The Demand for Risky Assets Under Uncertain Inflation

by

Irwin Friend*
Yoram Landskroner*
and
Etienne Losq*

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RODNEY L. WHITE CENTER
FOR FINANCIAL RESEARCH

University of Pennsylvania

The Wharton School

Philadelphia, Pa. 19174

1. Introduction and Summary

In several recent papers by a colleague and ourselves, we have derived and empirically tested utility functions for individual households which were then combined to construct an aggregate demand function for risky assets.[1] The main conclusions of this earlier work are: The assumption of constant proportional risk aversion for households is as a first approximation a fairly accurate description of the market place. Second, regardless of their wealth level, the coefficients of proportional risk aversion for households are on average well in excess of one and probably in excess of two, so that households are more risk averse than would be implied by a log utility function. Third, under the assumption of constant proportional risk aversion, a simple form of the aggregate equilibrium relationship between the relative demand for risky assets and the market price of risk (MPR) can be (and has been) developed.

The determination in this earlier work of the aggregate relationship between the demand for risky assets and the MPR is based on a number of different models ranging from the simplest case where a risk-free asset in nominal terms exists, the separation theorem is valid, all assets are readily marketable, and there is no taxation to the more complex case where there is no risk-free asset, the separation theorem is not applicable, and both non-marketable assets and taxation exist. However, none of these models explicitly considers the effect of inflation on the demand for risky assets.

The subsequent analysis, where the variables are defined in nominal terms, will use the analytical framework and results developed in our previous work to obtain new results on the effect of inflation on the MPR and on the pricing of individual risky assets (as specified by the familiar capital asset pricing model). To summarize our new results:

(1) The traditional capital asset pricing model measured in nominal terms (CAPM) understates the MPR if an uncertain inflation is expected and if there is a positive covariance between the rate of return on the market and the rate of inflation. (The reverse is true if an uncertain deflation is expected or if there is a negative covariance between market return and inflation). (2) The CAPM overstates the risk of an asset under expectations of uncertain inflation if there is a positive covariance between the rate of return on the asset and the rate of inflation. (3) With inflationary expectations, the CAPM overstates the required rate of return on a risky asset if the correlation between the rate of return on a risky asset and the rate of inflation ($\rho_{i\pi}$) is greater than the product of the correlation between the rate of inflation and the rate of return on the market ($\rho_{\pi m}$) and the correlation between the rate of return on the risky asset and the rate of return on the market (ρ_{im}), and if all correlations are positive.

That the first and third of these results are not obvious is indicated by the fact that the only similar earlier analysis of the effect of uncertain inflation on the MPR and on the CAPM that we have seen arrives at quite different and, as we shall show, incorrect conclusions. That analysis by Chen and Boness[2] is less general than ours (using a specific utility function, namely quadratic, employing an approximation for the rate of inflation which at least theoretically could affect the results significantly, and assuming all assets to be marketable), but even so we shall show that properly interpreted it would lead to the same qualitative conclusions as those derived in this paper. These new results may provide at least a partial explanation of two puzzling empirical findings in the literature on capital asset pricing: the higher estimates of the MPR which have been obtained from the bond than the stock market; and the absence of a linear relationship (predicted by the traditional CAPM) among the nominal risk-free rate, the rate of return on bonds, and the rate of return on stocks.

In Part 2 of the paper we shall develop the theoretical effect of uncertain inflation on the MPR and the CAPM when all assets are marketable. Part 3 will expand the analysis to include nonmarketable as well as marketable assets. Finally in Part 4, we shall discuss the theoretical implications of our findings, explain the reasons for the different conclusions reached by Chen and Boness, and incorporate some empirical results into our theoretical framework.

2. Theoretical Analysis: All Assets are Marketable

The analysis in this paper is based on a continuous time model of capital asset pricing which assumes an infinitesimal planning horizon and no finite changes in value in an infinitesimal period. We derive here the effects of uncertain inflation on the demand for risky assets, i.e. on the MPR, the CAPM and the equilibrium demand relation for risky assets. In this part, following the traditional CAPM we assume that all assets are marketable (perfectly liquid). We distinguish between a risk-free asset (in nominal terms) and n risky assets. We also assume homogeneous expectations for all investors.

First, assume that the random nominal rate of return on the i^{th} asset, r_i, is generated by a continuous Gaussian (Wiener) process such as[3]

$$r_i dt = E(r_i)dt + \sigma_i dz_i = E(r_i)dt + \sigma_i y_i \sqrt{dt} \tag{1}$$

In this equation r_i, $E(r_i)$, and σ_i are the instantaneous rate of return, its expected value and its standard deviation respectively; $dz = y\sqrt{dt}$, where z is a stochastic process with independent movements; and y is a purely random process, that is y_t and y_{t+s} are identically and independently distributed, and by construction $E(y) = 0$ and $E(y^2) = 1$; dt is an infinitestimal period of time. In a similar way we assume that the random rate of inflation, r_π, is generated by a Guassian process.[4]

$$r_\pi dt = E(r_\pi)dt + \sigma_\pi y_\pi \sqrt{dt} \tag{2}$$

The real wealth dynamics for the k^{th} investor may be written in a stochastic differential equation form.[5]

$$W_{k,t+dt} = W_{k,t}\left[1+\frac{(r_f - r_\pi)dt}{1+r_\pi dt} + \sum_j \alpha_{jk} \frac{(r_j - r_f)dt}{1+ r_\pi dt}\right] \tag{3}$$

where W_{kt} is the wealth of the k'th investor at a point of time t; r_f is the nominal risk-free rate of return; α_{jk} the proportion of wealth invested in

Expressing $\frac{1}{1+r_\pi dt}$ as

the j^{th} asset by the k^{th} investor; and the budget constraint is

$$\sum_j \alpha_{jk} + \alpha_{fk} = 1$$

Substituting (1) and (2) into (3) and differentiating the expected utility of final real wealth, $W_{k,t+dt}$, with respect to α_{ik}, we derive the first order condition for a maximum

$$E\left[U'(W_{k,t+dt})W_{k,t}\left\{\frac{E(r_i - r_f)dt + \sigma_i y_i \sqrt{dt}}{1 + r_\pi dt}\right\}\right] = 0 \quad i = 1,\ldots,n \qquad (5)$$

... and eliminating terms of order (dt) ... yields

$$U' \cdot E[(r_i - r_f - \sigma_{i\pi})dt] + U'' \cdot W_{k,t} E[-\sigma_{i\pi} dt + \sum_j \alpha_{jk} \sigma_{ij} dt] = 0 \qquad (8)$$

where $\sigma_{i\pi} = \sigma_i \sigma_\pi y_i y_\pi$ and $\sigma_{ij} = \sigma_i \sigma_j y_i y_j$ are covariance terms from (1) and (2). Defining the Arrow-Pratt measure of relative risk aversion

$$c_k = - \frac{U''(W_{k,t})W_{k,t}}{U'(W_{k,t})}$$

Substituting and rearranging (8) we obtain

$$E(r_i)-r_f-\sigma_{i\pi} = c_k\left[\sum_j \alpha_{jk}\sigma_{ij} - \sigma_{i\pi}\right] \tag{9}$$

Aggregating over investors, weighted by their wealth relative γ_k, an equilibrium relation for the i^{th} asset is derived:

$$E(r_i)-r_f-\sigma_{i\pi} = C\left[\sum_{kj}\sum\gamma_k\alpha_{jk}\sigma_{ij} - \sigma_{i\pi}\right] = \alpha C\left[\sigma_{im} - \frac{\sigma_{i\pi}}{\alpha}\right] \tag{10}$$

where m is the market portfolio (containing all risky assets weighted by their relative market value); $\alpha = \sum_{kj}\sum\gamma_k\alpha_{jk}$ is the ratio of risky to total value of all assets. The term in the brackets and $\sigma_{i\pi}$ are terms unique to each asset; the second term $\alpha C = \alpha\left[\Sigma\gamma_k/c_k\right]^{-1}$, a weighted harmonic mean times α, is a factor common to all assets and equal to the MPR. This can be seen by aggregating (10) over all assets weighted by $X_{im} = \frac{\alpha_i}{\alpha}$, where α_i is the ratio of the i^{th} risky asset to the total value of risky assets, obtaining

$$\alpha C = \frac{E(r_m)-r_f-\sigma_{m\pi}}{\sigma_m^2 - \frac{\sigma_{m\pi}}{\alpha}} \tag{11}$$

When the utility function of each investor is characterized by constant proportional risk aversion as indicated by our earlier work, C becomes a simple harmonic mean of the individual c_k's and does not change over time unless the c_k's change.

Substituting (11) into (10) we obtain the capital asset pricing model adjusted for inflation, in a similar form to the original CAPM,[6]

$$E(r_i) = r_f + \sigma_{i\pi} + \left[\frac{E(r_m)-r_f-\sigma_{m\pi}}{\sigma_m^2 - \frac{\sigma_{m\pi}}{\alpha}}\right](\sigma_{im} - \frac{\sigma_{i\pi}}{\alpha}) \tag{12}$$

Adjusting for personal income taxes, and assuming for simplicity that r_m and r_f

of investors are subject to the same tax rates, we obtain the CAPM adjusted for the effect of taxation as well as uncertain inflation

$$E(r_i) = r_f + \sigma_{i\pi} + \left[\frac{E(r_m) - r_f - \sigma_{m\pi}}{\sigma_m^2 - \frac{\sigma_{m\pi}}{\alpha(1-t)}} \right] \left(\sigma_{im} - \frac{\sigma_{i\pi}}{\alpha(1-t)}\right) \tag{12a}$$

where t is a weighted average tax rate.[7] The expression in the brackets on the RHS of (12) and (12a) is the market price of risk adjusted for uncertain inflation, a factor common to all risky assets. The term in the parentheses on the RHS of the equations and $\sigma_{i\pi}$ are the terms which are unique to each asset.[8] The comparable expression for the traditional CAPM is

$$E(r_i) = r_f + \left[\frac{E(r_m - r_f)}{\sigma_m^2} \right] (\sigma_{im}) \tag{13}$$

The major differences between (12) and (13) are first in the covariance terms with rate of inflation which appear both in the market price of risk and in the measure of the asset's risk, and second in the separate $\sigma_{i\pi}$ term which would differentiate the two equations even if investors were risk neutral. These terms show the effects of uncertain inflation on the pricing of capital assets.[9]

3. Theoretical Analysis: Nonmarketable Assets

In this part we extend our model to include nonmarketable assets, specifically human capital, as well as marketable assets.[10] It is assumed that the rate of return on human capital, similar to the rates of return on marketable assets and the inflation rate, is generated by a Gaussian process.

The return dynamic for the k^{th} investor's human capital is of the form

$$r_{kh}dt = E(r_{kh})dt + \sigma_{kh}y_{kh}\sqrt{dt} \tag{14}$$

k ... proportion of the investor's total wealth invested in human capital, which is assumed to be exogenously given because it is largely independent of the investor's decisions; and α_{jk} redefined as the proportion of the j^{th} marketable asset in the marketable wealth of the k^{th} investor. As in Part 2, differentiating $E\left[U(W_{k,t+dt})\right]$ with respect to α_{ik} yields the first order conditions for a maximum,

$$E\left[U'(W_{k,t+dt})W_{k,t}\left\{\frac{E(r_i-r_f)dt+\sigma_i y_i\sqrt{dt}}{1+r_\pi dt}\right\}\right] = 0 \qquad i = 1,\ldots,n \tag{16}$$

Expanding the marginal utility function in a Taylor series, taking expectations and eliminating higher order terms yields

$$E(r_i)-r_f-\sigma_{i\pi} = c_k\left[(1-h_k)\sum_j \alpha_{jk}\sigma_{ij} + h_k\sigma_{i,kh} - \sigma_{i\pi}\right] \tag{17}$$

where $\sigma_{i,kh} = \text{cov}(r_i, r_{kh})$.

Aggregating (17) over investors, we obtain the equilibrium relation for the i^{th} asset

$$E(r_i)-r_f - \sigma_{i\pi} = \alpha C\left[(1-h)\sigma_{im} + \frac{h}{\alpha}\sigma_{ih} - \frac{1}{\alpha}\sigma_{i\pi}\right]. \tag{18}$$

Now aggregating (18) over all assets we can derive explicitly the market price of risk under the extended definition of wealth and adjusted for uncertain inflation:

$$\alpha C = \frac{E(r)_m - r_f - \sigma_{m\pi}}{(1-h)\sigma_m^2 + \frac{h}{\alpha}\sigma_{mh} - \frac{\sigma_{m\pi}}{\alpha}} \tag{19}$$

where h is the proportion of aggregate human capital to total wealth, r_h is the aggregate return on human capital, and $\sigma_{mh} = \text{cov}(r_m, r_h)$. Substituting (19) into (18) and allowing for taxes, the equilibrium relation for the i^{th} asset in a form similar to the traditional CAPM becomes

$$E(r_i) = r_f + \sigma_{i\pi} + \left[\frac{E(r)_m - r_f - \sigma_{m\pi}}{(1-h)\sigma_m^2 + \frac{h}{\alpha}\sigma_{mh} - \frac{\sigma_{m\pi}}{\alpha(1-t)}}\right]\left[(1-h)\sigma_{im} + \frac{h}{\alpha}\sigma_{ih} - \frac{\sigma_{i\pi}}{\alpha(1-t)}\right] \tag{20}$$

As in Part 2, the return differential on a risky asset $E(r_i - r_f)$ is determined by the market price of risk, the first term in brackets common to all assets, by the systematic risk of the asset, the second term in brackets, and by an inflation adjustment factor $(\sigma_{i\pi})$. The additional effects here are those involving human capital, i.e. σ_{mh} and σ_{ih}.[11] It is interesting to note that $E(r_i)$ does not depend upon the covariability of r_π and r_h.

4. Implications of Findings

The analysis in Parts 2 and 3 shows that the traditional capital asset pricing model (CAPM) understates the market price of risk (MPR) if an uncertain inflation is expected and if there is a positive correlation between the rate of return on the market and the rate of inflation, i.e. $\rho_{m\pi} > 0$. Thus, in the simplest case where it is assumed that a risk-free asset in nominal terms exists, all assets are readily marketable and there is no taxation, the MPR is changed from $MPR_1 = \frac{E(r_m - r_f)}{\sigma_m^2}$ under the traditional CAPM to $MPR_2 = \frac{E(r_m - r_f) - \sigma_{m\pi}}{\sigma_m^2 - \frac{\sigma_{m\pi}}{\alpha}}$ under inflationary expectations, where it will be recalled $\sigma_{m\pi}$ is the covariance between the rate of return on the market and r_π the rate of inflation, and α is the ratio of risky to total assets. Then, $MPR_2 > MPR_1$ since $E(r_m - r_f)$ is known to be in excess of σ_m^2 (i.e., $MPR_1 > 1$) and since $\alpha < 1$. Relaxing the simplifying assumptions does not change the qualitative relationship between MPR_1 and MPR_2.

Again using the simplest case for convenience, it is easily seen that the traditional CAPM overstates the risk of the i^{th} asset if uncertain inflation is expected and if $\rho_{i\pi} > 0$. Thus, the asset's risk under the traditional CAPM, which is measured by σ_{im}, is transformed into $\sigma_{im} - \frac{\sigma_{i\pi}}{\alpha}$ under inflationary expectations.

The required rate of return on a risky asset under these assumptions is the sum of the risk-free rate, the covariance of the return between the return on the risky asset and the rate of inflation, and the product of the asset's risk and the MPR. Substitution of the above expressions for the MPR and the asset's risk into Equations (12) and (13) shows that the relationship between the required rates of return without and with inflationary expectations ($E_1(r_i)$ and $E_2(r_i)$ respectively) depends on the values of the correlations between the rate of return on a risky asset and the rate of inflation ($\rho_{i\pi}$), between the rate of return on the risky asset and the rate of return on the market (ρ_{im}), and between the rate of inflation and the rate of return on the market ($\rho_{m\pi}$). Specifically if all ρ's are positive, and if $MPR_1 > \alpha$ and $\sigma_m^2 > \frac{\sigma_{m\pi}}{\alpha}$ both of which are known to be true, $E_1(r_i) \gtreqless E_2(r_i)$ depending on whether $\rho_{i\pi} \gtreqless \rho_{im}\rho_{m\pi}$.

Of these results, the first relating to the effect of inflation on the MPR, and (as a result) the third relating to the effect on the required rate of return on a risky asset, may appear surprising. Indeed as will be shown below they have led to some confusion in the literature. It may be useful therefore to clarify the meaning of the first finding that, under inflationary expectations and a positive covariation between market return and inflation, $MPR_1 < MPR_2$. This finding simply means that under the indicated circumstances the traditional measure of the MPR understates its true value. It does not mean that the MPR is raised by inflation so long as the covariation between market return and inflation is positive, since the nominal values appearing in MPR_1 without inflation are not the same as those in MPR_2 with inflation.

Two other implications of our analysis are of some interest. First, under equilibrium conditions, the results in Equations (11) and (12) imply that if investors are risk neutral (so that C=0),the expected rate of return on a risky asset ($E(r_i)$) is not equal to the risk-free return (r_f) but to the risk-free return plus the covariance ($\sigma_{i\pi}$) between the return on the risky asset and the rate of inflation. Second, for a log utility function (where C=1) the expected rate of return on a risky asset is independent of the covariance between the return on that asset and the rate of inflation.

As contrasted with the conclusions reached above, Chen and Boness ([3]), assuming a quadratic utility function, have drawn conclusions from their analysis of the expected rate of return on a risky asset under uncertain inflation which are quite different even though their results are analogous to those in Equation (12). Their results are not identical with ours since as a result of the approximation procedures they follow, they lose two terms in Equation (12) (the $\sigma_{i\pi}$ in the second term on the right-hand side of the equation and the $\sigma_{m\pi}$ in the numerator of the market price of risk) but properly interpreted their findings would lead to the same qualitative conclusions as ours. However, their interpretation of their results is incorrect. They state that "the traditional capital asset pricing model overstates the market price of risk if an uncertain inflation is expected; and it understates the market price of risk if an uncertain deflation is expected" (p. 474). As indicated in Equation (11), the reverse is true in theory (disregarding the terms they omit).

The reason for the incorrect conclusion by Chen and Boness is that they assume in maximizing expected utility of terminal wealth that the coefficients of their quadratic utility function are the same regardless of whether wealth is measured in real or nominal terms. Thus, in their utility function with real wealth as the argument, $U_Y(\tilde{Y}_k) = \tilde{Y}_k - C_k\tilde{Y}_k^2$ where Y is real wealth. They then assume that, under the traditional version of the capital asset pricing model which has not explicitly taken account of uncertain inflation, $U_Z(\tilde{Z}_k) = \tilde{Z}_k - C_k\tilde{Z}_k^2$ where Z is nominal wealth. They state that the market

$$MPR_Y = [\sum_k \frac{1}{2C_k} - \sum_k E_k(\tilde{Y}_k)]^{-1}$$ with allowance for inflation and

$$MPR_Z = [\sum_k \frac{1}{2C_k} - \sum_k E_k(\tilde{Z}_k)]^{-1}$$ without such an allowance, thus concluding that $MPR_Z > MPR_Y$ with positive inflation since then $Z_k > Y_k$. In fact, $U_Z(\tilde{Z}_k) = \tilde{Z}_k - C_k\tilde{Z}_k^2 = \tilde{r}_\pi\tilde{Y} - C_k\tilde{r}_\pi^2\tilde{Y}^2$, and it is no longer possible to draw their conclusion which is based on the assumed identity of the nominal and real wealth coefficients in $U_Y(Y_k)$ and $U_Z(Z_k)$.

It is interesting to note that in commenting on these results, Hendershott

([7]), p. 506) similarly (and incorrectly) concludes that with a positive co-variation between rates of return on risky assets and the rate of inflation "the traditional capital-asset pricing model that ignores uncertainty regarding inflation...overstates the market price of risk when (uncertain) inflation is expected and understates the price if deflation is expected. These are interesting and important results."

To conclude our analysis of the effect of inflation on the demand for risky assets, the magnitude of the difference between the true MPR in the presence of uncertain inflation and the MPR as usually measured will depend largely on $\sigma_{m\pi}$, with the rate of return on the market (r_m) customarily measured from returns on New York Stock Exchange stocks. The magnitude of the difference between the true required or expected rate of return on an individual risky asset and the rate as measured by the CAPM will depend as well on $\sigma_{i\pi}$.

According to competitive economic theory, with rising prices the correlation over time between $E(r_\pi)$ and both $E(r_m)$ and $E(r_i)$ generally should be strongly positive and, unless σ_π is negligible,[12] the value of $\sigma_{m\pi}$ and $\sigma_{i\pi}$ would be expected to be substantially greater than zero. In fact, the correlation between actual annual or quarterly rates of inflation and actual contemporaneous returns on New York Stock Exchange stocks as a whole for any extensive period tested back to the latter part of the 19th Century is either statistically insignificantly different from zero or more commonly slightly (though significantly) negative.[13]

As a result, the $\sigma_{m\pi}$ term in Equations (11) and (12) is negligible in relation to the σ_m^2 term (and even more so in relation to $E(\bar{r}_m - r_f)$) so that the usual measure of the MPR is virtually unaffected by inflation.[14] It should be noted that this result largely reflects the effect on stock returns of unanticipated changes in the inflation rate, and that inflation may be better anticipated in the future.

The fact that the usual measure of the MPR would not be appreciably affected by an adjustment for inflation does not mean that the market price of risk estimated from other risky assets such as bonds (or, preferably, from all marketable risky assets combined) would be similarly unaffected. Since σ_{im} measured from bond returns is much smaller than the corresponding σ_m^2 measure from stock returns (Friend and Blume [6]) while $\rho_{i\pi}$ may be substantially negative for bonds unlike the virtual absence of any correlation for stocks,[15] an adjustment for inflation may greatly affect the MPR estimated from bond returns. This may help to explain at least in part the higher estimates of the MPR which have been obtained from the bond than from the stock market.

Thus, estimating the MPR from corporate bond data for 1902 to 1971, without any adjustment for inflation, points to a figure of 2.48 as compared to 1.70 obtained from the data for stocks.[16] The MPR estimate from the stock returns would be virtually unaffected by an adjustment for inflation while that from bonds would be reduced substantially. The revised MPR estimates are 1.66 for stocks and 2.07 for bonds.

The above results may also help to explain the fact that if a traditional CAPM is tested by relating empirical (cross-section) data on the risk-free rate, rates of return on bonds and rates of return on stocks to the asset's non-diversifiable risk, the relationship is not linear as it is for stocks alone (Blume and Friend [2]). The risk-free rate and the returns on bonds are generally significantly below the level which would be implied by the linear relationship (both observed and predicted by the CAPM) between return and risk for stocks. While the MPR--the factor in brackets in Equations (12) and (12a)--might not be affected appreciably by inflation if r_m were measured, as it should be, from the entire portfolio of marketable risky assets,[17] the observed non-linearity in the traditional fitting of the CAPM to all classes of securities might reflect a positive correlation between $\sigma_{i\pi}$ and σ_{im} in the CAPM corrected for inflation (Eq. (12) or (12a)). Thus, in the presence of inflation the usual beta measure may be an understatement of risk for low beta assets and an overstatement of risk for other assets, helping to explain at least part of the observed non-linearity in the relationship between return and non-diversifiable risk.[18]

FOOTNOTES

*Richard K. Mellon Professor of Finance and Economics, Assistant Professor of Finance, and Ph.D. student in Finance, University of Pennsylvania.

[1]Friend [5]; Friend and Blume [6]; and Landskroner [8]. All these papers assume homogeneous expectations. An extension to the case of heterogeneous expectations appears in Blume and Friend [2].

[2]Chen and Boness [3]. Hendershott [7] in that same issue of the Journal (p. 506) agrees with the Chen-Boness conclusions. Roll [14] also explores the effect of inflation on the CAPM, but like Chen and Boness uses a quadratic utility function and further makes the restrictive assumption that a riskless asset in real terms exists. Long [10] has studied the effect of inflation on capital asset prices in a multi-period framework. Fischer [4] derives the demand for index bonds under uncertain inflation, using a continuous time model.

[3]For a discussion and previous economic application of the Wiener process see Merton [12] and Ross [15]. It should be noted that the use of Itô's lemma (see Merton) would allow for a somewhat more sophisticated derivation of the results given below.

[4]The rate of inflation $r_\pi dt = \frac{dp}{p}$ where p is the general price level.

[5]Equation (3) is obtained from the definition

$$W_{k,t+dt} = W_{k,t}\left[1 + \alpha_{fk}R_f dt + \sum_j \alpha_{jk}R_j dt\right]$$

where R_j is the real rate of return on asset j, i.e.

$$R_j = \frac{(r_j - r_\pi)}{1 + r_\pi} .$$

[6]An alternative form of the CAPM adjusted for inflation is

$$E(r_i) = r_f + \sigma_{i\pi} + \left[\frac{E(r_m) - r_f - \sigma_{m\pi}}{\alpha\sigma_m^2 - \sigma_{m\pi}}\right](\alpha\sigma_{im} - \sigma_{i\pi}).$$

Here the definitions of the MPR and the systematic risk of the individual asset are slightly changed. This, however, would not affect our results in comparison to the original CAPM.

[7]$H_t = \left[\Sigma\gamma_k/(1-t_k)\right]^{-1}$ is a weighted harmonic mean of investor tax rates t_k. With a relatively small effective tax rate t_k, $H_t \simeq (1-t)$, where t is the weighted average personal tax rate. Thus,

$$H_t = \left[\Sigma\gamma_k(1+t_k+t_k^2 + \ldots)\right]^{-1} \simeq \left[\Sigma\gamma_k(1+t_k)\right]^{-1}$$

$$= (1+t)^{-1} \simeq (1-t).$$

[8]The CAPM in the absence of a risk-free asset, i.e. where a zero beta portfolio replaces the risk-free asset, seems of quite limited economic usefulness once inflation is explicitly allowed for.

[9]A third equilibrium relation, the demand for risky assets by the k^{th} investor, can be obtained by aggregating (9) over all assets and adjusting for income taxes

$$\alpha_k(1-t_k) = \frac{E(r_m) - r_f - \sigma_{m\pi}}{\sigma_m^2}\left(\frac{1}{c_k}\right) + \frac{\sigma_{m\pi}}{\sigma_m^2}$$

where t_k is a personal income tax rate assumed to be applicable to r_m and r_f. The results of the explicit consideration of uncertain inflation here are the additional terms on the RHS, which again involve the covariance of the inflation rate $\sigma_{m\pi}$.

[10]A first step in this direction was provided by Mayers [11] who derives the CAPM inclusive of nonmarketable assets.

[11]Aggregating (17) over all assets, adjusting for taxes we obtain the equilibrium demand relation

$$\alpha_k(1-t)_k = \frac{E(r)_m - r_f - \sigma_{m\pi}}{(1-h_k)\sigma_{mk,m}}\left(\frac{1}{c_k}\right) - \frac{h_k(1-t_k)}{(1-h_k)} \cdot \frac{\sigma_{kh,m}}{\sigma_{mk,m}} + \frac{\sigma_{m\pi}}{(1-h_k)\sigma_{mk,m}}$$

where $\sigma_{mk,m} = \text{cov}(r_{mk} r_m)$, r_{mk} is the rate of return on the k^{th} investor's portfolio of marketable assets. This expression is substituted for σ_m^2 in the equation in footnote 9, since when nonmarketable assets are included the separation theorem will no longer hold, i.e. investors will not be expected to hold the same portfolio of risky marketable assets. See Mayers [11] and Landskroner [8].

[12]σ_m is known to be substantial. See Friend and Blume [5].

[13]Lintner [9]; Oudet [13]. This does not mean of course that investment in stocks has not proved superior to investment in bonds as a hedge against inflation.

[14]Unpublished work by Blume and Friend suggest that when past as well as current rates of inflation are allowed for, the overall correlation between inflation and return on stocks becomes positive, though weakly so, and stock risk in the longer run is somewhat reduced by inflation.

[15]The correlation between quarterly rates of inflation and the contemporaneous returns on outstanding bonds has been -0.70 for the USA in the period after World War II.

[16]These estimates are based on the annual data for stocks and bonds summarized in Tables 6 and 7 of Friend and Blume [5], a covariance of -.0009 between the rate of inflation and return on bonds in the 1902-1971 period, a covariance of -.0017 between inflation and return on stocks, and an estimated α of 0.9 and (1-t) of 0.9 in Equation (12a). The MPR from corporate bond data without an adjustment for inflation is estimated by deriving the risk differential on the market $(E(r_m)-r_f)$ from the CAPM where the yield to maturity on bonds is $E(r_i)$ and β_{im} on bonds is estimated by regressing bond returns (r_i) on stock market returns. Thus the MPR without adjustment for inflation is estimated as $\frac{E(r_i)-r_f}{\sigma_{im}}$ where $\sigma_{im} = \beta_{im}\sigma_m^2$. With adjustment for inflation, the MPR from bond data is estimated as

$$\frac{E(r_i)-r_f - \sigma_{i\pi}}{\sigma_{im} - \frac{\sigma_{i\pi}}{\alpha(1-t)}} .$$

[17]This assumption seems plausible since $\sigma_{m\pi}$ is probably quite close to zero, and small when compared to $E(r_m - r_f)$ and σ_m^2. The sign of $\rho_{m\pi}$ is not clear. Thus, $\rho_{m\pi}$ estimated from stocks is normally slightly negative; for bonds it is more substantially negative; and for houses it is probably moderately positive.

[18]It is planned to estimate $\sigma_{i\pi}$ and σ_{im} for stocks only, over various time periods, so that we can test how much of the observed non-linearity in the relationship between return and non-diversifiable risk is explained by our modified CAPM.

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