WHEN IS CORPORATE FINANCIAL POLICY RELEVANT?

by

Simon Benninga*

Working Paper No. 11-78

Revised, August, 1978

*The Wharton School,

University of Pennsylvania

University of Tel-Aviv

RODNEY L. WHITE CENTER
FOR FINANCIAL RESEARCH

University of Pennsylvania

The Wharton School

Philadelphia, Pa. 19104

WHEN IS CORPORATE FINANCIAL POLICY RELEVANT?

In this paper we construct a multiperiod general equilibrium model with uncertainty. The purpose of the model is to test certain propositions about the relevance of corporate financial policy which have hitherto been established only under the assumption of no firm bankruptcy or in a one-period framework (usually both). Prominent papers on this topic are the original work of Modigliani and Miller (1958), papers by Stiglitz (1969,1974), Fama and Miller (1972), and Baron (1976). The two general statements of the Modigliani and Miller irrelevance proposition are those of Stiglitz: He showed that--assuming no bankruptcy of firms--a financial equilibrium is not unique. Stiglitz and a number of other authors have alos observed that in a complete market framework, firm valuation may be written without describing firm financial policy, but thus far no one has proven a nonuniqueness theorem for financial equilibria with firm bankruptcy.

Starting with the assumption that the market explicitly prices consumption for all time-state pairs, we establish a number of results about corporate financial policy. First, the issuance of new corporate equity is irrelevant to a model which describes corporate financial policy (Proposition 2). Second, an equilibrium in a corporate financial framework is not unique, provided that changes made in financial policy do not change the feasible consumption set of consumers. This applies to both complete and incomplete market models (Theorem 7 and Section 8); however, in contrast to "irrelevance" theorems with no firm bankruptcy, a change in firm financial policies may require shareholders and bondholders to change their whole portfolios. The no-bankruptcy models, on the other hand, show that small changes in firm financial policies require changes only in consumer holdings of the firm's shares and in holdings of the riskless asset. Part of the power of the MM results derives from these limited changes: If we regard perfect market models as being an approximation to markets with imperfections such as transactions costs, then the true "irrelevance" of firm financial policies derives from the fact that (given non-bankruptcy of borrowers) firms can change them so easily, with only minimal compensating changes by consumers. In Theorem 8 we show that this result only rarely extends to the bankruptcy case; there we characterize the changes in financial policy which lead shareholders and bondholders to change at most their holdings in the firm's shares, bonds, and the riskless security. This theorem, too, holds in both complete and incomplete markets (Section 8).

Thus most theorems about financial policy hold in both complete and incomplete markets. The only exception--discussed in Section 8--is when a change in financial policy produces a different feasible set of consumption opportunities. In this case equilibrium will, in general, be unique, even in the sense of Theorem 7.

The structure of the paper is as follows: In Sections 1,2, and 3 we discuss various aspects of the model, including production, bankruptcy, and the consumer choice problem. In Section 4 we show that, given complete market pricing and no restrictions in short sales, all consumers wish every firm to maximize its net present value. In Section 5 we define equilibrium, and in the next section we show what changes need to be made in the standard equilibrium proofs in order account for the special feastures of our model. Section 7 discusses the relevance theorems for financial equilibria, and in Section 8 we conclude by showing which of our results hold with incomplete markets.

1. <u>The Time-State Structure, Prices, Commodities, Bonds, and Shares</u>.

Many of the features of the model are adapted from Radner (1972). There are H commodities, I consumers, and J firms. There are T periods, $t = 1,\ldots,T$. At each t, there is a partition of events S_t, with the partition S_{t+1} being at least as fine as the partition S_t. The combination (t,A) of a date t and an event $A \varepsilon S_t$ is called a <u>date-event pair</u> and will be denoted $m = (t,A)$. Given $m = (t,A)$, $A \varepsilon S_t$, we shall say of another date-event pair $n = (u,B)$ that $n > m$ if $u > t$ and A B. The set M is the set of all date-event pairs.

Given $m = (t,A)$, we define the set of all <u>successor</u> time-state pairs of m by $m^+ = (t+1,B)$, where B A. The <u>predecessor</u> of $m = (t,A)$ will be denoted by $m^- = (t-1,B)$, where B A. Because of the structure of the model, each m may have several successors, but only one predecessor.

At each $m \varepsilon M$ there is production and trade of the H commodities. The price of commodity h at $m \varepsilon M$ will be denoted by p_m^{ch}, and a vector of commodity prices for each $m \varepsilon M$ will be denoted by $p^c = (p_m^c)$. Prices are assumed to be non-negative. At each $m \varepsilon M$, consumers may buy bonds, whose income accrues to the producers who sell them. A bond b bought by a consumer i at $m = (t,A)$ reduces his budget constraint by $p_m^b b$, $0 \leq p_m^b \leq 1$ being the price of the bond, and raises the constraints by b at $m^+ = (t+1,B)$, B A, if at B the bond's issuer does not go bankrupt. Consumers do not go bankrupt; what happens when firms go bankrupt will be described in Section 2. A vector of bond prices at $m \varepsilon M$ is denoted by $p_m^b = (p_{om}^b, p_{1m}^b, \ldots, p_{Jm}^b)$, where the subscript $j = 1,\ldots,J$, indicates the bond price of producer j, and the subscript o indicates the price of loans between consumers.

2. Firms, production and bankruptcy

There are J firms. Each firm j, $j = 1,\ldots,J$, possesses a (stochastic) production technology, $\phi_j = (\phi_{jm})$, where $\phi_{jm}\colon R^H_+ \to R^H_+$ is strictly concave, continuous, and increasing production function such that $\phi_{jm}(0) = 0$.

Part of the uncertainty on the firm's part derives from the need to purchase inputs for production one period prior to their use. At the predecessor m^- of each time-state pair m, each firm j must decide on inputs z_{jm-}; these inputs will then be stored and used one period later to produce outputs. Thus production at m is described by

$$(1) \qquad y_{jm} = \phi_{jm}(z_{jm-})$$

Net firm income from production at each $m \in M$ is thus given by

$$(2) \qquad p^c_m(y_{jm} - z_{jm}).$$

(Note that in the last period no inputs are purchased, so that for $m=(T,A)$, $z_{jm} = 0$).

In addition, consumers may buy shares of each firm j, $j=1,\ldots,J$. Given $m \in M$, the price of firm j on the stock market is p^s_{jm}. A vector of share prices at $m \in M$ will be indicated by $p^s_m = (p^s_{1m},\ldots,p^s_{Jm})$. The bulk of the paper will be concerned with the case of "complete markets": We shall assume that in every time-state pair m returns of financial assets in successors n of m are priced according to a set of prices (p_n). The exact way in which the complete mrket prices determine the values of debt and equity will be specified in section 2. We shall assume that the complete market prices are given by a "market maker" (the "invisible hand").

At m = 0 (the present), the firm produces with a vector of inputs $\bar{z}_j >> 0$. It is convenient to asssume (and leads to no loss in generality) that firms start at m = 0 with shareholders but with no bondholders. Thus $y_{jo} = \phi_{jo}(\bar{z}_j)$ and z_{jo} denote inputs purchased in the present and used to produce in the next period.[1]

Financing: In order to allow firm j to finance the purchase of inputs, we shall allow the firm to sell new equity and one-period bonds at each time-state pair m. The bonds may be risky; i.e., we shall allow the possibility of firm bankruptcy.

The set of complete prices for financial assets allows us to determine how the prices of equity and debt (and, indeed, of the firm) are determined. To illustrate, we first consider any time-state pair m in the next-to-last period; i.e., m = (T-1,A). Let $n=m^+$ be any successor of m. Then if firm j issued b_{jm} of bonds at m, the payoff to these bonds at n is given by

$$(3) \quad r^b_{jn} = \begin{cases} b_{jm} & \text{if } p^c_n y_{jn} - b_{jm} \geq 0 \\ p^c_n y_{jn} & \text{otherwise} \end{cases}$$

r^b_{jn} is the total return at n to consumers who purchased bonds in j at m. Note that the top line of r^b_{jn} occurs if firm j is not bankrupt at n, and that the bottom line indicates firm bankruptcy. Expression (3) is relatively simple, since in the last period, firms liquidate and therefore issue no new debt or equity, or indeed purchase no inputs.

The payoff to shareholders in time-state pair n is given by

$$(4) \quad r^s_{jn} = \begin{cases} p^c_n y_{jn} - b_{jm} & \text{if } p^c_n y_{jn} \geq b_{jn} \\ 0 & \text{otherwise} \end{cases}$$

As before, the top line represents the no-bankruptcy case and the bottom line that of bankruptcy.

Since we have assumed complete market prices, we may determine the value of the equity of j at m by

$$(5) \qquad p^s_{jm} = \sum_m p_n r^s_{jn}, \qquad n = (T,A)$$

Correspondingly, the value of the firm's debt issued at m is

$$(6) \qquad \sum_m p_n r^b_{jn} \qquad n = (T,A)$$

It is convenient to denote the latter expression by

$$(7) \qquad p^b_{jm} b_{jm} = \sum_m p_n r^b_{jn}$$

The firm's issuance of b_{jm} of bonds at m may be viewed as the promise (except for the risk of bankruptcy) to repay the holders of the debt b_{jm} in any successor time-state pair of m. p^b_{jm} is then the market discount factor for this promise. If the debt issued by firm j is riskless (i.e., $r^b_{jn} = b_{jm}$ for every $n = m^+$), p^b_{jm} is "one over one-plus-the-riskless-interest-rate."

Now consider some time-state pair $\ell = (T-2,A)$, and let $m = \ell^+$. If firm j issues bonds $b_{j\ell}$, then for any $m = \ell^+$ funds available to repay the bonds come from three sources:

a. net production income: $p^c_m(y_{jm} - z_{jm})$

b. bonds sold at m: $p^b_{jm} b_{jm}$

c. new equity sold by the firm at m: we shall denote this by $\alpha_{jm} p^s_{jm}$, where $0 \leq \alpha_{jm} < 1$, in order to make the point that new equity sold by the firm is just another way of the firm selling some of its equity at m instead of shareholders who bought shares at ℓ.[2]

To determine the return to bonds sold in ℓ at any $m = \ell^+$, we now differentiate among three cases

case 1: $p^c_m(y_{jm} - z_{jm}) + p^b_{jm}b_{jm} + \alpha_{jm}p^s_{jm} \geq b_{j\ell}$

In this case, the firm has sufficient funds to repay bondholders, and there is no bankruptcy.

case 2: Suppose the firm does not have sufficient funds to repay bondholders (i.e., case 1 doesn't hold), but there exists some fraction δ_{jm} of the remaining equity so that $0 \leq \delta_{jm} \leq 1$

(8) $$p^c_m(y_{jm} - z_{jm}) + p^b_{jm}b_{jm} + \alpha_{jm}p^s_{jm} + \delta_{jm}(1-\alpha_{jm})p^s_{jm} = b_{j\ell}$$

That is, although the firm is formally bankrupt, shareholders who bought shares at ℓ can, by transferring some of their equity to bondholders, assure bondholders full repayment of their bonds.

case 3: Neither of the two cases above hold. The firm is unable to repay its obligations to the bondholders, and all the equity in the firm passes to them.

We have thus defined the return to bondholders in every successor state of ℓ. To summarize:

(9) $$r^b_{jm} = \begin{cases} b_{j\ell} & \text{case 1} \\ b_{j\ell} & \text{case 2} \\ p^c_m(y_{jm}-z_{jm}) + p^b_{jm}b_{jm} + p^s_{jm} & \text{case 3} \end{cases}$$

The three cases outlined above serve also to define the total return of equity holders who purchased shares in firm j at ℓ. Any individual who purchased equity in j at gets part of his return as a payout from the firm (dividends) and part from selling his equity (if, after bankruptcy and after the firm has sold α_{jm} there is any equity left). Adding the dividend and sale of remining equity, we get the total return of equity holders at m:

$$(10)\quad r^s_{jm} = \begin{cases} p^c_m(y_{jm}-z_{jm}) + p^b_{jm}b_{jm} + p^s_{jm} - b_{j\ell} & \text{case 1} \\ (1-\delta_{jm})(1-\alpha_{jm})p^s_{jm} & \text{case 2} \\ 0 & \text{case 3} \end{cases}$$

It is easy to simplify further case 2 of r^s_{jm}: Note that

$$(1-\delta_{jm})(1-\alpha_{jm})p^s_{jm} + p^c_m(y_{jm}-z_{jm}) + p^b_{jm}b_{jm} +$$
$$\alpha_{jm}p^s_{jm} + \delta_{jm}(1-\alpha_{jm})p^s_{jm} =$$
$$p^s_{jm} + p^b_{jm}b_{jm} + p^c_m(y_{jm}-z_{jm}).$$

But then by (8) it follows that

$$(1-\delta_{jm})(1-\alpha_{jm})p^s_{jm} = p^s_{jm} + p^b_{jm}b_{jm} + p^c_m(y_{jm}-z_{jm}) - b_{j\ell}$$

We may now rewrite (10) as

$$(11)\quad r^s_{jm} = \begin{cases} p^c_m(y_{jm}-z_{jm}) + p^b_{jm}b_{jm} + p^s_{jm} - b_{j\ell} & \text{cases 1 and 2} \\ 0 & \text{case 3} \end{cases}$$

Note that since there is no initial debt, the payout to initial shareholders (that is, individuals who come into m=0 owning shares of the firm) is given by

$$(12)\quad r^s_{jo} = p^c_o(y_{jo} - z_{jo}) + p^b_{jo}b_{jo} + p^s_{jo}$$

By our assumption of complete market pricing, expressions (9) and (11) serve to establish the value of firm j's bonds and equity at ℓ. Let p_m be the set of prices for financial returns in successor time-state pairs of ℓ. Then

$$(13)\qquad p^s_j = \sum_m p_m r^s_{jm}$$

$$(14)\qquad p^b_{jm}b_{jm} = \sum_m p_m r^b_{jm}$$

It is now obvious that we may proceed recursively, thus establishing the value of debt and equity for each firm in every time-state pair.

The following two facts are immediate:

Proposition 1: $r^b_{jm} + r^s_{jm} = p^c_m(y_{jm} - z_{jm}) + p^b_{jm}b_{jm} + p^s_{jm}$.

Proof:

This follows from adding the relevant lines of (9) and (11).

Proposition 2: No generality is lost by assuming that $\alpha_{jm} = 0$ for all m; i.e., we may assume that firm j never issues new equity.

Proof:

The proof of this is essentially contained in the restatement of (10) to give (11). Note that in neither (9) nor (11) does α_{jm} appear.

In light of Proposition 2, we shall assume that firms never issue new equity.[3] Thus at each $m \in M$, firm j has but two decisions to make: It must decide how many inputs z_{jm} to purchase, and it must decide how much debt b_{jm} to issue. We shall denote this firm decision by $k_j = (z_j, b_j)$.

3. Consumers

At each $m \in M$ each of I consumers must make three choices. Letting i represent a typical consumer, these choices are:

(i) $x_{im} = (x_{i1m}, x_{i2m}, \ldots, x_{iHm})$; a consumption vector, $x_{im} \geq 0$.

(ii) $b_{im} = (b_{iom}, b_{i1m}, \ldots, b_{iJm})$; a vector of bonds. We shall take b_{ijm}, $j=1,\ldots,J$, to represent the proportion of firm j's bonds purchased by consumer i. b_{iom} will represent loans to other consumers; if $b_{iom} < 0$, i is a net borrower, and if $b_{iom} > 0$, he is a net lender.

(iii) $f_{im} = (f_{i1m}, \ldots, f_{iJm})$; a stock portfolio. f_{ijm} represents the fraction of firm j's shares purchased by consumer i.

We shall assume that each consumer i starts off with an initial endowment of shares, $\bar{f}_i = (\bar{f}_{ij})$, where

$$\sum_i \bar{f}_{ij} = 1, \; j=1,\ldots,J.$$

In addition, we shall assume that for every $m \in M$, i receives a commodity endowment $\bar{\omega}_{im} >> 0$. Given prices p and firm plans k_j for each j, consumer i's budget constraints are given by:

$$(15) \quad p_m^c x_{im} + \sum_{j=1}^{J} b_{ijm} p_{jm}^b b_{jm} + \sum_{j=1}^{J} f_{ijm} p_{jm}^s + p_{om}^b b_{iom} \leq$$

$$p_m^c \bar{\omega}_{im} + \sum_{j=1}^{J} b_{ij\ell} r_{jm}^b + \sum_{j=1}^{J} f_{ij\ell} r_{jm}^s + b_{io\ell},$$

where $\ell = m-$

and for $m=0$, $b_{ij\ell} = 0$, $f_{ij\ell} = \bar{f}_{ij}$.

We shall assume that each consumer i maximizes a concave, increasing, and twice differentiable utility function U_i, where U_i is defined on consumption vectors x_i.

Note that we require that inter-consumer loans be riskless; a mechanism for consumer bankruptcy similar to that developed for firms could, however, easily be added to the model.[4] Finally, note that we allow f_{ijm} and b_{ijm} to be negative; there are no restrictions as to short sales.

4. The firm objective function

The purpose of this section is to show that with complete market pricing and no restrictions in short sales, all consumers prefer that firms maximize a single objective function, namely the net present value of the firm.

Given market prices p and consumer plans $e_i = (x_i, f_i, b_i)$, we first define implicit prices ρ_n^i by

$$(16) \qquad \rho_n^i = \frac{p_{mh}^c}{p_{nh'}^c} \frac{\partial U_i/\partial x_{inh'}}{\partial U_i/\partial x_{imh}}, \quad p_{mh}^c, p_{mh'}^c > 0,$$

where the partial derivatives are evaluated at x_i. (We shall, in section 6, normalize prices so that ρ_n^i is always well defined.) The following lemma shows that implicit prices play the role of state prices for each consumer.[5]

Lemma 3: If consumer i is not at a corner maximum given firm plans k_j and prices p, then

$$(17) \qquad p_{jm}^s = \sum_n \rho_n^i r_{jn}^s, \; n = m^+$$

$$(18) \qquad p_{jm}^b b_{jm} = \sum_n \rho_n^i r_{jn}^b, \; n=m^+$$

Proof:

Without loss in generality suppose that for every m, $p_m^{c1}>0$. Letting $\ell = m-$, we get

$$(19)\quad x_{i1m} = \frac{1}{p_m^{c1}} \{p_m^{c}\bar{\omega}_{im} + \sum_j b_{ij\ell}r_{jm}^b + \sum_j f_{ij\ell}r_{jm}^s + b_{oi\ell}$$

$$- \sum_j f_{ijm}p_{jm}^s - \sum_j b_{ijm}p_{jm}^b b_{jm} - p_{om}^b b_{iom}\}$$

Given prices p and firm plans k_j, the utility of consumer i is a function of

$$(f_{im},b_{im},x_{i2m},\ldots,x_{iHm}),\ m \in M$$

and since a maximum is obtained at x_{im}, we may thus write

$$U_i(x_i) = \bar{U}_i(f_{im},b_{im},x_{i2m},\ldots,x_{iHm}),\ m \in M.$$

Since at a maximum

$$\frac{\partial \bar{U}_i}{\partial f_{ijm}} = \frac{\partial \bar{U}_i}{\partial b_{ijm}} = 0,$$

the result follows.

QED

Lemma 4: A necessary and sufficient condition for firms to maximize consumer welfare with respect to their choices of inputs is that

$$(20)\qquad \sum_n p_n \frac{\partial (p_n^c y_{jn})}{\partial z_{jm}^h} = p_m^{ch}$$

for every j,h,m, and n=m+.

Proof: We wish to take the derivative of $\bar{U}_i$ with respect to z^h_{hm}. Assume that ℓ is the predecessor of m, and that n is any successor of m. First define A as the set of n where firm j is not bankrupt; let B be the set of bankrupt time-state pairs n. Utilizing (19),

$$\frac{\partial \bar{U}_i}{\partial z^h_{jm}} = \frac{\partial \bar{U}_i}{\partial x_{ilm}} \frac{\partial x_{ilm}}{\partial z^h_{jm}} + \sum_n \frac{\partial \bar{U}_i}{\partial x_{iln}} \frac{\partial x_{iln}}{\partial z^h_{jm}}$$

$$= \frac{\partial \bar{U}_i}{\partial x_{ilm}} \cdot \frac{1}{p^{cl}_m} \{f_{ij\ell} [- p^{ch}_m + \frac{\partial}{\partial z^h_{jm}} (p^s_{jm} + p^b_{jm} b_{jm})]\}$$

$$+ \quad \frac{\partial \bar{U}_i}{\partial x_{ilm}} \cdot \frac{1}{pcl_m} \{ - f_{ijm} (\sum_A p_n \frac{\partial (p^c_n y_{jn})}{\partial z^h_{jm}})$$

$$- b_{ijm} (\sum_B p_n \frac{\partial (p^c_n y_{jn})}{\partial z^h_{jm}})\}$$

$$+ \sum_A \frac{\partial \bar{U}_i}{\partial x_{iln}} \cdot \frac{1}{p^{cl}_n} \{ f_{ijm} \frac{\partial (p^c_n y_{jn})}{\partial z^h_{jm}}\}$$

$$+ \sum_B \frac{\partial \bar{U}_i}{\partial x_{iln}} \cdot \frac{1}{p^{cl}_n} \{b_{ijm} \frac{\partial (p^c_n y_{jn})}{\partial z^h_{jm}}\}$$

Setting this whole expression equal to zero and dividing through by $\frac{\partial \bar{U}_i}{\partial x_{ilm}} \cdot \frac{1}{p^{cl}_m}$ we see (using the results of Lemma 3) that all terms but the first term cancel, so that

$$(21) \qquad \frac{\partial}{\partial z^h_{jm}} (p^s_{jm} + p^b_{jm} b_{jm}) = p^{ch}_m$$

is a necessary condition for maximization of consumer welfare with respect to firm inputs. Since (Proposition 1)

$$p^s_{jm} + p^b_{jm} b_{jm} = \sum_n p_n (p^c_n (y_{jn} - z_{jn}) + p^b_{jn} b_{jn} + p^s_{jn})$$

and since y_{jn} is a concave function of z_{jm}, we see that it is also a sufficient condition.

Finally, note that (21) implies

$$\sum_n p_n \frac{\partial (p^c_n y_{jn})}{\partial z^h_{jm}} = p^{ch}_m$$

for every h, j, m.

Lemma 5: Consumers have no preferences over firm debt-equity choices, provided (17) and (18) hold for all i,j,m.

Proof:

By a method similar to that of Lemma 4, we may show that

$$\frac{\partial U_i}{\partial b_{jm}} = 0.$$

The details are in every respect similar to the method of proof used in Lemma 4, and we will not give them here.

Finally, note that by Proposition 1 the return to the initial shareholders of firm j is

$$(22)\quad r^s_{jo} = p^c_o(y_{jo} - z_{jo}) + p^b_{jo}b_{jo} + p^b_{jo}$$

$$= p^c_o(y_{jo} - z_{jo}) + \sum_m p_m\{p^c_m(y_{jm} - z_{jm}) + p^b_{jm}b_{jm} + p^s_{jm}\}$$

$$= p^c_o(y_{jo} - z_{jo}) + \sum_m p_m \{p^c_m(y_{jm} - z_{jm}) +$$

$$\sum_n p_n \{p^c_n(y_{jn} - z_{jn}) + p^b_{jn} + p^s_{jn}\}\}, \text{ etc.}$$

where $m = o^+$, $n = m^+$

That is, the value of the returns to initial shareholders is the discounted (by state prices) sum of all net production income in all time-state pairs. We call this sum the net present value of the firm.[6] The preceding lemmas prove:

Theorem 6: Given prices p and assuming no restrictions on short sales, all consumers wish each firm j to maximize its net present value.

Finally, note that even without complete market pricing we can define a meaningful concept of personalized net present value, but substituting implicit prices ρ^i_m for p_m in (22). If we are willing to regard $\partial\rho^i_n/\partial z^h_{jm} = 0$, then Lemma 4 will hold even in the absence of complete market pricing, and it then follows that all consumers wish firms to maximize their personalized net present value.[7] We return to this concept in section 8.

5. The definition of equilibrium

Prices $p^* = (p^{c*}_m, p^*_m)$ for every $m \in M$, firm plans $k^*_j = (z^*_j, b^*_j)$ for $j = 1,\ldots,J$, and consumer plans $e^*_i = (x^*_i, f^*_i, b^*_i)$ for $i = 1,\ldots,I$, will be said to consititute an equilibrium if the following conditions are fulfilled:

(E.1) k_j^* feasible for every j, and k_j^* maximizes firm net present value (22) given prices p^*.

(E.2) e_i^* feasible for every i (15), and e_i^* maximizes U_i over all feasible consumption plans.

(E.3) Demand and supply of goods are equal:

$$\sum_i x_{im}^* - \sum_j y_{jm}^* + \sum_j z_{jm}^* = \sum_i \bar{\omega}_{im}, \qquad m \varepsilon M$$

(E.4) Demand and supply for bonds are equal:

$$\sum_i b_{ijm}^* = 1, \; j=1,\ldots,J; \quad m \varepsilon M$$

$$\sum_i b_{iom}^* = 0, \qquad m \varepsilon M$$

(E.5) Demand and supply of shares are equal:

$$\sum_i f_{ijm}^* = 1 \qquad m \varepsilon M; \; j=1,\ldots,J.$$

6. The existence of equilibrium

In this section we shall not prove the existence of equilibrium. Instead, we shall outline the major elements of the proof; given a few changes to account for special features of the model, the existence of equilibrium follows readily from standard arguments (see, for example Debreu (1959) or Arrow and Hahn (1971)).

We start by establishing the form of the price sets: For each time-state pair $m \varepsilon M$, let commodity prices be drawn from the set

$$(23) \quad P_m^c = \{p_m^c = (p_m^{c1},\ldots,p_m^{cH}) \Big| \sum_h p_m^{ch} = 1, \; p_m^{ch} \geq 0\}$$

Given $m \varepsilon M$, let state prices for successors n of m come from the set

$$(24)\quad P_m^S = \{(p_n),\ n = m^+ ;\ \sum_n p_n \leq 1,\ p_n \geq 0\}$$

Note that P_m^S is not in the usual form for price sets; we do not require that state prices for successor states of m sum to one. The reason for this is that the state prices do not merely fix relative prices for future consumption in each of the states, but that they also determine the intra-period rate of interest. To see this, note that it follows from (14) that the one-period risk-free interest rate at m is $1/\sum_n p_n - 1$.

The normalization for state prices assures that this interest rate can vary; if we were to normalize state prices by summing them to one, the result would be a risk-free rate of zero.

The next step is to bound consumption, production and input vectors. We do this by noting that total production in the economy must be less than

$$(25)\quad \sum_{jm}\sum \phi_{jm}(\sum_{im}\sum \bar{\omega}_{im} + \bar{z}_j) = C = (C^1,\ldots,C^H)$$

Consequently, we may assume that for all m and all i and j,

$$(26)\quad 0 \leq x_{im},\ z_{jm},\ y_{jm} \leq C.$$

Note that C places an effective bound on "reasonable" borrowing in the economy. Indicate by D the maximum value of C,

$$D \leq C^1 + \ldots + C^H$$

Then no firm j can promise to repay more than D in the last period, since otherwise it is surely bankrupt and by (14) the value of its debt not change. That is, for $m = (T-1,A)$,

$$0 \leq b_{jm} \leq D$$

Going back one period, we see that in the predecessor ℓ of m, firm j cannot reasonably borrow more than 2D, since otherwise it will surely be bankrupt at m. This establishes a bound for firm borrowing at every $m \in M$. Since we are restricting consumers to be riskless borrowers, a similar argument will apply to them.

In order to bound security purchases (or short sales) of consumers, note that

$$(27) \quad -D \leq f_{ij\ell} r^{s}_{jm} \leq D$$

where m is any time-state pair at the last period, and ℓ is its predecessor. Thus, effective payments by firms to consumers (in the case of "long" purchases at ℓ) or payments by consumers in lieu of firms (for "short" sales at ℓ) are bounded at m. The budget set of consumers may thus be bounded, as well as the feasible set of firms, and we may not use a standard argument to establish the existence of equilibrium.[8]

7. The relevance of corporate financial policy

The voluminous literature relating to the Modigliani and Miller (1959) irrelevance proposition has largely dealt with the case where corporate bankruptcy was specifically excluded. Where bankruptcy has been included, writers have generally limited themselves to observing that the value of the firm with complete market pricing is independent of the firm's debt equity ratio; this is the case for Stiglitz (1969) and Fama and Miller (1972). Baron (1976) extended this to the observation that even without complete market pricing, the lack of corner maximization would suffice to make individuals view corporate value as independent of the debt-equity ratio. All of these observations were made in a one-period framework; equation (22)--which follows from Proposition 1--is a natural multi-period extension of these results; as noted, (22) is valid even when there is not complete market pricing, if we substitute implicit prices.

In this section we establish two theorems relating to the "relevance" (or "irrelevance") of corporate financial policy. The first theorem shows that in a multi-period world, a financial equilibrium may be non-unique. The non-uniqueness holds even if firms are risky borrowers; however, it may require individuals to readjust all of their portfolio holdings. This first theorem extends results of Stiglitz (1969, 1974) to the case where bankruptcy is possible.

While Theorem 7 shows that financial equilibria are not, in general, unique, it is a less powerful theorem than it appears. Part of the power of the original MM results for the case where no bankruptcy is possible is their simplicity: Any firm may change its debt-equity ratio by asking its shareholders to make only two changes in their portfolios--namely in their holdings of the firm's shares and in their holding of the riskless security. It is natural to ask whether this result may be extended to the bankruptcy case. Given that a firm makes some change in its debt-equity ratio, can we give conditions which guarantee that its shareholders and bondholders have to make only minimal changes in their portfolios? Theorem 8 characterizes such changes in debt-equity ratios; it is evident that--except in a limited number of cases--debt-equity is very "relevant" from this point of view: Only rarely will debt-equity changes of one firm be accompanied by only minimal changes in individual portfolios. If we consider models of the kind considered here as approximations to a real world where, for example, transactions costs are widespread, then Theorem 8 assumes even greater importance.

<u>Theorem 7</u>: Let (p^*, k_j^*, e_i^*) be an equilibrium, and suppose that the return vectors $\{r_{jn}^{s*}, r_{jn}^{b*}\}$ span the space of the successors of m for every $m\varepsilon M$. Then the debt-equity ratios of the firms are not unique in the following sense: If $k_j = (z_j^*, \hat{b}_j)$, where $\hat{b}_j$ is sufficiently close to b_j^*, then there exists another equilibrium $(\hat{p}, \hat{k}_j, \hat{x}_i)$ having the following properties:

(i) $p^{c*} = \hat{p}^c$; $p_m^* = \hat{p}_m$; that is, commodity and state prices are unchanged.

(ii) $\hat{x}_i = x_i^*$; that is, consumers' consumption is unchanged.

<u>Proof</u>:

Without loss in generality we may assume that the return vectors $\{r_{jn}^{s*}, r_{jn}^{b*}\}$ for $n = m^+$ not only span the successors of m for all $m\varepsilon M$, but that these vectors are also independent. A small change in the b_j^* will not affect this independence. To prove the theorem, consider first consumption in the last period; letting $n = (T,A)$, we have

$$p_n^{c*} x_{in} = \sum_j f_{ijm}^* r_{jn}^{s*} + \sum_j b_{ijm}^* r_{jn}^{b*} + p_n^{c*} \bar{\omega}_{im}$$

where m is the predecessor of n and equality is obtained from the fact that U_i is increasing. Denoting the new return vectors by $\{\hat{r}^s_j, \hat{r}^b_j\}$, we may find--since the new return vectors are also independent--a portfolio $(\hat{f}_{im}, \hat{b}_{im})$ at m such that

$$p_n^{c*} x_{in}^{*} = \sum_j \hat{f}_{ijm} \hat{r}_{jn}^{s} + \sum_j \hat{b}_{ijm} \hat{r}_{jn}^{b} + p_n^{c*} \bar{\omega}_{im}$$

Next note that the new portfolio costs the same as the old portfolio, since

$$\sum_n p_n^{*} (\sum \hat{f}_{ijm} \hat{r}_{jn}^{s} + \sum_j \hat{b}_{ijm} \hat{b}_{jn},$$

$$= \sum p_n^{c*} (x_{in}^{*} - \bar{\omega}_{in})$$

and

$$\sum_n p_n^{*} (\sum_j f_{ijm}^{*} r_{jn}^{s*} + \sum_j b_{ijm}^{*} r_{jn}^{b*}) = \sum_n p_n^{c*} (x_{in}^{*} - \bar{\omega}_{in})$$

Finally, we show that the change in debt-equity ratios has not affected total demand for securities. Let $W_{jm}^{s} = \sum_i f_{ijm}$ and $W_{jm}^{b} = \sum_i b_{ijm}$ be total security demand at m (stars or hats will, of course, denote the demands appropriate to the various debt-equity structures). We wish to show that $\hat{W}_{jm}^{s} = \hat{W}_{jm}^{b} = 1$. First note that

$$(28) \quad \sum_j r_{jn}^{s} \hat{W}_{jm}^{s} + \sum_j r_{jn}^{b} \hat{W}_{jm}^{b} = \sum_i p_n^{c} (x_{in}^{*} - \bar{\omega}_{in})$$

$$\sum_j r_{jn}^{s*} W_{jm}^{s*} + \sum_j r_{jn}^{b*} W_{jm}^{b*} = \sum_i p_n^{c} (x_{in}^{*} - \bar{\omega}_{in})$$

It now follows that $\hat{W}_{jm}^{s} = \hat{W}_{jm}^{b} = 1$ solves these equations, since

a. $W_{jm}^{s*} = W_{jm}^{b*} = 1$

b. for every j, Proposition 1 shows that

$$r_{jm}^{b*} + r_{jm}^{s*} = r_{jm}^{b} + r_{jm}^{s}$$

Furthermore, since the $\{\hat{r}_{jm}^{s}, \hat{r}_{jm}^{b}\}$ are independent, it follows that $\hat{W}_{jm}^{s} = \hat{W}_{jm}^{b} = 1$ is the unique solution to (28). To complete the proof, note that the change in b_j^{*} may require a change in the portfolio of the predecessor of m. Having established that the new portfolio at m costs the same as the old portfolio, we may now replicate the above argument for the predecessor of m, and so on until we reach the initial period.

QED

Note that it is not strictly necessary in the proof of Theorem 7 that the return vectors span the space of successors of m for every $m \in M$. It is sufficient that the return vectors $\{\hat{r}^s_{jm}, \hat{r}^b_{jm}\}$ span the same space as the return vectors $\{r^{s*}_{jm}, r^{b*}_{jm}\}$. We return to this point in Section 8.

We now turn to the second theorem of this section. Let (p^*, k^*_j, e^*_i) be an equilibrium. Then we shall say that the financial policy of firm j is <u>strongly irrelevant</u> at m if a small change in b^*_{jm} leads to a new equilibrium in which shareholders of j have at most to change their holdings of j's bonds, j's shares, and the riskless security.

It has been shown by Stiglitz (1974) that if firm j is a riskless borrower at m, then the financial policy of j is strongly irrelevant. In the next theorem we characterize such equilibria.

<u>Theorem 8</u>: Let (p^*, k^*_j, e^*_i) be an equilibrium. Then the financial policy of j at m is strongly irrelevant if and only if (i) or (ii) hold:

(i) $f^*_{ijm} = b^*_{ijm}$ for every individual i.

(ii) There exist $\alpha, \beta, \gamma, \delta, \varepsilon, \zeta$ such that for every $n = m^+$

$$r^{s*}_{jn} = \alpha\, \hat{r}^s_{jn} + \beta\, \hat{r}^b_{jn} + \gamma$$

$$r^{b*}_{jn} = \delta\, \hat{r}^s_{jn} + \varepsilon\, \hat{r}^b_{jn} + \zeta$$

<u>Proof</u>:

<u>Sufficiency</u>: Clearly (i) is sufficient. To see that (ii) is sufficient, define

$$\hat{f}_{ijm} = \alpha\, f^{*}_{ijm} + \gamma\, b^{*}_{ijm}$$

$$\hat{b}_{ijm} = \beta\, f^{*}_{ijm} + \varepsilon\, b^{*}_{ijm}$$

$$\hat{b}_{iom} = \gamma\, f^{*}_{ijm} + \zeta\, b^{*}_{ijm} + b^{*}_{iom}$$

Then it follows from (ii) that for every $n = m^{+}$,

$$\hat{f}_{ijm}\hat{r}^{s}_{jn} + \hat{b}_{ijm}\hat{r}^{b}_{jn} + \hat{b}_{iom} =$$

$$f^{*}_{ijm}(\alpha\, \hat{r}^{s}_{jn} + \beta\, \hat{r}^{b}_{jn} + \gamma) + b^{*}_{ijm}(\delta\, r^{s}_{jn} + \varepsilon\, r^{b}_{jn} + \zeta)$$

$$+ b^{*}_{iom} = f^{*}_{ijm} r^{s*}_{jn} + b^{*}_{ijm} r^{b*}_{jn} + b^{*}_{iom}$$

By an argument similar to that used in the proof of Theorem 7 it follows that the cost of the new portfolio is the same as that of the old and that equilibrium is preserved. We have thus defined a new portfolio $(\hat{f}_{im}, \hat{b}_{im})$ for each individual i, where

$$\hat{f}_{ihm} = \begin{cases} f^{*}_{ihm} & h \neq j \\ \hat{f}_{ijm} & h = j \end{cases}$$

and

$$\hat{b}_{ihm} = \begin{cases} b^{*}_{ihm} & h \neq j,0 \\ \hat{b}_{ijm} & h = j,0 \end{cases}$$

From equations (9) and (11) it follows that no further changes in the portfolio are needed.

Necessity: Assume first that the return vectors at $n = m^+$ span the state space of the successors of m. For the purposes of this part of the proof only, write r_h^s, h=1,...,J, for the column vector of returns at m^+, i.e.,

$$r_h^s = \begin{pmatrix} r_{A_1}^s \\ \cdot \\ \cdot \\ \cdot \\ r_{A_n}^s \end{pmatrix}$$

, where $A_1, \ldots, A_n$ are states at some successor of m.

Then by the assumption that the return vectors span the space of successors of m, we have

$$r_j^{s*} = \alpha \hat{r}_j^s + \beta \hat{r}_j^b + \gamma + \sum_h \theta_h r_h^{s*} + \sum_h \lambda_h r_h^{b*}$$

and

$$r_j^{b*} = \delta \hat{r}_j^s + \varepsilon \hat{r}_j^b + \zeta + \sum_h \theta_h' r_h^{s*} + \sum_h \lambda_h' r_h^{b*}$$

where the r_h^{s*} and r_h^{b*} together form an independent set. Since $\hat{r}_j^s + \hat{r}_j^b = r_j^{s*} + r_j^{b*}$ (see Proposition 1), we may conclude that

$$\begin{aligned} \alpha + \delta &= 1 \\ \beta + \varepsilon &= 1 \\ \gamma + \zeta &= 0 \\ \theta_h + \theta_h' &= 0 \\ \lambda_h + \lambda_h' &= 0 \end{aligned}$$

Now suppose that

$$f^{*}_{ijm} r^{s*}_{j} + b^{*}_{ijm} r^{b*}_{j} + b^{*}_{iom} = \hat{f}_{ijm} \hat{r}^{s}_{j} + \hat{b}_{ijm} \hat{r}^{b}_{j} + \hat{b}_{iom}$$

Then it follows that for all h,

$$\theta_h f^{*}_{ijm} = - \theta'_h b^{*}_{ijm} \quad \text{and} \quad \lambda_h f^{*}_{ijm} = -\lambda'_h b^{*}_{ijm}$$

and this must imply either (i) or (ii) of the original claims of the theorem, since either $f^{*}_{ijm} = b^{*}_{ijm}$ or $\theta_h = \theta'_h = 0 = \lambda_h = \lambda'_h$.

If the return vectors do not span the state space of successors of n, we may simply take a maximally independent set from among them (as long as it includes r^s_j, r^b_j, and the vector all whose coordinates are 1; the latter to represent the return vector of the riskless asset) and add to this set vectors sufficient to span the space. The same proof then goes through.

QED

We note two special cases which fit into the conditions listed in Theorem 8: First, if all consumers choose their portfolios only on the basis of portfolio mean and variance, then it may be shown (Mossin (1966)) that in equilibrium every consumer holds equal proportions of all risky assets. Thus condition (i) of Theorem 8 is automatically fulfilled, and the financial policy of every firm is strongly irrelevant. Second, consider the case where firm j is a riskless borrower at m. Then a small change in b^{*}_{jm}, say to $\hat{b}_{jm} = b^{*}_{jm} + \psi$ will produce a constant change in returns to shareholders for every successor state of m. We may thus write

$$r^{s*}_{jn} = \hat{r}^{s}_{jn} - \psi$$
$$r^{b*}_{jn} = \hat{r}^{b}_{jn} + \psi$$

Again, financial policy is strongly irrelevant.[9] Thus Theorem 8 covers previously known cases of strong irrelevance (Hamada (1969) has shown that the MM propositions hold in a mean-variance framework if firms are assumed to be riskless borrowers. The theorem thus extends his results.)

8. Incomplete markets

Suppose that the market gave only commodity prices p^c_m, a vector of equity prices $p^s_m = (p^s_{1m},\ldots,p^s_{Jm})$, and a vector of discounted bond prices $p^b_m = (p^b_{1m},\ldots,p^b_{Jm})$, for every $m \varepsilon M$. In such a market it may not be possible to find complete market prices.[10] If, at equilibrium, the returns of the securities at m span the successors of m for every $m \varepsilon M$, then a set of complete market prices (equal to the implicit prices of individuals) may be found; markets are then complete, and Theorems 7 and 8 hold.

In the paragraph following Theorem 7, we have indicated that even in the absence of spanning, the theorem may be true if the return vectors obtained as a result of changes in financial policies span the same space as the original return vectors. The key to this result is that all individuals will agree on how to price the combinations of returns given by the market. In effect, although individual implicit prices may not be equal, certain linear combinations of these prices (corresponding to the linear--incomplete--combinations of returns offered by the firms) will be equal for all individuals. Individuals will thus agree about pricing the new return vectors, and again Theorem 7 will hold.[11]

Thus the only difficult case for incomplete markets is where a change in financial policies produces a change in the set of consumption possibilities spanned in the market. In this case there are two difficulties: 1. There is no guarantee that individuals will not wish to change their consumption; indeed, they may be forced to change their consumption vectors by the change in financial policies. 2. Even if individuals do not wish to change their consumption, there is no guarantee that they will agree to price the new financial assets created in the same way. Each individual--provided no one changed his consumption pattern from that of the original equilibrium--would be willing to price firms' debt and equity at his own implicit prices. But if implicit prices are different and the new consumption set is different from the old, these individual valuations may not be the same. In general, then, an incomplete a market equilibrium will be unique if: 1. The financial assets do not span the space of successors for some time-state pair m; and 2. Any change in debt-equity ratios will produce a new consumption set, different from the old.

FOOTNOTES

[1]As we shall see later in this section, the inclusion in the model of initial bondholders would lead to one of two outcomes at $m = 0$: Either the firm would not be bankrupt, and a fixed charge would be subtracted from payments to shareholders, or the firm would be bankrupt and bondholders would be the shareholders. Clearly, in the initial period this would make no difference.

[2]Suppose we were to allow the firm to issue a_{jm} new shares in every time-state pair m. Assuming that there is originally one share of stock outstanding in the firm (this one share being fractionally distributed among consumers). Then the total amount of equity outstanding in the firm at any $m \varepsilon M$ would be $1 + \sum_{\ell \leq m} a_{j\ell}$. Suppose this equity were fully purchased by shareholders at m, and suppose that for n some successor of m, the firm issued a_{jn} new shares. Then consumers who had purchased shares at m would now hold only

$$\frac{1 + \sum_{\ell \leq m} a_{j\ell}}{1 + \sum_{\ell \leq n} a_{j\ell}}$$

of the firm's equity, and

$$\alpha_{jn} = \frac{a_{jn}}{1 + \sum_{\ell \leq n} a_{j\ell}}$$

[3]Another way of viewing Proposition 2 is as a restatement of the Modigliani and Miller (1961) theorem on the irrelevance of dividends. If the firm does not go bankrupt at m, the dividend paid to shareholders is

$$p_m^c(y_{jm} - z_{jm}) + p_{jm}^b b_{jm} + \alpha_{jm} p_{jm}^s - b_j$$

In both cases of bankruptcy, the dividend paid to shareholders is zero.

But the total payout to shareholders (given by r_{jm}^s) is invariant to the dividend decision.

[4]The drawback to such a mechanism is informational: Interconsumer loans would then have risk characteristics associated with each borrower, and the informational demands on consumer lenders would be enormous. Since our purpose is to study the effect of firm bankruptcy, such an addition would serve little purpose.

[5]The importance of implicit prices in a two-period model has been explored by Baron (1978).

[6]This concept differs somewhat from the standard concept of net present value which can be found in any basic finance text (see, for example, Van Horne (1974)). As usually defined, net present value discounts some average return for each period at a rate representative of the risk of achieving that return, whereas in (22) we are discounting the return in each time-state pair by its price. A second (albeit very minor) differnece is that (22) includes an expression for current income $p_o^c y_{jo}$, which is usually not included in net present value.

[7]Hart (1977) has shown that in large economies this is approximately true.

[8]Both Hart (1974) and Green (1973) have objected to the Radner rational expectations model on the ground that it may be unreasonable to bound borrowing. If consumers have knowledge of how much firm production income will be, however, they will also be able to perceive an upper bound on borrowing which is "reasonable," in the sense that any more borrowing would surely bankrupt the firm and that returns to bondholders would not be changed. A similar point has been made by Milne (1977).

[9]An exception would be if firm j were a riskless borrower at m, but where any more borrowing would force it into bankruptcy. In that case, the conditions of the theorem hold only for $\psi \leq 0$.

[10]Another difficulty in incomplete markets is establishing a firm objective function. One way out of this problem is to substitute individual implicit prices into (22), and then have the firm maximize its average (weighted by shareholdings) implicit valuation. In financial market theory, this approach was first used by Dreze (1974). For an application to markets with bankruptcy, see Benninga (1979).

[11]The argument in this paragraph parallels the "unanimity" arguments of Ekern and Wilson (1974), Leland (1974), and Radner (1974), with respect to the firm's investment decision.

Bibliography

Arrow, K.J. and F. H. Hahn (1971). General Competitive Analysis. San Francisco: Holden-Day, Inc.

Baron, D.P. (1976). "Default Risk and the Modigliani-Miller Theorem: A Synthesis." American Economic Review 66, 204-212.

___________ (1978). "On the Relationship between Complete and Incomplete Financial Market Models." International Economic Review, forthcoming.

Benninga, S.Z. (1979). "Competitive Equilibrium with Bankruptcy in a Sequence of Markets." International Economic Review, forthcoming.

Debreu, G. (1959). Theory of Value. New York: John Wiley and Sons.

Dreze, J. (1974). "Investment Under Private Ownership: Optimality, Equilibrium, and Stability." in J. Dreze (ed.), Allocation under Uncertainty. London: Macmillan.

Ekern, S. and R. Wilson (1974). "On the Theory of the Firm in an Economy with Incomplete Markets." Bell Journal 5, 171-80.

Fama, E. and M. H. Miller (1972). The Theory of Finance. New York: Holt, Rinehart, and Winston.

Green, J. (1973). "Temporary General Equilibrium in a Sequential Trading Model with Spot and Future Transactions." Econometrica 41, 1103-23.

Hamada, R.S. (1969). "Portfolio Analysis, Market Equilibrium, and Corporation Finance." J. of Finance 24, 13-31.

Hart, O. (1975). "On the Optimality of Equilibrium when the Market Structure is Incomplete." J. Economic Theory 10,

Hart, O. (1977). "Monopolistic Competition in a Large Economy with Differentiated Commodities." Review of Economic Studies, forthcoming.

Leland, H.E. (1974). "Production Theory and the Stock Market." Bell Journal 5, 125-44.

Milne, F. (1977). "Short-Selling, Default Risk, and the Existence of Equilibrium in a Securities Model." Working Paper, Australian National University.

Modigliani, F. and M. H. Miller (1958). "The Cost of Capital, Corporation Finance, and the Theory of Investment." American Economic Review 48, 261-97.

__________ (1961). "Dividend Policy, Growth, and the Valuation of Shares." J. of Business 34, 411-33.

Mossin, J. (1966). "Equilibrium in a Capital Asset Market." Econometrica 34, 768-83.

Radner, R. (1972). "Existence of Equilibrium of Plans, Prices, and Price Expectations in a Sequence of Markets." Econometrica 40, 289-304.

__________ (1974). "A Reformulation of the Ekern-Wilson Model in Terms of Arrow-Debreu." Bell J. of Economics 5, 101-4.

Stiglitz, J.E. "A Re-Examination of the Modigliani-Miller Theorem." American Economic Review 59 784-793.

__________ (1974). "On the Irrelevance of Corporate Financial Policy." American Economic Review 64, 851-67.

Van Horne, J.C. (1974). Financial Management and Policy. London: Prentice-Hall International.