A NOTE ON THE MODIGLIANI-MILLER THEOREM AND BANKRUPTCY*

by

Simon Benninga

Working Paper No. 10-77

RODNEY L. WHITE CENTER
FOR FINANCIAL RESEARCH

University of Pennsylvania

The Wharton School

Philadelphia, Pa. 19104

*This note grew out of my Ph.D. thesis at the Faculty of Management, Tel-Aviv University. I owe an enormous debt to Dr. Elhanan Helpman, my advisor, whose help and advice were most vital. The specific idea for this note grew out of a conversation with David Baron, and cogniscenti will recognize the similarity of my basic model to two of Baron's [1], [2]. Mistakes are, of course, mine.

1. INTRODUCTION

Since their proposition on the irrelevance of a firm's debt/equity ratio was the first proposed, Modigliani and Miller's proof [3] has been repeated with numerous variations. Nearly all the proofs have assumed that firms do not face bankruptcy, and Stiglitz [4], [5] showed that if this assumption holds, the MM results may be proven even if the returns of the firms are uncertain. The critical element in the Stiglitz proofs (as in many other proofs) is that shareholders in each firm can "undo" changes in the financial structure of the firm.

In this note we shall define an "offsetting" change in the debt/equity structure of a firm to be a reduction in bonds issued and a corresponding increase in shares issued, which, at market prices prevailing before the offsetting, exactly balance one another. Deriving consumers' implicit prices for firm bonds and shares, we shall then examine the change in these prices with respect to marginal changes in firm debt and equity. If the change in prices with respect to such changes is zero for consumers who currently hold firm bonds and/or equity, the offsetting changes can go through and the MM theorem holds. We shall show that a necessary and sufficient condition exists which characterizes the non-change of share and bond prices in the face of offsetting changes in capital structure, in fact this condition is the same for both bond and share prices, and it is "very nearly" equivalent to non-bankruptcy. For all practical purposes, then, non-bankruptcy is both a necessary and a sufficient condition for this particular MM proof.

2. THE MODEL

There are two periods; by 1 we denote the first period, and by m denote events at the second period. In both periods, a single physical good is available for both consumption and production; for simplicity we shall take its price as unity.

In the first period each of J firms purchases inputs from which outputs will be produced by the firm in the second period. For firm j, denote the inputs by z_j. Production is stochastic, and the outputs produced by j at event m if inputs of z_j from the first period are available will be denoted by

$$y_{mj} = y_{mj}(z_j) \tag{1}$$

In addition to purchasing outputs, each firm j may engage in two kinds of financial activities in the first period: It may sell bonds, which are to be repaid in the second period, and it may sell new equity. Let $p^s=(p_1^s,\ldots p_J^s)$ be the vector of share prices in the market, where p_j^s is the price of firm j's equity in the first period. Furthermore, let $p^b = (p_o^b,p_1^b,\ldots,p_J^b)$ be the vector of bond prices which prevail. The subscript zero denotes the price of inter-consumer loans (to be discussed shortly), and the other subscripts denote the price of firm bonds. If the firm j issues b_j nominal bonds, its net receipts from these bonds will be $p_j^b b_j$. Barring bankruptcy, it will owe the bondholders the face value of the bonds at every event in the second period. We shall assume that all bond prices lie between zero and one; there are no negative interest rates.

We shall assume that initially there is one share of firm stock outstanding and this its price is p_j^s. Each of the I consumers has an initial endowment of the shares of each firm; the endowment of consumer i of firm j's shares will be denoted by $\bar{f}_{ij}$, $0 \leq \bar{f}_{ij} \leq 1$. The initial endowments of shares in any firm are assumed to sum to unity, i.e.,

$$\bar{f}_{ij} = 1 \tag{2}$$

A firm j which issues new equity a_j in period 1 derives the income from this equity, $p_j^s a_j$. Thus net firm income in the first period is

$$p_j^s a_j + p_j^b b_j - z_j \tag{3}$$

It is convenient to assume that firms are constrained to set this net income equal to zero (this is equivalent to ignoring first period dividends and production, and in no way affects the generality of the arguments to come).

The dividends the firm pays in the second period depend on its second period revenues. Given event m and firm j, the revenue of j at m will be denoted by r_{mj}, where

$$r_{mj} = y_{mj} - b_j. \tag{4}$$

Firm j will make a positive payoff to stockholders only if r_{mj} is non-negative. Otherwise, the firm is bankrupt, and what income there is from production goes to bondholders. Denote by r_{mj}^s the dividend to stockholders, and by r_{mj}^b the returns of bondholders. Then

$$r_{mj}^s = \begin{cases} r_{mj} & \text{if } r_{mj} \geq 0 \\ 0 & \text{otherwise} \end{cases} \tag{5}$$

and

$$r^b_{mj} = \begin{cases} b_j & \text{if } r_{mj} \geq 0 \\ r_{mj} + b_j & \text{otherwise} \end{cases} \tag{6}$$

3. Consumers

Each consumer, in addition to having an endowment of shares, has an endowment of the physical good in both the first period and every state of the second period; we denote this endowment by (w_{1i}, w_{mi}). Given share prices p^s and bond prices p^b, each consumer i must make three choices:

(i) He must choose a consumption vector (x_{1i}, x_{mi}), where x_{1i} denotes consumption of the physical good in state m of the second period.

(ii) He must choose a share portfolio, $f_i = (f_{i1}, \ldots, f_{iJ})$. We shall assume that $f_{ij} \geq 0$ is consumer i's share in firm j.

(iii) He must choose a bond portfolio, $b_i = (b_{io}, b_{i1}, \ldots, b_{iJ})$. Here b_{ij}, j=1, ..., J, is the proportion of firm j's nominal bonds purchased by consumer i. b_{io} will represent consumer i's net lending or borrowing from other consumers; if $b_{io} \geq 0$, consumer i is a net lender to other consumers.

Constraints on consumption are given by the following equations:

$$x_{1i} \leq \sum_j p^s_j(\bar{f}_{ij} - f_{ij}) - \sum_j b_{1j} b_j p^b_j - p^b_o b_{io} + w_{1i} \tag{7}$$

$$x_{mi} \leq \sum_j f_{ij}\left(\frac{r^s_{mj}}{1+a_j}\right) + \sum_j b_{ij} r^b_{mj} + b_{io} + w_{mi} \tag{8}$$

We shall assume that consumer i maximizes a state-dependent utility function of his consumption at both periods of the form:

$$U^i(x_{1i}, x_{mi}) = U_1^i(x_{1i}) + \sum_m \Pi_m^i U_m^i(x_{mi}) \tag{9}$$

Here π_m^i represents i's assessment of the probability of the occurrence of state m; we shall assume that all such probabilities are strictly positive and sum to one for each i:

$$\Pi_m^i > 0, \quad \sum_m \Pi_m^i = 1, \quad i=1,\ldots,I. \tag{10}$$

Further more, we assume everywhere positive marginal utilities:

$$\frac{dU_1^i}{dx_{1i}} > 0, \qquad \frac{dU_m^i}{dx_{mi}} > 0 \tag{11}$$

Now assume that given prices p^s and p^b and firm plans (y_{mj}, z_j, a_j, b_j) for j=1, ... , J, consumer i maximizes U^i at (x_i^*, f_i^*, b_i^*). Then strict equality must exist in the equations (7) and (8) above, and we may thus write U^i as a function of the portfolios,

$$U^i(x_{1i}^*, x_{mi}^*) = \bar{U}^i(f_i^*, b_i^*) \tag{12}$$

If $f_{ij}^* > 0$ and $b_{ij}^* > 0$, we may differentiate $\bar{U}^i$ with respect to the portfolios, getting

$$\frac{d\bar{U}^i}{df_{ij}} = 0 \Rightarrow \frac{-dU_1^i}{dx_{1i}} p_j^s + \sum_m \Pi_m^i \frac{dU_m^i}{dx_{mi}} \frac{r_{mj}^s}{1+a_j} = 0 \tag{13}$$

$$\frac{d\bar{U}^i}{db_{ij}} = 0 \Rightarrow \frac{-dU_1^i}{dx_{1i}} p_j^b b_j + \sum_m \Pi_m^i \frac{dU_m^i}{dx_{mi}} r_{mj}^b = 0 \tag{14}$$

Solving the above two equations for prices p_j^b and p_j^s, we get

$$p_{1j}^s = \frac{1}{1+a_j}\left[\sum_m \Pi_m^i \frac{dU_m^i|dx_{mi}}{dU_1^i|dx_{1i}} r_{mj}^s\right] = \frac{1}{1+a_j} \sum_m \Pi_m^i \rho_m^i r_{mj}^s \tag{15}$$

$$p_j^b b_j = \sum_m \Pi_m^i \rho_m^i r_{mj}^b \quad , \quad j=1,\ldots,J \tag{16}$$

$$p_o^b = \sum_m \Pi_m^i \rho_m^i \tag{17}$$

Since inter-consumer lending is riskless, equation (17) represents the riskless interest rate in terms of consumer i's marginal rates of substitution between periods.

4. The value of a riskless (non-bankrupt) firm

In what follows we shall assume that all consumers evaluate changes in firm plans by their marginal utilities. The fundamental equations to which we shall refer will be equations (15) - (17). It is worthwhile repeating the meaning of these equations: The right-hand side of the equations is the shadow price to consumer i of the firm's shares, its debt, and consumer debt, respectively. The left-hand side is the market price. Equality need not obtain; only if a share or bond of firm j is held in the portfolio does equality obtain in equations (15) and (16) respectively.

The one exception to this rule is equation (17), as proved in the following lemma:

Lemma 1: Given prices p^s and p^b, all individuals have the same implicit evaluation of the riskless interest rate. I.e.,

$$p_o^b = \sum_m \Pi_m^i \rho_m^i \quad , \text{ for every } i. \tag{18}$$

Proof:

Since individuals may both borrow and lend risklessly, they will borrow if the left-hand side above exceeds the right-hand side, and they will lend otherwise, until equality is attained.

Q.E.D.

We now return to the Stiglitz proof. Given firm j plans (y_{mj}, z_j, a_j, b_j) and prices p_j^s and p_j^b, the value of firm j in period 1 is

$$(1 + a_j)p_j^s \; + \; p_j^b b_j. \tag{19}$$

Define an offsetting change in the debt/equity structure of the firm as follows: In an offsetting change, debt b_j is reduced by $\alpha > 0$, and equity is increased by $\frac{\alpha p_j^b}{p_j^s}$. It is obvious that if current prices continue to hold after the offsetting change, there will be no change in the value of the firm. This, indeed, is the substance of Stiglitz's proof (and a great many others). In Stiglitz's proof, the invariance of prices under offsetting changes is established by showing that consumer's consumption sets do not change when current shareholders "offset" the offsetting changes. Thus, if firm j is not risky, then current shareholders and bondholders will readjust to financial offsetting, at the current market prices; in particular they will be willing to purchase the new amounts of shareholders and bondholders.

Now suppose that current shareholders and bondholders evaluate changes in firm plans at their implicit valuations. We may now ask the converse question to the MM theorem: If current share and bond holders readjust to financial offsetting at market prices, what is implied about firm bankruptcy? To answer this question, denote by $\bar{A}$ those second period events m for which firm j will be bankrupt at current plans. Taking the derivatives of p_j^s and p_j^b (equations (15) and (16) above), we get:

$$\frac{\partial p_j^s}{\partial a_j} = \frac{-1}{(1+a_j)^2} \sum_{A} \Pi_m^i \rho_m^i (y_{mj} - b_j) \tag{20}$$

$$\frac{\partial p_j^s}{\partial b_j} = \frac{-1}{1+a_j} \sum_{A} \Pi_m^i \rho_m^i \tag{21}$$

$$\frac{\partial p_j^b}{\partial a_j} = 0 \tag{22}$$

$$\frac{\partial p_j^b}{\partial b_j} = \frac{-1}{(b_j)^2} \sum_{\bar{A}} \Pi_m^i \rho_m^i y_m, \quad \text{Since } p_j^b = \frac{1}{b_j}\Big(\sum_{A} \Pi_m^i \rho_m^i b_j + \sum_{\bar{A}} \Pi_m^i \rho_m^i y_{mj}\Big) \tag{23}$$

Simple algebra may now be used to show:

Theorem 2: If shareholders are indifferent to offsetting changes in financing, then either:

(i) $y_{mj} = 0$ for every $m \in \bar{A}$

or

(ii) $\bar{A}$ is empty (firm j does not go bankrupt), and hence $p_o^b = p_j^b$

Proof:

As above, let $\alpha > 0$ be a small reduction in nominal bonds issued, and let the firm issue new shares in the amount $\frac{\alpha p_j^b}{p_j^s}$. Then the change in p_j^s is (dropping superscript i)

$$\frac{\partial p^s}{\partial a_j} \cdot \frac{\alpha p_j^b}{p_j^s} - \frac{\partial p_j^s}{\partial b_j} \alpha = \frac{-1}{(1+a_j)^2} \sum_A \Pi_m \rho_m (y_{mj} - b_j) \frac{\alpha p_j^b}{p_j^s} + \frac{\alpha}{1+a_j} \sum_A \Pi_m \rho_m$$

$$= \frac{-1}{(1+a_j)^2} \sum_A \Pi_m \rho_m (y_{mj} - b_j) \frac{(1+a_j)\alpha p_j^b}{\sum_A \Pi_m \rho_m (y_{mj} - b_j)} + \frac{\alpha}{(1+a_j)} \sum_A \Pi_m \rho_m$$

$$= \frac{\alpha}{1+a_j} \left[\sum_A \Pi_m \rho_m - p_j^b \right]$$

This is zero iff

$$\sum_A \Pi_m \rho_m - p_j^b = 0$$

and this implies (Multiplying by b_j)

$$\sum_A \Pi_m \rho_m b_j = p_j^b b_j = \sum_A \Pi_m \rho_m b_j + \sum_{\bar{A}} \Pi_m \rho_m y_{mj}$$

which gives the desired result.

Q.E.D.

Similar results hold for bondholders:

Theorem 3: If bondholders are indifferent to offsetting changes in financing, then either

(i) $y_{mj} = 0$ for every $m \in \bar{A}$

or

(ii) $\bar{A}$ is empty.

Proof:

By (11)

$$\frac{\partial p_j^b}{\partial b_j} = \frac{-1}{(b_j)^2} \sum_{\bar{A}} \Pi_m \rho_m y_{mj}$$

The result is thus immediate.

Q.E.D.

Thus Stigilitz's proof of MM theorem holds iff (i) there is no bankruptcy or (ii) for every state at the second period in which the firm goes bankrupt, its income from production is zero.

BIBLIOGRAPHY

1. Baron, D. P. "Default Risk and the Modigliani-Miller Theorem: A Synthesis." <u>American Economic Review</u> 66 (1976), pp. 204-212.

2. Baron, D. P. "On the Relationship between Complete and Incomplete Financial Market Models." Discussion Paper No. 241, Center for Math. Studies in Economics and Management Science, Northwestern University, Sept., 1976.

3. Modigliani, F. and M. H. Miller. "The Cost of Capital, Corporation Finance, and the Theory of Investment." <u>American Economic Review</u> 48 (1958), pp. 261-97.

4. Stiglitz, J. E. "A Re-Examination of the Modigliani-Miller Theorem." <u>American Economic Review</u> 59 (1969), pp. 784-93.

5. Stiglitz, J. E. "On the Irrelevance of Corporate Financial Policy." <u>American Economic Review</u> 64 (1974), pp. 851-67.