MULTIPERIOD STOCHASTIC DOMINANCE
WITH RISKLESS ASSETS

by

Haim Levy* and Azriel Levy**

Working Paper No. 24-79

RODNEY L. WHITE CENTER
FOR FINANCIAL RESEARCH

University of Pennsylvania

The Wharton School

Philadelphia, PA 19104

*University of Pennsylvania and the Hebrew University of Jerusalem, Israel
**The Hebrew University of Jerusalem, Israel

Multiperiod Stochastic Dominance
With Riskless Assets

Haim Levy* and Azriel Levy**

Ordering risky options is carried out mainly in the well-known Mean-Variance framework, which was developed by Markowitz [8], [9] and Tobin [11]. Employing the mean-variance rule, one must make some assumptions either with respect to the investor's utility function (i.e., a quadratic utility function) or to the shape of the statistical distribution of the risky options. Stochastic Dominance rules, on the other hand, require very weak assumptions with respect to the investor's utility function and no assumptions at all on the distribution functions. First, second, and third degree stochastic dominance (FSD, SSD and TSD) have been developed by Quirk & Saposnik [10], Fishburn [2], Hadar & Russell [3], Hanoch & Levy [4], Whitmore [12], and others.

In a recent paper, Levy & Kroll [7] extended the stochastic dominance rules to the case where the investor can mix each of the two risky options under comparison with a riskless asset. The technical difficulty with such an extension is that once all mixtures of the riskless asset and the risky option are permitted, there is an infinite number of comparison to carry out. In [7] Levy and Kroll present a way of circumventing this difficulty, and show that in fact only one comparison is required.

---

* University of Pennsylvania and the Hebrew University of Jerusalem, Israel.

**The Hebrew University of Jerusalem, Israel.

In the present paper we develop multiperiod, first, second and third degree stochastic dominance with riskless assets. The investor is assumed to maximize the expected utility defined on his terminal wealth but he is allowed at the end of each period to decide on the optimal proportion invested in the riskless asset. Thus, if $x_1, x_2 \ldots x_n$ stand for the returns on the first option in n periods and y_1, $y_2 \ldots y_n$ denote the returns on the second option, we develop in this paper conditions for dominance of, say,

$$\prod_{i=1}^{n} x_i(\alpha_i) \quad \text{over} \quad \prod_{i=1}^{n} y_i(\beta_i) \quad \text{where,}$$

$x_i(\alpha_i)$ and $y_i(\beta_i)$ denote combinations of the risky options and the risky assets, e.g. $x_i(\alpha_i) = \alpha_i x_i + (1-\alpha_i)r_i$ when r_i stands for the return on riskless asset in period i. Note that we do not assume a constant interest rate in each period, hence we use r_i rather then r for the interest rate. Levy [6] developed conditions for dominance of

$$\prod_{i=1}^{n} x_i \quad \text{over} \quad \prod_{i=1}^{n} y_i$$

namely, of one multiperiod distribution over the other when investing in the riskless asset is not allowed. Thus, this paper can be conceived either as an extension of the paper by Levy and Kroll (which deals with one-period stochastic dominance with a riskless asset) to the multiperiod case, or as an extension of the multiperiod paper by Levy to the case where investment in riskless assets in each period is allowed. In the next section some definitions and a brief review of theorems that we shall use in this paper are given. The conditions for multiperiod dominance with a riskless asset are given in section III. Concluding remarks and suggestions for further research are given in section IV.

II Definitions and A Review

Let F (X) and G(X) be the cumulative distributions of two risky options with density functions f(x) and g(x), and let r stand for the riskless interest rate. Denote by X_α the mix of the random variables X and the riskless interest rate, $X_\alpha = \alpha X + (1-\alpha) r$ when $0 < \alpha < 1$[1]. $\{F_\alpha\}$ and $\{G_\alpha\}$ stand for the infinite sets of all combinations of F and r and G and r, respectively.

We deal in this paper with three classes of utility functions $U_k (k = 1, 2, 3)$, where $U \varepsilon U_1$ if $U' \geq 0$; $U \varepsilon U_2$ if $U' \geq 0$ and $U'' \leq 0$; and $U \varepsilon U_3$ if $U' \geq 0$, $U'' \leq 0$, and $U''' \geq 0$.

The decision rules appropriate for the classes U_k (k = 1, 2, 3) are known as first, second and third degree stochastic dominance (FSD, SSD and TSD, respectively).

Theorem 1: Let F and G be the cumulative distributions of two distinct, uncertain options X and Y. Then F dominates G (FDG) by FSD, SSD, and TSD, denoted by FD^1G, FD^2G, and FD^3G, respectively, iff,

(1) $F(x) \leq G(x)$ for all x FSD

(2) $\int_{-\infty}^{x} [G(t) - F(t)]\, dt \geq 0$ for all x SSD

(3) $\int_{-\infty}^{x}\int_{-\infty}^{v} [G(t) - F(t)]\, dt dv \geq 0$ for all x and $E_F(x) \geq E_G(x)$ TSD.

(Recall that at least one strict inequality must hold in all cases).

For proof of FSD, SSD and TSD see Quirk and Saposnik [10], Hadar and Russell [2], Hanoch and Levy [4] Whitmore [12] and others.

---

[1]We would like to avoid the trivial cases when α=0 and α=1 which imply that the investor does not diversify between the riskless asset and the risky asset. Also note that α>1 is not permitted in the multiperiod framework. If α>1, one may get a negative wealth at the end of a given period which makes the reinvestment procedures meaningless.

Theorem 2: Let $F^n(x)$ and $G^n(x)$ be two n-period cumulative distributions (i.e., the distributions of $\prod_{i=1}^{n} x_i$). Then a sufficient condition for $E_{F^n}U(x) > E_{G^n}U(x)$ for all $U\varepsilon\ U_k$ (k = 1, 2, 3) is that $F_i D^k G_i$ for all periods i, i = 1, 2,...n, for k = 1, 2, 3, respectively. (when F_i and G_i are the cumulative distributions in period i)

For proof see Levy [6][2]

Definition: We say that the set $\{F_\alpha\}$ dominates the set $\{G_\alpha\}$ if for every element $G_\beta\ \varepsilon\ \{G_\beta\}$ there is at least one element $F_\alpha\ \varepsilon\ \{F_\alpha\}$ which dominates it.

Theorem 3: $\{F_\alpha\}D^k\ \{G_\beta\}$ (k = 1,2, 3) if and only if there is at least one $0 < \alpha$ such that $F_\alpha\ D^k\ G$ (k = 1, 2, 3). For proof see Levy & Kroll [7].

Using the above definitions and theorems let us turn to the investigation of the multiperiod stochastic dominance rules with riskless assets.

III: Multiperiod Framework

We will start by proving some claims concerning the two-period case, and then by induction generalize our results to the n-period case. We first derive the multiperiod distributions of terminal wealth, and then use the derived density function in order to derive the expected utility of the terminal wealth. The expected utility formula is then employed in deriving stochastic dominance rules for the multiperiod case when investing in riskless assets in each period is allowed.

---

[2]Levy proved this theorem only for k = 1, 2. But the extension to i = 3, i.e. for TSD, is immediate. Also see Huang, Vertinsky, and Ziemba [5].

Let us assume that in period i the investor invests $(1-\alpha_i)$ of each dollar in the riskless asset, and α_i in the risky asset $(0 < \alpha_i < 1)$

Let,

R_i = the rate of return on the risky asset in period i

r_i' = the rate of return on the riskless asset in period i (the riskless interest rate).

Thus in period i, the return on a combination of the risky asset and the riskless asset is, therefore:

$$1 + (1 - \alpha_i)r_i' + \alpha_i R_i = (1 - \alpha_i)\ (1 + r_i') + \alpha_i(1 + R_i)\ .$$

Denote, $1 + R_i = X_i$

$1 + r_i' = r_i$ (one plus the rate of interest)

Thus, the final wealth at the end of period i, resulting from the investment of one dollar at the beginning of the period is given by:

$$Z_i(\alpha) = (1 - \alpha_i)\ r_i + \alpha_i X_i$$

From these definitions it is obvious that $0 \leq X_i < \infty$

while $(1 - \alpha_i)r_i \leq Z_i(\alpha_i) < \infty$

For the first and the second periods this definition turns out to be:

$$Z_1(\alpha_1) = (1 - \alpha_1)r_1 + \alpha_1 X_1$$

$$Z_2(\alpha_2) = (1 - \alpha_2)r_2 + \alpha_2 X_2$$

Hence, the cumulative distribution of the terminal wealth $Z_1(\alpha_1) \cdot Z_2(\alpha_2)$ is given by:[3]

$$F^{\alpha_1\alpha_2}(x) = \Pr\{Z_1(\alpha_1) \cdot Z_2(\alpha_2) \leq x\} =$$
$$= \int_{(1-\alpha_1)r_1}^{\infty} \int_{(1-\alpha_2)r_2}^{x/t_1} f_1^{(\alpha_1)}(t_1) f_2^{(\alpha_2)}(t_2) dt_1 dt_2$$

where f_1 and f_2 are of the desity functions of x_1 and x_2 respectively and $f_1^{(\alpha_1)}$ and $f_2^{(\alpha_2)}$ are the desity functions of $Z_1(\alpha_1)$ and $Z_2(\alpha_2)$, respectively. Thus, the last equation can be rewritten as follows,[4]

$$(4) \quad F^{\alpha_1,\alpha_2}(x) = \int_{(1-\alpha_1)r_1}^{\infty} F_2^{(\alpha_2)}\left(\frac{x}{t_1}\right) f_1^{\alpha_1}(t_1) dt_1$$

However, by definition of the cumulative distribution $F_2^{(\alpha_2)}(\frac{x}{t_1})$ and the cumulative distribution of the risky asset F_2 we obtain the following relationship:

---

[3]Throughout the paper we assume that rates of return are independent over time.

[4]Note that the forumulas are developed for $\alpha_1, \alpha_2 > 0$. However, all the results hold also for the case $\alpha_1 = 0$, $\alpha_2 = 0$. For example, if $\alpha_1 = 0$, we get

$$\Pr\{r_1 \cdot [(1-\alpha_2)r_2 + \alpha_2 X_2] \leq x\} =$$
$$= \Pr\left\{X_2 \leq \frac{x/r_1 - (1-\alpha_2)r_2}{\alpha_2}\right\} = F_2\left(\frac{x/r_1 - (1-\alpha_2)r_2}{\alpha_2}\right)$$

If both $\alpha_1 = \alpha_2 = 0$, the result is $r_1 \cdot r_2$ with certainty. However, we are interested in this paper in a case where diverification between the riskless assets and the risky option takes place. The preceeding forumulas are for $\alpha_1, \alpha_2 > 0$. However, as we shall see below, all results hold also for $\alpha_1 = 0$ and/or $\alpha_2 = 0$.

(5) $$F_2^{(\alpha_2)}\left(\frac{x}{t_1}\right) = P_r\{(1-\alpha_2)r_2 + \alpha_2 x_2 \le \frac{x}{t_1}\} =$$

$$= Pr\ \{x_2 \le \frac{\frac{x}{t_1} - (1-\alpha_2)r_2}{\alpha_2}\} = F_2\left(\frac{\frac{x}{t_1} - (1-\alpha_2)r_2}{\alpha_2}\right)$$

Similarly, $F_1^{(\alpha_1)}(x) = P_r\ \{(1-\alpha_1)r_1 + \alpha_1 x_1 \le t_1\} = F_1\left(\frac{t_1-(1-\alpha_1)r_1}{\alpha_1}\right)$

Hence the density function of $z_1(\alpha_1)$ is given by,

(6) $$f_1^{(\alpha_1)}(t_1) = \frac{\partial F_1^{(\alpha_1)}(t_1)}{\partial t_1} = \frac{\partial F_1\left(\frac{t_1-(1-\alpha_1)r_1}{\alpha_1}\right)}{\partial t_1} = f_1\left(\frac{t_1-(1-\alpha_1)r_1}{\alpha_1}\right)\frac{1}{\alpha_1}$$

Substituting (6) and (5) in (4), we get:

(7) $$F^{\alpha_1,\alpha_2}(x) = \frac{1}{\alpha_1}\int_{(1-\alpha_1)r_1}^{\infty} F_2\left(\frac{\frac{x}{t_1} - (1-\alpha_2)r_2}{\alpha_2}\right) f_1\left(\frac{t_1-(1-\alpha_1)r_1}{\alpha_1}\right) dt_1$$

In the same manner, the general n-period probability distributon function is:

(8) $$F^{(\alpha_1,\alpha_2\ldots\alpha_n)}(x) = \int_{(1-\alpha_1)r_1}^{\infty}\ldots\int_{(1-\alpha_n)r_n}^{M} f_1^{(\alpha_1)}(t_1)\ldots f_n^{(\alpha_n)}(t_n)\, dt_1\ldots dt_n$$

where $M \equiv x/\prod_{i=1}^{n-1} t_i$

However, since the cumulative distribution of the i^{th} period is given by,

$$F_i^{(\alpha_i)}(t_i) = P_r\ \{(1 - \alpha_i)r_i + \alpha_i X_i \leq t_i\} =$$

$$= F_i(\frac{t_i-(1-\alpha_i)r_i}{\alpha_i})$$

we have,

$$f_i^{(\alpha_i)}\ (t_i) = \frac{1}{\alpha_i}\ f_i\ (\frac{t_i-(1-\alpha_i)r_i}{\alpha_i})$$

Therefore, equation (8) can be written as follows:

$$(9)\quad F^{(\alpha_1,\alpha_2..\alpha_n)}(x) = \frac{1}{\alpha_1} \cdots \frac{1}{\alpha_{n-1}}\ [\ \int_{(1-\alpha_1)r_1}^{\infty} \cdots \int_{(1-\alpha_{n-1})r_{n-1}}^{\infty} F_n\ (\frac{M - (1-\alpha_n)r_n}{\alpha_n}).$$

$$\cdot\ f_1\ (\frac{t_1 - (1-\alpha_n)r_1}{\alpha_1}) \ldots f_{n-1}\ (\frac{t_{n-1}-(1-\alpha_n)r_{n-1}}{\alpha_n})\ ]\ dt_1 \ldots dt_n$$

Using the above formulae for the multiperiod density functions let us turn to the expansion of the expected utility function $U(X) \varepsilon U_k$ (k=1,2,3) where EU is defined on terminal wealth. We first start with the two-period case, where the investor is assumed to maximize the expected utility defined on his terminal wealth. We shall develop the expected utility formulae for the case $1 \geq \alpha_1 > 0$ and $1 \geq \alpha_2 > 0$. However, in the case that either α_1,α_2 or both are equal to zero, the results are straightforward, as we shall see below.

The two period density function is:

$$(10)\quad \frac{\partial F^{(\alpha_1\alpha_2)}(x)}{\partial x} = f^{(\alpha_1,\alpha_2)}(x) = \frac{1}{\alpha_1\cdot\alpha_2}\left[\int_{(1-\alpha_1)r_1}^{\infty} f_2\left(\frac{x/t_1 - (1-\alpha_2)r_2}{\alpha_2}\right)\frac{1}{t_1}\cdot\right.$$

$$\left.\cdot\, f_1\left(\frac{t_1-(1-\alpha_1)r_1}{\alpha_1}\right)\right]dt_1$$

where $f_1(\cdot)$ and $f_2(\cdot)$ are the first and second period density functions, respectively. Since $\alpha_1 \leq 1$ and $\alpha_2 \leq 1$, the range of the terminal value x is given by $(1 - \alpha_1)r_1(1 - \alpha_2)r_2 \leq X < \infty$

Thus,

$$(11)\qquad E\,U\,(X) = \int_{(1-\alpha_1)(1-\alpha_2)r_1\cdot r_2}^{\infty} U\,(X)\, f^{(\alpha_1,\alpha_2)}(x)\, dx =$$

$$= \frac{1}{\alpha_1\cdot\alpha_2}\int_{(1-\alpha_1)(1-\alpha_2)r_1\,\cdot\, r_2}^{\infty} U\,(X)\left[\int_{(1-\alpha_1)r_1}^{\infty} f_2\left(\frac{x/t_1-(1-\alpha_2)r_2}{\alpha_2}\right)\frac{1}{t_1}\right.$$

$$\left.\cdot\, f_1\left(\frac{t_1-(1-\alpha_1)r_1}{\alpha_1}\right) dt_1\right] dx$$

Eq. (11) can be rewritten as,

$$(12)\ E\,U\,(X) = \frac{1}{\alpha_1\cdot\alpha_2}\int_{(1-\alpha_1)r_1}^{\infty}\left\{\left[\int_{(1-\alpha_1)(1-\alpha_2)r_1\cdot r_2}^{\infty} U\,(X)\ f_2\left(\frac{x/t_1-(1-\alpha_2)r_2}{\alpha_2}\right)\frac{1}{t_1}\,dx\right]\right.$$

$$\left.\cdot\, f_1\left(\frac{t_1-(1-\alpha_1)r_1}{\alpha_1}\right)\right\} dt_1$$

Make the transformation $\frac{x}{t_1} = y$, (with $dx = t_1 dy$), to obtain:

$$(13)\ E\,U\,(X) = \frac{1}{\alpha_1 \cdot \alpha_2} \int_{(1-\alpha_1)r_1}^{\infty} \left[\int_{a_1}^{\infty} U\,(t_1 y) \cdot f_2\left(\frac{y-(1-\alpha_2)r_2}{\alpha_2}\right) dy \right].$$

$$\cdot\ f_1\left(\frac{t_1-(1-\alpha_1)r_1}{\alpha_1}\right) dt_1$$

where $a_1 \equiv \dfrac{(1-\alpha_1)(1-\alpha_2)r_1 \cdot r_2}{t_1}$

Once again make the transformation, $\dfrac{y-(1-\alpha_2)r_2}{\alpha_2} = z$ (with $dy = \alpha_2 dz$) to obtain,

$$(14)\ E\,U\,(X) = \frac{1}{\alpha_1} \int_{(1-\alpha_1)r_1}^{\infty} \left[\int_{a_2}^{\infty} U\,\{t_1(\alpha_2\, z + (1-\alpha_2)\, r_2)\} \cdot f_2\,(z)\, d\, z \right] \cdot$$

$$\cdot\ f_1\left(\frac{t_1-(1-\alpha_1)r_1}{\alpha_1}\right) dt_1$$

where we define $a_2 = \dfrac{\dfrac{(1-\alpha_1)(1-\alpha_2)r_1 r_2}{t_1} - (1-\alpha_2)r_2}{\alpha_2}$

Finally, define, $w = \dfrac{t_1-(1-\alpha_1)r_1}{\alpha_1}$ (with $dt_1 = \alpha_1 dw$) and make this transformation in (14), to obtain,

$$(15)\qquad E\,U\,(X) = \int_{0}^{\infty} \left[\int_{a_3}^{\infty} U\,\{[(1-\alpha_1)r_1 + \alpha_1 w][(1-\alpha_2)r_2 + \alpha_2 z)]\} \cdot f_2\,(z)\, dz \right] \cdot$$

$$\cdot\ f_1\,(w)\, d\, w$$

where a_3 is given by $a_3 = \dfrac{\dfrac{(1-\alpha_1)(1-\alpha_2)r_1 r_2}{w\alpha_1 + (1-\alpha_1)\, r_1} - (1-\alpha_2)r_2}{\alpha_2}$

The lower limit of the second integral in (15) (a_3) is a function of w. The maximal value of this limit, as a function of w, is reached when w is minimal, i.e. when w = 0. In this case, the lower limit is exactly 0. For any w greater than zero, the value of the lower limit is negative. Since for $z < 0$, $f_2(z) = 0$[5], we can substitute for this lower limit the value 0.

As a final result, the expected utility defined on terminal wealth is given by:

(16) $$E\,U\,(X) = \int_0^\infty \int_0^\infty U\,\{[(1-\alpha_1)r_1 + \alpha_1\,w][(1-\alpha_2)r_2 + \alpha_2 z)]\} f_1(w) f_2(z)\,dwdz$$

where X is the terminal two period wealth ($X = [(1-\alpha_1)r_1 + \alpha_1 X_1][(1-\alpha_2)r_2 + \alpha_2 X_2]$)

Note that formula (16) holds also for the case when $\alpha_1 = 0$ or $\alpha_2 = 0$ or both are equal to zero. For example, if $\alpha_1 = 0$, $U\,(x) = U\,(r_1 \cdot [(1-\alpha_2)r_2 + \alpha_2 X_2])$ and, $EU\,(x) = E\,U\,(r_1 \cdot [(1-\alpha_2)\,r_2 + \alpha_2\,X_2]) =$

(16a) $$= \int_0^\infty U\,\{(r_1[(1-\alpha_2)\,r_2 + \alpha_2 X_2]\}\,f_2(X_2)dx_2.$$

If both $\alpha_1 = 0$, and $\alpha_2 = 0$, we have the degenerate case and the expected utility is $U(r_1 \cdot r_2)$. We see, therefore, that equation (16) is completely general for $0 \leq \alpha_1 \leq 1$. Substituting $\alpha_1 = 0$ brings us to equation (16a). Substituting both $\alpha_1 = 0$ and $\alpha_2 = 0$, results in $U(r_1 \cdot r_2)$ with certainty.

---

[5]Note that $z = \dfrac{y-(1-\alpha_2)r_2}{\alpha_2} = \dfrac{x/t_1-(1-\alpha_2)r_2}{\alpha_2}$ since $t_1 \geq (1-\alpha_1)r_1$ and $x \geq (1-\alpha_1)(1-\alpha_2)r_1 \cdot r_2$, $z \geq 0$, and for all $z < 0$, $f(z) = 0$.

By similar means one can generalize the expected utility formula for the n-period case:

$$(17)\quad E\,U\,(X) = \int_0^{\infty} \cdots\cdots \int_0^{\infty} U\,\{[(1-\alpha_1)r_1+\alpha_1 y_1]\ldots[(1-\alpha_n)r_n+\alpha_n y_n]\}f_1(y_1)\ldots f_n(y_n)\cdot \cdot dt_1\ldots dt_n.$$

where y_1, $y_2 \ldots y_n$ are random variables of the risky returns in the corresponding periods. It is interesting to note that equation (16) (or eq (17)) is symetrical; if one assumes stationarity over time of the distribution functions $f_1 = f_2$, and if the interest rate stays constant over time $r_1 = r_2$, then the optimal diversification strategy in each period is also independent over time, i.e., $\alpha_1 = \alpha_2$. An interesting issue is the interrelationship between three factors: the distribution functions f_1, $f_2 .. f_n$, the interest rates, r_1, r_2, $..r_n$, and the optimal diversification strategy α_1, $\alpha_2 .. \alpha_n$, for various classes of utility function. These economic issues are quite complicated and require a considerable amount of space. Thus, we will turn to analyze stochastic dominance conditions and leave the above mentioned issues to a separate paper.

We shall use (16) in order to find stochastic dominance conditions for dominance of one option over the other with riskless assets. However, before doing so we need the following lemma:

Lemma:

Let: $T(w) = \int_0^{\infty} U\,\{[(1-\beta_1)\ r_1+\beta_1 w]\cdot[(1-\beta_2)r_2+\beta_2 z]\}g_2(z)dz$

where $g_2(z)$ is some density function with $g_2(z) = 0$ for $z < 0$, $\beta_1,\beta_2 \geq 0$. If $U\,(\cdot) \in U_k$ $(k = 1,2,\ 3)$ then $T(w) \in U_k$.

Proof:

(a) $T'(w) = \int_0^\infty U'\{[(1-\beta_1)r_1+\beta_1 w][(1-\beta_2)r_2+\beta_2 z]\}\ \beta_1[(1-\beta_2)r_2+\beta_2 z]\ g_2(z)dz \geq 0$

Since $\beta_1,\beta_2 \geq 0$, $U'(\cdot) \geq 0$ implies that $T'(w) \geq 0$.

(b) $T''(w) = \int_0^\infty U''\{[(1-\beta_1)r_1+\beta_1 w][(1-\beta_2)r_2+\beta_2 z]\}\ \beta_1^2[(1-\beta_2)r_2+\beta_2 z]^2\ g_2(z)dz \leq 0$

Once again, $U''(\cdot) \leq 0 \Rightarrow T''(w) \leq 0$.

(c) $T'''(w) = \int_0^\infty U'''\{[(1-\beta_1)r_1+\beta_1 w][(1-\beta_2)r_2+\beta_2 z]\}\ \beta_1^3[(1-\beta_2)r_2+\beta_2 z]^3\ g_2(z)dz \geq 0$

And by the same argument, if $U''' \geq 0$ then also $T'''(w) \geq 0$.

We use the expected utility as defined by (16) (and (17)), and the above lemma is proving the dominance conditions given in the next theorem.

Theorem 4

Let F_1, F_2 and G_1, G_2 be the cumulative distributions in periods 1 and 2 of the two distinct options. F_{λ_1,λ_2} and G_{β_1,β_2} stand for cummulative distributions of the two-period terminal wealth when investing in riskless assets is allowed. Then a sufficient condition for $\{F_{\lambda_1,\lambda_2}\}\ D^k\ \{G_{\beta_1,\beta_2}\}$ $(k = 1, 2, 3)$, $(0 \leq \lambda_i \leq 1$ and $0 \leq \beta_i \leq 1)$ is that there exist some (α_1,α_2), $(0 \leq \alpha_1, \alpha_2 \leq 1)$ such that[6]

$$F_1^{(\alpha_1)} D^k G_1$$

$$F_2^{(\alpha_2)} D^k G_2 \qquad k = 1, 2, 3.$$

[6]In the case of stationarity of the density function in all periods, dominance in only one period implies dominance in the multiperiod case.

Proof:

We have to show that for each (β_1,β_2) $0 < \beta_i \leq 1$ there exist (λ_1, λ_2) $0 < \lambda_i \leq 1$, such that $F^2_{\lambda_1,\lambda_2} D^k G^2_{\beta_1,\beta_2}$ (for k = 1, 2, 3). We shall show that for each $U(x) \varepsilon U_k$ (18) holds,

$$(18) \qquad E_{F^2_{\lambda_1,\lambda_2}} U(x) \geq E_{G^2_{\beta_1,\beta_2}} U(x).$$

By definition,

$$(19) \qquad E_{F^2_{\lambda_1\lambda_2}} U(x) = \int_{(1-\lambda_1)(1-\lambda_2)r_1r_2}^{\infty} U(x)\, f^2_{\lambda_1\lambda_2}(x)\, dx$$

Using the definition of $f_{\lambda_1\lambda_2}$ we obtain

$$E_{F_{\lambda_1,\lambda_2}} U(x) = \frac{1}{\lambda_1 \cdot \lambda_2} \int_{(1-\lambda_1)(1-\lambda_2)r_1 \cdot r_2}^{\infty} U(x) \left[\int_{(1-\lambda_1)r_1}^{\infty} f_2 \left(\frac{x/t_1 - (1-\lambda_2)r_2}{\lambda_2} \right) \frac{1}{t_1} \cdot \right.$$

$$\left. \cdot \; f_1 \left(\frac{t_1 - (1-\lambda_1)r_1}{\lambda_1} \right) \right] dt_1 dx$$

$$= \frac{1}{\lambda_1 \cdot \lambda_2} \int_{(1-\lambda_1)r_1}^{\infty} \left[\int_{(1-\lambda_1)(1-\lambda_2)r_1,r_2}^{\infty} U(x)\, f_2 \left(\frac{t_1 - (1-\lambda_2)r_2}{\lambda_2} \right) dx \right].$$

$$\cdot \; f_1 \left(\frac{t_1 - (1-\lambda_1)r_1}{\lambda_1} \right) \frac{1}{t_1}\, dt_1$$

Making the following transformation: $\frac{x}{t_1} = y$ we obtain,

$$(20)\ E_{F^2_{\lambda_1,\lambda_2}} U(x) = \frac{1}{\lambda_1 \cdot \lambda_2} \int_{(1-\lambda_1)r_1}^{\infty} \left[\int_{b_1}^{\infty} U(t_1 y)\ f_2\left(\frac{y-(1-\lambda_2)r_2}{\lambda_2}\right) dy\right] \cdot$$

$$\cdot\ f_1\left(\frac{t_1-(1-\lambda_1)r_1}{\lambda_1}\right)] dt_1$$

where, $b_1 = \frac{(1-\lambda_1)(1-\lambda_2 r_1 \cdot r_2}{t_1}$

Recall that by the conditions of the theorem there are α_1 and α_2 such that $F_1^{(\alpha_1)}$ and $F_2^{(\alpha_2)}$ dominate G_1 and G_2 respectively. Suppose now that we choose arbitrary mixes $G_1^{(\beta_1)}$ and $G_2^{(\beta_2)}$.

Then choose λ_1 and λ_2 such that, $\lambda_1 = \alpha_1\beta_1$; $\lambda_2 = \alpha_2\beta_2$.
Then, equation (20) can be rewritten as,

$$(21)\ E_{F^2_{\lambda_1,\lambda_2}} U(x) = \frac{1}{\lambda_1 \cdot \lambda_2} \int_{(1-\lambda_2)r_1}^{\infty} \left[\int_{b_1}^{\infty} U(t_1 y)\ f_2\left(\frac{\frac{y-(1-\beta_2)r_2}{\beta_2} - (1-\alpha_2)r_2}{\alpha_2}\right) dy\right] \cdot$$

$$\cdot\ f_1\left(\frac{\frac{t_1-(1-\beta_1)r_1}{\beta_1} - (1-\alpha_1)r_1}{\alpha_1}\right) dt_1$$

Make the following two transformations:

$$z = \frac{y-(1-\beta_2)r_2}{\beta_2} \qquad (\beta_2 dz = dy)$$

$$w = \frac{t-(1-\beta_1)r_1}{\beta_1} \qquad (\beta_1 dw = dt_1)$$

and use the fact that $\lambda_1 = \alpha_1\beta_1$ and $\lambda_2 = \alpha_2\beta_2$ to obtain,

$$(22)\quad E_{F^2_{\lambda_1,\lambda_2}} U(x) = \frac{1}{\alpha_1 \cdot \alpha_2} \int_{(1-\alpha_1)r_1}^{\infty} \left[\int_{b_2}^{\infty} u\,\{[(1-\beta_1)r_2+\beta_1 w][(1-\beta_2)r_2 + \beta_2 z]\} \cdot \right.$$

$$\left. \cdot\, f_2\left(\frac{z-(1-\alpha_2)r_2}{\alpha_2}\right) dz\right]\; f_1\left(\frac{w-(1-\alpha_1)r_1}{\alpha_1}\right) dw$$

when b_2 is defined as $b_2 = \dfrac{\dfrac{(1-\lambda_1)(1-\lambda_2)r_1 r_2}{w\beta_1 + (1-\beta_1)r_1} - (1-\beta_2)r_2}{\beta_2}$

The largest value of the lower limit of the second integral is reached when w is minimal, i.e. when $w = (1-\alpha_1)r_1$. In this case, the value of the lower limit is $(1-\alpha_2)r_2$. Since $f\left(\dfrac{z-(1-\alpha_2)r_2}{\alpha_2}\right) = 0$ for $z < (1 - \alpha_2)r_2$, we can substitute the value of the lower limit with $(1-\alpha_2)r_2$. After this substitution one can determine that the following inequality holds,

$$(23)\quad E_{F^2_{\lambda_1,\lambda_2}} U(x) = \frac{1}{\alpha_1} \int_{(1-\alpha_1)r_1}^{\infty} \int_{(1-\alpha_2)r_2}^{\infty} U\,\{[(1-\beta_1)r_1+\beta_1 w]\ [(1-\beta_2)r_2+\beta_2 z]\}\, \frac{1}{\alpha_2} \cdot$$

$$\cdot\, f_2\left(\frac{z-(1-\alpha_2)r_2}{\alpha_2}\right) dz] \;\; f_1\left(\frac{w-(1-\alpha_1)r_1}{\alpha_1}\right) dw$$

$$\geq \frac{1}{\alpha_1} \int_{(1-\alpha_1)r_1}^{\infty} \left[\int_{0}^{\infty} U\ \{[(1-\beta_1)r_1+\beta_1 w]\cdot[(1-\beta_2)r_2+\beta_2 z]\} g_2(z)\ dz\right] \cdot$$

$$\cdot f_1\left(\frac{w-(1-\alpha_1)r_1}{\alpha_1}\right) dw$$

Note that inequality (23) is straight forward once one shows that the following holds,

$$(24) \qquad \int_{(1-\alpha_2)r}^{\infty} U\,[(1-\beta_1)r_1 + \beta_1 w][(1-\beta_2)r_2 + \beta_2\, z]\,\frac{1}{\alpha_2}\, f_2\,(\frac{z-(1-\alpha_2)r_2}{\alpha_2})dz \geq$$

$$\int_{0}^{\infty} U\,[(1-\beta_1)r_1 + \beta_1 w][(1-\beta_2)r_2 + \beta_2\, z]\, g_2\,(z)\, dz$$

(Since the other terms of both sides of (23) are identical).

However the above inequality stems from the second condition of the theorem which asserts $F_2 \, {}^{\alpha_2}{}_2 D^k G_2$, and from the fact that

$U\,[(1-\beta_1)r_1 + \beta_1 w][(1-\beta_2)r_2 + \beta_2\, z] \equiv U(\cdot) \,\varepsilon\, U_k$, $(k = 1,2,3)$ as shown by the lemma.[7]

So far we used only one condition of the theorem. However, since the second term of inequality (23) can be rewritten as,

$$(25) \qquad \frac{1}{\alpha_1} \int_{(1-\alpha_1)r_1}^{\infty} T\,(w) \quad f_1\,(\frac{w-(1-\alpha_1)r_1}{\alpha_1})^1 dw$$

where $\quad T(w) = \int_0^{\infty} U\,\{[(1-\beta_1)r_1 + \beta_1 w][(1-\beta_2)r_2 + \beta_2\, z]\, g_2\, dz$

and $\quad T(w)\, \varepsilon\, U_k$, $k = 1,2,3$. (See the Lemma)

we can use the condition $F_1^{\alpha_1}\, D^k G_1$ in order to conclude that, also the following inequality holds,

$$(26) \qquad \frac{1}{\alpha_1} \int_{(1-\alpha_1)r_1}^{\infty} T(w) \quad f_1\,(\frac{w-(1-\alpha_1)}{\alpha_1} dw \geq \int_0^{\infty} T\,(w)\; g_1\,(w)\; dw$$

Since $T(w)\varepsilon U_k$ and $F_1^{(\alpha_1)} D^k G$. However, since (see eq. (16))

---

[7]Note that $f_2^{(\alpha_2)}(z) = \frac{1}{\alpha_2}\; f_2\,(\frac{z-(1-\alpha_2)r}{\alpha_2})$ and $f_2^{(\alpha_2)}(z) = 0$ for all vaules smaller than $(1-\alpha_2)\, r_2$. Also note that since $g_2(z)$ is non-negative, also the assertion of the lemma holds by substituting 1 for $g_2(z)$,.

$$(27)\ \int_0^\infty T(w)g_1\ (w)\ dw = \int_0^\infty\int_0^\infty U\ \{[(1-\beta_1)r_1+\beta_1 w]\ [(1-\beta_2)r_2+\beta_2 z]\}\ g_1(w)g_2(z)\ dw\ dz$$

$$= E_{G^2_{\beta_1,\beta_2}}\ U(x)$$

We can conclude from (24), (25) and (26) that for each β_1, β_2, there exist λ_1 and λ_2 such that,

$$E_{F_{\lambda_1,\lambda_2}}\ U(x) \geq E_{G_{\beta_1,\beta_2}}\ U(x) \quad \text{for all } U\varepsilon U_k\ (k = 1,2,3)$$

Q.E.D.

<u>Theorem 5</u>

Let $F^n_{\alpha_1 \ldots \alpha_n}$ and $G^n_{\alpha_1 \ldots \alpha_n}$ be the cumulative distributions of n period risks, where α_i represents the proportion invested in the risky asset. A sufficient condition for $\{F^n_{\alpha_1 \ldots \alpha_n}\}\ D^k\ \{G^n_{\alpha_1 \ldots \alpha_n}\}$ (k = 1,2,3) is that there exist $\alpha_1 \ldots \alpha_n$, $0 \leq \alpha_i \leq 1$ such that $F_i^{(\alpha_i)}(\alpha_i)D^k G_i$ for all periods i = 1, 2,...n, for k = 1,2,3 respectivity.

<u>Proof</u>: Let us choose vector $\beta_1,\beta_2 \ldots \beta_n$ which denotes the diversification strategy of G_1, $G_2 \ldots G_n$ and the riskless assets $r_1,..r_n$. We have to show that for each $\beta_1,\ldots,\beta_n$ $(0 \leq \beta_i \leq 1,)$ which we select there exist $0 \leq \lambda_1,\ldots,\lambda_n \leq 1$, such that $F^n_{\lambda_1 \ldots \lambda_n}\ D^k G^n_{\beta_1 \ldots \beta_n}$, (k = 1,2,3) The proof is by induction. For n=2, we know from theorem (4) that $F_i^{\alpha_i}\ D^k G_i$ for i = 1,2 implies that for each pair (β_1,β_2) $0 \leq \beta_i \leq 1$, there exists a pair (λ_1,λ_2), $\lambda_i = \alpha_i\beta_i$, such that $F^2_{\lambda_1\lambda_2}\ \ D^k\ \ G^2_{\beta_1\beta_2}$ when the superscript means the two-period terminal wealth. Let us assume that the theorem is true for n-1 periods, i.e.

$F_i^{(\alpha_i)}\ D^k G_i$ i=1, 2,...,n-1 implies that for each vector of β's $(\beta_1 \ldots \beta_{n-1})$, $0 \leq \beta_i \leq 1$, there exists a vector of λ's $(\lambda_1 \ldots \lambda_{n-1})$, $0 \leq \lambda_i \leq 1$, such that $F^{n-1}_{\lambda_1 \ldots \lambda_{n-1}}(t)\ D^k G^{n-1}_{\beta_1 \ldots \beta_{n-1}}(t)$, and $\lambda_i = \alpha_i \beta_i$. We also know by the condition of the theorem that for some α_n, $F_n^{(\alpha_n)}\ D^k G_n$. Thus it is given that[8] $F^{n-1}_{\lambda_1 \ldots \lambda_{n-1}}\ D^k G^{n-1}_{\beta_1 \ldots \beta_{n-1}}$ and that $F_n^{(\alpha_n)}\ D^k G_n$ (for k = 1,2, 3), and we have to prove that $F^n_{\lambda_1 \ldots \lambda_n}\ D^k G^n_{\beta_1 \ldots \beta_n}$,(k = 1,2, 3)

Let,

$$\phi \equiv \prod_{i=1}^{n-1} [(1-\lambda_i)r_i + \lambda_i X_i] \ ; \ \Psi \equiv \prod_{i=1}^{n-1} [(1-\beta_i)r_i + \beta_i X_i]$$

$$F^{n-1}_{\lambda_1 \ldots \lambda_{n-1}}(t) = P \left\{ \prod_{i=1}^{n-1} [(1-\lambda_i)r_i + \lambda_i X_i] \leq t \right\} = P \{\phi \leq t\} = F_\phi(t)$$

$$G^{n-1}_{\beta_1 \ldots \beta_{n-1}}(t) = P \left\{ \prod_{i=1}^{n-1} [(1-\beta_i)r_i + \lambda_i X_i] \leq t \right\} = P \{\psi \leq t\} = G_\psi(t)$$

It is given (by the induction assumption) that $F_\phi(t)\ D^k G_\psi(t)$, for k=1,2,3 (a particular case where for the n-1 period distribution α=1, β=1 and hence

---

[8]Note by F_n we denote the n-th period distribution and by F^n the n-period distribution of terminal wealth. When ever we use an superscript to denote the period this means a multiperiod distribution.

$\lambda=\alpha\beta=1$) and also that $F_n^{(\alpha_n)} D^k G_n$. Considering $F_\phi(t)$ and $G_\Psi(t)$ as the first period distributions we can apply theorem 4 and conclude that for each pair $(1, \beta_n)$ there exists some $(1, \lambda_n)$ (for the first period: $\alpha_1 = 1$ and we take $\beta_1 = 1$, getting $\lambda_1 = 1 \cdot 1 = 1$; for the second period $\alpha_2 = \alpha_n$, $\beta_2 = \beta_n$ and $\lambda_n = \alpha_n \beta_n$) such that: $F^2_{1\lambda_n} \; D^k \; G^2_{1\beta_n}$ for k = 1, 2, 3

Thus,

$$F^2_{1\lambda_n}(t) = \Pr\{\phi \cdot [(1-\lambda_n)r_n + \lambda_n X_n] \leq t\} \equiv F^n_{\lambda_1 \ldots \lambda_n}(t)$$

$$G^2_{1\beta_n}(t) = \Pr\{\Psi \cdot [(1-\beta_n)r_n + \beta_n X_n] \leq t\} \equiv G^n_{\beta_1 \ldots \beta_n}(t)$$

We can conclude from theorem (4) that $F^n_{(\lambda_1, \lambda_2 \ldots \lambda_n)} \; D^k \; G^n_{(\beta_1, \beta_2 \ldots \beta_n)}$ (k = 1,2,3). This last assertion is true since by definition, the n-period two random variables are given by

$$\prod_{i=1}^{n} [\,(1-\beta_i)r + \beta_i X_i] = \Psi\, [(1-\beta_n)r_n + \beta_n X_n] \text{ and } \prod_{i=1}^{n} [(1-\lambda_i)r + \lambda_i X_i] =$$

$$= \phi\, [(1-\lambda_n)r_n + \beta_n X_n] \text{ respectively.}$$

Q.E.D.

Corollary

If one assumes a stationarity over time of the distribution of the risky option and a constant interest rate in all periods, it is sufficient to find $F_1^{\alpha} D G_1$ in order to conclude that $\{F^n_{\lambda_1, \ldots \lambda_n}\}$ dominates the set $\{G^n_{\alpha_1, \ldots \alpha_n}\}$ by stochastic dominance of order k; where k = 1,2, 3.

In theorems 4 and 5 we established dominance conditions for multiperiod distribution with riskless assets. The condition for multiperiod dominance is that there is domiance in each one period. However, there is no need to look for many values of α_i in order to determine if indeed $F_i^{(\alpha_i)} D G_i$. Levy and Kroll [7] have shown that for the one-period case one inequality has to be checked in order to determine if such α_i indeed exists.

IV. Concluding Remarks

We investigated in this paper the relationship between one-period and multiperiod first, second and third degrees stochastic dominance, when lending at a riskless interest rate is allowed. The interest rate does not have to be constant over-time. The general result is that dominance in each period implies dominance in the multiperiod case. However, this relationship does not work both ways and dominance in n-period does not imply dominance in each period. It is possible that F^n does not dominate G^n when riskless asset is not allowed, and by allowing to lend money at some riskless interest rates $r_1, \ldots r_n$, one may find dominance among various multiperiod mixtures of the risky options and the riskless assets.

Finally, this paper provides a framework for further research investigating the following economic issues:

(a) How risk-averters allocate their income between the risky options and the riskless asset, when the riskless interest rate varies over time.

(b) What is the impact of change in the intial investor's wealth on the allocation of his investment between the risky option and the riskless asset, in a multiperiod framework.

(c) What are the multiperiod dominance conditions to specific statistical distributions, e.g., normal distributions. Some of these issues have been dealt with by Arrow [1] for the one period case. The invetigation of these issues in the multiperiod framework is under preparation.

REFERENCES

[1] Arrow, K.J. Aspects of the Theory of Risk Bearing. Yrjö Jahnssonin Säätio, Helsinki, 1965.

[2] Fishburn, P.S. Decision and Value Theory. Wiley, New York, 1964.

[3] Hadar, J., and W. R. Russell. "Rules for Ordering Uncertain Prospects," American Economic Review, 59 (1969), 25-34.

[4] Hanoch, G., and H. Levy. "The Efficiency Analysis of Choices Involving Risk," Review of Economic Studies, 36 (1969), 335-46.

[5] Huang, C. C., Vertinsky, I., and Ziemba, W.T., "On Multiperiod Stochastic Dominance" Journal of Financial and Quantitative Analysis (1978), 1-13

[6] Levy, H., Stochastic Dominance, "Efficiency Criteria, and Efficient Portfolios: The Multi-Period Case," American Economic Review, 63 (1973), 986-94.

[7] Levy, H., and Y. Kroll. "Ordering Uncertain Options with Borrowing and Lending," Journal of Finance, 33 (1978) 553-73.

[8] Markowitz, H.M. "Portfolio Selection," Journal of Finance, 7 (1952), 77-91.

[9] Markowitz, H.M. "Investment for the Long Run: New Evidence for an Old Rule," Journal of Finance 31 (1976), 1273-86.

[10] Quirk, J.P., and R. Saposnik. "Admissibility and Measurable Utility Functions," Review of Economic Studies, 29 (1962), 140-46.

[11] Tobin, J. "Liquidity Preference as Behavior Towards Risk" Review of Economic Studies, 25 (1958), 65-86.

[12] Whitmore, G.A. "Third Degree Stochastic Dominance, American Economic Review, 60 (1970), 457-59.