MAJORITY CHOICE AND THE OBJECTIVE FUNCTION OF THE FIRM UNDER UNCERTAINTY*

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Abstract

A set of production vectors is said to fulfill the "current spanning" condition if any firm's current production vector is spanned by the rest of the market. Our major theorem shows that given current spanning a production plan may be found which will always win the approval of a majority of the firm's initial shareholders. Furthermore, these shareholders are shown to be the only ones concerned with the firm's production plans. Finally, it is shown that an equilibrium exists in which all firms choose plans approved by a majority of their shareholders.

MAJORITY CHOICE AND THE OBJECTIVE FUNCTION OF THE FIRM

UNDER UNCERTAINTY

Since the publication of papers by Jensen and Long (1972) and Stiglitz (1972), it has been known that value maximization may be neither a clearly defined firm objective nor one preferred by a majority (or indeed any) of the firm's shareholders. Subsequent papers by Ekern and Wilson (1974) Leland (1974), and Radner (1974) showed that shareholders might be unanimous as to preferences over marginal changes in firm production plans as long as these were made in a limited subspace of all possible plans. Specifically, shareholder unanimity holds if any new production plan does "not alter the state-distributions of returns available in the economy" (E-W, 1974, p.175).

The above condition, which we call the E-W spanning condition, is extremely restrictive. To see this, consider the following example: Suppose there are three states and three firms. Assume that current state-dependent return vectors are:

Firm 1: (3,3,3)

Firm 2: (1,2,2)

Firm 3: (2,1,1)

The E-W spanning condition requires that any change in production must be such that the last two components of the new vector of returns are equal. The restrictiveness of the E-W condition is such that it is obvious that in practical situations unanimity will rarely exist; see for example Satterthwaite (1977). In this paper we introduce a different spanning condition, which we call current spanning. This condition requires that each firm's current production vector is spanned by the rest of the market. The current spanning condition guarantees a production plan which will always win the approval of a majority of the firm's shareholders.

Thus even in incomplete markets, where the E-W spanning condition does not hold, the firm's objective may be established by a majority vote. In the above example current spanning holds; thus with no restrictions on the production technologies (as required by the E-W spanning condition) majority voting may be used to establish a firm production plan. Moreover, shareholders will have no incentive to misrepresent their preferences in this voting procedure. The literature thus far has dealt exclusively with unanimous shareholder decisions.[1] Clearly majority rule has a much wider range of application.

The structure of the paper is as follows: We examine the problem which faces a firm in a typical period of an n-period model. The firm must decide how much of its current production income to distribute as dividends and how much to retain for the purchase of firm inputs in the next period. We examine a situation where individuals have made optimal portfolio decisions given current market prices and firm plans. We then ask what will be the individual's preferences over retained earnings. Given current spanning, initial shareholders are shown to have single-peaked preferences over the firm's retained earnings, and furthermore, these shareholders will be the only ones concerned with the firm's production plans. In this setting no individual shareholder will find it advantageous to misrepresent his preferences. Thus each shareholder will vote sincerely and majority rule will yield a single (and stable) outcome. Finally, it is shown that an equilibrium exists in which all firms choose plans approved by the majority of their shareholders.

1. The model

We suppose there to be only one physical good, whose price will always one. The assumption about the number of goods is made for simplicity only; as we show in section 6, the generalization of the model to many goods and many periods is straightforward.

There are I individuals (consumers) and J firms. There are two periods; where the need exists, "today" will be subscripted by 0, and states tomorrow will be indicated by m, $m=1,\ldots,M$. Thus, for example, the consumption of consumer i will be denoted by $x_i = (x_{io}, x_{i1}, \ldots, x_{iM})$, where the first component of the vector denotes today's consumption, and the other components denote consumption tomorrow if state m obtains. The securities of each of the J firms are traded today (tomorrow the firms liquidate, and there is consequently no trade in their securities). The vector of security prices will be denoted by $p = (p_b, p_1, \ldots, p_J)$, where $p_1, \ldots, p_J$ are the prices of shares of firms 1, ... , J respectively, and p_b is the price of a riskless bond.

Firms: Each of the J firms enters period 0 with an endowment of bonds $\bar{b}_j$. As mentioned in the introduction, our two periods are to be thought of as two "typical" periods in an n-period model; thus the endowment of bonds should be thought of as the earnings from last period retained to the current period in the form of bonds. This endowment is now used to purchase inputs z_{jo} for the purpose of producing outputs $y_{jo}(z_{jo})$ in period 0. Since the price of the physical good is one, $y_{jo}(z_{jo})$ is the total production income for firm j in period 0.

In addition to producing in period 0, firm j pays dividends from its production income and retains the remainder of its earnings (if any) in the form of bonds which are redeemed in the next period.[2] If the price of the riskless bond is p_b, and if the firm buys b_j (nominal value) of these riskless bonds, the total amount spent on riskless bonds by firm j will be $p_b b_j$; it is this quantity which we shall call the retained earnings of firm j in

Production tomorrow is assumed to be stochastic; having retained earnings in the amount of $p_b b_j$ in period 0, firm j will produce $y_{jm}(z_{jm})$ tomorrow if state m occurs, where $z_{jm} = b_j$. In the multi-good model, the firm will maximize its production income in state m subject to the constraint that the costs of the inputs will not exceed the amount b_j. (The function y_{jm} will be assumed to be real-valued, non-negative, bounded, and concave.)

Firm j may pay dividends to its shareholders in both period 0 and in the first period. There is, however, a difference between dividends paid in period 0 and those paid in the first period: The former are paid to initial shareholders (those individuals i for whom $\overline{f}_{ij} > 0$), while the latter are paid to those whose shareholdings in firm j at the end of period 0 are positive (those i for whom $f_{ij} > 0$). The dividends paid by firm j in the present are $y_{jo}(z_{jo}) - p_b b_j$. In the first period dividends will depend on the earnings retained by the firm in period 0. If the firm has purchased inputs z_{jm} in state m out of retained earnings, dividends in state m will be $y_{jm}(z_{jm})$.

It is convenient to place one budget constraint on the firm. The reason for doing so will become apparent in the proof of the existence of equilibrium in section 5. We require that the firm's retained earnings come out of production income in period 0.[3] Thus we require

(1) $$y_{jo}(z_{jo}) - p_b b_j \geq 0$$

Consumers: Each consumer receives an endowment of bonds $\overline{b}_i$ in period zero. As in the case of the firm (since we are considering two periods in an n-period model) $\overline{b}_i$ should be thought of as consumer i's debt (if $\overline{b}_i < 0$) from having sold bonds in the "previous" period. If $\overline{b}_i > 0$, it represents repayment of debt from bonds purchased in the "previous" period. In addition each consumer receives an endowment of the commodity in both period zero and every state of the first period. For consumer i, denote [...] $h = 0, 1, \ldots, M$. We shall assume that all endowments are strictly positive. In addition to his endowment, consumer i will be assumed to possess an initial portfolio of shares at the beginning of the first period. Denoting this portfolio by $\bar{f}_i = (\bar{f}_{i1}, \dots , \bar{f}_{iJ})$ we shall assume:

(2) $\bar{f}_{ij} \geq 0$, for every i and every j,

(3) $\sum_i \bar{f}_{ij} = 1$, for every j.

Given firm retained earnings, $p_b b_j$, for each j, consumer i chooses a consumption vector $x_i = (x_{io}, x_{i1}, \dots , x_{iM})$, a share portfolio $f_i = (f_{i1}, \dots , f_{iJ})$ and a nominal amount of bonds b_i. Budget constraints for consumer i are given by

(4) $$x_{io} \leq \sum_j p_j(\bar{f}_{ij} - f_{ij}) + \sum_j \bar{f}_{ij}(y_{jo} - p_b b_j) + \bar{\omega}_{io}$$

$$+ \bar{b}_i - p_b b_i$$

(5) $$x_{im} \leq \sum_j f_{ij} y_{jm} + b_i + \bar{\omega}_{im}$$

Note that we have placed no positivity constraint on consumer i's purchases of new shares. Thus we allow f_{ij} to be either positive or negative; $f_{ij} < 0$ denotes "short sales" by consumer i of firm j's shares.

2. Consumer maximization and implicit prices[4]

Each consumer i maximizes a utility function defined on his consumption vector x_i. We shall assume that this function $U_i(x_i)$ is twice continuous, strictly concave, and increasing in each argument. Furthermore, we assume that

(6) $\partial U_i/\partial x_{ih} \to \infty$ as $x_{ih} \to 0$, for all $h=0,1, \dots , M$.

Now assume that given prices $p = (p_b, p_1, \dots , p_J)$ and firm retained earnings, that consumer i maximizes U_i at (x_i^*, f_i^*, b_i^*). Then strict

... in the budget constraint equations, and we may thus write U_i as a function of the securities portfolio,

(7) $$U_i(x_i^*) = \overline{U}_i(f_i^*, b_i^*)$$

Differentiating to obtain the first order maximum conditions, we obtain

(8) $$\frac{\partial \overline{U}_i}{\partial f_{ij}} = 0 \quad \Longrightarrow \quad p_j = \sum_m \frac{\partial U_i/\partial x_{im}}{\partial U_i/\partial x_{ic}} y_{jm}$$

where the partial derivatives are evaluated at x_i^*. Letting q_m^i denote the quotient of the partial derivatives so defined yields:

(9) $$p_j = \sum_m q_m^i y_{jm}(z_{jm}), \; j = 1, \ldots , J.$$

(10) $$p_b = \sum_m q_m^i$$

q_m^i are known as the individual's implicit prices for state m. See Baron (1978). Note that the q's depend on specific firm production plans.

3. Current spanning and consumer preferences over firm plans

In this section we introduce the notion of current spanning. As previously noted, our purpose is to explain firm behavior in incomplete markets where the E-W spanning property is not satisfied. While C-spanning and E-W spanning are disjoint conditions, the implications of the C-spanning condition are less radical than the E-W spanning. Our result is less radical as well; instead of shareholder unanimity we shall show that there exists majority agreement among shareholders on production plans.

Current spanning is satisfied when any firm's present vector of returns is spanned by the rest of the firms' current return vectors. Thus, letting $y_h = (y_{h1}, \ldots , y_{hm})$ denote firm h's vector of returns, we have

(11) $$y_j = \sum_{h \neq j} \alpha_h y_h \; ; \text{ for all j.}$$

On the other hand the E-W spanning condition states that any new production plan, obtained by a small change in the current production plan, is spanned by current return vectors. Clearly, barring second-order effects, the condition can be stated as

$$y_j' = \sum_h \alpha_h y_h.$$

That is, the marginal production is spanned by current return vectors.

We now proceed to give two examples which show that the two conditions are disjoint. This will also illustrate the intuition behind current spanning.

<u>Ex 1</u>: <u>E-W spanning and <u>not</u> C-spanning:</u>

Let there be three states and three firms. Assume that at current bond-sales b_1, b_2, b_3 return vectors are

$$y_1(b_1) = (3,3,3)$$
$$y_2(b_2) = (1,2,2)$$
$$y_3(b_3) = (1,2,2)$$

The E-W condition requires that if $y_1\ (b_1 + \Delta b_1)$ is the new production vector when Δb_1 is added to inputs, that

$$y_1(b_1 + \Delta b_1) = (3 + \lambda_1 \Delta b_1, 3+\lambda_2 \Delta b_1,\ 3 + \lambda_2 \Delta b_1)$$

Thus as long as changes in production are equal in the last two coordinates the market satisfies E-W spanning. Note that in this example C-spanning does not exist, since y_1 cannot be written as a linear combination of y_2 and y_3.

<u>Ex 2</u>: <u>C-spanning and not E-W spanning</u>

Let there be 3 firms and 3 states. Assume that at current bond sales b_1, b_2, b_3, return vectors are

$$y_1(b_1) = (3,3,3)$$
$$y_2(b_2) = (1,2,2)$$

Clearly, since $y_1 = y_2 + y_3$, C-spanning exists. On the other hand, suppose $y_1\ (b_1 + \Delta b_1) = (3,3 + \lambda\Delta b_1,\ 3 + \delta\Delta b_1)$. Then $y_1(b_1 + \Delta b_1)$ cannot be spanned by y_1, y_2, y_3 as long as $\lambda \neq \delta$. Note that the C-spanning condition requires linear dependence of the vectors of returns. However, linear dependence is not a sufficient condition; what is required is that condition (11) hold for every j. The following example shows that this is stronger than linear dependence: Suppose we add a fourth firm to the present example

$$y_4(b_4) = (1,1,2)$$

Then while the set (y_1,y_2,y_3,y_4) is linearly dependent, y_4 cannot be written as a linear combination of the first three. Finally, note that current spanning does not imply completeness; for example if a fourth zero component is added to the vector of returns of the above example, the market would still satisfy C-spanning but will be incomplete.

<u>Theorem 1</u>: Suppose that firm j, $1 \leq j \leq J$, chooses b_j^* and that $(x_i^*,\ f_i^*,\ b_i^*)$ maximizes U_i, $1 \leq i \leq I$. Then if current spanning holds, an individual i who holds shares of the firm at period zero (i.e. $\bar{f}ij > 0$) will prefer that firm j choose retained earnings such that b_j maximize the net present value of the firm:

$$(12) \qquad p_j - p_b b_j.$$

Furthermore the maximization of (12) is equivalent to the maximization of

$$(13) \qquad \sum_m q_m^i\, y_{jm}(z_{jm}(b_j)) - p_b b_j$$

with the implicit prices $\{q_m^i\}$ taken to be constant. Note that we do not assume that the implicit prices $\{q_m^i\}$ are constant. Rather we shall prove that the maximization of the net-present value when current spanning holds is equivalent to maximizing (13) <u>as if</u> the $\{q_m^i\}$ were constant.

Proof:

Since p_b is positive, maximization with respect to the retained earnings is equivalent to maximization with respect to the bonds b_j. Differentiating U_i yields the first-order conditions

$$(14) \qquad \frac{\partial U_i}{\partial b_j} = \frac{\partial U_i}{\partial x_{io}} \frac{\partial x_{io}}{\partial b_j} + \sum_m \frac{\partial U_i}{\partial x_{im}} \frac{\partial x_{im}}{\partial b_j} = 0$$

By the current spanning assumption we may assume that

$$(15) \qquad x_{io} = \sum_j \bar{f}_{ij} (p_j - p_b b_j + y_{jo}) - \sum_{h \neq j} f_{ih} p_h + \bar{\omega}_{io}$$

$$+ \bar{b}_i - p_b b_i$$

$$(16) \qquad x_{im} = \sum_{h \neq j} f_{ih} y_{hm} + b_i + \bar{\omega}_{im}$$

Therefore, (14) becomes

$$(17) \qquad \frac{\partial U_i}{\partial b_j} = \frac{\partial U_i}{\partial x_o} \bar{f}_{ij} \left(\frac{\partial p_j}{\partial b_j} - p_b\right) = 0$$

Thus if $\bar{f}_{ij} = 0$ (17) always holds. If $\bar{f}_{ij} > 0$ (17) is the necessary condition for the maximization of (12).

To establish the second part of the theorem, recall equation (9); We now have

$$(18) \qquad \frac{\partial p_j}{\partial b_j} - p_b = \frac{\partial}{\partial b_j} \left(\sum_m q_m^i y_{jm}\right) - p_b$$

$$= \sum_m q_m^i \frac{dy_{jm}}{db_j} + \sum_m y_{jm} \frac{\partial q_m^i}{\partial b_j} - p_b$$

Recalling (15) and (16) we have

$$(19)\quad \frac{\partial p_j}{\partial b_j} - p_b = \sum_m q_m^i \frac{dy_{jm}}{db_j}$$

$$\sum_m y_{jm} \left[\frac{\overline{f}_{ij}}{U_o^2} \left(\frac{\partial p_j}{\partial b_j} - p_b \right) \left(U_o \frac{\partial U_m}{\partial x_{io}} - U_m \frac{\partial U_o}{\partial x_{io}} \right) \right] - p_b$$

where U_o and U_m denote the appropriate partial derivatives of U_i.

Therefore,

$$(20)\quad \frac{\partial p_j}{\partial b_j} - p_b = \frac{1}{c} \left[\sum_m q_m^i \frac{dy_{jm}}{db_j} - p_b \right]$$

where

$$c = 1 - \sum_m y_{jm} \left(U_o \frac{\partial U_m}{\partial x_{io}} - U_m \frac{\partial U_o}{\partial x_{io}} \right) \frac{\overline{f}_{ij}}{U_o^2}$$

By checking the second order condition, it is clear that equating (20) to zero will yield a sufficient as well as necessary condition for maximization of (13) or equivalently (12). This completes the proof.[5]

QED

The implicit prices q_m^i may be viewed as consumer i's present value for one unit of production in state m. Thus Theorem 1 states that each initial shareholder prefers that firms in which he holds shares maximize the (personalized) net present value of production. It is intuitively obvious that shareholders may fail to agree about firm objectives simply because their implicit prices differ. This conflict was long ago resolved by Arrow (1964). Define a market to be <u>complete</u> if $q_m^i = q_m^h$ for every i, h, and m. Market completeness will always be assured if the returns of securities span the set of states; to see this, assume that for each state m there exists a firm j whose dividends in state m are 1 and whose dividends in every other state are 0. Then it follows from (9) that the market will be complete. This is the concept of completeness which is implicit in Arrow's discussion. It forms the basis for the assumption (made by Arrow and by many others) that the market mechanism starts with state prices, and not with security prices. As is evident from section 2 of this paper, state prices may be derived from security prices.

Corollary: If the market is complete, all initial shareholders in firm j are unanimous in their preferences over b_j.

<u>Proof</u>:

If markets are complete, all initial shareholders want the firm to maximize the same objective function (12).

QED

That markets are not, in general, complete makes it difficult for us to claim that shareholders will, in most cases, be unanimous as to firm choice. In the next section we show that there always exists a choice of retained earnings for each firm which will be supported by a majority of the initial shareholders.

4. Majority choice

Theorem 2: Suppose that firm j chooses its retained earnings $p_b b_j^*$ and that (x_i^*, f_i^*, b_i^*) maximizes U_i for i = 1, ... , I. Then there exists $p_b \hat{b}_j$ which is preferred by a majority of initial shareholders (weighted by their initial shareholdings) to any other choice of retained earnings $p_b b_j$. Moreover, in this setting, no individual shareholder will find it advantageous to misrepresent his preferences, and thus each shareholder will vote sincerely.

Before we prove the theorem, several points should be noted. First, $\hat{b}_j$ may be equal to b_j^*. If this is the case for every firm j and if furthermore demands and supplies of all commodities and shares are equal, we have an equilibrium. In the next section we shall show that such an equilibrium (which we call a majority choice production equilibrium) exists. Second, it follows trivially from the corollary and from our proof of Theorem 2 that if markets are complete, the criterion of majority choice is equivalent to the Arrow-Debreu criterion. Third, although we assume that the production functions are concave, this is a sufficient condition which is not necessary. In the one-good case discussed here, it is easy to construct an example of an S-shaped production function which will yield the same result.

Finally, majority rule in general works as follows: Individual voters (shareholders, in our case) consider each pair of alternatives and vote between them. The alternative agreed upon as the group's choice is that which has a majority over every other in a pair-wise comparison. However, in our case this cumbersome procedure can be replaced by a simpler one in which each individual reports his most preferred alternative and the group choice is the median of all these reported best choices.

Proof of Theorem 2:

We first consider the symmetric case. I.e., we assume that each initial shareholder has the same number of shares in the firm and thus is weighted equally. By Theorem 1, each shareholder maximizes the expression (13):

$$\sum_m q_m^i \, y_{jm}(b_j)) - p_b b_j$$

with respect to b_j (or $p_b b_j$). Since (13) is a non-negative combination of concave functions combined with a linear function, it is concave. It now follows from our assumption that there exists a finite solution to the maximization problem of each consumer, that his preferences over the retained earnings of the firm are single-peaked.[6]

It is well known that with single-peaked preferences and an odd number of voters, majority rule is a well-behaved procedure, in the sense that it yields a transitive social ordering and thus a single alternative as the outcome. This single alternative has the property that it cannot be defeated, in a pair-wise comparison, by any other alternative. (For a basic reference, see Arrow's original work (1963) or Black (1948)). In the single-peaked case, each voter reports his most preferred alternative and the group's choice is the median of all these reported most favored choices. This procedure is clearly non-manipulable, as the only way a voter can change the outcome is in a direction which is lower than his preference.

With an even number of voters we add artificially an additional person (or, alternatively, a set of preferences, or a single most preferred alternative) to act as a tie-breaker. This will assure us of an odd number of voters in each case and so a transitive social ordering, with the resulting non-manipulable voting procedure. Other possible tie-breaking rules are discussed in the paragraph following the end of the proof.

We may now relax the symmetry assumption and regard the general case in which individual shareholders do not necessarily have an equal number of shares. Since $\overline{f}_{ij}$ is the fraction of the firm owned by individual i, we take the least common denominator (LCD) of $\overline{f}_{ij}$ over all individuals i and count each reported peak LCD.$\overline{f}_{ij}$ times. It is easily checked that the tie-breaking rule has to be applied only in the case where the LCD is even. Thus the previous arguments hold about the form of the tie-breaking rule. We may now apply the rule of the symmetric case: If the LCD is odd, no tie-breaking rule is used. If the LCD is even, a tie-breaking rule is used to establish the unique, non-manipulable majority choice.

QED

Examples of other possible tie-breaking rules are the following: Consider the simple case of an even number of voters where the "middle" falls between two individuals' peaks. We can then break the tie by taking the left (or the right) of these two peaks. These rules are as arbitrary as the one in the proof, and for our purposes will be just as effective.[7] Taking the mean of these two peaks, however, will yield a manipulable procedure, since both individuals will then find it advantagious to report a different peak. Thus each will "pull" to his more preferred direction.

The majority choice which we have shown to exist in Theorem 2 consists of choosing the median of the expressed preferences of initial shareholders (with the possible addition of an extra preference where the LCD of the $\overline{f}_{ij}$ is even). Since the median of an array of numbers is a continuous function of the elements of the array, we have the following corollary:

Corollary: The outcome of the above voting procedure is a continuous function of the individual reported (and sincere) preferences over the retained earnings $p_b b_j$.

5. The existence of a majority choice production equilibrium

The aim of this section is to establish the existence of an equilibrium in which every firm chooses retained earnings which have the approval of the majority of its shareholders. The proof follows traditional equilibrium proofs, and we shall give details sparingly.

Definition: p^*, (x_i^*, f_i^*, b_i^*), (z_j^*, b_j^*) is an equilibrium if

(E.1) $$\sum_i \bar{\omega}_{io} + \sum_j y_{jo}(z_{jo}^*) - \sum_i x_{io}^* - \sum_j z_{jo}^* = 0$$

(E.2) $$\sum_i \bar{\omega}_{im} + \sum_j y_{jm}(z_{jm}^*) - \sum_i x_{im}^* - \sum_j z_{jm}^* = 0$$

(E.3) $$\sum_i f_{ij}^* = 1, \text{ for all } j.$$

(E.4) $$\sum_i b_i^* + \sum_j b_j^* = 0$$

Lemma: (E.3) and (E.4) $\Rightarrow$ (E.1) and (E.2).

Proof:

By assumption, consumers are insatiable. We may therefore assume equality in the budget constraint equations (4) and (5) for each consumer i. Summing these equations, and assuming that there was equilibrium in the previous period gives the desired result.[8]

QED

By the above lemma we need consider only one market, namely the securities market, in our search for equilibrium.

We now define a bounded economy as follows: Let A_o be a bound on feasible consumption and input vectors

$$A_o = \sum_i \overline{\omega}_{io}$$

Similarly, we may bound consumption in state m by A_m, where

$$A_m = \sum_i \overline{\omega}_{im} + \sum_j y_{jm}(A_o)$$

We bound securities vectors by assuming that for every i and for every j,

$$-1 \leq f_{ij} \leq 1$$

Finally, note that the bounds on consumption and input vectors together with our constraints on short sales effectively put a bound on b_i and b_j, as long as we constrain x_{io}, $x_{im} \geq 0$ for all i.

The above bounds correspond to the maximal amounts of consumption and portfolio holdings which we shall allow. While the first two bounds are standard, the third requires some explanation: The upper bound on f_{ij} is equivalent to the assumption that no individual may consider purchasing more than all the stock of any firm. The lower bound corresponds to the assumption that no individual may short-sell more than all the stock of any firm. This bound (on short sales) may be taken as equivalent to the standard market procedure whereby a short-seller has to borrow the stock which he sells short.

We now define a permissible vector (x_i, f_i, b_i) for individual i to be a vector obeying the budget constraints (4) and (5) and falling within the bounds defined above. Similarly, a permissible vector for firm j is defined to be a choice of retained earnings obeying (1) and falling within the above bounds.

We are now in a position to define the preference functions of individuals and firms. For individual i, define

$$P_i(p, (z_j)) = P_i(p, z_1, \ldots , z_J) = \{(x_i, f_i, b_i) \mid x_i \text{ maximizes } U_i \text{ over all permissible vectors, and } (x_i, f_i, b_i) \text{ is permissible } \}.$$

For firm j define

$$P_j(p, (z_h), (x_i)) = P_j(p, z_1, \ldots , z_J, x_1, \ldots , x_I) = \{ \hat{z}_j \mid \hat{z}_j \text{ is permissible and is the majority decision given } x_1, \ldots , x_I \}.$$

It is easily established that P_i is convex. By the results of the previous section, P_j is single-valued and continuous in all its arguments, provided we assume that that the utility functions U_i have continuous derivatives. We shall henceforth so assume.

Theorem 3: There exists a majority choice production equilibrium.

Proof:

The proof uses the Kakutani fixed point theorem, and its details are so well known (see, for example, Arrow and Hahn (1971) or Debreu (1959)), that we shall only sketch them. We first designate the simplex of security prices by S,

$$S = \{(p_b, p_1, \ldots , p_J) \mid p_b + p_1 + \ldots + p_J = 1, \; p_b, p_j \geq 0 \}$$

Next, designate the bounded choice sets by B_i for consumers and by B_j for firms. Now define a function

$$\Psi: \; S \times \prod_j B_j \times \prod_i B_i \rightarrow S \times \prod_j B_j \times \prod_i B_i$$

where

$$\Psi(p, (z_j), (x_i, f_i, b_i)) = (\hat{p}, (\hat{z}_j), P_i (p, (z_j)))$$

and where

$$\hat{z}_j = P_j\,(p,\ (z_h),\ (x_i)\ \ )$$

$$\hat{p}_b = \frac{p_b + \max\left(\sum_i b_i + \sum_j b_j,\ 0\right)}{1 + \max\left(\sum_i b_i + \sum_j b_j,\ 0\right) + \sum_j \max\left(\sum_i f_{ij} - 1,\ 0\right)}$$

$$\hat{p}_j = \frac{p_j + \max\left(\sum_i f_{ij} - 1,\ 0\right)}{1 + \max\left(\sum_i b_i + \sum_j b_j,\ 0\right) + \sum_j \max\left(\sum_i f_{ij} - 1,\ 0\right)}$$

It now follows from continuity and Kakutani's theorem that a fixed point exists. Call this fixed point p^*. As is well known, $p_j^* = 0$ iff there exists an excess supply of the security j in question. If this be the case, designate one consumer, call him i_o, and assign all the excess supply of securities to that consumer. We thus have an equilibrium.

QED

6. Extensions and conclusions

The extension of our model to more than two periods is readily achieved. In a rational-expectations framework such as that of Radner (1972), the firm has to announce its plan for all time-event pairs given prices and consumer plans. By a procedure similar to that used in the proof of Theorem 1, it may be shown that in such a framework, preferences about firm plans in any period t are held only by shareholders who purchased shares of the firm in the previous period t-1. But since this is exactly the case discussed in our paper, it follows that our procedure can remain essentially unchanged for the multi-period case. For example, for a discussion of the multi-period rational-expectations model with securities' markets, see Benninga (1979).

The model is also straightforwardly extended to the n-commodity case. Since the firm maximizes the first period production income subject to the constraint that the costs of the inputs may not exceed the retained earnings, the maximized income is concave in the retained earnings. Thus initial shareholders will have preferences on the retained earnings which are both concave and single-peaked in this case as well.

To conclude: We have shown that under the current spanning assumption shareholders have single-peaked preferences over the firm's retained earnings. Their expressed preferences over the retained earnings will be their true preferences, and majority rule will yield a single (and stable) outcome. The initial shareholders will be the only individuals to have strong preferences (i.e., be non-indifferent) over the retained earnings, and each one of them prefers that firms in which he holds shares maximize the net present value of production. Moreover, an equilibrium exists in which all firms choose plans approved by a majority of their shareholders.

Footnotes

[1]An exception is Gevers (1974), but the questions dealt with there are different from ours.

[2]Although the introduction of bonds might seem an unnessary complication, it makes the generalization to many goods a straightforward one.

[3]Equation (1) follows the usual logic of a corporate society: Shareholders in a firm need not be required to pay for inputs purchased by the firm in excess of its production income. If markets were complete, it might be that all shareholders would be willing to pay for such inputs; since we are primarily concerned with incomplete markets, however, it seems more logical to eliminate this possibility.

[4]The analysis of this section owes much to Baron (1978).

[5]In the proof we should have dealt with the case where c = 0. However, this case may be solved trivially by one of the following methods: A suitable change in the initial condition $\overline{F}_{ij}$ and $\overline{\omega}_{io}$ will guarantee that $c \neq 0$ without changing the individual's optimum. Alternatively, assume that for at least one state m, the second derivative $U_{mo} = 0$. This is true, for example, for all states m if U is a Von-Neumann-Morgenstern utility function. Then c = 0 becomes a cardinal property and may be changed by a suitable transformation of the utility function.

[6]The existence of a finite solution is guaranteed by our assumption that the production functions are bounded (Section 1).

[7] One difference results from the possible intransitivity of the majority rule in case of an even number of voters. In our case, since all we are interested in is a single outcome, acyclicity which is guaranteed with the tie-breaking rule will suffice.

[8] Implicit in all our arguments are the following two equalities:

$$\sum_i \bar{b}_i + \sum_j \bar{b}_j = 0$$

$$\bar{\omega}_{io} + b_i > 0, \text{ for all } i.$$

The first of these conditions may be thought of as an equilibrium condition we shall need to guarantee that individual i has the possibility of consuming $x_{io} > 0$. In a multi-period model, both these conditions would automatically be fulfilled.

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