Mutual Fund Separation in Financial Theory--The Separating Distributions

by

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Abstract

This paper finds necessary and sufficient conditions on the stochastic structure of asset returns for portfolio choice to be equivalent to choice among a limited number of mutual funds of assets, independent of investors' preferences. This type of separation result is central to much of modern financial theory and, as a consequence, the distributions which satisfy these conditions, the separating distributions, form the underlying basis for much of this theory.

Introduction

Modern financial theory derives much of its analytic power from a few strong assumptions it imposes on the models it develops. Without such assumptions the special problems of the financial theorist, e.g., the comparative statics of portfolio positions, the pricing of financial instruments and, quite generally, the behavior of speculative markets would be as intractible and empirically empty in finance as they are in general equilibrium theory. Perhaps the most successful of the theoretical assumptions employed has been that of separability. Roughly speaking, separation occurs in a portfolio problem of choice among many risky assets when that choice is simplified to that of choosing amongst combinations of subsets, or funds, of these assets.

The first rigorous separation results in portfolio theory were due to Markowitz and Tobin [1958], but the intuition if not the rigor of separation had long played an important role in the neoclassical literature. Earlier work by Fellner and Hicks had made it clear that the relevant parameters in asset choice problems were those of return and risk, and it was well understood that the problem of valuing risky assets in equilibrium was essentially the problem of determining the risk premium, i.e., the differential anticipated return of the risky asset over that of a sure asset. In their development of the mean variance analysis in portfolio theory, Markowitz and Tobin were the first to put the tradeoff between return and risk on a solid analytic footing.

In a mean variance analysis the investor is concerned with only two parameters of the probability distribution of total returns on investment, the mean return and the variance of the return. For a risk averse investor the latter is a "bad" to be traded off against higher mean returns; risk in such an analysis is equivalent to variance. Aside from simply putting earlier notions into mathematical notation within this framework, Markowitz and Tobin obtained a number of important results. Most notably, the analysis stressed the role played by covariance, or, more generally, corelations, among assets in determining the optimal portfolio proportions. In addition, the authors obtained the first separation theorem. In a world with a riskless asset they showed that an investor could separate his portfolio decision into two stages. In the first stage an efficient portfolio or fund, M, of risky assets could be chosen and in the second the investor's attitudes towards risk could be introduced to determine the optimal allocation of wealth between the riskless asset and the efficient portfolio of risky assets. The important simplification is that all efficient portfolios are simply combinations of the same fund, M, of risky assets and the riskless asset.

The extension of the mean variance theory to an equilibrium theory completed the neoclassical analysis and was first accomplished by Sharpe and Lintner. Since the separation principle holds in a mean variance world, all investors with the same ex ante beliefs, regardless of their attitudes towards risk, must hold the same fund of risky assets. Sharpe and Lintner recognized that this must imply that the efficient fund of risky assets, M, is the same as the market portfolio of risky assets and from the conditions which guarantee the efficiency of the market portfolio they derived the mean variance capital asset pricing model that forms the core of much of modern finance. In an important contribution, Black generalized their results to show that separation would also obtain between two efficient funds of risky assets in a mean variance world without a riskless asset.

The simplicity and intuitive appeal of the mean variance portfolio and equilibrium results attracted a great deal of attention and much effort has been directed at determining their generality. Both Markowitz and Tobin noted that if investors were von-Neumann Morgenstern expected utility maximizers, then a mean variance analysis could be justified either by assuming that utility functions were quadratic or by assuming that asset returns were distributed by a multivariate normal distribution. Tobin further remarked that "any 'two parameter' family of random variables" would be sufficient to justify a two parameter risk-return theory.

The use of quadratic utility functions, even in their monotone range, however, to justify the mean variance approach has become somewhat unfashionable. This is due largely to Arrow's observation that the quadratic utility function exhibits increasing absolute risk aversion which implies that risky assets are inferior assets in a portfolio problem. Efforts to find acceptable, tractible classes of utility functions for financial work have proved most successful in intertemporal work. Here separation theorems of a somewhat different kind have been obtained by Leland, Mossin, Hakansson [1971], and Ross [1974], who were concerned with questions of the intertemporal stationarity of optimal portfolio policies and found that the constant relative risk aversion utility functions played a pivotal role; under certain circumstances they implied the separation result that portfolio composition was wealth independent. In a definitive paper, Cass and Stiglitz thoroughly examined the utility function approach to separation. They were able to completely characterize the classes of utility functions that would permit separation in any stochastic environment, in the sense that for all wealth levels an investor with a utility function in one of their classes would divide his wealth between a limited number of specific funds of the assets; the composition of the funds being independent of wealth.

The dual side to this research, i.e., the delineation of the classes of stochastic processes that permit separation for all utility functions, has also been the object of research--perhaps spurred to some extent by Tobin's rather cryptic remark. In fact, it soon became clear that just "any 'two parameter' family of random variables" would not do and that further restrictions had to be imposed. Feldstein, for example, showed that lognormal random variables, while defined by two parameters, would not admit of separation. Both Merton and Ross [1975], however, demonstrated that the use of the continuous time lognormal or Wiener processes would allow separation.

The additional criterion that seemed to be required was that of closure of the random law under addition. The well known Pareto-Levy class of stable distributions not only served as limiting laws in central limit theorems, but also were defined in terms of a type of closure under addition. Mandelbrot introduced this class of distributions into financial work and Fama examined the portfolio implications of stability and proved a separation theorem for these distributions. This work led Cass and Stiglitz to conjecture that the stable distributions were both necessary and sufficient for separation.

In fact, however, there exist a number of counterexamples to this conjecture. Agnew displayed an example of a multivariate distribution which was not stable (and, a fortiori, not normal) yet for which all risk averse individuals would choose mean variance efficient portfolios and, hence, obey the earlier Markowitz and Tobin separation rules. Somewhat less idiosyncratically, a body of what might be called symmetry results has been collected. For example, Samuelson pointed out that if the multivariate distribution function of the random assets is unchanged by permuting the assets then all risk averse investors will allocate their wealth equally across the assets.

The intent of this paper is to resolve the question of what distributions permit separation and, in so doing, to tie together a number of the results cited above. Section I introduces the formal definition of separation and describes the class of distributions with the property that all investors choose the same optimal portfolio.

Section II proves a two fund separability result which will provide the desired generalization of the mean variance and two parameter theories. Section III analyzes the analogous K fund case, and Section IV discusses some further extensions. Section V summarizes and concludes the paper. Portions of the proofs of a technical and supportive nature are contained in an appendix.

I. One Fund Separability

This section introduces the concept of separability and provides necessary and sufficient conditions for the simplest example, one fund separability. A number of previous results in the literature are then examined with the help of these equivalent conditions. A word on notation is in order.

Throughout, $\tilde{X}$ will denote the n-vector, $(\tilde{X}_1,\ldots,\tilde{X}_n)$, of individual random returns. Tildes over variables indicate that they are random, and if one of the assets (or a portfolio) is explicitly assumed to be riskless, it will be taken to be the o^{th} asset. Lower case Greek letters will denote portfolios, n vectors which sum to unity, i.e., α is a portfolio if and only if

$$\sum_i \alpha_i \equiv \alpha e = 1,$$

where

$$e \equiv (1,\ldots,1).$$

(We will permit free short sales, i.e., $\alpha_i < 0.$) The only exception to this rule will be the vector η which will denote an arbitrage portfolio which uses no wealth, i.e.,

$$\sum_i \eta_i \equiv \eta e = 0.$$

Separability, as looked at from the distributional side, is a somewhat ambiguous concept and a variety of possible definitions suggest themselves. We will deal explicitly with only the two that seem most natural, but several other formulations can be shown to be equivalent. A particularly strong form of separation can be defined in terms of the principle of stochastic dominance. Let U denote the set of all monotone, increasing, concave (utility) functions on R. A random return $\tilde{y}$ is said to stochastically dominate an alternative, $\tilde{w}$, written $\tilde{y} \gtrsim \tilde{w}$, iff ($\forall \varepsilon$ U).

$$E\{U[\tilde{y}]\} \geq E\{U[\tilde{w}]\}, \tag{1}$$

where $E\{\cdot\}$ denotes the expectation operator. In other words, by (1), no risk averse investor prefers $\tilde{w}$ to $\tilde{y}$.

In the mathematics literature, Strassen, and in economics, Ross [1971], have independently demonstrated that the statement $\tilde{y} \gtrsim \tilde{w}$ is equivalent to asserting the existence of two random variables $\tilde{z}$ and $\tilde{\varepsilon}$ with

$$\tilde{w} \sim \tilde{y} + \tilde{z} + \tilde{\varepsilon}, \tag{2}$$

where "$\sim$" is read "is distributed as", $\tilde{z} \leq 0$, is a nonpositive random return and $\tilde{\varepsilon}$ is a noise term, i.e.,

$$E\{\tilde{\varepsilon} | \tilde{y} + \tilde{z}\} = 0.$$

It is important to recognize that (2) does not say that $\tilde{w}$ and $\tilde{y} + \tilde{z} + \tilde{\varepsilon}$ are equal, only that they are identically distributed. For example, the number of atoms of radium that decay in a second might have the same distribution as the number of telephone calls made to a central exchange, but the two variables are not necessarily equal.

The sufficiency of (2) for (1) should be clear since (2) simply asserts that $\tilde{w}$ can be constructed from $\tilde{y}$ by shifting some probability mass downward and adding noise, changes that make $\tilde{w}$ of less value for all $u \in U$.

What is somewhat surprising is that the two conditions are actually equivalent. Notice, too, that wealth has been suppressed in the utility functions in (1) since we are regarding the same utility function at different wealth levels as different members of U.

The following definition of separability uses the concept of stochastic dominance.

Definition 1:

A set of returns, $\tilde{X}$, is said to exhibit strong k-fund separability, skfs, iff there exist k mutual funds of the n assets, $\alpha^1,\ldots,\alpha^k$, such that for any portfolio β, there exists a portfolio α,

$$\alpha = a_1\alpha^1 + \ldots + a_k\alpha^k,$$

with

$$\alpha\tilde{X} \succsim \beta\tilde{X},$$

where a_i is the weight given fund α^i and $\sum_{i=1}^{k} a_i = 1$. In addition, there exists some $u \in U$ for which $E\{U(\alpha\tilde{X})\}$ is maximized at some α^*.

The definition of skfs requires that for any portfolio, β, there is a portfolio of the k mutual funds that stochastically dominates β. Trivially, the property has force only when $k < n$, and it is really only useful when k is substantially less than n. The last part of the definition is a boundedness assumption on the returns and it only insures that there be some utility function for which the portfolio problem has a solution. For example, if there were two riskless assets with different returns, this requirement would be violated, and it would be artificial to permit separation in such a case.

The concept of separability can also be stated in a somewhat weaker form than Definition 1. Formally, we can define weak separability as follows.

Definition 2:

A set of returns, $\tilde{X}$, is said to exhibit weak k-fund separability, wkfs, iff there exist k mutual funds of the n assets, $\alpha^1, \ldots, \alpha^k$, such that for any portfolio β and any $u \varepsilon U$, there exists a portfolio α,

$$\alpha = a_1\alpha^1 + \ldots + a_k\alpha^k,$$

with

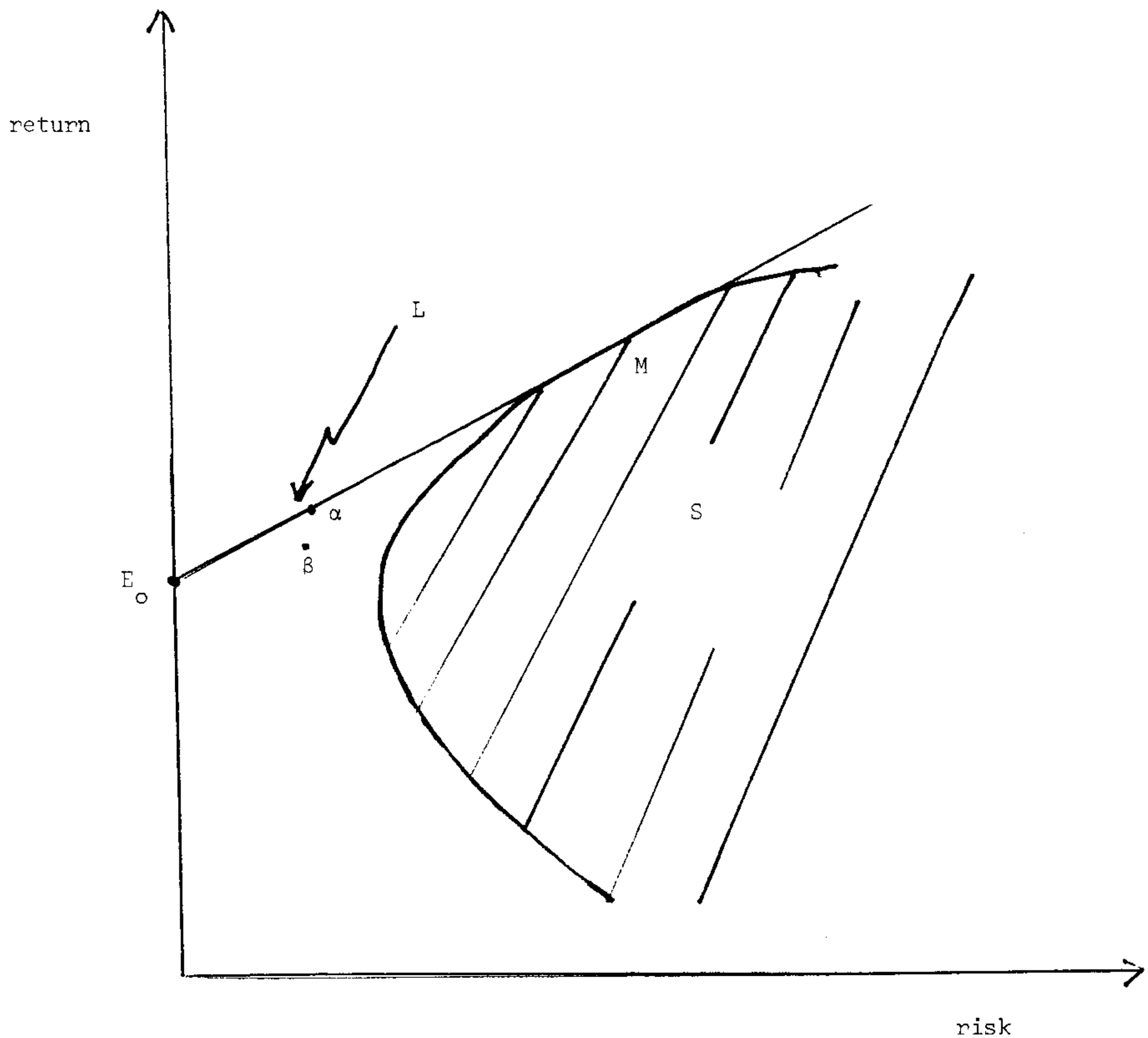
$$E\{U(\alpha\tilde{X})\} \geq E\{U(\beta\tilde{X})\}$$

(if the expectations exist). In addition, there exists some $u \varepsilon U$ for which $E\{U(\alpha\tilde{X})\}$ is maximized at some α^*.

Notice that for wkfs we do not require $\alpha\tilde{X}$ to stochastically dominate $\beta\tilde{X}$, rather we permit the choice of α to depend on U, the particular utility function under consideration. It is clear from the definition that skfs implies wkfs, i.e., if $\tilde{X}$ is strongly k-fund separable then it is weakly k-fund separable. We shall, in fact, show below that these two definitions are equivalent.

It might be useful, at this stage, to examine these definitions in the traditional mean variance or general two-parameter case. Figure I illustrates the familiar geometry with a riskless asset with return E_o. The set S is the set of (return, risk) pairs that can be obtained by forming portfolios of the risky assets alone, and we will assume that S is strictly convex. The efficient frontier is the set of pairs with maximum return for a given level of risk. If we permit free borrowing and lending in the riskless asset, the efficient frontier will be the line L formed by investment at E_o and investment in M, a unique and efficient fund of risky assets. This is an illustration of a two fund separation theorem, since all risk averse investors will choose

Figure I



positions along L, i.e., portfolios made up of investment in a fund consisting of only the riskless asset and a fund, M, of risky assets.

Furthermore, this type of separation is s2fs, since for any choice of β there is a point, α, on L that dominates β for all choices of $u \varepsilon U$; such a point will have the same risk level and a higher return. For w2fs and not s2fs we would need a portfolio β such that different points on L bested it for different utility functions, but no single point, α, was best for all utility functions. The geometry of Figure I makes it clear that such a situation cannot occur.

The two concepts come to the same thing, of course, if $k = 1$, i.e., w1fs and s1fs are equivalent, since in both cases we are requiring that there is a single dominant portfolio for all $u \varepsilon U$. This is a statement of stochastic dominance and in Theorem 1, below, we will use the stochastic dominance results cited above to establish necessary and sufficient conditions on $\tilde{X}$ for 1fs. Some of the technical aspects of the proof have been put in an appendix.

Theorem 1:

A vector of asset returns, $\tilde{X}$, exhibits 1fs if and only if the following conditions are satisfied:

$(\exists \tilde{z}, \tilde{\varepsilon})$

(i) $\tilde{X}_i = \tilde{z} + \tilde{\varepsilon}_i$

(ii) $E\{\tilde{\varepsilon}_i | \tilde{z}\} = 0$ (C1)

and

(iii) $(\exists \alpha)$ $\alpha\tilde{\varepsilon} \equiv 0.$

Proof:

We can verify the sufficiency of (C1) by showing that the portfolio α stochastically dominates any alternative portfolio β. Define η by

$$\beta = \alpha + \eta, \ \eta e = 0.$$

From (i) and (iii)

$$\alpha\tilde{X} = \alpha(\tilde{z} + \tilde{\varepsilon}) = \tilde{z}.$$

Now,

$$E\{\beta\tilde{X}|\alpha\tilde{X}\} = E\{(\alpha + \eta)(\tilde{z} + \tilde{\varepsilon})|\tilde{z}\}$$

$$= E\{\tilde{z} + \eta\tilde{\varepsilon}|\tilde{z}\}$$

$$= \tilde{z}.$$

It follows from (2) that $\alpha\tilde{X}$ stochastically dominates $\beta\tilde{X}$.

The difficult part of the proof is necessity. Suppose that $\tilde{X}$ exhibits lfs. Let α be the dominant portfolio and define

$$\tilde{z} \equiv \alpha\tilde{X}.$$

Since α is the dominant portfolio, for all η with $\eta e = 0$, we must have

$$\alpha\tilde{X} + \eta\tilde{X} \sim \alpha\tilde{X} + \tilde{\varepsilon}_\eta \qquad (3)$$

or

$$\tilde{z} + \eta\tilde{X} \sim \tilde{z} + \tilde{\varepsilon}_\eta,$$

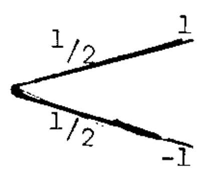
where $\tilde{\varepsilon}_\eta$ is a noise term that depends on η, i.e.,

$$E\{\tilde{\varepsilon}_\eta|\alpha\tilde{X}\} = 0. \qquad (4)$$

Notice that, since U contains only monotone increasing u, we could have $E\{\tilde{\varepsilon}_\eta | \alpha\tilde{X}\} < 0$ (on a set of positive measure) and still satisfy stochastic dominance, but then $E\{\eta\tilde{X}\} < 0$ and we must have $E\{\tilde{\varepsilon}_{-\eta} | \alpha\tilde{X}\} > 0$ which would be a violation. Since (4) must hold for arbitrary choices of η ($\eta e = 0$) we can apply Theorem A2 in the appendix to show that

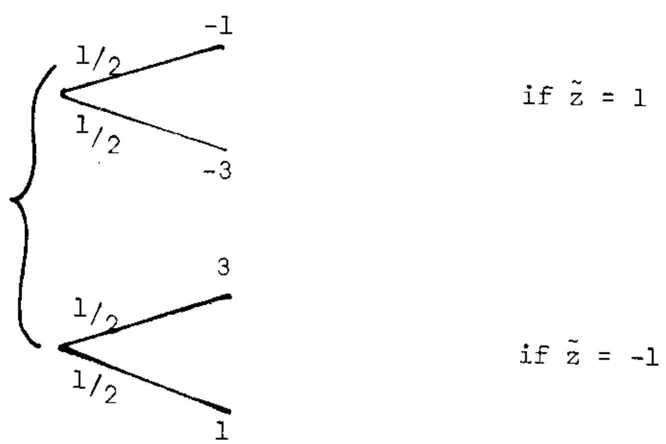
$$E\{\eta\tilde{X} | \tilde{z}\} = 0. \tag{5}$$

Notice that this result is not immediate from (3) and (4) and it is here that the distinction between equality and distributed equality, "$\sim$", becomes important. Consider the following example. Let

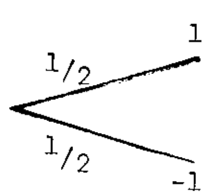
$\tilde{z}$ = 

,

and

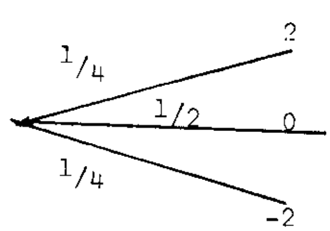
$\eta\tilde{X}$ = 

If we take

$\tilde{\varepsilon}_\eta$ = 

,

then

$\tilde{z} + \eta\tilde{X} = \tilde{z} + \tilde{\varepsilon}_\eta$ = 

and

$$E\{\tilde{\varepsilon}_{\eta} | \tilde{z}\} = 0,$$

but

$$E\{\eta\tilde{X} | \tilde{z}\} = \pm 2 \neq 0.$$

Demonstrating that (5) holds is the technically difficult portion of the proof which is done in the appendix. Given (5), though, the remainder of the argument is straightforward. Letting

$$\eta^{i} \equiv e_{i} - \alpha,$$

where e_i is the i^{th} unit vector we can define

$$\tilde{\varepsilon}_{i} \equiv \eta^{i}\tilde{X} = \tilde{X}_{i} - \tilde{z} \quad .$$

Clearly,

$$\begin{aligned} \tilde{X}_{i} &= \tilde{z} + (\tilde{X}_{i} - \tilde{z}) \\ &= \tilde{z} + \tilde{\varepsilon}_{i}, \end{aligned}$$

and from (5)

$$E\{\tilde{\varepsilon}_{i} | \tilde{z}\} = 0.$$

Q.E.D.

The conditions (C1) provide a constructive characterization that easily permits one to see how general lfs may be. By picking an arbitrary $\tilde{z}$ random return and n-1 arbitrarily chosen (conditionally mean zero random variables) and by defining the n^{th}, $\tilde{\varepsilon}_n$, so as to satisfy (C1)(iii) for some α, the resulting $\tilde{X}$ will exhibit lfs. Of course, it follows from (C1) that a necessary condition for lfs is that all n assets have the same expected return, $E\{\tilde{z}\}$.

In addition, if the $\tilde{X}_i$ have finite variances, then α will simply be the minimum variance portfolio. (This is true even if the $\tilde{X}_i$ do not have compact support, and, consequently, no monotone quadratic utility function can be defined on their range.) To see this, note that the variance of any alternative portfolio, $\beta = \alpha + \eta$, is given by

$$\begin{aligned} \text{Var}\{(\alpha + \eta)\tilde{X}\} &= \text{Var}\{\tilde{z}\} + \eta V \eta \\ &\geq \text{Var}\{\tilde{z}\}, \end{aligned}$$

where V is the covariance matrix of $\tilde{\varepsilon}$.

Conditions (C1) and the sufficiency portion of Theorem 1 also contain a number of previous results as special cases. Rothschild and Stiglitz observed that if the $\tilde{X}_i$ were identically independently distributed, then the unique optimal portfolio would be the equal weight portfolio, $(^1/_n, \ldots, ^1/_n)$. Samuelson generalized this result to the case where the distribution function, $F(X_1, \ldots, X_n)$, is unaltered under permutations of the variables. Suppose that Samuelson's condition is satisfied. Define

$$\tilde{\varepsilon}_i \equiv \tilde{X}_i - \frac{1}{n} \sum_i \tilde{X}_i,$$

and

$$\tilde{z} \equiv \frac{1}{n} \sum_i \tilde{X}_i.$$

Clearly,

$$E\{\tilde{\varepsilon}_i | \tilde{z}\} = E\{\tilde{X}_i - \frac{1}{n} \sum_i \tilde{X}_i | \frac{1}{n} \sum_i \tilde{X}_i\}$$

$$= E\{\tilde{X}_i | \frac{1}{n} \sum_i \tilde{X}_i\} - \frac{1}{n} \sum_i \tilde{X}_i$$

$$= \frac{1}{n} \sum_i \tilde{X}_i - \frac{1}{n} \sum_i \tilde{X}_i$$

$$= 0,$$

where we have used the symmetry of the distribution function to conclude that the conditional expectation of any $\tilde{X}_i$, given the average of the $\tilde{X}_i$'s must be the average.[1/]

These results are too specialized to be terribly important in their own right, but they are useful preliminary results and they do serve to illustrate that lfs can occur with a variety of different distributions. In particular, there is no necessity that the $\tilde{X}_i$ follow a multivariate stable distribution.[2/] Finding necessary conditions on distribution functions equivalent to (C1), though, is difficult and since it also does not seem to be a natural way to pose the problem of separation we will not consider the question of equivalent conditions on distribution functions further. As we shall see, what matters for separation is not the marginal distributions of the returns, but, rather, their co-relations which are properties of the joint distribution.

In the next section we will develop the notion of 2fs and it is with this concept that we will be able to generalize the traditional two parameter portfolio theory.[3/]

II. Two Fund Separability

Two fund separability is, of course, the central theme of modern portfolio and capital asset pricing theory.

Theorem 2:

A vector of asset returns, X, exhibits w2fs if and only if the following conditions are satisfied:

$$(\exists \tilde{y}, \tilde{z}, \tilde{\varepsilon})$$

(i) $\tilde{X}_i = E_i + \tilde{y} + b_i\tilde{z} + \tilde{\varepsilon}_i$,

where

$$E_i \equiv E\{\tilde{X}_i\} = a_o + a_1 b_i,$$

(ii) $(\forall\lambda)$ $\quad E\{\tilde{\varepsilon}_i | \lambda\tilde{y} + (1-\lambda)\tilde{z}\} = 0,$ (C2)

(iii) $(\exists\alpha, \beta)$ $\quad \alpha\tilde{\varepsilon} = \beta\tilde{\varepsilon} \equiv 0,$

and if b is not a constant vector, then

$$\alpha b \neq \beta b.$$

Proof:

The proof of sufficiency is still straightforward, as in the case of 1fs. Let γ be an alternative portfolio to one that is a linear combination of α and β. By (C2)(i) and (iii) we can define λ such that

$$\gamma b = \lambda\alpha b + (1-\lambda)\beta b$$

and

$$\gamma E = \lambda\alpha E + (1-\lambda)\beta E,$$

and η is defined by

$$\gamma = \lambda\alpha + (1-\lambda)\beta + \eta.$$

Clearly, $\eta E = 0$, and therefore

$$E\{\eta\tilde{X}|\lambda\alpha\tilde{X} + (1-\lambda)\beta\tilde{X}\} = E\{\eta E + \eta E\tilde{z} + \eta\tilde{\varepsilon}\,|$$
$$\lambda(\alpha E + \tilde{y} + (\alpha E)\tilde{z}) + (1-\lambda)(\beta E + \tilde{y} + (\beta E)\tilde{z})$$
$$= 0,$$

by (C2)(ii). This verifies that $(\lambda\alpha + (1-\lambda)\beta)\tilde{X}$ stochastically dominates $\gamma\tilde{X}$ and, consequently, (C2) is sufficient for s2fs and, a fortiori, for w2fs as well.

The proof of necessity, again, is more difficult. Suppose that $\tilde{X}$ does exhibit w2fs. By definition, $(\forall u, \gamma)(\exists\lambda)$

$$E\{U[\lambda\alpha\tilde{X} + (1-\lambda)\beta\tilde{X}]\} \geq E\{U[\gamma\tilde{X}]\}.$$

To put the point somewhat differently, for every $u \in U$, the optimum is attained at a portfolio that is a linear combination (simplicial) of two portfolios α and β. If $u(\cdot)$ is chosen to be everywhere differentiable and if it has an internal optimum, then $\lambda\alpha + (1-\lambda)\beta$ must satisfy the first order conditions for an optimum at some value of λ.[4/] The first order conditions are

$$E\{U'[\gamma\tilde{X}](\tilde{X}_i - X_j)\} = 0; \quad \text{all } i, j. \tag{6}$$

On the other hand, we can find the optimal value of λ by the first order condition

$$E\{U'[\lambda\alpha\tilde{X} + (1-\lambda)\beta\tilde{X}](\alpha\tilde{X} - \beta\tilde{X})\} = 0. \tag{7}$$

This condition, (7), then, must imply that (6) is satisfied for $\gamma = \lambda\alpha + (1-\lambda)\beta$. Furthermore, given λ, this implication must hold for all positive, monotone declining $U'(\cdot)$.

Picking a particular λ value and defining

$$\tilde{q} \equiv \lambda\alpha\tilde{X} + (1-\lambda)\beta\tilde{X},$$

$$v(\tilde{q}) \equiv E\{\alpha\tilde{X} - \beta\tilde{X}|\tilde{q}\},$$

and

$$S_i(\tilde{q}) \equiv E\{\tilde{X}_i|\tilde{q}\},$$

it must be the case that for all positive, monotone declining functions $h(\cdot)$,

$$E\{h(\tilde{q})v(\tilde{q})\} = 0,$$

implies that

$$E\{h(\tilde{q})[S_i(\tilde{q}) - S_j(\tilde{q})]\} = 0; \text{ all } i, j.$$

Applying Theorem A1 in the appendix we have that $(\forall i)(\exists b_i)$

$$S_i(\tilde{q}) = S_1(\tilde{q}) = b_i V(\tilde{q}), \text{ a.e.},$$

where we set $b = 0$ if $V(\tilde{q}) \equiv 0$.

Taking expectations over the conditioning variable, $\tilde{q}$, in (8) we have that

$$E_i - E_1 = b_i(\alpha E - \beta E), \text{ for all } i, \tag{9}$$

and if $V(\tilde{q}) \not\equiv 0$, (8) also implies that

$$\alpha b - \beta b = 1. \tag{10}$$

Now, define $\tilde{y}$ and $\tilde{z}$ by the two equation system

$$\alpha\tilde{X} = \alpha E + \tilde{y} + (\alpha b)\tilde{z}$$

and (11)

$$\beta\tilde{X} = \beta E + \tilde{y} + (\beta b)\tilde{z},$$

and let

$$\tilde{\varepsilon}_i \equiv \tilde{X}_i - [E_i + \tilde{y} + b_i\tilde{z}]. \tag{12}$$

From (8), (9), and (10) we have that for all λ

$$E\{\tilde{\varepsilon}_i - \tilde{\varepsilon}_1 | \lambda\tilde{y} + (1-\lambda)\tilde{z}\} =$$

$$E\{\tilde{X}_i - \tilde{X}_1 - [E_i - E_1 + (b_i - b_1)\tilde{z}] | \lambda\tilde{y} + (1-\lambda)\tilde{z}\}$$

$$= E\{\tilde{X}_i - \tilde{X}_1 - b_i(\alpha\tilde{X} - \beta\tilde{X}) | \lambda\tilde{y} + (1-\lambda)\tilde{z}\}$$

$$= 0,$$

and, from (11), $\alpha\tilde{\varepsilon} \equiv 0$ implies that

$$E\{\tilde{\varepsilon}_i | \lambda\tilde{y} + (1-\lambda)\tilde{z}\} = 0.$$

Combining (9) and (12) we have

$$\tilde{X}_i = E_i + \tilde{y} + b_i\tilde{z} + \tilde{\varepsilon}_i$$

and (C2)(i)

$$E_i = a_o + a_1 b_i,$$

where

$$a_o = E_1,$$

and

$$a_1 = \alpha E - \beta E.$$

Notice that in the case of $V(\tilde{q}) \equiv 0$, the problem reduces to that of one fund separability.

Q.E.D.

While the form of (C2), particularly (C2)(i), might seem both specialized and somewhat odd, in so far as the literature goes in this subject, it is quite general. In the first place, conditions (C2), as with (C1), offer a constructive approach to the separation problem. We are free to pick $\tilde{y}$ and $\tilde{z}$ arbitrarily and any n-2 random variables $\tilde{\varepsilon}_i$ which satisfy (C2)(ii),[5/] and then we can define $\tilde{\varepsilon}_{n-1}$ and $\tilde{\varepsilon}_n$ to satisfy (C2)(iii) for some choice of α and β. Moreover, the thrust of the theorem is that (C2) represents the <u>most</u> general set of conditions that can be found which permit the usual development of portfolio theory. To see this more clearly we can use Theorem 2 to examine a number of alternative theoretical developments of portfolio separation. To facilitate the exposition, we will implicitly be considering cases where E is not a constant vector.

Two Parameter Models

The two parameter models introduced by Tobin require that expected utility be a function of only two parameters, the mean return, m, and a risk variable, σ, for all choices of a portfolio. For a given risk level, σ, then, the objective is to maximize the return, m. If this procedure implies a 2 fund separation result for all choices of a utility function, then by Theorem 2 the random distribution must be of the form of (C2).[6/] We can, however, say a bit more than this.

Consider any portfolio, γ, with an expected return, $\gamma E = m$. If we choose λ so that

$$\lambda\alpha E + (1-\lambda)\beta E = m,$$

then

$$\gamma\tilde{X} = (\lambda\alpha + (1-\lambda)\beta)\tilde{X} + \eta\tilde{X}, \tag{13}$$

where

$$\eta e = \eta E = 0 \text{ and, consequently,}$$

$$E\{\eta\tilde{X} | (\lambda\alpha + (1-\lambda)\beta)\tilde{X}\} = 0.$$

In other words, whatever spread parameter, σ, is used the portfolio $\lambda\alpha + (1-\lambda)\beta$ has minimum spread for the given expected return. This permits us to compute the two funds, α and β, when separation occurs.

Furthermore, suppose that the random variables $\tilde{X}_i$ possess variances. It is clear from (13) that $\lambda\alpha + (1-\lambda)\beta$ must be the minimum variance portfolio with the given return $\lambda\alpha E + (1-\lambda)\beta E$. The portfolios α and β then will be two portfolios which span the mean variance efficient frontier. In fact any two such portfolios can be chosen for separation, and this illustrates an important general point about separation theorems.

In (C2) we did not require that α and β be unique. Rather, all we can say is that α and β span a space in which any two (independent) members can serve as separating funds. For example, choosing $\lambda_1 \neq \lambda_2$ we can define

$$\gamma^1 = \lambda^1\alpha + (1-\lambda^1)\beta,$$

and

$$\gamma^2 = \lambda^2\alpha + (1-\lambda^2)\beta,$$

and the portfolios γ^1 and γ^2 will be separating funds. In addition, if we pick γ^2 (say to be the market portfolio in a pricing model) then γ^1 can be chosen to be uncorrelated with γ^2 (e.g., as a "zero beta" portfolio in a mean variance pricing model.[7/])

In the two parameter case, then, the spread parameter can be taken to be the variance (if it exists). It should be stressed though that this two parameter evaluation is valid if and only if the random returns are of the form of (C2). In summary, if a two parameter return-risk tradeoff can be taken and yields separation, then (C2) must be satisfied. Conversely, if (C2) holds, i.e., given w2fs, then the relevant separating portfolios will be minimum spread portfolios and this will permit a two parameter interpretation.

Normally Distributed Returns

What of the normal distribution then, or the continuous time version, the Wiener motion? Since we know that these distributions exhibit two fund separation, it must follow that all multivariate normal random variables take the form of (C2). It is instructive, though, to demonstrate this directly since (C2) is, at least at first appearances, a somewhat restrictive form.

If the covariance matrix, V, is singular, then a riskless portfolio can be formed and for simplicity of exposition alone we will assume that V is nonsingular and explicitly assume a riskless asset with return E_o. If (C2) is to be satisfied the riskless asset must also take the form of (C2)(i) and as a consequence $\tilde{\varepsilon}_o \equiv 0$ and

$$\tilde{y} + E_o\tilde{z} \equiv 0.$$

In other words, scaling $\tilde{z}$ so that $a_1=1$ we must be able to write the risky assets in the form

$$\tilde{X}_i - E_i = (E_i - E_o)\tilde{z} + \tilde{\varepsilon}_i,$$

where

$$E\{\tilde{\varepsilon}_i | \tilde{z}\} = 0. \qquad (14)$$

To do this we choose $\tilde{z}$ to be the excess random return on the separating portfolio, α, defined by

$$\alpha = kV^{-1}(E - E_o),$$

where E_o now denotes a constant vector with E_o in all entries, and k is a constant chosen to normalize α to be a portfolio, $\alpha e = 1$. Thus,

$$\begin{aligned}\tilde{z} &\equiv \alpha\tilde{X} - \alpha E \\ &= k(E - E_o)V^{-1}(\tilde{X} - E).\end{aligned}$$

Letting γ be any vector it is well known that a sufficient condition for conditional independence, (14), of normal random variables is that they be uncorrelated. Since

$$\begin{aligned}&\text{covariance } (\gamma'\tilde{\varepsilon}, \tilde{z}) \\ &= \text{covariance } (\gamma[\tilde{X} - E - (E - E_o)\tilde{z}], \tilde{z}) \\ &= \gamma'[I - k(E - E_o) \cdot (E - E_o)'V^{-1}]V \cdot kV^{-1}(E - E_o) \\ &= k\gamma'(E - E_o) - k\gamma'(E - E_o) \cdot k(E - E_o)'V^{-1}(E - E_o) \\ &= k\gamma'(E - E_o) - k\gamma'(E - E_o) \\ &= 0,\end{aligned}$$

it follows that all multivariate normal random variables satisfy the conditions of (C2).

The same is true of the stable distributions, but since the use of stable distributions in portfolio theory is generally closely linked with the use of market factor models (see, e.g., Fama), it is more instructive to treat them as special cases of the factor model approach.

Market Factor Models

The general one factor generating model is written in the form

$$\tilde{X}_i = E_i + b_i\tilde{z} + \tilde{\varepsilon}_i \tag{15}$$

or with a common risk as

$$\tilde{X}_i \equiv E_i + \tilde{y} + b_i\tilde{z} + \tilde{\varepsilon}_i, \tag{16}$$

where

$$E\{\tilde{\varepsilon}_i | \tilde{y}, \tilde{z}\} = 0.$$

Generally, $\tilde{z}$ is interpreted as a systematic on market risk and $\tilde{\varepsilon}$ is the unsystematic risk of the asset.

Until something more is said about the random factors and $\tilde{\varepsilon}$ terms, though, these models are without content, i.e., all random variables can be put into these forms. For example, if $\tilde{\varepsilon} \equiv 0$, then these can be easily understood as generating mechanisms which restrict the rank of the state space tableau of returns (see Ross (1973) and (1976)). More usually, $\tilde{\varepsilon}$ is not identically zero and it is required that there exist α such that

$$\alpha b = 1,$$

and

$$\alpha \tilde{\varepsilon} \equiv 0,$$

but it should be noted that this, too, imposes no further restrictions on the random variables. A meaningful restriction is obtained when we require the $\tilde{\varepsilon}_i$ to have a degree of independence (e.g., mutually uncorrelated).

Naturally enough, without any restrictions there can be no separation results, but unfortunately, the requirement that the $\tilde{\varepsilon}_i$ be linearly independent, while a strong and interesting assumption, is also not sufficient for two fund separation. We will say more about this case in Sections III and IV, but for the moment we can easily demonstrate that either model (15) or (16) with $\tilde{\varepsilon} \equiv 0$ is sufficient for two fund separation. This should be clear since the only discretionary source of risk in the two models is the amount of market risk, $\tilde{z}$, borne in the portfolio and this implies that a two parameter model is applicable. Directly, though, with $\tilde{\varepsilon} \equiv 0$, if arbitrage is not possible then both (15) and (16) must actually be written in the form of (C2).[8/]

Suppose now, that $\tilde{\varepsilon}$ is not identically zero. Suppose, too, that there exist portfolios α and β for which

$$\alpha b \neq \beta b,$$

and

$$\alpha\tilde{\varepsilon} \equiv \beta\tilde{\varepsilon} \equiv 0.$$

Given any portfolio, γ, it is possible to combine α and β so that the resulting portfolio has the same systematic, $\tilde{z}$, risk as the γ portfolio and no $\tilde{\varepsilon}$ risk, but there is no assurance that such a portfolio combination will have as high an expected return as that of the γ portfolio. In other words, unless (C2)(i) holds and E_i is a linear function of b_i, we cannot have 2fs in the market model. We will return to the problem of separation with market models in Section III below.

<u>Capital Asset Pricing Theory</u>

The traditional results of the two parameter capital asset pricing models follow in a straightforward fashion from w2fs. This can be shown in any of a number of ways, but it is most instructive to derive the theory directly from the separating form of (C2). For expositional purposes, suppose that there is a riskless asset. From (C2) w2fs implies that

$$\tilde{X}_i = E_i + b_i\tilde{z} + \tilde{\varepsilon}_i,$$

and

$$E_i - E_o = a_1 b_i . \tag{18}$$

Since all efficient portfolios have no $\tilde{\varepsilon}$ risk, the market portfolio must also have no $\tilde{\varepsilon}$ risk and takes the form

$$\tilde{X}_m = E_m + b_m\tilde{z}, \tag{19}$$

and scaling $\tilde{z}$ so that $b_m = 1$ we have the familiar Sharpe-Lintner pricing result

$$E_i - E_o = (E_m - E_o)b_i . \tag{20}$$

This derivation is deceptively simple and it is worth recapitulating. Once we know that the distribution permits w2fs, then from (C2)(i) it is clear that the market portfolio is efficient and can be taken to be one of the separating funds. The basic pricing result (20) is an immediate consequence.

Stochastic Dominance Theory

There has long been a hope that the principles of stochastic dominance could be applied in portfolio problems to eliminate the need for strong distributional assumptions. To date, the results have been somewhat weak and Theorem 2 offers some explanation. Conditions (C2) are required of any portfolio theory that obtains separation results and this makes any attempt to search for weaker or more distribution free results unrewarding. This is not to say that work on stochastic dominance is futile, but, rather, that results will be forthcoming only in special cases.[9/]

Before generalizing Theorem 2 to the case of kfs, we will end this section with a simple corollary that establishes the promised equivalence between w2fs and s2fs.

Corollary:

w2fs and s2fs are equivalent.

Proof:

As we observed before s2fs implies w2fs. From Theorem 2, though, w2fs is equivalent to (C2) and by the proof of sufficiency for Theorem 2, (C2) implies s2fs.

Q.E.D.

In what follows we will drop the distinction between strong and weak separability.

III. K-Fund Separability:

The following theorem generalizes Theorem 2 to the case of kfs. As is to be anticipated, the k funds are not unique and the funds given in the theorem, as with the 2fs result, should be interpreted as defining a k-1 dimensional space within which all optimal portfolios lie. Any basis for this space will be a set of k separating funds.

Theorem 3:

A vector of asset returns, $\tilde{X}$, exhibits kfs if and only if the following conditions are satisfied

$$(\exists \tilde{y}, \tilde{z}^1, \ldots, \tilde{z}^{k-2}, \tilde{\varepsilon})$$

(i) $\tilde{X}_i = E_i + \tilde{y} + \sum_{j=1}^{k-1} b_{ij}\tilde{z}^j + \tilde{\varepsilon}_i,$

where $E_i = a_o + \sum_{j=1}^{k-1} a_j b_{ij},$

(ii) $(\forall \lambda) \quad E\{\tilde{\varepsilon}_i | \lambda_k \tilde{y} + \sum_{j=1}^{k-1} \lambda_j \tilde{z}^j\} = 0,$ (CK)

(iii) $(\exists \alpha^1, \ldots, \alpha k)$ with $\alpha^1 \tilde{\varepsilon} = \ldots = \alpha^k \tilde{\varepsilon} \equiv 0,$

and the two matrices

$$\left[e \; \vdots \; B \right] \qquad \text{and} \qquad \left[e \; \vdots \; [\alpha^i B^j], \right]$$

where

$$B \equiv [b_{ij}],$$

have identical rank on all submatrices formed from corresponding columns.

Proof:

The proof is a straightforward generalization of that for Theorem 2.

Q.E.D.

Given (CK) it is easy to verify the corollary that wkfs and skfs are equivalent, and we have omitted the distinction above. Distributions which satisfy (CK) for some k will be termed separating distributions.

The generalization to kfs is of some interest beyond that of the two fund separation result. In particular, CK permits the presence of systematic factors, $\tilde{z}^j$ with arbitrary coefficients, b_{ij}. The rank conditions of (CK)(iii) do not interfere with this; they only insure that separation obtains in situations where the factors cannot be combined into a fewer number of factors. This is necessary since while kfs is sufficient for k'fs when k' > k the latter is, of course, not true. The analogous situation arises with (C2)(iii) in the case of 2fs. If E is not a constant vector, then $[e \vdots E]$ is of full rank 2 and we require that $\alpha E \neq \beta E$ for the two separating funds α and β to span all returns. In the k fund case we need to span all feasible factor weights as well.

The conditions (CK) also indicate the pivotal importance of general linear factor models for separation. For k-fund separation it is both necessary and sufficient that returns be generated by a k factor generating mechanism of the form of (CK)(i). Furthermore, the expected returns, E_i, cannot be arbitrary, but must, in fact, be linearly dependent on the factor beta weights. Once these factors are identified as observable portfolios, the separability conditions become testable capital market equilibrium conditions.

IV. Some Extensions

One of the consequences of the above representations of separability is that separation occurs only under certain restrictive conditions. If exact separation does not occur, though, there remains the possibility of approximate separation. Suppose, for example, that returns are generated by the simple one factor model

$$\tilde{X}_i = E_i + b_i\tilde{w} + \tilde{\varepsilon}_i, \tag{15}$$

where the $\tilde{\varepsilon}_i$ are linearly independent. As we have seen, this model, generally, fails to satisfy the separation conditions

Suppose, though, that by the law of large numbers portfolios, α, can be formed that diversify away the nonsystematic $\tilde{\varepsilon}$ risk so that

$$\alpha\tilde{\varepsilon} \approx 0.$$

For such portfolios, then, the generating model is essentially of the form

$$\tilde{X}_i \cong E_i + b_i\tilde{w},$$

and we have already shown that (if arbitrage possibilities are eliminated) this model exhibits 2fs. These concepts are developed into the basis of capital pricing theory in Ross [1973] and [1976].

V. Summary and Conclusion

This paper has described the conditions on the stochastic distributions of asset returns that are both necessary and sufficient for separation for all risk averse utility functions. In this sense it is the dual to the papers by Cass and Stiglitz, Hakansson, Leland, Mossin and others who have developed the theory of utility functions which permit separation for all stochastic environments. The results may be somewhat surprising in that certain common distributions, in particular, the Paretian stable distributions do not emerge as central to the analysis. Rather, what we are concerned with in portfolio theory is not so much the marginal distributions of asset returns as the interrelationships amongst random assets. For the problems of separation and, therefore, for portfolio theory in general, the linear factor models occupy a central canonical role in describing these correlations.

Footnotes

1/ We should also note, at this stage, the work of Hadar and Russell, and Fishburn. Fishburn has derived a number of conditions which are equivalent to one portfolio dominating another in a stochastic sense. For example, in our notation, he derived the result that a portfolio, α, would dominate a portfolio β (in U) if and only if for every choice of $u \in U$ there is one investment in the α portfolio which is preferred to one in the β portfolio. This result can then be applied to the question of whether diversification as opposed to "plunging" is optimal. This latter question has occupied Hadar and Russell in their work. Our concern is somewhat different; we are interested in restrictions on the underlying distributions which guarantee separability in the k-fund sense defined above. That is, we are asking under what conditions the class of optimal portfolios may be simply characterized.

2/ On the other hand, while it is possible to use Theorem 1 to verify the sufficiency for lfs of a multivariate stable distribution with identical means, to do so requires the development of some multivariate distribution theory for these laws that does not appear to be readily available in the literature. In the normal case, for example, it is well known that if

$$\eta' v\alpha = 0,$$

then

$$E\{\eta\tilde{X}|\alpha\tilde{X}\} = 0,$$

where V is the covariance matrix of $\tilde{X}$. What is required for other stable variables is a similar notion of conditional independence given zero cospread.

3/ Before concluding, it might be useful to briefly consider the notion of monotone separation, M, i.e., separation when we drop the concavity requirement. Since monotone separation implies U separation, a fortiori (C1) remains a necessary condition. For sufficiency, though, we require $\text{Prob}\{\tilde{\varepsilon}_\eta > 0|\alpha\tilde{X}\} = 0$ (a.e.) and this implies that $\tilde{\varepsilon}_\eta = 0$, a.e.. In other words, a necessary and sufficient condition for M separation (with no short sales restrictions) is that $\tilde{X}_i = \tilde{z}$, all i.

4/ Such a $U(\cdot)$ exists by assumption.

5/ It might be of interest to note that (C2)(ii) can be restated in a slightly different form. Requiring that $E\{\tilde{\varepsilon}_i | \tilde{y}, \tilde{z}\} = 0$ clearly implies (C2)(ii), but I am as yet unsure whether it is actually stronger than (C2)(ii).

6/ Notice that, a priori, the problem of maximizing αE subject to a constraint on $\sigma(\alpha)$ does not generally yield a separation result without some restrictions on the spread function, σ.

7/ If there is a riskless asset with return E_o it can be chosen as one of the separating funds. This follows from the requirement that a riskless asset must also satisfy (C2). Hence, $\tilde{\varepsilon}_o \equiv 0$ and

$$\tilde{y} + b_o\tilde{z} \equiv 0.$$

For all risky assets, then scaling $\tilde{z}$ yields

$$\tilde{X}_i = E_i + (E_i - E_o)\tilde{z} + \tilde{\varepsilon}_i.$$

Since $\alpha\tilde{X}$ and $\beta\tilde{X}$ are now dependent only on $\tilde{z}$, they span the riskless asset, and it can be chosen as one of the separating funds.

8/ Consider (15), for example, with $\tilde{\varepsilon} = 0$. If b is constant, then all portfolios have the same risk, $b\tilde{z}$, and unless E is also constant it will be possible to have arbitrarily high returns at no cost or risk with an arbitrage portfolio η. If b is not constant, to prevent the same sort of arbitrage all portfolios with $\eta e = \eta b = 0$, must also have $\eta E = 0$, hence there must exist constants a_o and a_1 such that

$$E_i = a_o + a_1 b_i.$$

9/ For example, it is possible to use the results of stochastic dominance, together with Theorem 2, to derive some interesting theorems about multivariate distribution theory. Theorem 2 permits a characterization of those distributions beyond the normal with the property that under specified linear restrictions, lack of correlation is equivalent to conditional independence.

Appendix

The purpose of this appendix is to establish the two results used in the proofs in the text.

Define M^+ to be the class of all nonnegative monotone declining functions and let L_a^b denote the class of functions that are continuous and piecewise linear on [a, b] and vanish elsewhere.

Theorem A1

If for all $h \varepsilon M^+$

$$\int hdF = 0 \Rightarrow \int hdG = 0, \qquad (1)$$

and if (1) holds for some strictly declining $\hat{h} \varepsilon M^+$, then $(\exists k)$

$$G = kF, \text{ a.e..} \qquad (2)$$

Proof:

First we show that (1) holds for all $h \varepsilon L_a^b$. If $h \varepsilon L_a^b$, then there exists a $\delta \neq 0$ and sufficiently small so that $\hat{h} + \delta h \varepsilon M^+$. Hence, if

$$\int hdF = 0,$$

then

$$\int (\hat{h} + \delta h)dF = 0,$$

which implies by (1) that

$$\int (\hat{h} + \delta h)dG = 0,$$

and, therefore,

$$\int hdG = 0.$$

Now, pick $g, h \varepsilon L_a^b$ and choose c such that

$$\int hdF = c\int gdF.$$

Since $h - cg \varepsilon L_a^b$,

$$\int (h - cg)dF = 0$$

implies that

$$\int (h - cg)dG = 0,$$

or

$$\int hdG = c\int gdG.$$

It follows, then, that $(\exists k)(\forall h \in L_a^b)$

$$\int hdF = k\int LdG, \qquad (3)$$

and it is easy to see that k is independent of the choice of interval [a, b]. We can now apply Proposition 8.17 in Breiman, for example, to verify from (3) that (2) must hold.

Q.E.D.

In applying Theorem A1 in the proof of Theorem 2, we take $dF = V(q)dQ$ and $dG = [S_i(q) - S_j(q)]dQ$, where Q is the distribution function of $\tilde{q}$. Notice that F and G are, themselves, not distribution functions, but that does not affect the proof and (8) in the text follows directly from A(2) above.

The next theorem, Theorem A2, is used in the proof of Theorem 1 in the text. Just as Theorem 1 is only a special case of Theorem 2, the proof of Theorem A2 is really just a specialization of the proof of Theorem 2 in the text. It has been separated out, though, both for expositional purposes and because it might be of some independent interest.

Theorem A2:

If $(\forall\lambda)$

$$\tilde{z} + \lambda\tilde{y} \sim \tilde{z} + \tilde{\varepsilon}(\lambda),$$

where

$$E\{\tilde{\varepsilon}(\lambda)|\tilde{z}\} = 0,$$

then

$$E\{\tilde{y}|\tilde{z}\} = 0.$$

Proof:

For all u ε U, Jensen's inequality implies that

$$E\{U[\tilde{z} + \lambda\tilde{y}]\} = E\{U[\tilde{z} + \tilde{\varepsilon}]\} \leq E\{U[\tilde{z}]\}. \quad (3)$$

Concavity also implies that

$$U[\tilde{z}] = U[\tilde{z} + \lambda\tilde{y} - \lambda\tilde{y}]$$

$$\leq U[\tilde{z} + \lambda\tilde{y}] + (-\lambda y)U'[\tilde{z} + \lambda\tilde{y}],$$

and taking expectations yields

$$E\{U[\tilde{z}]\} \leq E\{U[\tilde{z} + \lambda\tilde{y}]\} - E\{\lambda\tilde{y}U'[\tilde{z} + \lambda\tilde{y}]\}. \quad (4)$$

The inequalities (3) and (4) imply that $(\forall\lambda)$

$$E\{\lambda\tilde{y}U'[\tilde{z} + \lambda\tilde{y}]\} \leq E\{U[\tilde{z} + \lambda\tilde{y}]\} - E\{U[\tilde{z}]\} \quad (5)$$

$$\leq 0.$$

From (5) it follows that $(\forall\lambda > 0)$

$$E\{\tilde{y}U'[\tilde{z} + \lambda\tilde{y}]\} \leq 0 ,$$

which implies that

$$E\{\tilde{y}U'[\tilde{z}]\} \leq 0,$$

and $(\forall\lambda < 0)$

$$E\{\tilde{y}U'[\tilde{z} + \lambda\tilde{y}]\} \geq 0,$$

which implies that

$$E\{\tilde{y}U'[\tilde{z}]\} \geq 0,$$

hence

$$E\{\tilde{y}U'[\tilde{z}]\} = 0.$$

Since $u \in U$, U' is an arbitrary element of M^+ and by Theorem A1

$$E\{\tilde{y}|\tilde{z}\} = 0, \text{ a.e.}.$$

<u>Q.E.D.</u>

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