Some Notes on the Capital Asset Pricing Model (CAPM), Short-Sale Restrictions and Related Issues

by

Stephen A. Ross

Working Paper No. 25-73

Rodney L. White Center for Financial Research
The Wharton School
University of Pennsylvania
Philadelphia, Pennsylvania 19174

Some Notes on the Capital Asset Pricing Model (CAPM), Short-Sale Restrictions and Related Issues

by

Stephen A. Ross

The purpose of these notes is two fold. First, a very simple and straightforward approach to the CAPM will be taken. Essentially we will show that all of the familiar results follow directly from the observation that the market portfolio, α^m, is mean-variance efficient. Second, we will apply this viewpoint to gather together and hopefully resolve a number of desultory questions and problems that have arisen from consideration of the original CAPM of Sharpe [1964] and Lintner [1965]. This will enable us to better understand the strength or robustness of the CAPM.

(1) It should be clear that, irrespective of the form of market institutions, if the market portfolio is mean-variance efficient then the CAPM results hold. (In what follows V will denote the covariance matrix of risky assets and E the vector of expected returns.) Since all assets are held in positive amounts in the market portfolio, efficiency implies satisfaction of the first order conditions for maximizing return at a given variance level,

$$E_i = \theta + \lambda \operatorname{cov}\{\tilde{x}_i, \tilde{x}_m\} \equiv \theta + \lambda \sigma_{im}^2 , \qquad (1)$$

where θ and λ are Lagrange multipliers, $\tilde{x}_i$ is the random return on the ith asset and $\tilde{x}_m$ is the random return on the market portfolio. The first order conditions, however, merely restate the CAPM. Rearranging yields the familiar form

$$E_i - \theta = \lambda \sigma_{im}^2 , \qquad (2)$$

or since for the market portfolio,

$$E_m - \theta = \lambda\sigma_m^2 \quad ,$$

we have

$$E_i - \theta = \frac{\sigma_{im}^2}{\sigma_m^2} (E_m - \theta) \quad . \tag{3}$$

Thus, efficiency implies that the security line equation of the CAPM (3) holds. Conversely, if the market portfolio is inefficient the CAPM will not hold. In other words, the efficiency of the market portfolio and the CAPM are equivalent.

(2) It follows that the route to establishing the CAPM lies in simply proving the efficiency of the market portfolio. With this guide we can considerably weaken the usual assumptions and simplify many proofs. The following sections will illustrate this approach.

(3) Consider the CAPM without a riskless asset. Now, each individual portfolio, α^ν , is efficient and indexing agents by $\nu = 1,\dots, n$ we have

$$E_i - \theta^\nu = \lambda^\nu V\alpha^\nu \quad . \tag{4}$$

Letting ω^ν be the proportion of wealth held by the ν^{th} agent, the market portfolio is simply a convex combination of the individual portfolios, i.e.,

$$\alpha^m = \sum_\nu \omega^\nu \alpha^\nu \quad . \tag{5}$$

Hence, aggregating (4) yields

$$E_i - \theta = \lambda V\alpha^m \equiv \lambda\sigma_{im}^2 \quad , \tag{6}$$

where

$$\lambda \equiv [\sum_\nu \omega^\nu \frac{1}{\lambda^\nu}]^{-1} \tag{7}$$

and

$$\theta = [\sum_\nu \omega^\nu \frac{1}{\lambda^\nu}] \sum_\nu \omega^\nu \frac{\theta_\nu}{\lambda_\nu} \quad . \tag{8}$$

Of course α^m is indeed a portfolio and (6) simply restates the CAPM. Notice, too that if α is any portfolio uncorrelated with the market portfolio,

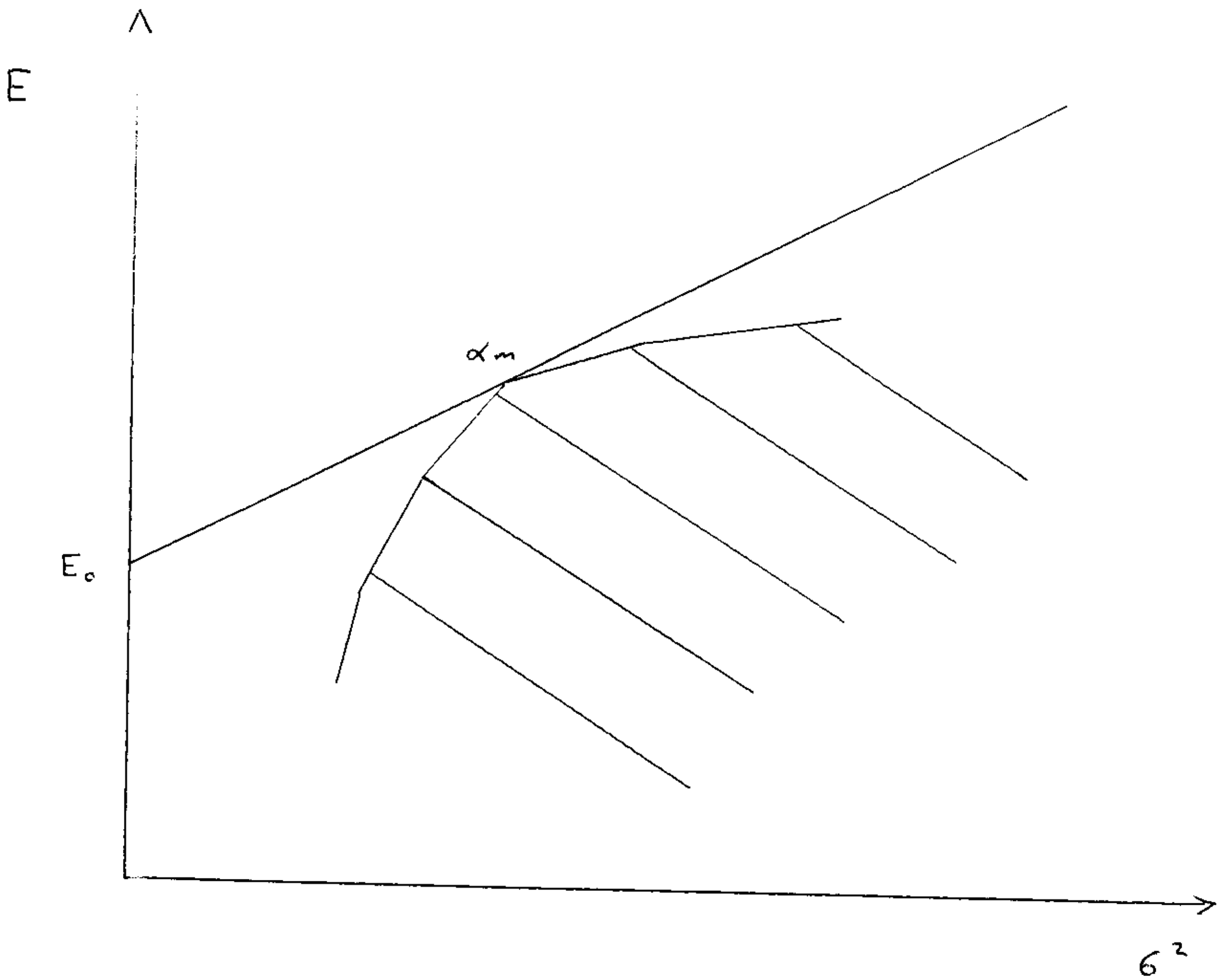
$$\sum_i \alpha_i E_i = \theta \quad , \tag{9}$$

so that θ can be interpreted as a zero-beta portfolio as in Black [1973] or Ross [1973]. With a riskless asset it is easy to see that θ is its rate of return.

(4) Now suppose that we do not permit short sales except on a single riskless asset. A geometric approach is best here, see Figure 1. Since all feasible risky portfolios are convex combination of assets, the feasible mean-variance set attainable solely by investments in risky assets must still be convex. It follows that separation obtains and all agents can achieve the efficient frontier by borrowing and lending against (in general) a single risky portfolio. In the familiar fashion, then, this single portfolio must be the market portfolio and since it is efficient the CAPM holds. Notice, of course, that while all assets are not necessarily represented in efficient portfolios of risky assets, with short sale restrictions they appear in positive amounts in the market portfolio and, therefore, the interior first-order conditions (1) must be satisfied.

(5) This naturally suggests that we might try verifying the CAPM if there exists one risky asset which can be shorted. Unfortunately, the following counterexample shows that this is not the case. Assume that the first asset may be freely shorted (observe, though, that no one in this example takes advantage of this possibility).

Figure 1



Let

$$V = \begin{bmatrix} 1 & 0 & 0 & 1/2 \\ 0 & 1 & 0 & 1/2 \\ 0 & 0 & 1 & 1/2 \\ 1/2 & 1/2 & 1/2 & 1 \end{bmatrix} \quad ; \quad E = (1, 2, 3, 4)$$

(It is easy to verify that V is positive definite and, therefore, an acceptable covariance matrix.) The first order conditions for optimality for an efficient portfolio with mean return 3 are given by

$$\beta_1 + 1/2\ \beta_4 = \theta_b + \lambda_b$$
$$\beta_2 + 1/2\ \beta_4 = \theta_b + 2\lambda_b$$
$$\beta_3 + 1/2\ \beta_4 = \theta_b + 3\lambda_b$$
$$1/2\ (\beta_1 + \beta_2 + \beta_3) + \beta_4 = \theta_b + 4\lambda_b \quad ,$$

which imply that

$$\beta_1 = \frac{1}{14}\ ; \quad \beta_2 = \frac{3}{14}\ ; \quad \beta_3 = \frac{5}{14}\ ; \quad \beta_4 = \frac{5}{14}$$

and

$$\theta_b = 3/28\ ; \quad \lambda_b = 1/7 \quad .$$

Now let the mean return be instead $2\ 1/2$. The first order conditions are satisfied by

$$\alpha_1 = \frac{7}{24}\ ; \quad \alpha_2 = \frac{8}{24}\ ; \quad \alpha_3 = \frac{9}{24}\ ; \quad \alpha_4 = 0$$

and

$$\theta_a = \frac{1}{4}\ ; \quad \lambda_a = \frac{1}{24} \quad .$$

Since

$$\sigma_a^2 = \frac{97}{288} < \frac{15}{28} = \sigma_b^2 \quad ,$$

the first order conditions guarantee that expected return is maximized at the respective variance levels. From Kuhn-Tucker conditions we also require

$$\frac{1}{2}(\alpha_1 + \alpha_2 + \alpha_3) + 0 > \theta_a + 4\lambda_a$$

or

$$\frac{1}{2} > \frac{5}{12}$$

to insure that $\alpha_4 = 0$ is optimal.

To construct an example violating the CAPM we only have to verify that the efficiency conditions, (4), are not satisfied by both portfolios. The β portfolio contains all four assets and the α portfolio only the first three. For β :

$$\frac{V_1\beta - V_2\beta}{V_1\beta - V_3\beta} = \frac{\beta_1 - \beta_2}{\beta_1 - \beta_3} = \frac{1}{2} = \frac{E_1 - E_2}{E_1 - E_3}$$

and

$$\frac{V_1\beta - V_4\beta}{V_1\beta - V_3\beta} = \frac{3}{2} = \frac{E_1 - E_4}{E_1 - E_3} ,$$

i.e., the β portfolio will satisfy (4) with the multipliers θ and λ eliminated. On the other hand, while

$$\frac{V_1\alpha - V_2\alpha}{V_1\alpha - V_3\alpha} = \frac{\alpha_1 - \alpha_2}{\alpha_1 - \alpha_3} = \frac{1}{2} ,$$

we also have

$$\frac{V_1\alpha - V_4\alpha}{V_1\alpha - V_3\alpha} = \frac{5}{2} \neq \frac{3}{2} = \frac{E_1 - E_4}{E_1 - E_3} ,$$

so that the α portfolio does not satisfy the efficiency conditions (4). Also, note that even for assets included in both portfolios

$$\frac{V_1\dot{\alpha}}{V_1\beta} = \frac{7}{6} ,$$

but

$$\frac{V_2\alpha}{V_2\beta} = \frac{28}{33} \quad .$$

In general, with short sales constraints the first order (Kuhn-Tucker) conditions take the form

$$V_i\alpha = \theta + \lambda E_i + \gamma_i \quad , \tag{10}$$

where V_i is i^{th} row of V and

$$\gamma_i > 0 \text{ implies that } \alpha_i = 0 \quad ,$$

and

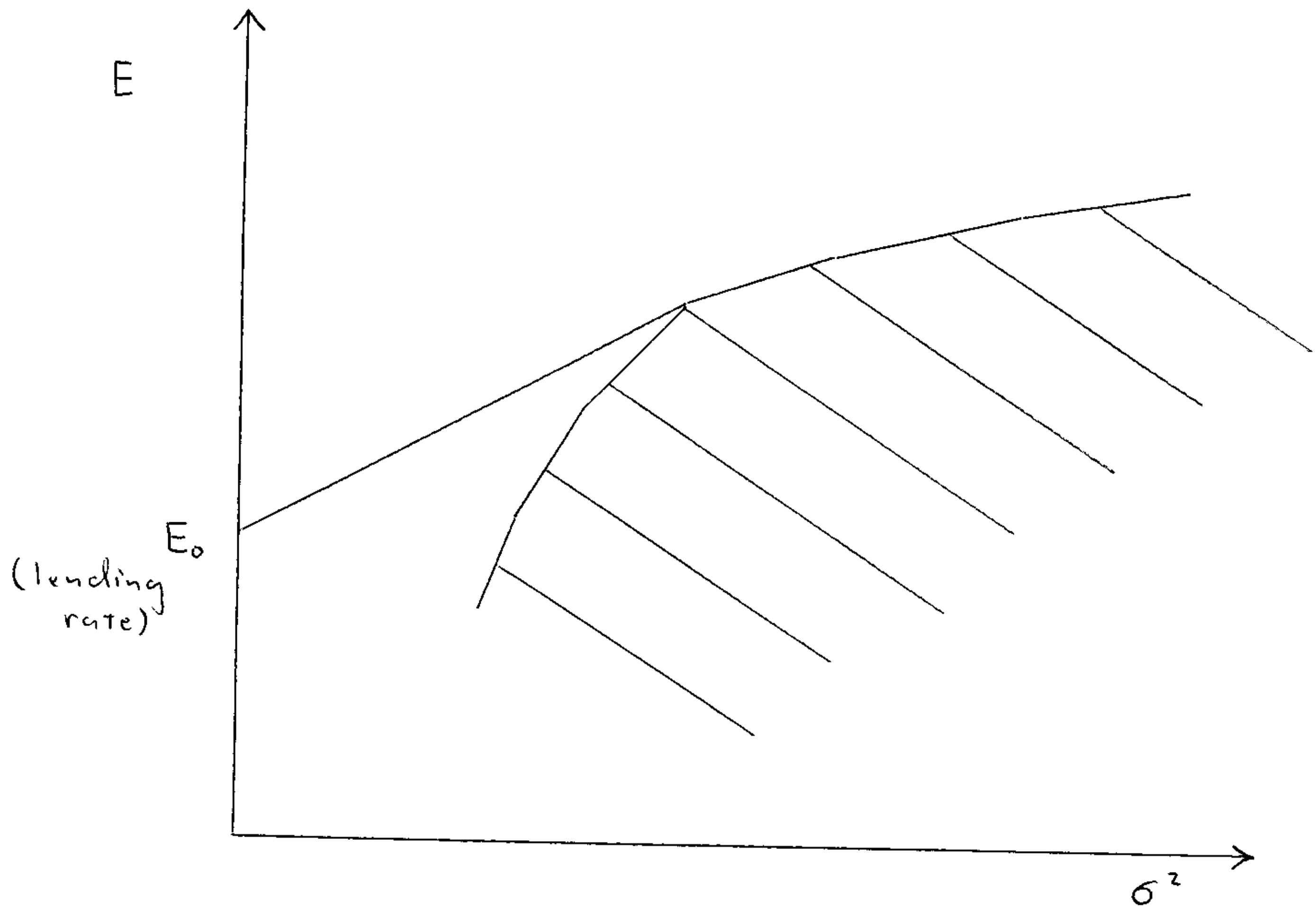
$$\alpha_i > 0 \text{ implies that } \gamma_i = 0 \quad .$$

If for asset i , $\alpha_i = 0$ and $\gamma_i > 0$, then the linearity relation does not hold. In fact, it is precisely because it fails to hold that asset i is not included in the portfolio.

(6) What happens if there is a riskless asset, but like other assets it cannot be shorted? Now the relevant figure is shown in Figure 2. Obviously, if all investors are long in the riskless asset the situation is as before and the CAPM holds. If not, then the range where the efficient set is not necessarily linear is relevant. We could now construct a counterexample to the CAPM just by adding to the example given above a riskless asset with a sufficiently low rate of return.

(7) Suppose now that short sales are not absolutely excluded, but rather are penalized as in the "real world". We fortunately don't have to be too

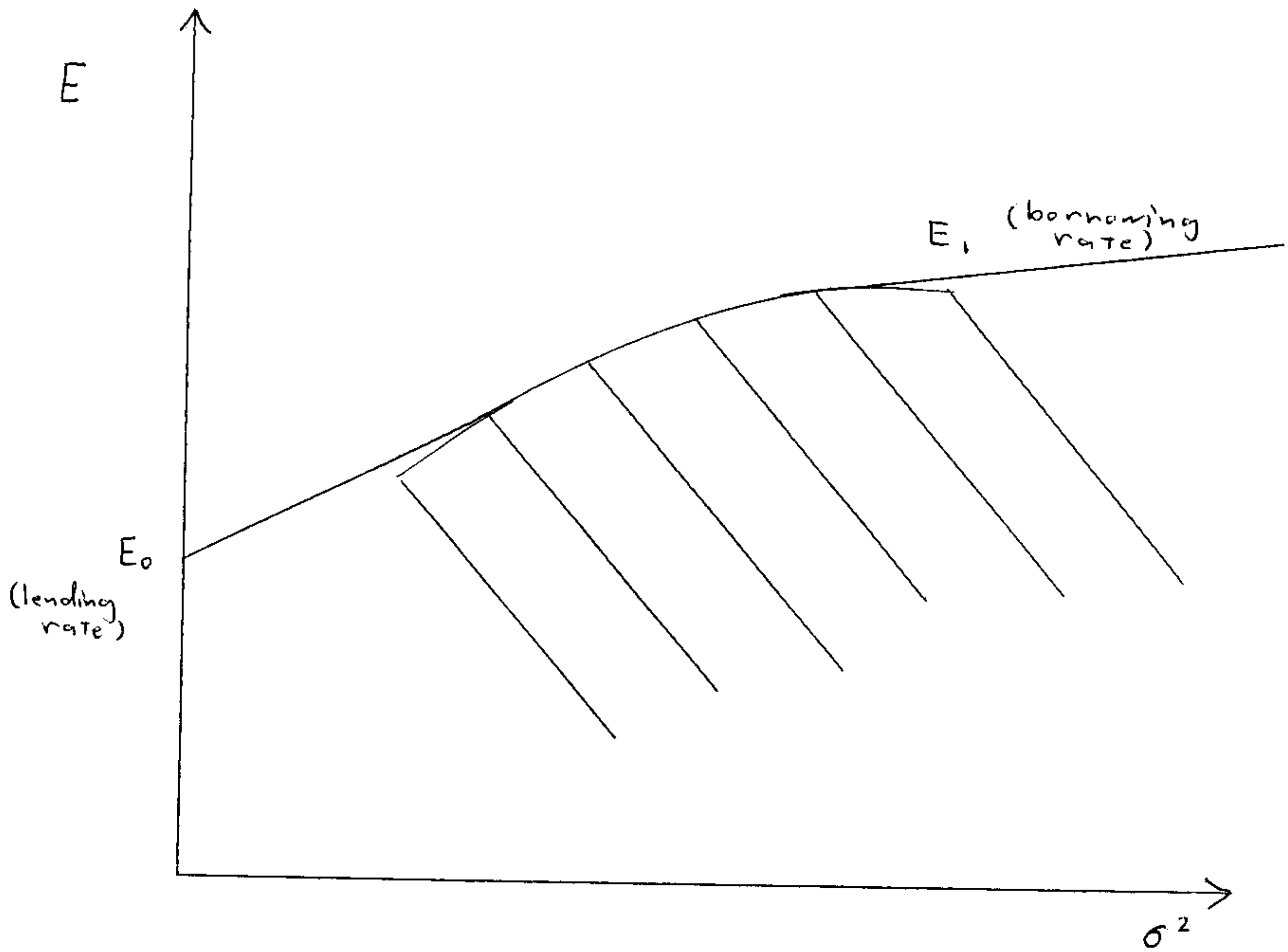
Figure 2



specific about the exact nature of the penalty; once again it may induce agents to hold portfolios that do not contain all assets. However, the analyses above remain valid and if, for example, there is a riskless asset that can be freely shorted, then separation obtains and as in (4) the CAPM holds. If the riskless asset can be shorted but only at a higher "penalty" rate, then the efficient frontier has two linear segments and the CAPM will, in general, fail. This case is treated in Blume and Friend [1973] and illustrated in Figure 3.

(8) One final case is interesting and worth treating. Suppose that returns are generated by a single factor market model with many assets, as in Ross [1973]. In this case nearly all well diversified portfolios will approximately be efficient and the CAPM may be expected to hold in an approximate sense. The area of approximation contains an important class of problems with genuine empirical significance, but they are beyond the scope of these notes. Some results may be found in Ross [1973].

Figure 3



Bibliography

Black, Fischer. "Capital Market Equilibrium with Restricted Borrowing." Journal of Business, March 1973.

Blume, Marshall and Friend, Irwin. "A New Look at the Capital Asset Pricing Model." Journal of Finance, March 1973.

Lintner, John. "The Valuation of Risk Assets and the Selection of Risky Investments in Stock Portfolios and Capital Budgets." Review of Economics and Statistics, February 1965, pp. 13-37.

Ross, Stephen A. "Return, Risk and Arbitrage." Rodney L. White Center for Financial Research, Working Paper No. 17-73a, 1973.

Sharpe, William. "Capital Asset Prices: A Theory of Market Equilibrium under Conditions of Risk." Journal of Finance, September 1964, pp. 425-442.