The Growth-Optimal General Equilibrium Market Model: A Reduced Form Expression for the Capital Market Line

by

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I. INTRODUCTION AND SUMMARY

Estimation of the capital market line is difficult for the practitioner. He is forced to utilize either of two general approaches. On the one hand, subjective estimates of market parameters can be made. The advantage here of course is that complex and current information can be brought to bear on the problem. The disadvantage is that this technique is exceedingly difficult and not easily teachable. Alternatively, ex post data can be utilized to construct a frequency distribution of market returns. This method offers the advantage of objectivity but either requires the highly restrictive assumption of stationarity of expected returns and variance or some provision for taking non-stationarity into account.

This paper derives a very simple objective method for estimation of the capital market line which requires only an estimate of the risk free rate of interest and the assumptions of growth optimal general equilibrium and stationarity of variance. The expected return of the market portfolio and therefore the market price of risk are found to be unique functions of the risk free rate of interest.

We will first briefly review the development of the partial equilibrium growth optimal model. We will then show how, in general equilibrium, this relationship gives a functional relationship between the risk free rate and the parameters of the market portfolio. Finally, under the assumption of stationarity of variance, we will show that the capital market line is a unique function of the risk free rate of interest and that the slope of the market line varies inversely with changes in the risk free rate.

II. THE MULTIPERIOD PARTIAL EQUILIBRIUM MODEL

The Sharpe (1964)-Lintner(1965)-Mossin(1966) capital asset pricing model (CAPM) gives us the following single period general equilibrium relationship:

$$\bar{r}_p = i + \lambda \sigma_p \tag{1}$$

where

$\bar{r}_p$ is the required return (and equilibrium expected return) of a perfectly diversified portfolio.

i is the risk free rate of interest

λ is the market price of risk and

σ_p is the standard deviation of the portfolio.

The market price of risk for this formulation of the CAPM is given by:

$$\lambda = \frac{\bar{r}_m - i}{\sigma_m} \tag{2}$$

where

$\bar{r}_m$ is the expected return on the total market portfolio of risky assets and

σ_m is the standard deviation of the market portfolio.

While this model (and its covariance counterpart for single securities) is extremely useful for explaining risk/return relationships in capital markets, its primary deficiency stems from its single period formulation. Many of the important aspects of finance disappear in a single period environment. There are no meaning to notions such as long run versus short run, business cycles, term structure, etcetera, etcetera in a single period world. Furthermore, there is no need to distinguish between arithemetic expected return and geometric expected return and there is no general basis for choosing an optimal combination of risk and return. The choice is a function of each individual's marginal utility of wealth.

There has lately been a flurry of research activity to determine if in a long run multiperiod environment there is a single, asymtotically optimal portfolio selection rule which dictates the appropriate amount of risk for long run investors regardless of their utility function. The results, so far, appear to be inconclusive.

For example, Latané (1959) and Markowitz (1977) have shown that under quite general conditions and a wide range of utility functions, a risk/return combination that maximizes geometric expected return will be most likely to maximize the long run expected utility of wealth. Markowitz (1977) states:

> Kelly (1956) and others, e.g., Latané (1957)(1959), Markowitz (1959, chapter 6) and Brieman (1960) (1961), have asserted that in selecting among probability distributions of return this period, the investor who continually reinvests for the long run should maximize the expected value of the logarithm of increase in wealth. Mossin (1968) and Samuelson (1963)(1969), on the other hand, have presented examples of games in which the investor reinvests continually for the long run, has any of a wide range of apparently plausible utility functions, yet definitely should not follow the aforementioned "expected log" rule.
>
> The argument of Kelly et al. is that, under the conditions considered, the investor who follows the expected log rule is almost sure to have a greater wealth in the long run than an investor who follows a distinctly different policy...On the other hand, Mossin and Samuelson argue that, for a wide range of plausible utility functions, the expected utility of the game as a whole provided by the expected log rule does not approach the expected utility provided by the optimum strategy, no matter how long the game is played...

> The conclusion of the present paper, however, is that utility analysis does not refute the expected log rule. Rather it confirms the rule, and provides a more satisfactory theoretical justification for it than has been available heretofore.

Markowitz's analysis and conclusions is premised on the relatively unrestrictive assumption that investors' utility functions are unbounded from above. Goldman (1974, p. 102), however, gives an example of a bounded utility function for which a growth optimal policy is far from optimal. Just how important his single exception to the universality of the near optimality of the growth optimal policy is not known. And of course in order to keep Goldman's result in perspective one must also consider Roll's (1973, p. 551) conclusion that "given temporally independent returns, a number of mean-variance efficient portfolios can be shown to bring complete ruin after an infinate sequence of re-investments".

Hakansson (1971) uses the Central-Limit Theorem to argue that for large T, the distribution of terminal wealth will be approximately log normal. This leads to definition of an asymptotic "efficient frontier" and, for a log normal distribution of terminal wealth, to determination of a growth optimal portfolio on this efficient frontier. Merton and Samuelson (1974), however, have shown that the above analysis is incorrect because of a mathematical error involving an illegitimate interchange of limits. Hakansson (1974) rejoins that this error "can, like scaffolding, be removed from my paper without consequence to the central assertions made there."

One should keep in mind that the theoretical issues above deal with the acceptability of using a growth optimal criteria for investors who do not have logarithmic utility functions. The whole problem could be somewhat moot if empirical evidence showed that logarithmic utility is a close approximation for most investors' utility functions. Fama and MacBeth (1974) are not able to reject the hypothesis that the process of price formation in the capital market is dominated by growth-optimizers. They do find, however, that the aggregate market portfolio is, in each period tested, less risky than the growth optimal. Friend and Blume (1975, pp. 900-901), concur: "Thus investors require a substantially larger premium to hold equities or other risky assets than they would if their attitude toward risk were described by logarithmic utility." Gordon, Paradish and Rorke (1972) report, however, that empirical tests obtained from experimential games suggest that a log utility function might characterize individuals with wealth in excess of $200,000.

A fair summary of the theoretical and empirical issues raised heretofore might be as follows: while the growth optimal rule has some especially attractive properties for long term investors it should be recognized that many investors appear to be willing to trade off some growth for less period to period fluctuation in their portfolios. For some of these investors this tradeoff may be irrational, but for others, with utility functions approximating the type described by Goldman (1974), the choice may be perfectly consistent with expected utility maximization.

The remainder of this paper will take the specific position that the market place is dominated by growth optimizers and that this strategy is appropriate for all investors. Then, following Latané and Young (1969) and Young and Trent (1969) it can be shown that the following is a good approximation of the relationship between the arithemetic and geometric means of a probability distribution which is not highly skewed:

$$\bar{G}^2 \approx \bar{R}^2 - \sigma^2 \qquad (3)$$

where G is one plus the geometric expected return,

R is one plus the arithemetric expected return, and

σ is the standard deviation of the distribution.

Latané (1972) uses this approximation and differential calculus to derive an expression for the growth optimal leverage of a perfectly diversified portfolio. The resulting expression is:

$$q^* \approx \frac{(1+i)\ (\bar{r}_m - i)}{\sigma_m^2 - (\bar{r}_m - i)^2} \qquad (4)$$

The variable q is defined as the portion of wealth invested in the market portfolio; (1-q) is therefore the proportion invested in the risk free asset. Values of q less than one correspond to net lending, while values of q greater than one represent net borrowing. Numerically, Latané's q is equivalent to Beta, the widely used measure of portfolio volatility. Therefore, the optimal leverage, q^*, computed from equation (4) is equivalent to the optimal risk level, β^*. Latané (1972) points out that expression (4) may be simplified by noting that (1+i) is close to one and $(\bar{r}_m - i)^2$ is close to zero. Making these approximations, equation (4) reduces to:

$$q^* = \beta^* \approx \frac{\bar{r}_m - i}{\sigma_m^2} \qquad (5)$$

Equation (5) is equal to Friend and Blume's (1975, pg. 903) equation 5 if Pratt's measure of relative risk aversion is equal to one, as it would be for logarithmic utility. Thus the discreet approximation in equation (5) equals the exact solution for continuous compounding.

The optimal level of risk given by equation (5) can be quite large for reasonable estimates of the market parameters and is quite sensitive to changes in the risk free rate of interest. For example, with the following market parameters ($\bar{r}_m = .12$, $\sigma_m = .18$ and $i = .06$), the optimal Beta computed from equation (5) is 1.85. If the risk free rate drops to .05, this optimal Beta increases to 2.16.

III. THE GENERAL EQUILIBRIUM CONSTRAINT

The previous section reviewed the extension of the general equilibrium, single period capital asset pricing model to a partial equilibrium, multiperiod model. The relationship expressed in equation (4), or the simpler approximation in equation (5), provides a theoretical relationship between single period market parameters and optimal portfolio leverage (or Beta) in a multiperiod world which consists of repetitive draws from the single period distribution. The relationship at this stage is a partial equilibrium one, since excess borrowing or lending in the securities markets is allowed.

It will be useful to study the implications of this model when it is constrained to a general equilibrium environment. In this situation the value of q^* or β^* must be exactly 1 since aggregate borrowing and lending must be equal. Making this substitution in equation (4) yields the following functional relationship among the market parameters:

$$\sigma_m^2 - \bar{r}_m + i + \bar{r}_m i - \bar{r}_m^2 = 0 \qquad (6)$$

We now have a theoretical mechanism for determining the impact of changes in the risk free rate of interest on the parameters of the market portfolio of risky assets.

IV. THE CAPITAL MARKET LINE AS A FUNCTION OF THE RISK FREE RATE OF INTEREST UNDER STATIONARITY OF VARIANCE

For a given change in the risk free rate of interest, prices of risky securities will adjust so that the return distribution of the market portfolio will satisfy the relation of equation (6). This adjustment may occur via adjustments in either the expected return parameter, the standard deviation or both. There are, therefore, a whole family of market parameter adjustments that would satisfy equation (6). Under the assumption of stationarity of variance, however, a unique relationship will exist between the risk free rate of interest and the capital market line.

Further insight can be obtained by solving equation (6) for $\bar{r}_m$ as a function of i and σ_m. This result is expressed below:

$$\bar{r}_m = \frac{-(1-i) + \sqrt{(1-i)^2 + 4(\sigma_m^2 + i)}}{2} \qquad (7)$$

We see that for the normal range of values for i, r_m and σ_m that the positive value of the square root will be relevant.

If $\bar{r}_m$ from equation (7) is substituted into equation (2) we obtain an expression for λ as a function of i and σ_m:

$$\lambda = \frac{-(1+i) + \sqrt{(1+i)^2 + 4\sigma_m^2}}{2\sigma_m} \qquad (8)$$

We can now write a new expression for the capital market line to replace equations (1) and (2):

$$r_p = i + \left[\frac{-(1+i) + \sqrt{(1+i)^2 + 4\sigma_m^2}}{2\sigma_m} \right] \sigma_p \qquad (9)$$

This reduced form expression is more complicated than equation (1) but will be easier to utilize since estimation of $\bar{r}_m$ is avoided.

If we now compute the partial of λ with respect to i, holding σ_m constant, we obtain:

$$\left.\frac{\partial \lambda}{\partial i}\right|_{\sigma_m \text{ fixed}} = -\frac{1}{2\sigma_m} + \frac{1}{2\sigma_m} \left(\frac{1+i}{\sqrt{(1+i)^2 + 4\sigma_m^2}} \right) \qquad (10)$$

The expression will be negative for all values of i as long as σ_m is greater than zero. So we see that as the level of the risk free interest rate increases, the market price of risk will decrease if variance is stationary and general equilibrium is maintained.

V. CONCLUSION AND DISCUSSION

Latané's relationship for growth optimal leverage has given us a means of obtaining a general equilibrium relationship between the three parameters of the security market line. This of course means that if any two parameters are known, the third is determined. We have used this relationship under the assumption of stationarity of variance to derive an expression for the capital market line which eliminates the need for estimating the expected return of the market portfolio. It has further been shown that the market price of risk will decrease as the level of the risk free interest rate increases.

It is seen from Table 1 that the change in slope can be quite large for the normal range of values of the risk free rate; the slope will increase by seven percent if the risk free rate decreases from twelve to four percent. Objective market line estimation techniques that do not take this into account can contain a serious source of error, especially when the market line estimate is being applied to high variance portfolios.

Table 1

Computed values of $\bar{r}_m$ and λ using equations (7) and (8) for various values of i. σ_m is assumed constant and equal to 0.20.

i	$\bar{r}_m$	λ
.04	.077	.185
.08	.116	.180
.12	.154	.173

FOOTNOTES

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