Factor Price Equalization Under Uncertainty

by

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INTRODUCTION

Extension of trade theory to the uncertain world is a recent venture. Nevertheless several different avenues have already been explored.

HELPMAN and RAZIN (1977), in a recent survey, have categorized the subject along three dimensions :

i) The type of uncertainty prevailing : price uncertainty, technological uncertainty and uncertainty in preferences. The first kind taken in isolation is inconsistent with a general equilibrium analysis. Of the last two we analyze here primarily technological uncertainty.

ii) The point in time at which international trade decisions are made : in some models, trading decisions are made before uncertainty is resolved. This is the assumption underlying the models of BRAINARD and COOPER (1968), BARDHAN (1971), BATRA and RUSSELL (1974) and RUFFIN (1974 a). Alternatively there exist models where the only commitment made by countries before uncertainty resolution is the allocation of factors of production. Following KEMP and LIVIATAN (1973), RUFFIN (1974 b), TURNOVSKY (1974), BARON and FORSYTHE (1977) and HELPMAN and RAZIN (1978) we shall study here a model of this latter kind.

iii) The type of financial markets in existence : when no financial markets or risk sharing arrangements prevail, firms maximize the expected utility of future profits, a type of model referred to as "entrepreneurial" by BARON & FORSYTHE. Although most of the articles quoted so far as well as the recent contributions of BATRA (1975), DAS (1977) and MAYER (1976) did use this kind of set up, we shall not follow this practice because such an objective function is not consistent with the main thrust of financial theory where it is assumed that owners of firms are not unconditional and stand ready to trade their shares.

To our knowledge only two articles have duly incorporated financial markets into the analysis : HELPMAN and RAZIN (1978) and BARON and FORSYTHE (1977).

The type of uncertainty considered was, however, very restricted, a point we shall delve upon below.

When capital markets do exist they may or may not be sufficiently rich to permit a Pareto optimal allocation of consumption between households, for given production plans. See NIELSEN (1975). When they do, we shall call them "complete" because one can show then that securities, pricing and firms' production decisions[1] are identical to what they would be in a market where there exist as many different securities as there are states of nature characterizing uncertainty.

In section 1 of this paper we shall draw on the joint-output trade-theory literature to prove Factor Price Equalization (FPE) under technological uncertainty, assuming that the international capital market is complete. Under this assumption it is well known that firms maximizing the market value of their shares will also unanimously satisfy capital market investors, provided price taking is assumed.

In section 2 we shall follow the lead of BARON and FORSYTHE (1977) and HELPMAN and RAZIN (1978) and prove FPE in an incomplete capital market setting, but using a fully general kind of technological uncertainty.

1 - TECHNOLOGICAL UNCERTAINTY AS A CASE OF JOINT OUTPUT

Let us describe a country's production set by a collection of M techniques or industries each one associated with an output commodity and characterized by the transformation function :

$$(1) \qquad t_j(y_{1j} \dots y_{sj} \dots y_{Sj}, - x_j)$$

where y_{sj} is a scalar amount of net output of the traded commodity j in state of nature s (s = 1 ... S) and x_j is a vector of non traded resources utilized by industry j. The production set of industry j is defined by $t_j(y_j, -x_j) \leqslant 0$ where $y_j = (y_{sj})$. The economy confronts a set of prices which are denoted by w for the vector of input prices and $p_s = (p_{sj})$ for the state-dependent output prices. If one wants to conceive of production as requiring a certain lapse of time, then p_s should be defined as the beginning-of-period prices of outputs contingent on state s occuring ; more specifically it is the future spot price vector which will prevail at the end of the period in state s <u>times</u> the beginning-of-period price of the ARROW-DEBREU contingent security pertaining to state s.[2] Hence $\sum_s p_s$ should be thought of as the futures price of the output discounted by the rate of interest.

The only assumption so far pertaining to the transformation functions t_j is that they are homogeneous of degree one ; production sets are closed cones containing the origin and assumed to be convex.

In the absence of joint production across states, it would be possible to rewrite the transformation functions in the following form :

$$(2) \qquad \begin{cases} y_{sj} \leqslant f_{sj}(x_{sj}) \\ \sum_s x_{sj} \leqslant x_j . \end{cases}$$

In all other cases, we shall say by definition that some joint production across states exists ; HIROTA and KUGA (1971) and SAMUELSON (1966) have provided criteria of joint production applicable to the transformation functions t_j.[3] Such criteria will not be satisfied by the technology :

$$(3) \qquad \underset{s}{\mathrm{Min}} \left[f_{sj}(x_j) - y_{sj} \right] \geqslant 0$$

or equivalently :

$$(4) \qquad y_{sj} \leqslant f_{sj}(x_j) \quad \forall_s \; ,$$

which we shall use frequently.

Although both the formulations (2) and (4) do provide for uncertainty, it would seem that (2) is not a realistic representation of production risk. Indeed (2) assumes that producers, by a proper utilization of their inputs, can exactly choose the amount to be produced in each state of nature ; if that were so, then obviously under the usual conditions (see for instance KUGA (1972)) factor price equalization (FPE) would prevail provided that the number of factors is less than or equal to the number of outputs times the number of states of nature. FPE would be likely to obtain even with a small number of output commodities.

By way of contrast formulation (4) drastically limits the freedom of producers to control production in each separate state of nature. The final output in each state is a function of the total industry input without any possibility of adaptation of the input structure in order to cope with a particular anticipated combination of states. Formulation (4) embodies some degree of fatalism which appears realistic. It is probably excessive, however, in that (3) or (4) permit absolutely <u>no degree of substitution</u> between output in different states once the total industry inputs are set. Because of these fixed output proportions for given input proportions, the production conedefined by (3) is polyhedral and does not display the smoothness assumed in neoclassical theory. Since most proofs of the FPE theorem assume that the production set is twice continously differentiable, it is clear that some problems may arise.

There is a special case, however, where FPE will clearly be verified despite the existence of joint production. Assume that (3) is actually :

$$(5) \qquad \underset{s}{\mathrm{Min}} \left[b_{sj} \; f_j(x_j) - y_{sj} \right] \geqslant 0$$

or :

$$(6) \qquad y_{sj} \leqslant b_{sj} \; f_j(x_j)$$

Then the production set can also be described by :

$$(7) \qquad \underset{s}{\mathrm{Min}} \left[f_j(x_j) - \frac{y_{sj}}{b_{sj}} \right] \geqslant 0$$

or :

$$(8) \qquad f_j(x_j) - \mathrm{Max} \left[\frac{y_{sj}}{b_{sj}} \right] \geqslant 0.$$

In that case outputs of the same commodity in different states for given inputs will always be produced in constant proportions independently of the input combination, and one can define a compound unit of output of commodity j as b_{1j} units of output in state 1 plus b_{2j} units of output in state 2, etc... The compound output is state independent and its price is :

$$p_j = \sum_s \; p_{sj} \, b_{sj} \, .$$

If the set of functions $\{f_j(x_j) \; ; \; j = 1 \ldots M\}$ satisfies the usual conditions for FPE under certainty, then FPE will also be satisfied in that case with p_j playing the role usually played by the certain output price. This very simple type of uncertainty is referred to in the finance literature as "scale uncertainty" ; it was introduced by DIAMOND (1967). HELPMAN and RAZIN (1978) and BARON and FORSYTHE (1977) utilized it to generalize international-trade theorems in the presence of incomplete capital markets.

By way of contrast, the task confronting us in this section is the extension of the FPE theorem to more general types of uncertainty but assuming a market where there are as many independent securities as there are states of nature. In the presence of multiple goods, commodities spot markets will nevertheless open in the second period of the analysis[4] ; we shall postulate that everyone has homogeneous expectations regarding the future value of spot prices in each state of nature.

Since uncertainty of the kind we consider here is a case of joint output, we naturally turn to the joint-output literature. Trade theorists (SAMUELSON (1953-54), LAND (1959)) have sometimes denied the possibility of FPE under joint output. The reason is clear : if there are as many outputs as there are factors and if two outputs or combinations of outputs must be produced in constant proportions, then their relative price plays no role in the firm's optimization calculus. As a result the situation is identical mathematically to a case of no joint output with one output fewer than factors ; one should therefore not expect FPE to prevail. KUGA (1972), however, has given a condition sufficient for FPE to prevail when there is one factor fewer than outputs ; he also provided an example of a transformation function which produces FPE although there are three outputs and two factors only.

In the present context, however, we are dealing with a special case of joint output. Equation (3) reflects a situation where various non overlapping subsets of outputs, namely those corresponding to the same commodity available in different states, are produced jointly. We have as many such subsets are there are commodities ; if the number of commodities is equal to the number of factors, the number of outputs (number of commodities times the number of states) is much larger than the number of factors. Hence FPE may still prevail on a general basis.

To formalize this intuition, recall the definition of an "industry" as the subset of the aggregate production set which produces one commodity only in different states, the transformation function of industry j being given by equation (4). Following WOODLAND (1977) and others[5], define industry j's value added as : [6]

(9) $\theta_j(p_j, x_j) \equiv \max_{y_j} \{p_j y_j \text{ ; subject to (4) and } y_j \geqslant 0\}$

which in this case boils down to :

$$\theta_j(p_j, x_j) = p_j f_j(x_j)$$

where f_j is a vector-valued production function. The value-added function is non negative, linearly homogeneous, convex and continuous in p_j, linearly homogeneous concave, continuous and non decreasing in x_j. Provided there exist some non empty set P_j of price vectors such that for p_j belonging to P_j, $\theta_j > 0$ for some $x_j > 0$, the value added function has all the properties of a production function.

Further define the unit value added cost function for industry j as :

$$(10)\quad C_j(p_j, w) \equiv \begin{cases} \min_{x_j} \{wx_j : \theta_j(p_j, x_j) \geqslant 1,\ x_j \geqslant 0\} & \text{if } p_j \in P_j \\ +\infty & \text{if } p_j \notin P_j \end{cases}$$

DIEWERT (1975) has proven the following properties[7] :

(i) $C_j(p, w)$ is a positive extended real valued function defined for all $w > 0$ and $p \geqslant 0$ and is infinite for $p_j \notin P_j$;

(ii) C_j is continuous from above and quasi-concave in (p, w) ;

(iii) C_j is homogeneous of degree - 1 in p ;

(iv) C_j is non decreasing and linearly homogeneous in w.

The function C_j (p, w) is analogous to a traditional unit output cost function. But in the present case, it depends on the output price vector since the latter serves to combine the several joint outputs.

In addition to DIEWERT's properties, we postulate that the unit value added cost function is continuously differentiable with respect to w. In what case, by SHEPHARD's (1953) lemma[8], we can compute the optimal input vector of industry j by :

$$x_j = \frac{\partial C_j}{\partial w}$$

we shall refer to this unit value added input requirement function as :

$$x_j = \frac{\partial C_j}{\partial w} \equiv a_j(p_j, w)$$

By the ~~definition~~ of C_j, we have :

(12) $\quad C_j = a_j(p_j, w)w$

which can also be viewed as an application of EULER's theorem to C_j which is linearly homogeneous in w.

So long as industry j functions at a positive level, i.e. so long as commodity j is produced in at least one state of nature, competitive equilibrium requires a zero profit[9] :

(13) $\quad C_j(p, w) = 1$

which can be written :

(14) $\quad a_j(p_j, w)w = 1.$

Postulating that there are as many industries as there are inputs, we can collect the row vectors a_j into a square matrix[10] :

$$A = \{a_j(p_j, w) \; ; \; j = 1\ldots M\}$$
$$= A(p, w)$$

which satisfies :

$$(15) \qquad \boxed{A(p, w)w = 1}$$

where 1 is now a vector of ones.

Consider two economies between which there is no trade in factors but there is perfect trade in commodities. These two economies are therefore confronted with the same vector of commodities p ; in addition they have the same social transformation function ; in a word they differ only by their endowments of factors of production. Consequently they both satisfy equation (15) with the same output price vector p. The question raised by FPE theory is whether or not (15) then implies that the input price vector w is the same for both economies ; a problem referred to mathematically as that of the "global univalence" of this equation with respect to the unknown w.[11]

McKENZIE (1955), SAMUELSON (1967) and KUGA (1972) have provided sets of sufficient conditions for global univalence to obtain.[12] KUGA's conditions applied to our problem are as follows :

Assumption 1

$\theta_j(p_j, x_j)$ is subject to constant returns to scale with respect to x_j and is strictly concave except along rays.

Assumption 2

$\theta_j(p_j, x_j)$ is twice continuously differentiable with respect to x_j.[13]

Assumption 3

The marginal productivities $\dfrac{\partial\, \theta_j(p_j, x_j)}{\partial\, x_j}$ are all positive.

Then the Factor Price Equalization theorem under uncertainty can be stated :

Theorem : So long as the two economies to be compared are completely diversified ($\theta_j > 0, \forall_j$), the competitive condition (15) ensures the equalization of factor prices provided that the Jacobian determinant $|A|$ of the unit value added cost function with respect to input prices, never vanishes along a path connecting the two economies, and provided assumptions 1-3 are met.

Generally speaking the same mathematical conditions which bring about FPE under certainty also produce it under uncertainty.[14] It remains, however, that those conditions may be satisfied less frequently than under certainty. For instance the condition $|A| \neq 0$, which in the case of two factors and two outputs implies the absence of factor-intensity reversal, is always satisfied by COBB-DOUGLAS production functions under certainty. It may not be satisfied by COBB-DOUGLAS functions under uncertainty. To demonstrate this, consider an economy with two factors, capital K and labor L, and two risky outputs, cloth c and wheat w. The optimal input decisions will be given by :

$$\max_{L_c, K_c, L_w, K_w} \quad \sum_s p_{cs} L_c^{\alpha_{cs}} K_c^{1-\alpha_{cs}} + \sum_s p_{ws} L_w^{\alpha_{ws}} K_w^{1-\alpha_{ws}}$$

subject to : $L_c + L_w = L$

$$K_c + K_w = K$$

with p_L and p_k being the multipliers pertaining to the two constraints. The relationship between factor prices and factor input combinations in the cloth industry will be given by :

$$\frac{p_L}{p_k} = \frac{\sum_s p_{cs}\, \alpha_{cs} (K_c/L_c)^{1-\alpha_{cs}}}{\sum_s p_{cs} (1-\alpha_{cs}) (K_c/L_c)^{-\alpha_{cs}}} \quad ;$$

it is graphed in figure 1 as a solid line. The slope of the asymptote is equal to :

$$\underline{\alpha}_c/(1-\underline{\alpha}_c) \quad \text{where} \quad \underline{\alpha}_c = \operatorname*{Min}_s(\alpha_{cs})$$

and the slope at the origin is :

$$\bar{\alpha}_c/(1-\bar{\alpha}_c) \quad \text{where} \quad \bar{\alpha}_c = \operatorname*{Max}_s(\alpha_{cs}).$$

The similar relationship for the wheat industry is represented as a dotted line. If :

$$\underset{s}{\mathrm{Min}}(\alpha_{ws}) < \underset{s}{\mathrm{Min}}(\alpha_{cs}) < \underset{s}{\mathrm{Max}}(\alpha_{cs}) < \underset{s}{\mathrm{Max}}(\alpha_{ws})$$

the two graphs will necessarily intersect, i.e. there will be a factor intensity reversal. As a result, two countries A and B which have different factor compositions such as $(K/L)^A$ and $(K/L)^B$ will experience different factor prices while they would not have under certainty or under pure scale uncertainty. Indeed the factor price-factor share relations would then have been straight lines.

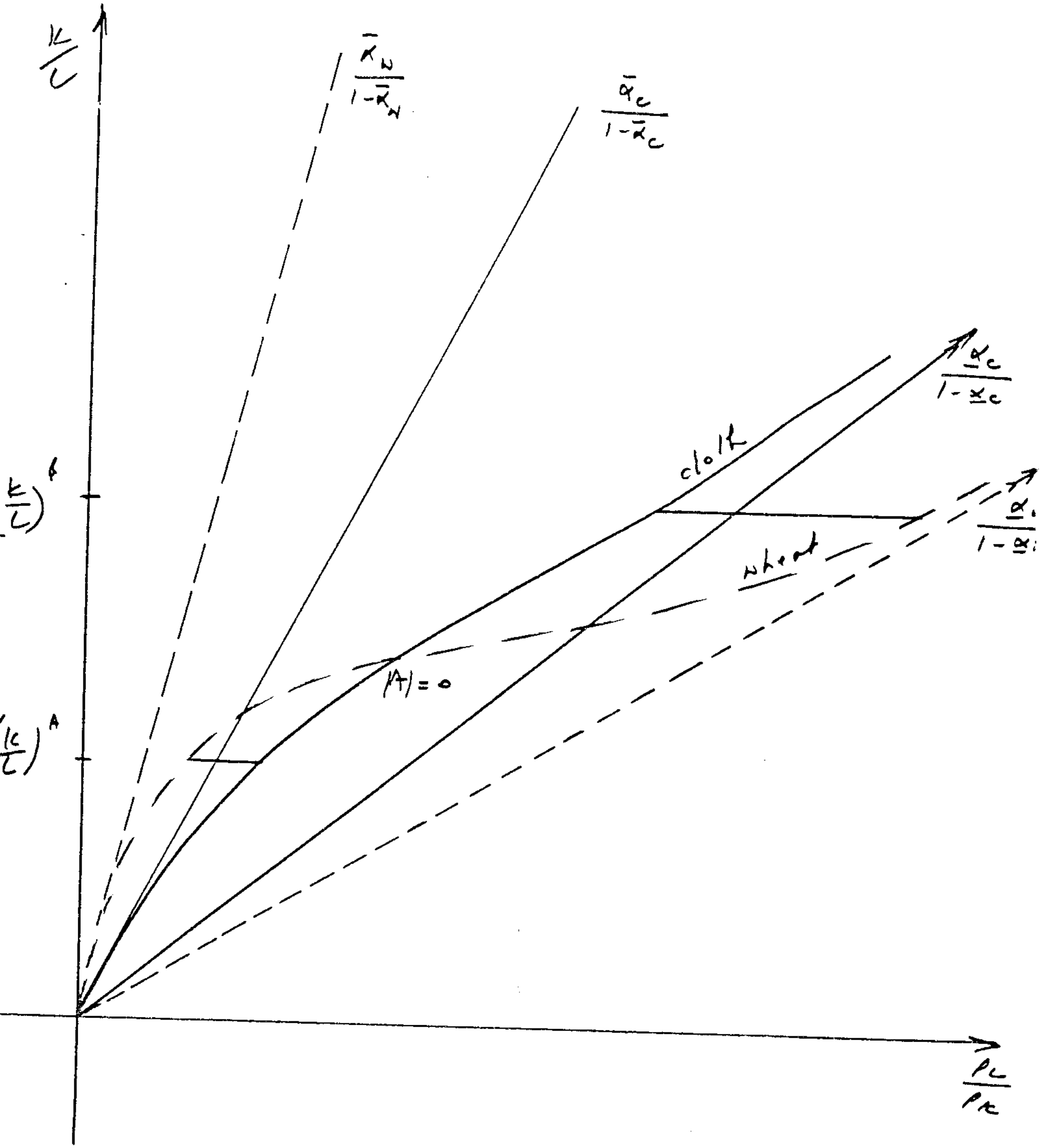

Figure 1. The possibility of non specialization

2 - FACTOR PRICE EQUALIZATION WITH AN INCOMPLETE CAPITAL MARKET

When the capital market is incomplete, an insufficient number of dimensions of trade in securities prevails to bring about the equality of different investors' rates of substitution between present and future consumption. We had so far denoted by p_s the vector of state-dependent output prices ; we had noted that they were actually future spot prices discounted back to the beginning of the period of production by means of the price of the ARROW-DEBREU elementary security pertaining to state s. Those prices had been used to compute each industry's value-added and cost function. When the equivalent of a set of ARROW-DEBREU securities does not exist, investors could only use their own personal rates of substitution to compute state-contingent discounted values. Hence value-added and cost functions would no longer be universal as they would depend on exactly which investor's welfare is being maximized by firms.

As a result the input decision would itself not be predicated on a unique criterion, and it is a priori impossible to indicate whether the decision ultimately to be made would guarantee Factor Price Equalization.

To solve this difficulty we must characterize explicitly the investors' decision making process. For this purpose, we introduce a slight change of notation : p_s will now refer to the future vector of spot prices not discounted back ; all investors hold identical anticipations regarding those prices. To simplify, it will be assumed that no consumption takes place at the beginning of the period of production so that all endowed resources are utilized as input. Finally the only securities in existence are the shares issued by each industry. They represent claims on the value of the future output of each commodity.

Let us introduce for each investor h his compensated demands for securities and future consumption. They are determined by computing his minimum present budget sufficient to ensure a required level of satisfaction derived from future consumption, equal to λ :

$$J^h(\phi, p, \lambda^h, x) = \underset{\gamma^h \, c^h}{\text{Min}} \left[\phi\gamma^h : \textstyle\sum_j \gamma_j^h \, p_{sj} \, f_{sj}(x_j) \geq p_s \, c_s^h \; ; \; U^h(c^h) = \lambda^h \right]$$

where :

ϕ = vector of current securities prices

γ^h = vector of fractional holdings of investor h in each industry or in each firm in each industry

c^h = investor h's vector of future consumption of each commodity in each state

c_s^h = the subvector pertaining to consumption in state s

$U^h(c^h)$ = some state-dependent utility function embodying the investor's attitude to risk and expectations regarding the probability of each state. This utility function is assumed to be continuous increasing twice continuously differentiable and concave.

The second constraint of this optimization problem ensures that the requisite level of utility is reached. This constraint applies ex ante, i.e. using the utility function objective as it is seen from the beginning of period. The first constraint is the budget requirement of the end of period ; the investment in shares must be large enough to produce returns in each state larger than the cost of the consumption basket sufficient to reach the target utility level. Since by assumption the market is incomplete, there are more states than securities and therefore some of these constraints will not be saturated ; as many will be saturated there are commodities.

The minimum budget function J, satisfies several remarkable properties :

(i) J is homogeneous of degree 0 in the price vector p_s of each state taken separately

(ii) J is linearly homogeneous, continuous and non decreasing in ϕ

(iii) J is continuous, homogeneous of degree -1 and non increasing in x, the vector of all inputs in all industries

(iv) For industry j taken separately, J is homogeneous of degree zero in (ϕ_j, x_j). Indeed if x_j is multiplied by λ and ϕ_j is also multiplied by λ, γ_j will be divided by λ and J will remain unchanged.

If, in addition, J is assumed to be differentiable with respect to x_j and ϕ_j, then property (iv) implies by EULER's theorem :

$$(16) \quad \boxed{\frac{\partial J}{\partial x_j} x_j = - \frac{\partial J}{\partial \phi_j} \phi_j} \quad \text{for all } j.$$

Furthermore, by SHEPHARD's lemma the optimality of the portfolio implies :

$$(17) \quad \boxed{\gamma_j = \frac{\partial J}{\partial \phi_j}} \quad \text{for all } j.$$

In order to determine the optimal input vector x, we examine investor h's beginning-of-period budget :

$$(18) \quad J^h - \phi \bar{\gamma}^h + \textstyle\sum_j \bar{\gamma}_j^h \, w \, x_j \leqslant w \, v^h$$

J^h is the investor's expenditure to purchase his new portfolio ; $\phi\bar{\gamma}^h$ represents the proceeds from his initial portfolio, $\bar{\gamma}^h$ being his vector of endowed shares ; wx_j is the cost of firm j's production plan to which investor h as an initial shareholder contributes an amount $\bar{\gamma}_j^h \, wx_j$; wv^h is the value of the initial endowment of factors of production. A natural objective for the firm is to minimize the left-hand side of (17) ; thereafter the minimum value can be confronted with the right-hand side to compute the level of utility λ^h attained by the investor.[15] In an economy with constant returns to scale, however, the scale of production will be determined by the amount of input resources socially available. At the level of the firm's calculus only the composition and not the scale of inputs can be determined. We therefore formulate the firms' optimization problem as follows :

$$H(\phi, p, \lambda^h, w) = \underset{\{x_j\}}{\text{Min}}\left[J^h - \phi\,\bar{\gamma}^h : \bar{\gamma}_j^h\, wx_j \leq 1,\ x_j \geq 0\right]$$

H could be called a "unit cost productive budget function". Clearly the decision x_j generally depends on the identity of h, i.e. of the particular investor the firm chooses to satisfy. So long as it does depend on the identity in this manner, the firm's decision will result from a political power struggle between investors, the outcome of which is not within the scope of the model. To eschew this difficulty one may examine the circumstances under which shareholders will be unanimous in supporting a production plan.

As was pointed out by LELAND (1973) and RADNER (1974), unanimity can be understood in either one of two ways. Ex ante unanimity describes a consensus situation between initial shareholders before any trading takes place in the capital market. Ex post unanimity is the kind which may prevail between the shareholders of record after the proposed production plan has been announced and trading has taken place in response.

The optimality conditions derived from the above optimization problem are :

$$\text{(18)}\qquad \mu_j^h \bar{\gamma}^h w = -\frac{dJ^h}{dx_j} + \sum_k \bar{\gamma}_k^h \frac{\partial \phi_k}{\partial x_j}$$

$$= -\frac{\partial J^h}{\partial x_j} - \sum_k \frac{\partial J^h}{\partial \phi_k}\frac{\partial \phi_k}{\partial x_j} + \sum_k \bar{\gamma}_k^h \frac{\partial \phi_k}{\partial x_j}$$

where the μj's are multipliers on the unit-cost constraint[16] and where account is taken of possible effects of firm j's decisions on its own market value and on that of other firms. It is assumed, however, that firm j's decision does not induce other firms to modify their own decisions. Substituting (16) into the above condition, we obtain :

$$\text{(19)}\qquad \mu_j^h \bar{\gamma}_j^h w = -\frac{\partial J^h}{\partial x_j} - \sum_k \gamma_k^h \frac{\partial \phi_h}{\partial x_j} + \sum_k \bar{\gamma}_k^h \frac{\partial \phi_k}{\partial x_j}\ .$$

In the ex post approach pioneered by EKERN and WILSON (1974), it is assumed that the decision to support or not to support the firm's plan is made after announcement (and proclaimed implementation) of the plan and after market trading. Hence the portfolios preceding the decisions were optimal :

$$\bar{\gamma}_k^h = \gamma_k^h \quad \text{for all } k$$

and the optimality condition reduces to :

$$(20) \qquad \boxed{\mu_j^h \bar{\gamma}_j^h w = - \frac{\partial J^h}{\partial x_j}}$$

which still does not imply unanimity. But suppose that the firm's production functions exhibited scalar uncertainty only :

$$f_{sj}(x_j) = b_{sj} \; f_j(x_j).$$

Then the optimality condition could simply be written :

$$\mu_j^h \; \bar{\gamma}_j^h \; w \;\; = - \frac{\partial J^h}{\partial f_j} \frac{\partial f_j}{\partial x_j} \; .$$

Since we are only trying to determine the composition of the inputs and not their scale, the latter being set by the constraints, it is clear that the scalar $- \frac{\partial J^h}{\partial f_j} / \mu_j^h \bar{\gamma}_j^h$ would play no role in the problem.[17] In this manner the influence of the particular investor h would be eliminated and unanimity would result.

Consider now the ex ante interpretation which was used by RADNER (1974), LELAND (1973), LE ROY (1976) and BARON (1978). Since in general $\gamma_k^h \neq \bar{\gamma}_k^h$, it is necessary in making his decision, that the investor forecast the effect of the proposed plan on the firm's value. The only information he possesses regarding the market is that pertaining to his own behavior, which is summarized in equations (16) and (17). Of these (16) can be used by the investor to obtain an estimate of the variations of the firm's share price :

$$\phi_j = - \frac{\partial J/\partial x_j}{\partial J/\partial \phi_j} \; x_j$$

$$\frac{\partial \phi_j}{\partial x_j} = - \frac{\partial J/\partial x_j}{\partial J/\partial \phi_j} - \frac{\partial \left(\frac{\partial J/\partial x_j}{\partial J/\partial \phi_j} \right)}{\partial x_j} \; x_j .$$

Suppose there are many firms in industry j and we consider changing the production plan of only one of them at a time. If the investor takes the prices of the other firms as given, then it must be so that :

$$(21) \qquad 0 = - \frac{\partial \left(\dfrac{\partial J/\partial x_j}{\partial J/\partial \phi_j} \right)}{\partial x_j} \; x_j ,$$

and therefore the security price variation, as anticipated by investor h, is :

$$(22) \qquad \frac{\partial \phi_j^h}{\partial x_j} = - \frac{\partial J^h/\partial x_j}{\partial J^h/\partial \phi_j} = - \frac{1}{\gamma_j^h} \frac{\partial J^h}{\partial x_j} .$$

Furthermore the investor also takes as given the prices of the firms outside industry j :

$$(23) \qquad 0 = \frac{\partial \phi_k}{\partial x_j} \quad \text{for } k \neq j .$$

Substituting (22) and (23) into (18), we obtain a new optimality condition :

$$\mu_j^h \bar{\gamma}_j^h w = - \frac{\partial J^h}{\partial x_j} + \frac{\partial J^h}{\partial x_j} \frac{\partial J^h/\partial x_j}{\partial J^h/\partial \phi_j} + \bar{\gamma}_j^h \frac{\partial \phi_j}{\partial x_j}$$

$$(24) \qquad \boxed{\mu_j^h w = \frac{\partial \phi_j^h}{\partial x_j} = - \frac{1}{\gamma_j^h} \frac{\partial J^h}{\partial x_j}}$$

Once again the scalar μ_j^h plays no role so long as it is uniformly positive. Hence the firm could just as well maximize its market value ϕ_j subject to $wx \leqslant 1$, and all investors would agree with this goal.

This conclusion does not imply, however, that they would unanimously support or reject any proposed change in the input decision. Indeed each investor may perceive or forecast differently the impact of the input change on the share price, as is clearly seen from equation (22) which depends on the identity of the investor. But note again that if the technology is characterized by scalar uncertainty, then :

$$\frac{\partial \phi_j}{\partial x_j} = - \frac{\partial J^h/\partial f_j}{\partial J^h/\partial \phi_j} \frac{\partial f_j}{\partial x_j}$$

and again the influence of the individual investor is confined to a scalar term so that unanimity will prevail.

In short, when a general risky technology is used, there is no reason to think that unanimity will prevail,[18] and the conflict between stockholders must somehow be resolved. Let us assume that this is done by means of side payments designed to compensate "minority" shareholders. This allows us to rewrite the input-choice problem as[19]:

$$H(\phi, p, \lambda, w) = \underset{\{x_j\}}{\text{Min}} \left[\sum_h J^h - \phi 1 : wx_j \leq 1,\ x_j > 0\ \forall_j \right]$$

where the constraints now refer to the total inputs made by all investors, and where use has been made of $\sum_h \bar{\gamma}^h = 1$. After proper compensation the input decision will be supported by all. Nevertheless it will remain a function of the utility level λ^h reached by each investor.

Here again decisions and compensations can be made <u>ex post</u> or <u>ex ante</u>. If they are made <u>ex post</u>, the optimality conditions will be analogous to equation (20) :

$$(25) \qquad \mu_j\, w = - \sum_h \frac{\partial J^h}{\partial x_j} .$$

If they are made <u>ex ante</u>, the conditions will be adapted from (24) :

$$(26) \qquad \mu_j\, w = \sum_h \bar{\gamma}_j^h \frac{\partial \phi_j^h}{\partial x_j} = - \sum_h \frac{\bar{\gamma}_j^h}{\gamma_j^h} \frac{\partial J^h}{\partial x_j} .$$

The functions μ_j and x_j of ϕ, p, λ and w would generally be different depending on the decision making process under consideration.[20]

Turning now to the problem of the optimal allocation of inputs <u>between</u> industries, we realize from equation (17) that optimality will be reached when :

$$\mu_j = 1 \qquad \forall_j \ .$$

This will be possible if no input reaches the corner solution of zero, i.e. if every commodity is produced. The unit value of μ_j implies in the <u>ex post</u> case, using (25) and (16) :

$$\text{(27)} \qquad wx_j(\phi, p, \lambda, w) = - \sum_h \frac{\partial J^h}{\partial x_j} x_j$$

$$= \sum_h \gamma_j^h \phi_j ;$$

$$= \phi_j$$

In the <u>ex ante</u> case we have from (26) :

$$\text{(28)} \qquad wx_j(\phi, p, \lambda, w) = - \sum_h \frac{\bar{\gamma}_j^h}{\gamma_j^h} \frac{\partial J^h}{\partial x_j} x_j$$

$$= \sum_h \bar{\gamma}_j^h \phi_j$$

$$= \phi_j$$

Note that in this latter case no use is made of the capital-market-equilibrium condition $\sum_h \gamma_j^h = 1$.

Equations (27) and (28) are potential foundations for a Factor Price Equalization theorem under uncertainty. Indeed in these sets of equations, there is one equation for each stock security and one factor price w for each factor. So long as there are at least as many stock securities traded internationally (and therefore as many commodities) as there are factors, we can expect FPE to prevail under proper mathematical conditions. We should be mindful, however, of the fact that the vectors p and $\lambda = \{\lambda^h\}$ appear in those equations. The presence of λ creates no difficulty as the utility levels reached by investors worldwide in a unified capital market would be identical whether we apply equations (27) and (28) to the inputs of one country or the other. The presence of the p vector forces us to add one sufficient condition : it is not enough to let the share securities be traded, the output commodities themselves must be in order for the p vector to be identical worldwide.[21]

In the context of ex ante decision making, it is equivalent for firms to satisfy (26) with $\mu j = 1$ or to maximize the quantity $\sum_h \bar{\gamma}^h \phi_j - wx_j$, where we must recall, however, that the variations of ϕ_j are anticipated differently by each investor in accordance with equation (22). This objective function allows us very easily to transpose KUGA's theorem (see section 1) as follows :

Theorem. Under ex ante decision making and side payments, factor price equalization obtains under uncertainty if :

i) Stock securities are traded worldwide
ii) Output commodities are traded worldwide
iii) There are at least as many stock securities as there are factors
iv) KUGA's assumptions 1-3 (section 1) applied to the function :

$$- \sum_h \bar{\gamma}^h_j \frac{\partial J^h / \partial x_j}{\partial J^h / \partial \phi_j} x_j$$

are satisfied for all j.

v) All commodities are produced in both countries.

vi) The Jacobian determinant of the cost functions wx in (28) with respect to w never vanishes along a path connecting the two countries (keeping ϕ, p and λ constant).

When the *ex post* decision making process is used, transposing KUGA's theorem is more problematic. This is because no single objective function has so far been identified which can be used by each firm on a decentralized basis, to reach the decision implied by optimality condition (25) with $\mu j = 1$. As a result, it is not clear which function, if any, should satisfy KUGA's assumptions 1-3. We might observe that :

- $\sum_h \frac{\partial J^h}{\partial x_j} x_j$ would serve the purpose *provided* that the firms in so doing constrain the variations of $\partial J/\partial x_j$ as follows :

$$0 = - \frac{\partial(\partial J/\partial x_j)}{\partial x_j} x_j.$$

This is tantamount to assuming that an investor's budget allocation $\gamma_j^h \phi_j$ to industry j is not affected by the decisions of a single firm j. It is not clear on what grounds such a constraint could be introduced.[22]

CONCLUSION

The Factor Price Equalization theorem has been generalized to uncertainty. Some of the sufficient conditions proposed here were analogous to those of the certainty case. Others were new ; in particular we assumed that :

- each industry issued shares securities which were traded internationally,
- when the capital market contained only those securities and was incomplete, side payments were allowed between shareholders at the time the input decisions were made,
- an *ex ante* decision making process was used ; i.e. input decisions and side payments took place *before* trading in shares in the capital market.

We were not able to find an FPE statement applicable to *ex post* decision making.

FOOTNOTES

1. Under a so-called "spanning condition". See LELAND (1974), EKERN and WILSON (1974 and RADNER (1974). See also below the case of scalar uncertainty.

2. If the capital market were incomplete, then different investors would discount future prices using different personalized rates of institutions between consumption now and consumption in state s. In that case, firms' decisions would not be predicated, as they will be in what follows, on uniquely defined securities prices but on the criterion that all investors should approve unanismously a marginal change in the allocation of inputs. These points will be developed in section 2.

3.

4. They would not, only if the numbers of securities were at least equal to the number of states times the number of commodities. We do not require here such capital market.

5. KHANG (1971), DIEWERT (1973, 1975), GORMAN (1968), JORGENSON and LAU (1974).

6. y_j is the vector of state-dependent outputs of a given commodity, produced therefore by a given industry. Similarly p_j is the vector of state-dependent prices of commodity j.

7. As quoted by WOODLAND (1977).

8. SHEPHARD's lemma is actually very similar to an older lemma of HOTTELING (1932). See DIEWERT (1973).

9. If that is so, then a lemma due to WOODLAND (1977), which is dual to SHEPHARD's lemma applied to the unit value added constraint, states that :

$$y_j = \frac{\partial C_j}{\partial p_j}$$

provided C_j is continuously differentiable with respect p_j. Defining :

$$y_j = \frac{\partial C_j}{\partial p_j} \equiv b_j(p_j, w),$$

we have also :

$$C_j = b_j(p_j, w)p_j$$

10. $|A|$ is the Jocabian determinant of the unit value added cost function with respect to the input-price vector.

11. One may want to look first for conditions under which a non negative solution vector w exists. See KUHN (1959).

12. Another set of sufficient conditions was contributed by GALE and NIKAIDO (1965). It differs markedly from the other three.

13. If production functions (4) are strictly concave (decreasing marginal productivity) and twice continuously differentiable so will be the value added function. The properties of value added functions were noted above ; e.g. they are all concave and continuous in x. But assumptions 1-3 impose requirements not met by all value added functions, e.g. strict concavity and double differentiability with continuous derivatives.

14. By way of contrast, the RYBCZYNSKI and STOLPER-SAMUELSON theorems would have to be drastically modified to take account of the multiplicity of joint outputs. If, however, the industry transformation functions are separable, i.e. :

$$t_j(y_j\ ;\ -\ x) = -\ f_j(x_j) + h_j(y_j) \leqslant 0$$

a condition obviously satisfied in the case of pure scale uncertainty, as is demonstrated by equation (8), then one can interpret $f_j(x_j)$ as a quantity of value added, or a quantity of compound output, and one can define an output price index (see WOODLAND) :

$$v_j(p_j) = \max_{y_j} \{p_j\ y_j\ :\ h_j(y_j) \leqslant 1\}.$$

The two theorems then hold using the quantities of compound outputs and the output price indexes in lieu of the usual output quantities and output prices. It is in this manner that BARON and FORSYTHE (1977) and HELPMAN and RAZIN (1978) generalized them.

15. This procedure is intuitively appealing but its mathematical equivalence with value maximization has not been established.

16. The forthcoming discussion assumes that the constraint $x_j \geqslant 0$ is satisfied by the solution of (18). As will be seen below, in the ex ante interpretation, (18) becomes (24) and the requirement $x_j \geqslant 0$ is satisfied whenever the function $-\frac{\partial J/\partial x_j}{\partial J/\partial \phi_j}\, x_j$ satisfies KUGA's assumptions 1-3 for all j. It is not clear what the corresponding necessary condition would be, in the ex post case.

17. So long as it is uniformly of the same sign (negative). Note that with this kind of uncertainty, J would be homogeneous of degree 0 in (f_j, ϕ_j) and therefore :

$$\frac{\partial J^h}{\partial f_j} f_j = - \frac{\partial J^h}{\partial \phi_j} \phi_j = - \gamma_j^h \phi_j \ .$$

Hence the optimality condition with respect to inputs would be :

$$w = \frac{1}{\mu_j^h} \frac{\gamma_j^h}{\bar{\gamma}_j^h} \frac{\phi_j}{f_j} \frac{\partial f_j}{\partial x_j}$$

The condition on the sign of the scalar boils down to a requirement that all voting ex post shareholders have a positive holding ($\gamma_j^h > 0$), and saturated unit-cost constraints ($\mu_j^h > 0$), that their endowed holdings be positive ($\bar{\gamma}_j^h > 0$), that the firm not be bankrupt ($\phi_j > 0$) or idle ($f_j > 0$).

18. We have only examined here conditions on the technology. EKERN (1975) and BARON (1978) have indicated that unanimity (ex post and ex ante respectively) still prevails with a general technology provided investors follow a mean-variance utility-of-wealth function. Factor Price Equalization would thus follow from such an assumption. NIELSEN (1975) has proven a more general result : ex post unanimity prevails whenever the market, although incomplete, permits a Pareto optimal allocation of risk bearing for given production plans. This is guaranteed when there exists a riskless asset and whenever investors' utility-of-consumption functions exhibit hyperbolic absolute risk aversion (HARA) (see RUBINSTEIN (1974) with identical cautiousness parameters, or whenever future returns satisfy some distributional characteristics indicated by ROSS (1976), a special case of which is that they be gaussian. The quadratic utility is a special case of HARA ; it is well known that it leads to a mean-variance utility-of-wealth function. So do gaussian distributions.

19. D.P. BARON pointed out that the negociation process needed to determine the requisite side payments may not converge and if it does, the result may not simply minimize the investors' summed willingnesses to pay. Indeed there will be an incentive for investors to misrepresent their preferences in order to obtain larger side compensations. The side-payment mechanism has been interpreted by some authors as a form of take-over bid.

20. Although several authors have made a statement to that effect, to our knowledge, no one has proven that the ex ante and ex post processes lead to the same equilibrium end result.

21. The trading of shares may presuppose the future trading of commodities. That issue is not perfectly settled. Of course the converse is not true. Hence, even if the commodities are assumed to be traded, we cannot dispose of the sufficiency condition that the world capital market be unified.

22. FAMA and LAFFER's "Reaction Principle" could be used to that effect. For a critique of this Principle, see BARON (1978).

REFERENCES

P.K. BARDHAN, "Uncertainty, Resource Allocation and Factor Shares in a Two-sector Model", MIT working paper, 1971.

D.P. BARON, "Investment Policy, Optimality and the Mean-Variance Methodology", Working paper, Katholieke Universiteit Leuven, 1978.

D.P. BARON and R. FORSYTHE, "Models of the Firm and International Trade Under Uncertainty", working paper, Northwestern University, 1977.

R.N. BATRA, "Production Uncertainty and the Heckscher-Ohlin Theorem", Review Economic Studies, April 1975, 42, 259-268.

R.N. BATRA and W.R. RUSSELL, "Gains from Trade Under Uncertainty", American Economic Review, Dec. 1974, 64, 1040-1048.

W.C. BRAINARD and R.N. COOPER, "Uncertainty and Diversification of International Trade", Food Research Institute Studies in Agricultural Economics, Trade and Development, 1968, 8, 257-85.

S.K. DAS, "Uncertainty and the Heckscher-Ohlin Theorem : A Comment", Review Economic Studies, Feb. 1977, XLIV (1), 136, 189-190.

P. DIAMOND, "The Role of a Stock Market in a General Equilibrium Model with Technological Uncertainty", American Economic Review, Sept. 1967, 57, 759-776.

W.E. DIEWERT, "Functional Forms for Profit and Transformation Functions", Journal of Economic Theory, June 1973, VI, 284-316.

W.E. DIEWERT, "Hicks' Aggregation Theorem and the Existence of a Real Value Added Function", October 1975, to appear in M. Fuss and D.Mc Fadden (Editors), An Econometric Approach to Production Theory.

S. EKERN, "On the Theory of the Firm in an Economy with Incomplete Markets : an Addendum," Bell Journal of Economics, Spring 1975, 6, 388-393.

E. FAMA and A.B. LAFFER, "The Number of Firms and Competition", American Economic Review, 47 (September 1972), 670-674.

S. EKERN and R. WILSON, "On the Theory of the Firm in an Economy with Incomplete Markets", Bell Journal of Economics, Spring 1974, 5, 171-180.

D. GALE and H. NIKAIDO, "The Jacobian Matrix and Global Univalence of Mappings", Mathematische Annalen, 1965, 49, 81-93.

W.M. GORMAN, "Measuring the Quantities of Fixed Factors", in Value, Capital and Growth, Papers in Honour of Sir John Hicks, edited by J.N. Wolfe, Edinburg University Press, 1968.

E. HELPMAN and A. RAZIN, "International Trade Under Uncertainty : a Select Survey of Recent Developments", working paper, Hebrew University, June 1977.

E. HELPMAN and A. RAZIN, "Uncertainty and International Trade in the Presence of Stock Markets", Review Economic Studies, February 1978.

M. HIROTA and K. KUGA, "On an Intrinsic Joint Production", International Economic Review, 1971, 12, 87-98.

H. HOTELLING, "Edgeworth's Taxation Paradox and the Nature of Demand and Supply Functions", Journal Political Economy, 1932, 40, 577-616.

M.C. KEMP and N. LIVIATAN, "Production and Trade Patterns Under Uncertainty", Economic Record, June 1973, 49, 215-27.

K. KUGA, "The Factor-Price Equalization Theorem", Econometrica, July 1972, 40, 723-736.

H.W. KUHN, "Factor Endowments and Factor Prices : Mathematical Appendix (to the A.H. LAND article)", Economica, May 1969, XXVI.

A.H. LAND, "Factor Endowments and Factor Prices", Economica, May 1959, XXVI, 137-142.

S.F. LE ROY, "Stock Market Optimality : Comment", Quarterly Journal of Economics, February 1976, 90, 150-155.

H.E. LELAND, "Capital Asset Markets, Production and Optimality : A Synthesis", Technical Report n° 115, Institute for Mathematical Studies in the Social Sciences, Stanford University, December 1973.

H.E. LELAND, "Production Theory and the Stock Market", Bell Journal of Economics, Spring 74, 5, n° 1, 125-144.

W. MAYER, "The Rybczynski, Stopler-Samuelson and Factor Price Equalization Theorems under Price Uncertainty", American Economic Review, Dec. 1976, 66, 797-8

L.W. McKENZIE, "Equality of Factor Prices in World Trade", Econometrica, July 1955, XXIII, 239-257.

N.C. NIELSEN, "The Firm as an Intermediary Between Consumers and Production Functions under Uncertainty", Ph.D. dissertation, Stanford University 1974.

R. RADNER, "A Note on Unanimity of Stockholders' Preferences among Alternative Production Plans : a Reformulation of the Ekern-Wilson Model", Bell Journal of Economics, Spring 1974, 5, n° 1, 181-186.

S.A. ROSS, "Mutual Fund Separation in Financial Theory -- The Separating Distributions", University of Pennsylvania, Working paper 1976.

M. RUBINSTEIN, "An Aggregation Theorem for Securities Markets", Journal Financial Economic, Sept. 1974, 1, 225-44.

R.J. RUFFIN, "International Trade Under Uncertainty", Journal of International Economy, August 1974, 4, 243-60.

R.J. RUFFIN, "Comparative Advantage under Uncertainty", Journal of International Economy, August 1974, 4, 261-274.

P.A. SAMUELSON, "Prices of Factors and Goods in General Equilibrium", Review of Economic Studies, 1953-54, XXI, 1-20.

P.A. SAMUELSON, "The Fundamental Singularity Theorem for Non-Joint Production", International Economic Review, 1966, 7, 34-41.

P.A. SAMUELSON, "Summary on Factor Price Equalization", International Economic Review, 1967, 8, 286-295.

R.W. SHEPHARD, "Cost and Production Function", Princeton : Princeton University Press, 1953.

S.J. TURNOVSKY, "Technological and Price Uncertainty in a Ricardian Model of International Trade", Review Economic Studies, April 1974, 41, 201-17.

A.D. WOODLAND, "Joint Outputs Intermediate Inputs and International Trade Theory", International Economic Review, October 1977, 18, 517-533.