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I. INTRODUCTION

Recent empirical tests of the traditional capital asset pricing model of Sharpe, Lintner and Mossin have found that this model is not fully consistent with observed returns on common stocks.[1] Partly in response, various authors have developed alternative pricing models. For instance, in a multi-period world, Merton postulates a changing investment opportunity set as a function of a stochastic interest rate. In another extension, Mayers allows for human capital. Many other extensions have been proposed, but any attempt to review everyone would required an article in itself. The purpose of this paper is to develop a general framework in which to interpret these various extensions and by implication their conceptual simularities.

The paper starts by showing how an investor facing a multiperiod consumption-investment problem can always reduce his problem to a seemingly one-period problem. In general, the associated one-period utility function would be defined not only over current consumption and end-of-period wealth but over other variables as well. While in many respects a review of prior results, this section does provide a clarification of the role of state variables in the derivation of seemingly one-period utility functions and an insight into the meaning of a changing investment opportunity set.

The main results of the paper follow: First, it will be shown that any seemingly one-period problem derived from a multi-period problem can be transformed, under the usual risk aversion assumptions, into an equivalent problem of minimizing the variance of return subject to various constraints--a generalization of the usual mean-variance portfolio problem of Markowitz. This generalized mean-variance problem turns out, however, to be so general that it would typically lack any economic content. To give the problem content, one must somehow restrict the generality of the multiperiod problem by making some assumption about the form of investors' multiperiod utility functions or about the stochastic structure of returns and other variables of concern to investors.

Whatever assumption is made, the formal structure of this generalized mean-variance problem remains unchanged. From this formal structure, it is possible to infer the general nature of the equilibrium which would obtain in the capital markets. This equilibrium will thus hold for any set of assumptions. In contrast, the often used technique of optimizing an investor's expected utility and aggregating the first order conditions, as in Merton, tends to obscure the conceptual similarities among different models.

II. THE SEEMINGLY ONE-PERIOD PROBLEM

It is widely known that dynamic programming techniques can be applied in an uncertain world to reduce an investor's consumption-investment problem to a seemingly one-period problem.[2] This section will review this application and introduce some notation.

While the discussion could be couched in a general multiperiod setting, it will serve the purposes of this paper to work with a two-period or, more precisely, a three-date world in which there is no labor income. In such a world, the investor's problem at time 0 is to determine how to use his current assets to provide optimally for current consumption and in some sense for uncertain consumption at time 1 and at time 2. At time 2, he will consume all his remaining wealth.

If the investor follows the expected utility maxim, his three-date consumption-investment problem can be formulated as: Let Q_t be the vector of the physical quantities of goods which he consumes at time t; Π_t be the vector of prices of these goods at time t; and c_t be the nominal value of consumption expenditures at time t for this investor. The investor's direct utility function U, defined over the vectors Q_0, Q_1, and Q_2 implies, as is well known, an equivalent indirect utility function V defined over nominal consumption expenditures and the prices of consumption goods.[3]

In terms of this indirect utility function V, the investor's problem is to

$$\max \quad E[V(c_0, \tilde{c}_1, \tilde{c}_2, \Pi_0, \tilde{\Pi}_1, \tilde{\Pi}_2)], \qquad (1)$$

where the tildes indicate random variables and the maximization is subject to any constraints required to maintain feasibility, such as constraints on short sales and the like. The results in the next section will require the assumption that V be strictly concave in nominal consumption expenditures c_0, c_1, and c_2 for given values of Π_0, Π_1, and Π_2.[4]

Dynamic programming uses the following observation to reduce (1) to a seemingly one-period problem: The optimal decisions to be made at time 1 will be dictated by the values of various variables as known to the investor at that time such as his wealth at that time, the prices of consumption goods, and so on. The values of these variables describe "the state of the system" and provide sufficient information for making an optimal decision at that time. The vector of everything known to the investor at time 1 obviously provides sufficient information but more parsimonious definitions are possible.

An investor in formulating his current decisions needs only therefore to concern himself directly with his immediate consumption and the potential impact of his current investment decisions upon the state of the system at time 1. Put more formally, there will exist, for an appropriately defined state of the system, a so-called recursive function defined over current consumption and the state of the system at time 1 which will yield a cardinal ranking of combinations of these two arguments. The investor's optimal current decisions are those which are feasible and maximize the expected value of this recursive function--a seemingly one-period problem.

Since the complexity of the macro-relationship to be developed below is partially related to the number of variables in the definition of the state of system, a parsimonious definition is desirable. A simple candidate would be just the nominal wealth of the investor at time 1 on the assumption that the greater his wealth, the better off he is. Even if prices of consumption goods were known, it is easy to construct counter-examples to show that this definition is not always adequate.[5]

In deriving a correct but parsimonious definition of the state of the system, the following symbols will be used:

- n the number of assets available, assumed for convenience to be the same at both time 0 and 1.
- v_{it} the dollar amount of asset i held by the investor at time t after any revision of the portfolio.
- w_t the wealth available to the investor at time t to be used for consumption and investment.
- r_{it} the total rate of return on asset i from time t-1 to t.
- R_t the vector of r_{it}'s at time t.

To highlight the logic in the subsequent argument, it will be assumed for the moment that Π_1 and Π_2 are known as of time 0. Consumption expenditures at time 0, c_0, will be the difference between initial wealth and that which is invested for future consumption, or $w_0 - \Sigma\, v_{i0}$. Likewise, c_1 will be $w_1 - \Sigma\, v_{i1}$. Because everything is consumed at time 2, c_2 will be $\Sigma\, v_{i1}(1+r_{i2})$.

Replacing c_0, c_1, and c_2 by these expressions, one can rewrite (1) as

$$\max \int_{R_1} \int_{R_2} V[w_0 - \Sigma\, v_{i0},\ w_1 - \Sigma\, v_{i1},\ \Sigma\, v_{i1}(1+r_{i2})]\, p(R_1,R_2)\, dR_2\, dR_1, \quad (2)$$

subject to any required constraints. The symbol $\int_{R_1}$ is to be interpreted as the multiple definite integral $\int_{r_{11}}\int_{r_{21}}\ldots\int_{r_{n1}}$, where r_{i1} symbolizes the region of integration; dR_1 is to be interpreted as $dr_{11}dr_{21}\ldots dr_{n1}$. The symbols $\int_{R_2}$ and dR_2 have similar interpretatations. The joint density function of the return vectors R_1 and R_2 is given by $p(R_1,R_2)$. As written, the joint density function $p(R_1,R_2)$ incorporates the implicit assumption that the investor cannot affect its form. The implications of relaxing this will be examined later.

Recognizing that the joint density $p(R_1,R_2)$ can be rewritten as the product of the marginal density $p(R_1)$ and the conditional density $p(R_2|R_1)$, one can rewrite (2), after some rearranging of terms, as

$$\max\int_{R_1}\left\{\left[\int_{R_2} V[w_0-\Sigma\, v_{i0},w_1-\Sigma\, v_{i1},\ \Sigma\, v_{i1}(1+r_{i2})]p(R_2|R_1)dR_2\right]\right. p(R_1)dR_1 \quad (3)$$

The expression in the large brackets is the expected utility as of time 0 from investing an amount v_{i1} in each asset i at time 1 conditional on current consumption $w_0 - \Sigma\, v_{i0}$ or c_0, end-of-period wealth w_1, and the vector of returns R_1, so that the state of the world at time 1 can be adequately described by w_1 and R_1.

The optimal values of the v_{i1}'s will therefore be functions of c_0, w_1, and R_1. Replacing the v_{i1}'s by these functions and performing the indicated integration with respect to R_2, one obtains a function, say f_0, of c_0, w_1, and R_1, which allows (3) to be rewritten as[6]

$$\max_{v_{0i}\text{'s}}\int_{R_1} f_0(c_0,w_1,R_1)p(R_1)dR_1 \quad (4)$$

If the return vectors R_1 and R_2 were independent, f_0 would only need to be defined over c_0 and w_1, the usual arguments in the traditional one-period world of Sharpe, Lintner, and Mossin. Except in one special but uninteresting case[7], Fama (1970) has demonstrated that as long as V is strictly concave in the consumption vector, f_0 will be strictly concave in c_0 and w_1, whether or not f_0 includes R_1 as an argument.[8]

Intuitively, dependencies in returns over time mean that an investor's perceived distribution of returns for any specific period changes as that period nears. In the two-period example, the distribution of returns in the second period as perceived at time 0 is the marginal distribution of R_2 or mathematically $\int_{R_1} p(R_1,R_2)dR_1$. At time 1, the perceived distribution is the conditional distribution $p(R_2|R_1)$. Following Merton, the literature has frequently referred to such changing distributions as "changing investment opportunity sets."

While the literature has used this term exclusively to describe these types of changing distributions, the term is in fact more general. There are certain types of assets whose very availability for investment, for instance, at time 1 hinges upon decisions made at time 0. One example of such an asset would be a two-period privately placed bond at time 0. If there were no secondary market for such bonds, the only way in which an investor could hold such a bond at time 1 would be to buy it at time 0. Of course, the purchase of such a bond would restrict the feasible strategies at time 1. Another example would be the purchase of an option to buy some asset like real estate at some future point. Such options may be non-transferable, either for reasons of law or for lack of a secondary market. Alternatively, transaction costs may result in different investment opportunities as a function of current decisions. With transaction costs, the net return on a two-period bond with one period to go would, for example, differ according to whether or not the bond was purchased when it was first issued.

A logical way to incorporate such changing investment opportunity sets is to redefine the density function in (2) so as to account for the effect that an investor's decision at time 0 might have upon the investment opportunities at time 1. Formally, the probability density in (2) can be redefined as $p(R_1,R_2,v_0)$ or $p(R_1)p(R_2|R_1,v_0)$, where v_0 is the vector of the v_{i0}'s. In this case, the function f_0 will be defined over c_0, w_1, R_1, as well as v_0, though the inclusion of w_1 is strictly speaking redundant. Thus, a seemingly one-period problem results even in this case, but with an important difference: Fama's proof that f_0 is strictly concave in c_0 and w_1 no longer applies since he implicitly assumes that the investor's decision cannot affect the probability distributions of returns. Thus, strict concavity of f_0 in c_0 and w_1 requires the assumption that investors themselves cannot affect the investment opportunities sets, as was implicitly assumed in deriving (4).

To allow for uncertainty as to future commodity prices, the indirect utility function V would have to be defined explicitly over both the vector of consumption expenditures and the vectors of commodities prices at time 1 and 2, Π_1 and Π_2. The probability density function in (2) would be replaced by $p(R_1,\Pi_1,R_2,\Pi_2)$ and in (3) by $p(R_1,\Pi_1)p(R_2,\Pi_2|R_1,\Pi_1)$. The recursive function f_0 would thus be defined over current consumption c_0, end-of-period wealth w_0, the vector of returns R_1, as well as the vector of commodity prices Π_1. Again if the price vectors like the return vectors are outside the control of the individual investor, the assumption that V is strictly concave in the consumption vector implies that f_0 will be strictly concave in current consumption expenditures and end-of-period wealth.

III. THE EFFICIENT SET AND EQUILIBRIUM

In 1952, Markowitz observed that any feasible portfolio would be one of two types: a portfolio which no risk averse investor would want to hold or a portfolio which would be suitable for some risk averse investor. The latter set of portfolios was termed "the efficient set." Introducing a riskfree asset, Sharpe developed an equilibrium pricing relationship for individual assets. Black developed a similar kind of relationship in the absence of a riskfree asset.

The first part of this section generalizes this concept of an efficient set to a multiperiod world, and the second part uses this generalization to develop an equilibrium pricing relationship for individual assets. This analysis will assume that an investor's actions cannot affect the probability distribution of returns, so that f_0 can be assumed strictly concave in c_0 and w_1.

A. The Generalized Efficient Set

If the prices of consumption goods are known, an investor's current decisions in a multiperiod world would be those which

$$\max_{v_i\text{'s}} \quad E[f_0(c_0,w_1,r_1,\ldots,r_i,\ldots,r_n)], \tag{5}$$

subject to any constraints required for feasibility. Problem (5) is the same as (4) except that the integrals have been replaced by the expected value operator and the vector of returns by the individual elements. Moreover, the time subscripts have been dropped when no confusion results. For the moment, it will be assumed that the returns are jointly normal.

The normality assumption allows $E[f_0]$ to be reexpressed as a function of c_0, the expected values and variances of the random variables w_1 and r_i, $i=1,\ldots,n$, and the covariances between each possible pair of these variables.[9] Intuitively, the expected values, variances, and covariances uniquely define the form of a joint normal distribution, and thus an investor should be able to evaluate a potential decision by its impact on c_0 and these parameters.

Since the investor cannot, by assumption, affect the expected values and the variances of the returns on the individual assets nor the covariances between any pair of these returns, he need only concern himself directly with the effect of his decisions on current consumption, the expected value and variance of end-of-period wealth, and the covariances of end-of-period wealth with each of the returns on the individual assets. Put more formally, $E[f_o]$ can be reduced to a function, say g, of those variables whose values an individual investor can influence conditional on those which he cannot, so that (5) can be restated as

$$\max_{v_i\text{'s}} \; g[c_0,E(w_1),\text{var}(w_1),\text{cov}(w_1,r_1),\ldots,\text{cov}(w_1,r_i),\ldots,\text{cov}(w_1,r_n)], \tag{6}$$

subject to any required feasibility constraints. To simplify the notation, conditioning variables known as of time 0 are not explicitly included in g.

The strict concavity of f_0 in c_0 and w_1 implies that the partial derivative of g with respect to $\text{var}(w_1)$ is negative.[10] In other words, the investor of this paper faced with the choice between two investment consumption strategies promising the same current consumption, the same expected end-of-period wealth, and the same covariances between end-of-period wealth and the returns on the individual assets would select that strategy with the smaller variance of end-of-period wealth. The same kind of proposition would hold in the one-period world of Markowitz except that there would be no reference to the covariances between wealth and returns since in this world the arguments of g would be only c_0, $E(w_1)$, and $var(w_1)$.

Thus, of all feasible consumption-investment strategies with the same values of c_0, $E(w_1)$, and $cov(w_1,r_i)$, $i=1,\ldots,n$, the investor of this paper would select that strategy with the smallest possible variance of end-of-period wealth. Likewise, of all feasible consumption-investment strategies with the same values of c_0, $var(w_1)$, and $cov(w_1,r_i)$, $i=1,\ldots,n$, the investor would select that strategy with the largest possible expected value of end-of-period wealth.[11] The set of all feasible portfolios satisfying both of these properties would correspond to a generalized version of Markowitz's efficient set; any other feasible portfolio would be inefficient.

At this point in the paper, it will prove convenient to distinguish explicitly between those variables unique to a specific investor and those common to all. Since w_0, c_0, the v_i's, and w_1 are unique to investor k, these variables will be superscripted by k. While equilibrium relationships can be derived under heterogeneous expectations, the resulting expressions have little intuitive economic appeal. Thus, the expected values, variances, and covariances of these returns, denoted henceforth by μ_i and σ_{ij}, will be assumed the same for all investors and will not be superscripted by k.

In this augmented notation, the first proposition implies for given values of c_0^k, $E(w_1^k)$, and $cov(w_1^k,r_i)$, $i=1,\ldots,n$, that the optimal v_i^k's will be given as the solution of

$$\min_{v_i\text{'s}} \quad \Sigma_i \, \Sigma_j \, v_i^k \, v_j^k \, \sigma_{ij} \tag{7a}$$

$$\text{s.t.} \quad w_0^k - c_0^k + \Sigma_j \, v_j^k \, \mu_j = E(w_1^k) \tag{7b}$$

$$\Sigma_j v_j^k = w_0^k - c_0^k \tag{7c}$$

$$\Sigma_j v_j^k \sigma_{ij} = \mathrm{cov}(w_1^k, r_i),\ i=1,\ldots,n, \tag{7d}$$

plus any additional constraints required to maintain feasibility. By varying c_o^k, $E(w_1^k)$, and $\mathrm{cov}(w_1^k, r_i)$, $i=1,\ldots,n$, over all feasible values, one would obtain possible candidates for the efficient set. Those candidates which also maximized $E(w_1^k)$ for given values of c_o^k, $\mathrm{var}(w_1^k)$, and $\mathrm{cov}(w_1^k, r_i)$, $i=1,\ldots,n$, would constitute the efficient set.[12]

Problem (7) contains (n+2) explicit constraints plus any additional constraints required to maintain feasibility. Since there are only n decision variables, at most n of these constraints can be linearly independent. If no two risky assets have perfectly positively or negatively correlated returns and if there is at most one riskfree asset, it is possible to show that a subset of n constraints of problem (7) will be linearly independent.[13]

Any constraints including feasibility constraints in excess of these n linearly independent constraints would be redundant and could be dropped from the constraint set of (7) without affecting the solution. Since n linearly independent constraints defined over n variables uniquely determine those n variables, the feasible set of portfolios, for given values of c_0^k, $E(w_1^k)$, and the relevant covariances, reduces to a single portfolio. The minimization part of problem (7) becomes trivial and its economic substance nil. Indeed, the objective of minimizing the variance of end-of-period wealth could just as well be replaced by one of maximizing this variance or for that matter by any other objective. Moreover, no feasible portfolio would be inefficient, making inefficiency a trivial concept.

In sum, problem (7) as formulated lacks economic substance. While macro relationships could be developed formally from (7) with the redundant constraints, it should not be surprising that such macro relationships would in general also lack economic substance.

Before exploring ways to introduce economic substance into (7), let us consider the effect of allowing uncertainty about end-of-period commodity prices. The function f_0 in (5) for investor k would now be defined over c_0, w_1^k, the vector R_1 as well as the vector of end-of-period commodity prices. If returns and prices are jointly normal, the function g, derivable from $E[f_0]$, would in turn be defined over c_0, $E(w_1^k)$, $var(w_1^k)$, $cov(w_1^k, r_i)$, $i=1,\ldots,n$, as well as the covariances of w_1^k with each end-of-period commodity price. As before, g would be conditioned by a large number of expected values, variances, and covariances outside the control of the individual investor.

The partial derivative of g with respect to $var(w_1^k)$ would be negative, so that, for given values of c_0, $E(w_1)$, and the relevant covariances, an investor would want to minimize $var(w_1^k)$. The effect on (7) would be to add for each commodity an additional constraint of the form

$$\Sigma_j \, v_j^k \, cov(r_j, \pi_\ell) = cov(w_1^k, \pi_\ell), \tag{8}$$

where π_ℓ is the end-of-period price of commodity ℓ. Since (7) already contains redundant constraints, these additional constraints would be redundant. Thus, (7) as formulated with constraints (7b), (7c), and (7d) will yield optimal investment decisions whether or not end-of-period commodity prices are known at the current time.

One way to provide economic substance to the objective of minimizing variance is to make some kind of an assumption about an investor's multiperiod problem which allows the constraint set of (7) to be expressed in less than n constraints. In this way, more than one feasible portfolio may be consistent with any specific set of constraints, so that the objective of minimizing variance becomes nontrivial.

With no prohibition on short sales or any other type of institutional constraint, the efficient set problem as originally conceived by Markowitz would require only constraints (7b) and (7c), so that the objective of minimizing variance would typically be non-trivial. A review of the logic of this paper shows that this traditional formulation is consistent with a multiperiod world in which future commodity prices are known and returns on assets are jointly normal within a period but independent between periods.

If there are institutional constraints, it may be necessary to consider them explicitly in reducing the number of constraints to less than n. When there are n linearly independent constraints, the parametization of c_0^k, $E(w_1^k)$, and the relevant covariances over only feasible values guarantees feasibility. When there are less than n linearly independent constraints, feasibility is no longer guaranteed.

For instance, consider a world in which short sales are prohibited, commodity prices are certain, and returns are normal but independent over time. For given values of c_0^k and $E(w_1^k)$, the minimum variance feasible portfolio may involve zero investments because of the short sale restriction. To account for this restriction, the two constraints (7b) and (7c) could be augmented by constraints of the form

$$v_\ell = 0, \tag{9}$$

where ℓ takes on the index values of those assets for which the short sale restriction is binding. Parenthetically, a non-marketable asset could be incorporated into the minimization problem by setting v_ℓ to some positive value instead of zero.[14]

If returns are dependent over time or if there is uncertainty as to commodity prices, the constraint set in (7) would typically contain n linearly independent constraints. To reduce the number of constraints to less than n might involve an approximating assumption. For example, it might be asserted that an investor is primarily concerned about dependencies in returns over time because of the relationship between returns and commodity prices. If the investor is willing to measure changes in commodity prices over time by a single index number, say π, the function f_0 might then be approximated by a function defined over c_0^k, w_1^k, and π. If π were jointly normal with the individual returns, the constraint set to (7) could be replaced by (7b), (7c) and the following

$$\Sigma_j \; v_j^k \; \mathrm{cov}(r_j,\pi) = \mathrm{cov}(w_1^k,\pi). \tag{10}$$

Again, explicit account of feasibility constraints may be necessary.

Merton has suggested the possibility that the changes in the riskfree rate from one period t the next, Δr_f, might capture the dependencies in returns over time which would be of concern to an investor.[15] If Δr_f were jointly normally distributed with the returns on other assets,[16] the constraint set to (7) could be replaced by (7b), (7c), and the following

$$\Sigma_j \; v_j^k \; \mathrm{cov}(r_j,\Delta r_f) = \mathrm{cov}(w_1^k,\Delta r_f) \tag{11}$$

Again, explicit account of feasibility constraints may be necessary.

The constraints do not have to be linear and could be non-linear. Such non-linear constraints enter quite naturally when returns are not normally distributed. An investor faced with a multiperiod problem in which his seemingly one-period utility function f_0 could be defined only over c_0^k and w_1^k might be willing to approximate $E[f_0]$ by a function g defined over c_0^k and the expected value, variance, and third moment of w_1^k. Arditti has shown under reasonable assumptions that the partial derivative of g with respect to $\mathrm{var}(w_1^k)$

is negative, so that for feasible values of c_o^k, $E(w_1^k)$, and the third moment, denoted $m_3(w_1^k)$, an investor would choose that portfolio which minimized $var(w_1^k)$. Thus, the constraint set to (7) could be replaced by (7b), (7c) and the following:

$$E[\Sigma_i (v_i r_i - E(\Sigma_j v_j r_j))^3] = m_3(w_1^k) \quad (12)$$

The use of moments greater than the third would allow finer approximations.[17]

B. Macro Relationships

In a completely general multiperiod consumption-investment problem, the objective of minimizing the variance of end-of-period wealth, which has proved such a useful concept in one-period models, has virtually no economic substance. In rough terms, the multiperiod nature of the problem forces so many constraints onto an investor that his portfolio choice is already made before he would even have the chance to minimize variance. One way to give economic meaning to the objective of minimizing variance in a multiperiod setting is to make some kind of an assumption about the world or some approximating assumption which allows the number of constraints to be reduced to less than the number of assets.

The constraints in such a reduced set can arise from any of the following three sources:

(a) the multiperiod nature of the problem itself,

(b) feasibility constraints, often associated with institutional restrictions, and

(c) departures from normality in the distribution of returns and other relevant variables.

These constraints are intimately connected to so-called "mutual fund theorems" and to the form of the equilibrium pricing relationship for individual assets.

The formal analysis will assume that three constraints are adequate to describe the portfolio selection process. The same logic applies to more than three constraints. Let the first two constraints be the usual ones of Markowitz, namely (7b) and (7c). The analysis will be carried out first assuming the third constraint to be linear and then non-linear. While not required, the n^{th} asset will be assumed to be riskfree in nominal dollars. As long as the government has the power to print money and does issue such riskfree assets, it is artificial to assume that such assets do not exist as is sometimes done.

The Linear Case. Let this third constraint in its linear form be given symbolically by

$$\Sigma_j \; v_j^k d_j = d^k, \tag{13}$$

where the d_j's are constants and the same for all investors. The specific values assigned to the d_j's would, of course, hinge upon the particular simplifying assumptions used in giving economic substance to the objective of minimizing variance.

For given values of c_0^k, $E(w_1^k)$, and d^k, the optimal amounts invested in each asset after revision will be those values of the v_i's which maximize

$$\Sigma_i \Sigma_j \; v_i^k v_j^k \sigma_{ij} - 2 \; \lambda_u^k [w_0^k - c_0^k + \Sigma_j \; v_j^k \; \mu_j - E(w_1^k)] \tag{14}$$

$$+ \; 2 \; \lambda_w^k [\Sigma_j \; v_j^k - w_0^k + c_0^k] + 2 \; \lambda_d^k [\Sigma_j \; v_j^k \; d_j - d^k],$$

where λ_μ^k, λ_w^k, and λ_d^k are Lagrange multipliers appropriate to investor k.

The first order conditions of (14) are

$$\Sigma_{j=1}^{n-1} \; \sigma_{ij} v_j^k - \lambda_\mu^k \; \mu_i + \lambda_w^h + \lambda_d^k \; d_i = 0, \; i=1,\ldots,n-1, \tag{15a}$$

$$- \; \lambda_\mu^k \; \mu_f + \lambda_w^k + \lambda_d^k \; d_f = 0, \tag{15b}$$

plus the three constraints of the original minimization problem. The parameters μ_n and d_n for the riskfree asset have been replaced by their more suggestive counterparts μ_f and d_f. Using (15b) to eliminate λ_w^k from (15a) and solving for the v_i^k's, one obtains

$$v_i^k = \lambda_\mu^k[\Sigma_{j=1}^{n-1} \sigma^{ij}(\mu_j-\mu_f)] - \lambda_d^k[\Sigma_{j=1}^{n-1} \sigma^{ij}(d_j-d_f)], \ i=1,\ldots,n-1, \quad (16)$$

where σ^{ij} are the elements of the inverse of the variance-covariance matrix of the (n-1) risky assets. The terms in the brackets do not have superscripts k and thus are the same for all investors.

The vector of amounts invested in each risky asset is therefore a weighted sum of two vectors which are the same for all investors; only the weights differ from one investor to another. The elements of these two vectors, which are simply numbers, could be interpreted as portfolio weights. Thus it could be said that all investors hold the same two portfolios of risky assets, differing only in the relative proportions of their wealth invested in each. Any wealth not invested in risky assets and not consumed would, of course, be placed in the riskfree asset. It has become fashionable to call such portfolios "mutual funds" and statements about the number and composition of such funds "mutual fund theorems."

Counting the riskfree asset as a separate portfolio, three mutual funds result from the three constraints of the original minimization problem--a "three fund theorem." In general, there will be an additional mutual fund with each additional constraint.[18] For example, in the traditional one-period world of Markowitz with no institutional constraints, there would be the two constraints (7b) and (7c) and thus two funds--usually taken to be the riskfree asset and the market portfolio of all risky assets. This two-fund theorem is also known as a "separation theorem."

The correspondence between the number of constraints and the number of mutual funds holds even if the number equals or exceeds the number of assets. In this case, the objective of minimizing variance would have no economic content as would any corresponding mutual fund theorem. For instance, with n assets and n linearly independent constraints, there would be n distinct or linearly independent portfolios. These n linearly independent portfolios as well as any other set of n linearly independent portfolios can with appropriate weights define any feasible portfolio. Thus, in this spanning sense, any two sets of n linearly independent portfolios are economically indistinguishable. Indeed, if there is one set of n linearly independent portfolios with no economic substance, it can be concluded that all sets of n linearly independent portfolios lack economic substance. One such set with clearly no economic substance consists of n portfolios in which each asset is assigned to one and only one portfolio.

The reason mutual fund theorems with fewer funds than assets may have economic meaning is the possibility that linear combinations of the funds may only be able to span a proper subset of the feasible portfolios. Thus, two mutual funds theorems with fewer funds than assets are economically distinguishable if the funds span different subsets of the feasible portfolios. This limitation on the number of economically meaningful funds does not appear to have been recognized explicitly in the literature because it is not immediately obvious from a straightforward optimization of expected utility functions.[19]

The appendix shows how (15) can be aggregated over all investors and manipulated to yield the following equilibrium relationship for the expected returns on individual assets:

$$\mu_i = r_f + (\mu_m - r_f)\beta_i + \gamma[(d_i - d_f) - \beta_i(d_m - d_f)], \qquad (17)$$

where μ_m is the expected return on the market portfolio of all assets including the riskfree asset, β_i is the usual beta coefficient of the capital asset pricing model defined as the ratio of $cov(r_i,r_m)$ to $\sigma^2(r_m)$, γ is the ratio of $\Sigma_k \lambda_d^k$ to $\Sigma_k \lambda_\mu^k$, and d_m is the average of the individual d_i's weighted in proportion to their market values. Without the last term generated by the third constraint, (17) would be the same as the traditional capital asset pricing model.

If there were four instead of three constraints, there would be a further additional term of the form $\delta[(e_i-e_f) - \beta_i(e_m-e_f)]$, the same form as the term associated with the third constraint. As in the first additional term, the e_i's represent the coefficients of a linear constraint and δ the ratio of appropriate sums of Lagrange multipliers.

In sum, the first two constraints of the minimization problem generate the traditional capital asset pricing model; each additional linear constraint generates a term in the form of the third term in (17). If there were n linear independent constraints, the equilibrium relationship, just as the objective of minimizing variance, would lack economic content. In this case, there would be the traditional risk premium plus (n-2) additional terms--a total of (n-1) risk premia. Any given vector of expected returns for (n-1) risky assets can always be expressed as the sum of the riskfree asset and (n-1) risk premia and coefficients as long as there are no restrictions on the coefficients.

To illustrate the meaning of the additional term in (17), consider an investor who is only concerned with the expected value and variance of w_1^k and the covariance between w_1^k and inflation, say π. The third constraint would be given by (10); and since $cov(r_f,\pi)$ is zero, the additional term would be

$$\gamma[cov(r_i,\pi) - \beta_i cov(r_m,\pi)] \qquad (18)$$

The expression in brackets measures the deviation of the $cov(r_i,\pi)$ from the market covariance adjusted for β_i. When zero, the expected return on the asset is given by the traditional capital asset pricing model. Thus, $(\mu_m-r_f)\beta_i$ implicitly assumes that $cov(r_i,\pi)$ is $\beta_i cov(r_m,\pi)$ with any deviation requiring an adjustment to the expected return in the form of (18). Preliminary work towards ascertaining the sign of γ suggests that some further assumptions may have to be made as to the specific form of individual utility functions, the distribution of wealth among individuals, as well as the kinds of productive assets available for investment.

As another illustration, consider a world in which returns are normally distributed and independent over time, commodity prices are known, but with the institutional restriction that assets cannot be sold short. Let us assume that, by chance, there happened to be only one asset which some investors would have wanted to sell short if they could. Letting it be the first asset, one could add for this group of investors a constraint setting v_1 to zero.[20] The equilibrium relationship would then be

$$\mu_i = r_f + (\mu_m-r_f)\beta_i + \gamma(d_i-\beta_i d_m), \tag{19}$$

where d_i is 1.0 when i is 1 and 0.0 otherwise and d_m is just the ratio of the market value of the first asset to the market value of all assets.

A careful analysis of the Lagrange multipliers shows in this case that γ is negative.[21] While $(d_i-\beta_i d_m)$ can theoretically take on any value for any asset, it would in practice probably be positive for the first asset and negative for the remaining risky assets and, in this example, zero for the riskfree asset. Beta coefficients for risky assets are usually positive and for markets with a large number of relatively equally sized assets, the product $\beta_i d_m$ would be less than 1.0. Thus, the effective short sale restriction on the first asset would result under reasonable values of β_i and d_m in a reduction in the expected returns on the first asset and an increase in the expected returns on the remaining risky assets.

Intuitively, the short sale restriction has reduced the demand for those risky assets which all investors would wish to hold long because some investors, if they could, would have wanted to sell short the first asset and invest at least a portion of the proceeds in the remaining assets. By the same token, the demand for the first asset has been increased because some investors, if they could, would want to hold a negative amount of this asset, not just a zero amount. Following Stiglitz, one would not expect such differences in expected returns to persist if they stemmed merely from the array of available financial assets. Firms, by themselves or through intermediaries, could realize profit by restructing the array of available financial assets. Such differences in expected returns however could persist over time if they stemmed from the characteristics of the real assets held in the economy.

In a more realistic setting, investors would probably wish to sell short more than one asset. To incorporate this possibility, the constraints (7b) and (7c) could be augmented with additional constraints setting the v_i^k's equal to zero where appropriate.[22] At the extreme, there would be n additional constraints--a total of n+2. Thus, a short sale restriction which restrains some investor in every asset would seem to result in an economically empty statement of equilibrium. With additional assumptions as to investor's utility functions, the distribution of wealth, and the nature of the investment assets, a useful equilibrium relationship might be developed.

<u>The Non-Linear Form</u>. It may happen because of non-normalities or institutional restrictions that the additional constraints would be non-linear. Let such a non-linear constraint be given symbolically by

$$d(v_1^k, v_2^k, \ldots, v_n^k) = d^k \tag{20}$$

As written, (20) assumes that the functional form d is the same for all investors--a kind of homogeneous expectation assumption. Corresponding to (14) would be a Lagrangian expression except that $\Sigma_j v_j^k d_j$ would be replaced by the function d.

In solving for the v_i^k's from the first order conditions of this Lagrangian expression, one would obtain as a kind of mutual fund theorem the expression:

$$v_i^k = \lambda_\mu^k[\Sigma_{j=1}^{n-1} \sigma^{ij}(\mu_i-\mu_f)] - \lambda_d^k[\Sigma_{j=1}^{n-1} \sigma^{ij}(d_j^k-d_f^k)], \quad i=1,\ldots,n-1, \tag{21}$$

The symbol d_j^k represents the partial derivative of d with respect to v_j^k evaluated at the optimal values of the v_i^k's for investor k. The symbol d_f^k referring to the n^{th} or riskfree asset is similarly defined.

Counting the riskfree asset as a mutual fund, (21) shows that any portfolio can be viewed as a linear combination of two portfolios which are the same for all investors and a portfolio unique to each investor. If the composition of this last portfolio were completely arbitrary, such a three-fund theorem would have little economic content. However, the composition of this third portfolio is not arbitrary but determined explicitly by the function d, so that such theorems may potentially be useful for both positive and normative purposes. As with linear constraints, each additional non-linear constraint would generate an additional fund.[23]

The equilibrium relationship for the μ_i's would have the same first two terms as (17) but a different third term, namely

$$\frac{1}{\Sigma_k \lambda_\mu^k} \{\Sigma_k \lambda_d^k d_i^k - \Sigma_k \lambda_d^k d_f^k - \beta_i[\Sigma_j w_j(\Sigma_k \lambda_d^k d_j^k - \Sigma_k \lambda_d^k d_f^k)]\}, \tag{22}$$

where the w_j are weights proportional to the market value of each asset j and summing to 1.0 over all assets including the riskfree asset. If the d_i^k's had the same value for all investors, the k superscripts could be dropped and $\Sigma_k \lambda_d^k$ factored out from the braces. Since γ is defined as the ratio of $\Sigma_k \lambda_d^k$ to $\Sigma_k \lambda_\mu^k$, the resulting expression would be the same as in the linear case. Thus, the differences in values of the d_i^k's from one investor to another differentiates the equilibrium relationships as between the linear and non-linear case.[24] As in the linear case, each additional non-linear constraint would generate an additional term of the same form as the third one in (22).

The economic difference is as follows: In the linear case, the third term is a product of a term which is intimately related to the preferences of individual investors as embodied in the Lagrange multipliers and a term which measures some characteristic of the individual asset. Depending upon the third constraint, one might be able to observe this last term without knowledge of the preferences of the individual investors. In the non-linear case, this separation between individual preferences and the characteristics of an individual security does not occur. Thus, from the point of view of estimation, approximations to the investors' problem using linear constraints would generally seem to be preferable to those involving non-linear constraints.

IV. Conclusion

This paper has shown how a general multiperiod consumption-investment problem can be recast as a one-period problem of minimizing over the available assets the variance of end-of-period wealth subject to various constraints--a generalization of Markowitz efficient set problem. The only really restrictive assumption was that investors could not affect, through their own actions, the probability distributions of future returns and commodity prices. As was shown, it was not even necessary to assume that these random variables were normal.

The rub however is that in general this minimization problem would have no economic content and therefore equilibrium relationships derived from this problem would also lack economic content. Intuitively, the multiperiod nature of the consumption-investment problem forces an investor to consider the relationship of his current decisions to so many things in subsequent periods that his current investment-consumption decisions are already made before he would even have an opportunity to minimize the variance of end-of-period wealth. The objective of minimizing variance makes no difference.

One way to introduce content is to make some assumption which would allow the problem of minimizing the variance of end-of-period wealth to be reformulated with fewer constraints than assets. While there are some assumptions which would allow such a reformulation with no approximation, most would typically involve some kind of approximation. Whether or not some assumption represents an adequate approximation is ultimately an empirical question and would depend upon one's purpose. What may be adequate for one purpose may be inadequate for another. With the results of this paper, a researcher can make an approximating assumption, directly write the equilibrium relationship for the expected returns of individual assets, and test its adequacy.

FOOTNOTES

[1]For example, see Blume and Friend for one test and references to others.

[2]E.g., Fama (1970)

[3]The indirect utility function V will be related to U by

$$V(c_0,c_1,c_2, \Pi_0,\Pi_1,\Pi_2) = \max_{Q_0,Q_1,Q_2} [U(Q_0,Q_1,Q_2)],$$

where the maximization is subject to the constraints that $Q_t \cdot \Pi_t \leq c_t$, $t=0,1,2$.

[4]It might be noted that this discrete formulation of the investor's decision problem is fundamentally different from the continuous time models which have been used by researchers like Merton. This discrete formulation does not in general reduce in the limit to these kinds of continuous time models. The continuous times models as used by Merton and others can be construed as limiting cases of discrete time models with additive utility functions. Additive utility functions and thus these continuous time models cannot allow explicitly for an individual's potential desire to smooth his consumption expenditures over time, whereas the discrete model presented here, in allowing for interaction among consumption levels over time, can incorporate such a desire.

[5]For example, an investor with little desire for consumption at time 1 might rationally prefer a lesser amount of wealth at time 1 if he could be guaranteed that the riskfree rate in the second period would be great enough. Thus, an investor might prefer at time 1 a state of the system in which his nominal wealth were $100 and the second period riskfree rate were 20 percent to a state in which his nominal wealth were $110 and the second period riskfree rate were 0 percent.

[6]That (4) will give the same optimal values as (2) is based upon the following type of argument. Let g(x,y) be a one-time differentiable strictly concave function with a maximum at some finite point (x,y) within the region of feasibility. Consider the problem $\max_y g(x,y)$. The function g will obtain its

maximum when $\partial g/\partial y = 0$. This last equality implicitly defines the optimal y as a function of x, say $y^*(x)$. Now, define $f(x) \equiv g[x, y^*(x)]$. The function f(x) is maximized for that x such that $\partial f/\partial x = 0$, say x^*. It is now possible to show that the point x^* and $y^*(x^*)$ is a maximum point since the two partial derivatives of g are both zero at that point. To extend this argument to cases in which g obtains its maximum on the edge of the feasible region, the reader is referred to Sethi.

[7]Cf. Fama (1976).

[8]In solving the multiperiod problem, Fama and MacBeth have developed a function similar to f_0 in (4) in that their function is defined over current consumption, end-of-period wealth, and state-of-the-system variables like the return vector R_1. In their development, one reason that the state variables enter into the seemingly one-period problem is that they define the original multiperiod utility function over the consumption vector and all state variables. What this paper shows is that even if the indirect multiperiod utility function is defined only over consumption expenditures, the optimization process itself could induce the inclusion of state variables into the seemingly one-period function f_0.

[9]Following the approach of Tobin, one would replace each of the random variables in f_o by its standardized equivalent. For example, w_1 would be replaced by $E(w_1) + \sigma(w_1)z(w_1)$, where $z(w_1)$ is the standardized normal variante associated with w_1. Performing the integration with respect to these standardized normal variates yields the desired result.

[10]The following establishes the proposition: define h as

$$h(\mu_x,\mu_y,\sigma_x,\sigma_y,\rho) = \int_{z_x}\left[\int_{z_y} f(\mu_x + \sigma_x z_x, \mu_y + \sigma_y z_y) n(z_x | z_y) dz\right] n(z_y) dz_y,$$

where x and y are jointly normally distributed with means μ_x and μ_y, standard deviations σ_x and σ_y, and correlation coefficient ρ and where z_x and z_y are standardized normal variates with correlation coefficient ρ. The function f is assumed strictly concave in its first argument for fixed values of the second. Take the partial of h with respect to σ_x to obtain

$$\frac{\partial h}{\partial \sigma_x} = \int_{z_x} \left[\int_{z_y} \frac{\partial f}{\partial\,(\mu + \sigma_x z_x)} \cdot z_x n(z_x | z_y) dz_x \right] n(z_y) dz_y .$$

The term in brackets may be recognized as the expectation of $z_x \partial f/\partial(\mu + \sigma_x z_x)$, conditional on z_y. Because of the concavity of f in its first argument and the symmetry of the normal distribution, this conditional expectation is negative. Integrating with respect to z_y preserves the negativity. The proposition in the text follows from the fact that $\partial h/\partial \sigma_x$ and $\partial h/\partial \sigma_x^2$ are of the same sign.

[11]Since this property will not be used in the derivation of the macrorelationships, a formal proof is omitted.

[12]As formulated, there would correspond an efficient set for each level of initial wealth w_0^k. By dividing each of the constraints by w_0^k and the objective function by the square of w_0^k, (7) can be reformulated in terms of proportions of initial wealth which are consumed and invested in each asset. Such an efficient set stated in terms of proportions would be the same for all investors regardless of their initial wealth. This formulation is more analogous to the usual presentation of the efficient set in a one-period world.

[13]In the absence of a riskfree asset, the assumed lack of perfect correlation between the return on any two assets assures that the variance-covariance matrix has full rank. Following Sharpe, it is well known that a zero-variance protfolio can be constructed from risky assets if and only if the returns of two or more of the assets are perfectly positively or negatively correlated. Since the return on a portfolio is a linear combination of the returns on the individual assets, it follows that the variance-covariance matrix has full rank. Thus, the n constraints given by (7d) are linearly independent. If there is a riskfree asset, the variance-covariance matrix of all assets will have rank (n-1). In this case, n linearly independent constraints can be formed from the (n-1) constraints of the form (7d) corresponding to the risky assets and either constraint (7b) or (7c).

[14]Mayers' paper on non-marketable assets can be interpreted in this framework.

[15]While Merton assumed commodity prices were known, the essential point is that, even if prices were uncertain, an investor could act as if his one-period recursive function, f_o, were defined only over c_o^k, w_1^k, and Δr_f.

[16]If the returns on all assets were jointly normal, Δr_f would not be normally distributed. To capture the essence of Merton's approximation, the first period return on a two-period discount instrument might be used instead of Δr_f. Such a return would be normally distributed and inversely related to Δr_f.

[17]If returns were dependent over time, the constraints of (7), augmented by (12) and a host of other third-order constraints, would still contain all the constraints of the form (7d). All of the third order constraints would therefore be redundant. This redundancy should not be surprising. Generalizing an economically meaningless model would not be expected to make it meaningful. What are needed are restrictions, not generalizations.

[18]Each additional constraint will generate an additional Lagrangian term in (14) which will in turn generate an additional portfolio in (16).

[19]Cf. John Long or Robert Merton.

[20]To preserve consistency in the interpretation of the Lagrange multipliers, one should also add constraints for the remaining investors setting v_1 to their optimal non-zero values. In this way, a Lagrange multiplier such as λ_μ^k would be interpreted for both groups as the rate of change in the minimal variance of end-of-period wealth associated with a change in μ^k, holding constant all the other constraints including the amount invested in the first asset.

[21]On the assumption that $E(w_1^k)$ is not at its maximum or minimum feasible value, an increase in this quantity, holding everything else constant, would require an increase in the minimum possible value of $\mathrm{var}(w_1^k)$, implying that λ_μ^k is positive and non-zero. For those investors truly constrained by the short sales restriction, λ_d^k would be negative and non-zero since a reduction in d^k to

a negative value would allow the investor to achieve a smaller variance for the same $E(w_1^k)$. On the assumption that an investor has not placed all of his wealth in one asset, an investor not constrained by the short sale constraint would choose that value of d^k which gave him the minimum variance portfolio for a given $E(w_1^k)$. Changes in d^k, either increases or decreases, would result in greater variances. Thus, λ_d^k would be zero. It follows immediately that γ, defined as $\Sigma_k \lambda_d^k / \Sigma_k \lambda_\mu^k$, would be negative as long as one investor is constrained by the short sale restricton.

[22]In the spirit of footnote 20, one could preserve consistency in the interpretation of the Lagrange multipliers as between individuals by including a constraint for each security which some investors would want to short sell in every minimization problem setting the appropriate v_i^k to zero or to its optimal positive value.

[23]Rubinstein has developed an equilibrium pricing relationship incorporating skewness. His formula is the same as one would obtain by combining (12) and (21).

[24]The same type of differences would occur if the third constraint were linear but the coefficients differing in value from one investor to another. Thus, there is a formal similarity between non-linear constraints and heterogeneous expectations.

APPENDIX

DERIVATION OF THE EQUILIBRIUM RELATIONSHIPS (17) AND (22)

Equation (17) is obtained as follows: Sum (15) over k and express the resulting equation as

$$\frac{1}{\lambda_\mu} \Sigma_{j=1}^{n} \sigma_{ij} V_j = \mu_i - \frac{\lambda_w}{\lambda_\mu} - \frac{\lambda_d}{\lambda_\mu} d_i, \ i=1,\ldots,n, \tag{A1}$$

where λ_x is defined as $\Sigma_k \lambda_x^k$ and V_j is the total market value of asset j given by $\Sigma_k v_j^k$. Letting V_m be the market value of all assets, the above equation can be multiplied by the ratio V_i/V_m and aggregated over i to give an expression involving the expected return on the market portfolio, μ_m:

$$\frac{1}{\lambda_\mu} \Sigma_{i=1}^{n} \frac{V_i}{V_m} \Sigma_{j=1}^{n} \sigma_{ij} V_j = \mu_m - \frac{\lambda_w}{\lambda_\mu} - \frac{\lambda_d}{\lambda_\mu} \Sigma_{i=1}^{n} \frac{V_i}{V_m} \cdot d_i \tag{A2}$$

The summation on the left hand side of (A1) may be recognized as V_m $cov(r_i, r_m)$, where r_m is the return on the market portfolio; and the double summation on the left hand side of (A2) as $V_m \ var(r_m)$. Dividing the second equation into the first yields after some simplification:

$$\mu_i - \frac{\lambda_w}{\lambda_\mu} - \frac{\lambda_d}{\lambda_\mu} d_i = \beta_i [\mu_m - \frac{\lambda_w}{\lambda_\mu} - \frac{\lambda_d}{\lambda_\mu} d_m], \tag{A3}$$

where β_i is defined as the ratio of $cov(r_i, r_m)$ to $var(r_m)$ and d_m as the average of the individual d_i's weighted by their market values.

Up to this point, the derivation has not explicitly used the assumption that asset n is riskfree. Thus, (A3) holds in the absence of a riskfree rate.

After aggregation, (15b) implies that the return on the riskfree asset, μ_n or r_f, satisfies

$$\frac{\lambda_w}{\lambda_\mu} = r_f + \frac{\lambda_d}{\lambda_\mu} \cdot d_f, \tag{A4}$$

where d_f denotes the coefficient on the n^{th} asset in the linear constraint. By substituting (A4) in (A3) and rearranging terms, one obtains equation (17) of the text.

To derive (22), replace d_i in (A1) by d_i^k, the partial derivative of d evaluated at the investor's optimal investment strategy. Repeating virtually the same mathematical manipulatons, one would obtain the desired results. The generalization of the equilibrium to four or more linear or non-linear constraints is straightforward.

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