire [1976], [1977], Norberg [1976]). These authors however derived the optimal policy under the assumption that the insured (or decision maker) is risk neutral. Here we obtain the optimal claims strategy for a risk averse decision maker, and analyze the economic properties of the optimal solution.

In section II we present our model and provide a method for computing the optimal claims strategy. In section III we investigate the effect of increased risk on the claims strategy of the decision maker and on his welfare. This effect however, depends on the chosen measure of increasing risk. We have used two alternative measures: Rothschild and Stiglitz's [1970] and Diamond and Stiglitz's [1974]. It is shown that increased risk (by Rothschild and Stiglitz's measure) increases the welfare of a risk neutral decision maker. An economic explanation for this (unexpected) result is provided. It is also shown that if the risk neutral decision maker is in the next to highest premium rate, his critical value rises with increased risk. The conclusions drawn for risk neutral decision makers hold true for risk averse decision makers if the decision maker exhibits decreasing (Arrow/Pratt) risk aversion and one employs Diamond and Stiglitz's (DS) measure of increasing risk. However, if Rothschild and Stiglitz's (RS) measure is employed, the effect of increasing risk on the decision maker cannot be predicted. A heuristic interpretation to this result is also provided.

In section IV we explore the effect of an increase in current (or transitory) income and the effect of an increase in permanent income (or wealth) on the optimal claims strategy. It is shown that an increase in current income tends to increase the critical values, whereas the effect of wealth cannot be predicted. In section V we present a numerical example illustrating the above notions and supporting the propositions stated and proved in the paper.

II. The Model

Suppose there are J insurance rates $r_1,\ldots,r_J$ applicable to the insured (decision maker). The rates satisfy $r_1 <\ldots< r_{J-1}$ and rate r_J means that the decision maker (DM) is denied insurance. Alternatively, we could assume that r_{J-1} is the maximal rate for which it is beneficial for the insured to buy insurance. It can then be assumed that for $j \leq J-1$, r_j is close enough to the expected damages so that the insured can indeed benefit from buying the insurance. Also, in many countries one is not allowed to drive without liability insurance. Hence the driver will buy such insurance even if the rates are significantly higher than the expected damages. If the insured has a rate r_j, $j < J$ and he claims a damage he is reimbursed the amount of his claim and he will be charged rate r_{j+1} starting next period.[1]

Assume that the multi-period utility function of the decision maker is additive of the form[2]

$$U(c_1,\ldots,c_t,\ldots) = \Sigma\beta^t u(c_t)$$

where c_t is consumption at time t, $u(\cdot)$ is the concave and twice differentiable one period utility at time t, and β is a discount factor. It is assumed that the net (from tax) income of the DM is A_t, and that whatever amount of money which is not spent on auto repairs (all damages are repaired) or insurance premiums is spent on consumption.[3] Let $F^j(t,\cdot)$ denote the subjective distribution function of the damages, Y, at time t of an insured with rate r_j. The distribution of damages as seen by the insurance company is not necessarily the same as the decision-maker's. In order to be able to use RS's and DS's measures of increased risk we shall assume that Y is distributed over an interval [0,M]. It will also be assumed that $M < A_t$ (see footnote 2).

Denote the damages not reimbursed by the insurance company by the random variable $\tilde{z}_t$, and the insurance rate by $\tilde{r}(t)$. It follows then that the infinite period utility is given by:

$$U(c_1,\ldots,c_t,\ldots) = \sum_{t=1}^{\infty} \beta^t u(A_t - \tilde{z}_t - \tilde{r}(t)). \tag{2.1}$$

The purpose of the DM is choosing a damage claims policy that maximizes the expected value of (2.1).

Denote by $V_{jt}(y)$ the maximum expected utility of the DM at time t conditional that he had a damage of amount y, that his current rate is r_j, and that he follows an optimal claims strategy.[4] Denote also by W_{jt} the expectation of $V_{jt}(y)$, i.e.

$$W_{jt} = E[V_{jt}(Y)]$$

where the expectation E is taken with respect to the distribution $F^j(t,.)$. W_{jt} thus represents the welfare (infinite-period expected utility) of the DM, assuming that he pursues an optimal claims policy, that his rate is r_j and that time is t. That is: $W_{jt} = \sum_{\tau=t}^{\infty} \beta^{\tau-t} E[u(A_\tau - \tilde{z}_\tau - \tilde{r}(\tau) | r(\tau) = r_j]$ given that the policy holder pursues an optimal claims policy.

If the DM had an accident of value y, he can either claim the damage or not claim it. If he claims, then his auto repair cost will be covered and his one-period utility at t will be $u(A_t - r_j)$. Since starting next period his rate will be r_{j+1}, his infinite period utility at $t+1$ will be $W_{j+1,t+1}$, and consequently his infinite-period utility at t is $u(A_t - r_j) + \beta W_{j+1,t+1}$. If he does not claim the damages, his one period utility at t will be $u(A_t - r_j - y)$, and his future utility $\beta W_{j,t+1}$, thus his infinite-period utility will be $u(A_t - r_j - y) + \beta W_{j,t+1}$.

It follows then, since $V_{jt}(y)$ represents an optimal policy, that

$$V_{jt}(y) = \max\{u(A_t - r_j - y) + \beta W_{j,t+1}, u(A_t - r_j) + \beta W_{j+1,t+1}\} \quad (2.2)$$

Thus the optimal policy of the DM can now be obtained from (2.2). A claim should be filed only if the damage y satisfies:

$$u(A_t - r_j - y) + \beta W_{j,t+1} < u(A_t - r_j) + \beta W_{j+1,t+1} \quad (2.3)$$

Since the derivative of u, u', satisfies $u' > 0$, it follows that there exists a critical value such that a claim should be made only if the damage exceeds this value. From the continuity of u it follows that the critical value, y^*_{jt}, satisfies

$$u(A_t - r_j - y^*_{jt}) = u(A_t - r_j) - \beta(W_{j,t+1} - W_{j+1,t+1}), \quad \begin{array}{l} j = 1,\ldots,J-1 \\ t = 1,\ldots \end{array} \quad (2.4)$$

It remains now to evaluate the $W_{j,t}$'s in order to obtain the $y^*_{j,t}$'s. The latter can be obtained, if the W_{jt}'s are known, by solving the implicit equation (2.4). In what follows we proceed under two alternative assumptions: 1) the planning horizon of the DM is finite, say T (henceforth: <u>the finite horizon assumption</u>,[5] and 2) the time horizon is infinite, and the functions $F^j(t,\cdot)$, and the parameters A_t are the same for all t (henceforth: <u>the stationarity assumption</u>).

We first present the relation between successive W_{jt}'s under the finite horizon assumption (this relation will then be used to present a method for computing the W_{jt}'s and the y^*_{jt}'s).

By the definition of W_{jt}

$$
\begin{aligned}
W_{jt} = E[V_{jt}(Y)] &= E[\max\{u(A_t - r_j - Y) + \beta W_{j,t+1},\ u(A_t - r_j) + \beta W_{j+1,t+1}\}] \\
&= E[\beta W_{j,t+1} + \max\{u(A_t - r_j - Y),\ s\}] \\
&= \beta W_{j,t+1} + E[\max\{u(A_t - r_j - Y),\ s\}] \\
&= \beta W_{j,t+1} + \int_0^{y^*} u(A_t - r_j - y)\,dF^j(t,y) \\
&\qquad + \int_{y^*}^{M} s\,dF^j(t,y), \qquad j = 1,\ldots,J-1 \\
&\qquad\qquad t = 1,\ldots,T-1
\end{aligned}
\tag{2.5}
$$

where[6]

$$
\begin{cases}
s = \beta(W_{j+1,t+1} - W_{j,t+1}) + u(A_t - r_j) \\
y^* \text{ satisfies: } u(A_t - r_j - y^*) = s.
\end{cases}
\tag{2.6}
$$

Thus, adding and subtracting $s \cdot \int_0^{y^*} dF(t,y)$ from (2.5) one obtains

$$
W_{jt} = u(A_t - r_j) + \beta W_{j+1,t+1} + \int_0^{y^*} [u(A_t - r_j - y) - s]\,dF(t,y) \qquad j = 1,\ldots,J-1 \quad t = 1,\ldots,T-1
\tag{2.7}
$$

where s and y^* are defined in (2.6). The critical values determining the strategy of claims are given by the y^*_{jt}'s. Thus if at time t the DM has rate r_j and suffers a damage higher than y^*_{jt} he will claim it, otherwise he will not. The method for computing the W_{jt}'s and y^*_{jt}'s is the following. The W_{jT}'s can be easily computed since at time T the DM will always file a claim (he cannot be penalized for that), hence

$$
\begin{aligned}
W_{jT} &= u(A_T - r_j) \qquad j = 1,\ldots,J-1 \\
W_{JT} &= E[u(A_T - Y)]
\end{aligned}
\tag{2.8}
$$

The rest of the W_{jt}'s and the y^*_{jt}'s can now be computed for $t < T$

The analysis under the stationarity assumption is similar. We note that under this assumption the W_{jt}'s, y^*_{jt}'s and V_{jt}'s are the same for all t. Hence, the index t is dropped from these variables, and the relation between successive W_j's can then be obtained from (2.7). It is given by

$$W_j = u(A-r_j) + \beta W_{j+1} + \int_0^{y^*} [u(A-r_j-y) - s]dF(y) \qquad j = 1,\ldots,J-1 \qquad (2.9)$$

where (see footnote 6)

$$\begin{cases} s = u(A-r_j) + \beta(W_{j+1} - W_j) \\ y^* \text{ is such that} \\ \qquad u(A - r_j - y^*) = s \end{cases} \qquad (2.10)$$

The W_j's can now be computed as follows. If the DM has rate r_J he is denied insurance hence

$$W_J = E[\sum_{\tau=0}^{\infty} \beta^{\tau} u(A-Y)] = \frac{1}{1-\beta} E[u(A-Y)]. \qquad (2.11)$$

Thus, we first evaluate W_J from (2.11). We then note that for $j = J-1$, (2.9) is an implicit equation of W_{J-1} with W_J as a parameter. Hence W_{J-1} can be obtained by solving this equation. The rest of the W_j's can then be obtained by proceeding in a similar way for $j = J-2, J-3,\ldots,1$.

III. Effects of changes in risk

In this section we investigate the effect of changes in risk on the optimal critical values y^* and on the welfare of the insured. We consider two alternative definitions of risk; the one of Rothschild and Stiglitz, and that of Diamond and Stiglitz. We briefly review these definitions, starting with the Rothschild and Stiglitz (RS) definition.[7]

Consider two distributions $F(z, R_1)$ and $F(z, R_2)$ on the interval $[0,M]$. We then say that $F(z,R_1)$ is riskier than $F(z, R_2)$ in the RS sense if

$$\int^{x} [F(z, R_1) - F(z, R_2)] > 0 \quad \text{for all} \quad 0 < x < M$$

In the sequel we shall consider distributions which are "close" to each other. We shall say that an increase in R (the "shift parameter") represents an increase in risk if

$$\begin{cases} \int_0^x F_R(z,R)dz \geq 0 \quad \text{for} \quad 0 \leq x \leq M \\ \int_0^M F_R(z,R)dz = 0 \end{cases} \tag{3.1}$$

where F_R denotes a partial derivative with respect to R.

In the Diamond and Stiglitz (DS) definition, one considers changes in the distribution while compensating for changes in utility. According to the DS definition an increase in R should be interpreted as an increase in risk if for a risk averse DM with a concave utility function $u(z)$ the following holds[8]

$$\begin{cases} \int_0^x u'(z)F_R(z,R)dz \geq 0 \qquad 0 \leq x \leq M \\ \int_0^M u'(z)F_R(z,R)dz = 0 \end{cases} \tag{3.2}$$

The results of this section are summarized in the following theorem and its corollary.

Theorem 3.1. A small increase in risk in the DS sense tends to increase W_j for $j < J$, and y^*_{J-1}, if u exhibits decreasing risk aversion.

Proof: Define the function $H(y^*, s, R)$ by

$$H(y^*, s, R) = \int_0^{y^*} [u(A - r_j - y) - s]dF(y,R) \tag{3.3}$$

where y^* and s are defined as in (2.10). It follows that (2.9) can be written as

$$W_j = u(A - r_j) + \beta W_{j+1} + H(y^*, s, R) \tag{3.4}$$

Consider now a small change dR in R, and the corresponding changes dW_j and dW_{j+1} in W_j and W_{j+1}. Taking the total differential of (3.4) one obtains on the right hand side of (3.5) are given by

$$H_{y*} = u(A - r_j - y*) - s = 0$$

since $y*$ satisfies $u(A - r_j - y*) = s$, and

$$H_s = -F(y*, R).$$

From the definition of s in (2.10) it follows that

$$ds = \beta(dW_{j+1} - dW_j),$$

hence (3.5) can be rewritten, after some rearrangement of terms as

$$[1 - \beta F(y*,R)]dW_j = \beta[1 - F(y*,R)]dW_{j+1} + H_R dR \quad j = 1,\ldots,J-1 \quad (3.6)$$

For H_R, note that using the definition of $y*$ and integration of (3.3) by parts yields

$$H(y*, s, R) = [u(A - r_j - y) - s]F(y,R)\Big|_0^{y*} + \int_0^{y*} u'(A - r_j - y)F(y,R)dy \quad (3.7)$$

$$= \int_0^{y*} u'(A - r_j - y)F(y,R)dy,$$

where the first term in the right hand side of (3.7) equals to zero since it follows from (2.6) that $[u(A - r_j - y*) - s] = 0$, and since $F(0,R) = 0$.[9] Thus, it follows from (3.2) that[10]

$$H_R = \int_0^{y*} u'(A - r_j - y)F_R(y,R)dy \geq 0 \quad (3.8)$$

From (2.11) it follows that

$$W_J = \frac{1}{1-\beta} E[u(A-Y)] = \frac{1}{1-\beta} \int_0^M u(A-y)dF(y,R)$$

$$= \frac{1}{1-\beta} [u(A-M) + \int_0^M u'(A-y)F(y,R)dy]$$

and therefore it follows from (3.2) that decreasing risk aversion implies:[11]

$$dW_J = [\partial W_J/\partial R]dR = [\frac{1}{1-\beta}\int_0^M u'(A-y)F_R(y,R)dy]dR \geq 0 \quad . \qquad (3.9)$$

The terms $[1 - \beta F(y^*,R)]$ and $\beta[1 - F(y^*,R)]$ appearing in (3.6) are positive because $0 < \beta < 1$ since β is a discount factor and $0 \leq F(y^*,R) \leq 1$, since $F(y^*,R)$ is a probability. It thus follows by induction from (3.6) and (3.8) that $dW_j \geq 0$, $j = 1,\ldots,J$. Hence for all $j < J$, W_j tends to increase as R increases.

The effect of a change in R on y^* can be analyzed by considering its definition. y^*, we recall from (2.10), satisfies

$$u(A - r_j - y^*) = u(A - r_j) + \beta(W_{j+1} - W_j).$$

An increase in R tends to increase both W_j and W_{j+1}, and we cannot determine (for all j), which is larger: dW_j or dW_{j+1}. However, if $j = J-1$ $dW_{j+1} = 0$, and in this case it follows that $dW_j \geq dW_{j+1}$, and therefore $\partial y^*/\partial R \geq 0$. Q. E. D.

We next investigate the effect of risk in the RS sense on the insured. For this we note that under RS's definition H_R in (3.8) is not necessarily non-negative. Hence it follows from (3.6) that the dW_j's may be negative or positive depending on the exact parameters of the problem.

As a corollary to Theorem (3.1) we next determine the effect of an increase in risk in the RS sense on a risk neutral decision maker.[12]

Corollary 3.1: An increase in risk in the RS sense tends to increase W_j, $j < J$, for a risk neutral decision maker. It also tends to increase y^*_{J-1}.

Proof: The RS definition of increasing risk is a special case of the DS definition for the case where $u(x) = x$ (i.e. the decision maker is risk neutral). This can be verified by inserting $u'(x)$ in (3.2), and comparing the result with (3.1). Q. E. D.

An economic interpretation for the corollary is the following. An increase in risk in the RS sense increases the probabilities of extreme value damages, both high and low. If claiming damages were not possible, the effect of having a higher probability for low damages would cancel the effect of having a higher probability for high damages, and the welfare of the DM would not change as a result of an increase in risk. Since claims can be made, the effect of the higher probabilities for high damages is partially offset. Hence the effect of higher probabilities of low damages is stronger than the effect of higher probabilities of high damages, and the net effect tends to increase the welfare of the DM.

Theorem 3.1 is now easy to interpret if one recalls that an increase in risk in the DS sense keeps expected utility fixed, while taking weight from the center of the distribution and moving it to the tails (whereas according to RS one keeps the mean of the distribution fixed). The same reasoning applied above to a risk neutral DM, can be applied to a risk averter replacing the words high-damages and low-damages by low-utility and high-utility, respectively (note that high-damage implies low-utility).

The reason that an increase in risk in the RS sense has an undetermined effect on the utility of the risk-averse DM is the following. An increase in risk in the RS sense has two effects. One discussed earlier in the case of a risk neutral DM, is the rise in the expected discounted net income (net from damages and insurance premiums) due to an increase in risk. The other is the adverse effect of uncertainty on the expected utility of the risk averse DM. Unless we make further assumptions about the utility function and/or the distribution function of the damages, it is impossible to predict which effect will dominate.

IV. Effects of changes in income and in permanent income

In consumption theory (see, e.g., Friedman [1957]) it has been argued that changes in permanent income (or wealth) have stronger effects on consumption than changes in current (or transitory) income. This theory has been challenged by many authors, and led many economists to investigate the difference between the impacts of permanent income and transitory income on other activities of the consumer. The investigation of these differences in impacts is important since the results can lead to major policy implications. Thus it would be interesting to explore whether in our case wealth affects the claims strategy differently than the transitory income does.

In what follows we consider a DM at time t and define a change in transitory income as a change in A_t, keeping other A_τ's fixed. The effect of permanent income is analyzed by considering the stationary case. A change in permanent income is defined as a change in the common (to all periods) income A. (We note that the sum of the discounted infinite horizon incomes of the DM is given by $(1-\beta)^{-1}A$).

The results concerning the effect of a change in transitory income on y^* are summarized in

<u>Theorem 4.1</u>. An increase in A_t while other A_τ, $\tau > t$ are kept fixed tends to increase the critical value y^*_{jt} of a risk averse DM.

<u>Proof</u>: We note from (2.6) that the critical value y^* satisfies

$$u(A_t - r_j - y^*) = \beta(W_{j+1,t+1} - W_{j,t+1}) + u(A_t - r_j) \qquad (4.1)$$

Consider now a small change dA in A_t, and the corresponding changes dy^*, $dW_{j,t+1}$, and $dW_{j+1,t+1}$ in y^*, $W_{j,t+1}$ and $W_{j+1,t+1}$, respectively. Since $W_{j,t+1}$ and $W_{j+1,t+1}$ do not depend on A_t, it follows that

$dW_{j,t+1} = dW_{j+1,t+1} = 0$, and therefore the total differential of (4.1) is given by

$$u'(A_t - r_j - y^*)dA_t - u'(A_t - r_j - y^*)dy^* = u'(A_t - r_j)dA_t. \tag{4.2}$$

Rearranging terms one obtains

$$dy^* = \{[u'(A_t - r_j - y^*) - u'(A_t - r_j)]/u'(A_t - r_j - y^*)\}dA_t \tag{4.3}$$

For a risk averse DM however, the term in the numerator of (4.3) must be positive, hence $dy^* \gtrless 0$ if $dA_t \gtrless 0$ and our theorem is proved.

It follows from the above theorem that when the (transitory) income increases, the DM is less likely to file a damage claim.

We next investigate the effect of permanent income on y^*. For this we note from (2.10) that y^* satisfies:

$$u(A - r_j) - \beta(W_j - W_{j+1}) = u(A - r_j - y^*).$$

Taking the total differential of this expression one obtains:

$$u'(A - r_j - y^*)dA - u'(A - r_j - y^*)dy^* = u'(A - r_j)dA - \beta(dW_j - dW_{j+1}),$$

and rearranging terms

$$dy^* = \{[u'(A - r_j - y^*) - u'(A - r_j)]/u'(A - r_j - y^*)\}dA + \beta[(dW_j - dW_{j+1})/u'(A - r_j - y^*)]. \tag{4.4}$$

Whereas the first term in the right hand side of (4.4) is positive whenever dA is positive and the DM risk averse, the sign of the second term cannot be determined. Thus, the direction of effect of changes in permanent income on y^* cannot be determined.

An intuitive explanation to the fact that an increase in current income increases y^*, whereas a change in permanent income has an unknown effect on y^* is the following. When the DM claims a damage he actually transfers consumption from the future (increased insurance premiums) to the present. The higher his current income, the lower is the marginal utility of current consumption (since consumption equals here to income and since he is risk averse), hence the less income he will want to transfer from the future to the present; thus he will set his critical value at a higher level. Changing his permanent income has an unknown effect on his substitution between present and future consumption.

V. An Example

In this section we present an example illustrating the notions appearing in the former sections. We also employ this example to infer on some additional properties of the optimal claims policy. We treat mainly the case where the time horizon is finite (we chose $T = 12$), and assume that the DM is risk neutral, and his income is $A = 20$ for all $t \leq T$. It is assumed that there are five rates (i.e. $J = 5$), and that the rates r_j are given by $r_j = (1.2)^j E(Y)$, where Y is a random variable denoting the damages of the DM. The distribution of Y does not change with t or j. We carry on our analysis by investigating the optimal claims policy and the welfare of the DM under the four distributions listed in Table 1. These distributions have the same mean and satisfy the RS conditions. For convenience we have indexed these distributions by their variance.

Table 1

In Table 2 we present the welfare of the DM and the optimal claims policy as functions of j and t. The results of the steady state case are presented in this Table under $t = \infty$. It follows from this table that y^*_{jt} decreases as the DM approaches T. This result is expected since the closer is the DM to T, the shorter the interval in which he will pay a penalty for claiming a damage, hence the more he will be willing to file a claim. One can conclude from this that older people should file claims more often than younger ones, because their T is smaller.

Table 2

In Table 3 we demonstrate the effect of increased uncertainty on the welfare and the claims policy of the DM (see footnote 7). It appears that the welfare of the DM tends to rise with increased uncertainty. The results are presented for the case $j = 1$, but the same results hold also for $j > 1$. It also happens that for most j, y^*_{jt} tends to increase with risk (the exception in row 9, table 3; we also recall that in section III we showed that only for $j = J-1$, y^*_j must necessarily increase with risk).

Table 3

In Table 4 we demonstrate the effect of increasing β on the claims policy.

Table 4

We observe in the above table that y^*_{jt} increases when β increases. This is expected since β, a discount factor, represents the preferences of the DM for future consumption vis-a-vis present consumption. Thus the higher is β the less willing will be the DM to file a claim.

Table 1

The distributions of damages

	$\sigma^2 = 0.234$	$\sigma^2 = 1.688$	$\sigma^2 = 3.5$	$\sigma^2 = 7.844$
y	p(y)	p(y)	p(y)	p(y)
0	.0078	.0156	.1250	.4063
1	.0078	.1250	.1250	.0625
2	.0156	.2031	.1250	.0156
3	.9375	.3125	.2500	.0313
4	.0156	.2031	.1250	.0156
5	.0078	.1250	.1250	.0625
6	.0078	.0156	.1250	.4063

Table 2*

The welfare of the DM and his optimal claims policy as functions of j and t

	j = 1		j = 2		j = 3		j = 4	
t	W_{1t}	y^*_{1t}	W_{2t}	y^*_{2t}	W_{3t}	y^*_{3t}	W_{4t}	y^*_{4t}
1	104.93	4	99.98	4	93.51	3	91.99	1
2	100.94	4	96.04	3	91.58	3	88.06	1
3	96.48	4	91.64	3	87.20	3	83.69	1
4	91.47	4	86.74	3	82.34	3	78.83	1
5	85.83	3	81.26	3	76.93	3	73.43	1
6	79.44	3	75.13	3	70.90	3	67.44	1
7	72.17	3	68.21	3	64.18	2	60.77	1
8	63.86	2	60.38	2	56.66	2	53.35	1
9	54.32	1	51.45	2	48.20	2	45.09	1
10	43.34	1	41.19	1	38.60	1	35.87	1
11	30.77	0	29.33	0	27.60	0	25.52	1
12	16.40	-	15.68	-	14.82	0	13.78	-
∞	140.40	4	135.40	4	130.92	3	127.39	1

$\sigma^2 = 7.84, \quad \beta = 0.9$

Table 3*

The welfare of the DM and his optimal claims policy as functions of risk

t	$\sigma^2 = 0.234$		$\sigma^2 = 1.688$		$\sigma^2 = 7.844$	
	W_{1t}	y^*_{1t}	W_{1t}	y^*_{1t}	W_{1t}	y^*_{1t}
1	99.51	2	100.99	3	104.93	4
2	95.58	2	97.04	3	100.94	4
3	91.21	2	92.64	3	96.48	4
4	86.35	2	87.74	3	91.47	4
5	80.96	2	82.28	3	85.83	3
6	74.96	2	76.17	3	79.44	3
7	68.30	2	69.32	3	72.17	3
8	60.89	2	61.62	2	63.86	2
9	52.61	2	52.84	2	54.32	1
10	42.53	1	42.59	1	43.34	1
11	30.52	0	30.52	0	30.77	0
12	16.40	-	16.40	-	16.40	-
∞	134.91	2	136.41	3	140.40	4

* $\beta = 0.9$

Table 4*

The welfare of the DM and his optimal claims policy as functions of β

t	$\beta = 0.8$		$\beta = 0.85$		$\beta = 0.90$	
	W_{1t}	y^*_{1t}	W_{1t}	y^*_{1t}	W_{1t}	y^*_{1t}
1	70.07	2	84.84	3	104.93	4
2	68.98	2	82.72	3	100.94	4
3	67.61	2	80.20	3	96.48	4
4	65.89	2	77.22	3	91.47	4
5	63.70	2	73.65	3	85.83	3
6	60.91	2	69.39	2	79.44	3
7	57.34	2	64.24	2	72.17	3
8	52.75	1	58.01	2	63.86	2
9	46.80	1	50.42	1	54.32	1
10	39.10	1	41.18	1	43.34	1
11	29.18	0	29.98	0	30.77	0
12	16.40	-	16.40	-	16.40	-
∞	74.38	2	96.76	3	140.40	4

* $\sigma^2 = 7.8$

Footnotes

1. For ease of exposition we assume that there is no deductible. This does not change the conclusions of the paper. Also, in some countries the rate of the insured decreases whenever he does not file a claim. We do not make such assumption here because it considerably complicates the mathematics, but does not alter the qualitative results (see the appendix).

2. In the economic literature, the additivity assumption is common; for example, in financial theory see Neave [1971], in environmental policy theory, see Keller, Spence and Zeckhauser [1972], and in growth theory see Arrow and Kurz [1970]. Conditions under which the assumption is allowed are given by Koopmans [1972]. These conditions differ from the usual Von Neumann and Morgenstern assumptions by the separability axiom.

3. The fact that u is defined on consumption only, may somewhat restrict the model. In the case where driving without insurance is not allowed, one would like to consider a utility function defined both on consumption and on the possibility or impossibility to drive. A similar problem also arises when the insured cannot cover his damages. One would like then to explicitly include the variable "bankruptcy" in the utility function.

4. We assume, for simplicity of exposition, that the DM can incur no more than one accident in a period.

5. If the time horizon is finite, the bounds of the summation in (2.1) are 1 and T instead of 1 and ∞.

6. For ease of notation we omit the indices j and t from s, y^*, and $F(\cdot)$, whenever this does not lead to confusion.

7. We consider here only the stationary case. An increase in risk means an increase in the risk of each $F^j(\cdot)$, $j = 1,\ldots,J$. The same results obtained here can be extended to the finite horizon case. For this one must define an increase in risk as the increase in the risk of each $F^j(t,\cdot)$, $j = 1,\ldots,J$, $t = 1,\ldots,T$. (see Levy and Paroush [1974] for a similar definition of increasing risk in a multi-period investment).

8. We say that the DM is risk averse if each period's utility function u is concave.

9. We assume, for exposition purposes, that F is a "smooth" distribution function possessing the property $F(0) = 0$. This assumption can be replaced, without affecting the results, by the following assumption: Suppose there is a probability $(1-\pi)$ that the DM will suffer an accident, and suppose that the distribution of the damages, conditional that an accident occurred is $G(x)$ where G is a "smooth" function. In this case $F(0) = \pi$, and $F(x) = \pi + (1-\pi)G(x)$ for $0 < x \leq M$. A change in risk will be interpreted as a change in the risk of G.

10. The DS paper (ibid., p. 340) has a random variable θ defined on an interval $[a,b]$. If $G(\theta,R)$ is a family of distribution functions of θ, the condition for R to be a risk parameter with respect to a utility $V(\theta,\alpha)$ is that

$$\int_a^\kappa V_\theta G_R \, d\theta \geq 0 \qquad a \leq \kappa < b$$
$$= 0 \qquad \kappa = b$$

Now let $\theta = -y$ and $\alpha = A-r_j$; so that $F(y,R) = 1 - G(\theta,R)$, and if $[a,b] = [-M,0]$ then the inequality above becomes

$$\int^w u'(A-r_j-y)\, F_R(y,R)dy \geq 0 \qquad 0 \leq w \leq M$$

11. This follows from part (i) of Theorem 3 of DS (ibid., p. 346). It is shown there that Mean Utility Preserving increases in risk are disliked by the more risk averse. Hence, since for $j = J$, DS's definition of increased risk requires that $\int_0^M u'(A-r_J-y)F_R(y,R)dy = 0$, it follows from the assumption that the insured exhibits decreasing risk aversion that $\int_0^M u'(A-y)F_R(y,R)dy \geq 0$. This stems from the fact that the argument of $u'(\cdot)$, in the former integral is lower than in the latter one. We also note that if J denotes the case where the insured is not allowed to drive, then clearly $dW_J = 0$.

12. A risk neutral may purchase insurance if his distribution is different from that of the insurance company. This case is important because it helps in providing an economic interpretation to Theorem 3.1, and since it helps in linking our model with the literature (which, as noted in the introduction, dealt with risk neutral decision-makers). In this case r_J has the same interpretation as the other r_j's.

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Appendix

Lemma A.1 The same results as in Proposition 1 hold true when there is a fixed deductible m_j for any rate j.

Proof: In this case (2.2) should be rewritten as:

$$V_{jt}(y) = \max\{u(A_t - r_j - y) + \beta W_{j,t+1},\ u(A_t - r_j - m_j) + \beta W_{j+1,t+1}\},$$

$$j = 1,\ldots,J-1$$

Hence, in this case equation (2.7) should be replaced by

$$W_{jt}(y) = u(A_t - r_j - m_j) + \beta W_{j+1,t+1} + \int_0^{y^*} [u(A_t - r_j - y) - s]dF(t,y)$$

where

$$s = \beta(W_{j+1,t+1} - W_{j,t+1}) + u(A_t - r_j - m_j)$$

y^* satisfies: $u(A_t - r_j - y^*) = s$

for $j = 1,\ldots,J-1$, $t = 1,\ldots,T-1$.

The rest of the analysis is similar to that in the main text. Q.E.D.

Lemma A.2 The same results as in Proposition 3.1 hold true when the rate of the insured decreases whenever he does not make a claim; and increases whenever he makes a claim (provided the rate can decrease or increase).

Proof: Here the case where j = 1, should be distinguished from the rest since in this case the rate cannot decrease. For j > 1, equation (2.2) should be replaced by

$$V_{jt}(y) = \max\{u(A_t-r_j-y) + \beta W_{j-1,t+1},$$

$$u(A_t-r_j) + \beta W_{j+1,t+1}\}, \quad j = 2,\ldots,J-1$$

and

$$V_{1t}(y) = \max\{u(A_t-r_1-y) + \beta W_{1,t+1}, \ u(A_t-r_1) + \beta W_{2,t+1}\}.$$

Proceeding as in section II, equation (2.7) should be replaced by

$$W_{jt} = u(A_t-r_j) + \beta W_{j+1,t+1} + \int_0^{y^*}[u(A_t-r_j-y) - s[dF(t,y) \tag{A.1}$$

$$j = 1,\ldots,J-1$$

where

$$s = \beta(W_{j+1,t+1} - W_{j-1,t+1}) + u(A_t-r_j) \quad j = 2,\ldots,J-1$$

$$y^* \text{ satisfies } u(A_t-r_j-y^*) = s. \quad j = 1,\ldots,J-1$$

and for j = 1

$$s = \beta(W_{2,t+1} - W_{1,t+1}) + u(A_t-r_1).$$

Taking the total differential of (A.1) and omitting t for the stationary case, one obtains:

$$dW_j = \beta dW_{j+1} + H_{y*}dy^* + H_s d_s + H_R dR_j, \qquad j = 1,\ldots,J-1$$

where H is defined as in (3.3). Now: $H_{y*} = 0$, $H_s = -F(y_j^*)$, $ds = \beta(dW_{j+1} - dW_{j-1})$ for $j = 2,\ldots,J-1$, and $ds = \beta(dW_{j+1} - dW_j)$ for $j = 1$. It therefore follows that

$$dW_j = \beta dW_{j+1} - \beta F(y_j^*)(dW_{j+1} - dW_{j-1}) + H_R dR_j, \qquad j = 2,\ldots,J-1$$

$$dW_1 = \beta dW_2 - \beta F(y_1^*)(dW_2 - dW_1) + H_R dR_1$$

and rearranging terms we obtain:

$$[1-\beta F(y_1^*)]dW_1 - \beta[1-F(y_1^*)]dW_2 = H_R dR_1$$

$$dW_j - \beta[1-F(y_j^*)]dW_{j+1} - \beta F(y_j^*)dW_{j-1} = H_R dR_j \qquad j = 2,\ldots,J-1. \tag{A.2}$$

The equations in (A.2) can be written in matrix notation as

$$QdW = h,$$

where $dW = (dW_1,\ldots,dW_{J-1})'$, $h = \{H_R dR_1,\ldots,H_R dR_{J-2}, H_R dR_{J-1} + [1-\beta F(y_{J-1}^*)]dW_J\}'$ and Q is a (J-1) x (J-1) matrix defined by:

$Q_{11} = 1 - \beta F(y_1^*)$

$Q_{jj} = 1$ for $j = 2,\ldots,J-1$

$Q_{j,j+1} = -[1-\beta F(y_j^*)] \qquad j = 1,\ldots,J-2$

$Q_{j,j-1} = -\beta F(y_j^*) \qquad j = 2,\ldots,J-1$

and $Q_{km} = 0$ whenever $|m-k| > 1$.

The diagonal elements of Q are positive and the off-diagonal elements of Q are nonpositive. Moreover, it can be verified that the row sums of Q are positive. It then follows from Bellman* [1970] p. 305, that Q^{-1} exists and contains nonnegative elements only. Since $dW = Q^{-1}h$, and since it has been shown in (3.8) and (3.9) that the elements, h_j, of h satisfy $h_j \geq 0$ for all j, it follows that $dW_j \geq 0$, $j = 2,\ldots,J-1$. Q.E.D.

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*Bellman, R., Introduction to Matrix Analysis, 2nd ed. New York: McGraw-Hill and Co., 1970.