Security Valuation Formulae: Their Relationship to Estimates of the Risk-Return Tradeoff

by

Daniel Rie

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RODNEY L. WHITE CENTER
FOR FINANCIAL RESEARCH

University of Pennsylvania

The Wharton School

Philadelphia, Pa. 19174

1. Introduction

This paper will show that a return generating process is the direct result of the parameters which control security valuation. Because the valuation process may be complex and respond to many external factors, a usefully simple return generating process may not exist. By beginning with a relatively simple valuation formula, a multiple market factor return generating process is developed and it is further shown that a single market factor model will produce errors which vary systematically with beta and which are explainable in terms of a few market wide variables. This theory is empirically tested and found to be appropriate in terms of the implied return generating process and also capable of giving a significant explanation of the residual errors of a single factor model. Unfortunately the difficulty in getting correct proxies for the independent variables weakens the results of the estimation. A comparison of these explained residual errors with those produced by the zero beta factor showed that the two are significantly related, but that the zero beta factor offered a significant improvement over the residuals generated from the available proxy variables.

2. Theoretical Development

The purpose of this paper is to examine the process which produces returns on assets in a market which is continuously in equilibrium, then to compare the prior expectations in such a market to the realized values of returns and to sample averages of realized returns. The focus of this examination on expected returns and sample mean values of returns is necessitated by the fact that in the various forms of the Capital Asset Pricing Model (CAPM) the adjustment process which moves the market into equilibrium is one of changing the expected return to an individual asset.[1] The CAPM describes the anticipated mean return to a security in terms of a single anticipated parameter of the distribution of returns to that security, known as beta, and two parameters of the overall market equilibrium at that time, the risk free rate, R_f, and the expected risk premium, $(\overline{R}_m - R_f)$. This has resulted in tests of the model's applicability to actual markets which use the distribution of realized returns on securities in a sample to gain inference about the mean of prior expected returns of the securities. To make such inferences, it is necessary to examine in more detail the relationship between realized and anticipated returns.

The primary difference between expected returns and realized returns is that expected returns involve looking forward from a single point in time, whereas, realized returns have a substantial component which comes from the change in what is perceived when looking forward (to determine price or anticipated return) from the beginning of the return period to what is perceived when looking forward at the end of the return period. To argue that such changes in perceptions will be uncorrelated over time would require an assertion that, when forming their prior distribution of returns, market participants are able to both forcast and assess the impact on their beliefs of such slowly varying forces of economic importance as changes in the size and age distribution of the population, changes in the nature of the distributions of income and wealth, changes in social customs, technological changes and changes in the overall stability of the economy. The processes which convert these basic factors into economic expectations do not seem to be sufficiently understood to warrant such an assertion. Therefore, rather than arguing that their impact will be neutralized by taking an average over a large number of successive observations, some explicit adjustment for secular changes in the parameters which underlie the valuation process is necessary. While this principle is by no means new, it has not been formulated previously in general terms. This will be done below.

The process which relates the observed returns to their prior mean value can be written as

$$\tilde{R}_{it} = E_{t-1} [\tilde{R}_{it}] + \tilde{\varepsilon}_{it} . \qquad (2\text{-}1)$$

If expectations are unbiased[2] then $\tilde{\varepsilon}_{it}$ has a mean of zero. If a series of such returns are observed and the mean return is computed in order to estimate the average of the prior expectations over the period the result is

$$\overline{R}_i = \frac{1}{T} \sum_{t=1}^{T} E_{t-1} [\tilde{R}_{it}] + \frac{1}{T} \sum_{t=1}^{T} \tilde{\varepsilon}_{it} \qquad (2\text{-}3)$$

and the resulting sample mean return, $\overline{R}_i$, will only equal the average of the prior means if the last term in (2-3), the average deviation from prior expectations, is zero. If the average deviation is not zero, it must be subtracted from the mean realized return to make any inferential statements about the average of prior expectations. Alternatively an estimate can be made of the value that $\tilde{\varepsilon}_{it}$ is expected to take on given the actual state of various exogenous factors, and this can be used to generate a conditional expectation for $\tilde{R}_{it}$. This process has been used by most recent authors e.g. Black, Jensen, and Scholes [2], and Blume and Friend [3].

Such conditional expectation, or adjustments for known drift in the average deviation from zero only push the problem back one level to the error term in the conditional expectation model. If the model which produces the conditional value of $\tilde{\varepsilon}_{it}$ were absolutely complete, the conditional expectation would be equal to the realized return. In practice only one or two external factors are considered, and very little effort has been expended in justifying the form of such models.[3] The proposition that the error terms of conditional expectation models will average out to zero remains a matter of speculation or assertion. The role of model imperfections in biasing conditional expectations is examined in the following section.

The process of finding the mean of prior expected returns by using a model of conditional expectations may be formalized in the following manner:

$\tilde{\varepsilon}_{it}$ = difference between realized and prior expected returns.

$E[\tilde{\varepsilon}_{it}|\emptyset_t]$ = Conditional expectation of $\tilde{\varepsilon}_{it}$ given state of the world at time t, $\emptyset_t$.

$E[\tilde{\varepsilon}_{it}|M_j(\emptyset_t)]$ = Conditional expectation of $\tilde{\varepsilon}_{it}$ given the state of the world at time t, $\emptyset_t$, and processing this information using <u>only</u> the model M_j.

although the definition of $\emptyset_t$ insures that

$$E[\tilde{\varepsilon}_{it}|\emptyset_t] = \tilde{\varepsilon}_{it} \tag{2-4}$$

the constraint that the conditional expectation will be formed by using the model M_j which relates the state of the world to $\tilde{\varepsilon}_{it}$, along with the assumption that M_j is a less than perfect model leads to

$$E[\tilde{\varepsilon}_{it}|M_j(\emptyset_t)]+{}_j\tilde{\mu}_{it}=\tilde{\varepsilon}_{it}$$
$$\sigma^2({}_j\mu_{it}) > 0 \tag{2-5}$$

If the model M_j is unbiased, then $E[{}_j\tilde{\mu}_{it}] = 0$

The problem of assertaining the sample mean value of prior expected returns using the model M_j becomes

$$\tilde{R}_{it} = E_{t-1}[\tilde{R}_{it}] + E[\tilde{\varepsilon}_{it}|M_j(\emptyset_t)] + \tilde{\mu}_{it} \tag{2-6}$$

and

$$\frac{1}{T}\sum_{t=1}^{T} E_{t-1}[\tilde{R}_{it}] + \frac{1}{T}\sum_{t=1}^{T}\tilde{\mu}_{it} = \frac{1}{T}\sum_{t=1}^{T}\tilde{R}_{it} - \frac{1}{T}[E(\tilde{\varepsilon}_{it}|M_j(\emptyset_t))] \tag{2-7}$$

To claim that the right hand side of equation (2-7) is equal to the average of prior expected returns is equivalent to asserting that the model M_j used to compute conditional expectations is sufficiently good that its errors, ${}_j\mu_{it}$ will not only have a prior mean of zero, but that the mean realized error will also be zero. If this error were some sort of noise which did not result from any systematic underlying phenomena, this might be a reasonable assumption. In fact the statistical properties of this error have generally been assumed until proven inadequate, at which time a new conditional expectation model is produced. From (2-7) it is clear that an examination of the errors cannot identify whether they have a non-zero mean, since the realized mean of ${}_j\tilde{\mu}_{it}$ is indistinguishable from the mean of prior expectations of return.

An alternative to a purely statistical approach to calculating the conditional expectation of returns can be found by looking at the definition of returns and the process of valuation. The return earned over the period between t-1 and t can be broken down into the component which comes from cash earnings which accrue to the holders of the asset and changes in the market value of the security. These changes in value can be further decomposed into expected and unexpected changes in value. It is these unanticipated changes in value which are of concern here because they may form an important part of the sample mean return for the reasons mentioned above. The deviation of single period cash earnings around anticipated mean values should be less susceptible to such long term secular drifts and will be ignored here. Realized returns can be written in terms of cash payments and the two equilibrium prices as:

$$\tilde{R}_{it} = (\tilde{C}_{it} + \tilde{P}_{it})/P_{it-1} \;\; -1 \qquad (2\text{-}8)$$

$\tilde{C}_{it}$ - cash payments received over the period ending at t

$\tilde{P}_{it}$ = price of the security at time t

$\tilde{R}_{it}$ = return earned by holding the security from t-1 to time t

Return to the holder of the security results from expectations plus unexpected changes in the valuation of the security and can be written

$$R_{it} = E_{t-1}[\tilde{R}_{it}] + \; (\tilde{C}_{it} - E_{t-1}[\tilde{C}_{it}]) + \tilde{RU}_{it}$$

where

$$\tilde{RU}_{it} = (\tilde{P}_{it} - E_{t-1}[P_{it}])/\tilde{P}_{it-1} \qquad (2\text{-}9)$$

so that RU_{it} is roughly equivalent to $\tilde{\varepsilon}_{it}$ in (2-1) and the deviation of return around its prior expected value will be a function of at least as many things as cause the price to vary around its expected value.

To identify just what factors may influence price requires a theory of valuation. To make the discussion of the role the valuation function more concrete the valuation function shown in (2-10) will be used. This pricing function has six arguments, the first three being specific to the particular asset and the last three representing market wide conditions which contribute to the determination of value of all assets. Any or all of these arguments may vary over time.

$$P_{it} = P(a_{it}, b_{it}, c_{it}, d_{mt}, e_{mt}, f_{mt})$$
$$\tilde{P}_{it+1} = P(\tilde{a}_{it+1}, \tilde{b}_{it+1}, \tilde{c}_{it+1}, \tilde{d}_{mt+1}, \tilde{e}_{mt+1}, \tilde{f}_{mt+1}) \qquad (2\text{-}10)$$

Returns for a particular asset will depend on the values that three company attributes, a_{it}, b_{it}, c_{it} and three market attributes d_{mt}, e_{mt}, f_{mt}, take on over time, and realized returns will depend on the joint distribution of all six items. What is of most concern here is the difference between prior expectations and realized returns. The problem of whether price expectations are realized on average cannot be reduced to the problem of whether the average realized values of the arguments of the valuation function are equal to their prior anticipated mean values, unless the valuation function (2-10) is linear. Even if linearity were assumed, it is unlikely that over reasonable estimation periods that sample averages of the parameters would average out to their prior mean values. Therefore, a substantial component of unanticipated return in overall returns can be expected even in portfolios which diversify out the portion of unanticipated return which comes from changes in company specific attributes. This is simply another way of saying that it is unlikely that the sample mean of the $\tilde{\varepsilon}_{it}$ values in (2-3) will be zero, and the non-zero mean is caused by changes in the market wide influences on asset valuvation. By using a valuation function which is sufficiently correct to have errors which are fully diversifiable and which contain no component of drift which will not average out to zero over reasonable estimation periods for diversified portfolios,we can determine what must be taken into account to form a conditional expectation which will not produce biased conditional values. In terms of (2-7), we require a valuation function which produces conditional expectations which are sufficiently correct that the sample average value of $\tilde{\mu}_{it}$ will be very close to zero. This condition is the requirement for making inferences about prior mean returns. Rather than attempt to determine what is required for a fully adequate valuation function, this study will proceed by examining the implications for analysis of observed data of using a fairly simplistic valuation model, and comparing the requirements imposed by using such a minimal model with other methods which have been used to examine observed returns and relate them to prior expectations.

The valuation model to be used is based on the premise that price represents the discounted value of future cash flows. Since the cash payments in future periods are uncertain, it will be assumed that this riskiness can be compensated for by applying an appropriate discount rate to the mean of the distribution of cash flows for each future period. As a further simplification it will be assumed that the mean of the currently believed distribution of cash flow per period is the same for each future period. This mean is subject to change over time, but the constant level for each future period is a type of no growth assumption for currently held beliefs about asset payments. Earnings in each period are treated as if they are fully paid out or alternatively that the valuation process is independent of the dividend decision. The intertemporal discount rates used to capitalize the means of the cash flow distributions are also assumed to be the same for each future period at each point in time. In other words, the term structure of capitalization rates is assumed to be flat at all times, but subject to changes in level over time. All compensation for riskiness in the distributions of cash flows is accomplished through the level of the capitalization rate. The impact of the distributions of cash flows on prices are otherwise summarized by their means. If the stream of payments is expected to last indefinitely then these assumptions lead to the valuation formula for a perpetual annuity which is given in (2-11).

$$P_{it} = \overline{C}_{it}/\rho_{it} \quad (\text{time} \geq t) \tag{2-11}$$

where $\overline{C}_{it}$ is the mean of the process generating the net cash payments per unit of time, and $1/\rho_{it}$ is the price per unit of mean cash payment at time t. At times prior to time t all the quantities in (2-11) are random variables so that (2-11) may be written

$$\tilde{P}_{it} = \tilde{\overline{C}}_{it}/\tilde{\rho}_{it} \quad (\text{time} < t) \tag{2-12}$$

The probability distributions of each of the right hand variables will change over time as information arrives, collapsing to a single point at time t. Equation (2-12) in combination with equation (2-8) can be used to arrive at the distribution of returns in terms of the distributions of mean cash payments and price per unit mean. In order to decompose the random return from t-1 to t into components which come from changes in expectations, $\tilde{\bar{C}}_{it}$, and changes in capitalization rates, ρ_{it}, it is useful to define new variables to normalize each realization of a random variable by expressing the outcome as the fractional deviation of the random variable around its previously anticipated mean.

$$\delta\tilde{C}_{it} \equiv (\tilde{C}_{it} - \bar{C}_{it-1})/\bar{C}_{it-1} \tag{2-13}$$

$$\delta\tilde{\bar{C}}_{it} \equiv (\tilde{\bar{C}}_{it} - \bar{C}_{it-1})/\bar{C}_{it-1} \tag{2-14}$$

$$\delta\tilde{\rho}_{it} \equiv (\tilde{\rho}_{it} - \rho_{it-1})/\rho_{it-1} \tag{2-15}$$

If the mean cash payment anticipated at time t-1 were one dollar and the actual payment turned out to be \$.80 then $\delta\tilde{C}_{it}$ would be -.2. If this were to result in a downward revision of the mean of future payments from \$1.00 down to \$.95 then $\delta\tilde{\bar{C}}_{it}$ would be -.05. By assumption each of the change variables defined in (2-13), (2-14), and (2-15) has a prior expected value of zero. Knowledge of the change variables is equivalent to knowledge of the uncertain levels if the previous level values are known. Therefore, (2-16), (2-17), and (2-18) can be used to substitute fractional changes for future levels in the pricing equation of (2-12) and (2-11), and (2-12) used to eliminate prices from the return equation (2-8), resulting in an expression for the value of realized return in terms of the fractional deviation of the components of price (and current net cash inflow) from their anticipated mean values.

$$\tilde{C}_{it} = (1 + \delta\tilde{C}_{it})\overline{C}_{it-1} \tag{2-16}$$

$$\tilde{\overline{C}}_{it} = (1 + \tilde{\overline{C}}_{it})\overline{C}_{it-1} \tag{2-17}$$

$$\tilde{\rho}_{it} = (1 + \delta\tilde{\rho}_{it})\rho_{it-1} \tag{2-18}$$

Substitution into (2-12) and (2-11)

$$\tilde{P}_{it} = P_{it-1}\ (1 + \delta\tilde{\overline{C}}_{it})/(1 + \delta\tilde{\rho}_{it}) \tag{2-19}$$

and substitution of (2-19), (2-16), and (2-11) into (2-8) yields:

$$\tilde{R}_{it} = (1 + \tilde{\overline{C}}_{it})/(1 + \delta\tilde{\rho}_{it}) - 1 + (1 + \delta\tilde{C}_{it})/\rho_{it-1} \tag{2-20}$$

by using the approximation

$$\frac{1}{1 + x} \approx 1 - X \text{ for } X << 1$$

and assuming that $\delta\tilde{\rho}_{it} << 1$

(2-20) becomes

$$\tilde{R}_{it} \approx (1 + \delta\tilde{\overline{C}}_{it})\ (1-\delta\tilde{\rho}_{it}) + (1+\delta\tilde{C}_{it})\rho_{it-1} - 1$$

$$= \delta\tilde{\overline{C}}_{it} - \delta\tilde{\rho}_{it} + \rho_{it-1}(1 + \delta\tilde{C}_{it}) + (\delta\tilde{\overline{C}}_{it})\ (\delta\tilde{\rho}_{it}) \tag{2-21}$$

Equation (2-21) decomposes returns into the components which come from fractional changes in expected net cash inflows ($\delta\tilde{\overline{C}}_{it}$), from fractional changes in capitalization rates ($\delta\tilde{\rho}_{it}$), from expected yield(ρ_{it-1}), from the deviation of the current cash yield from its anticipated value $(\rho_{it-1}\delta\tilde{C}_{it})$ and from the interaction of the change in expectation with the change in capitalization rate. By taking the expectation of both sides of (2-21) it can be reduced to the component of realized return which is due to prior expectations and the components which are unanticipated. The expectation of each of the delta terms is zero so that the expectation of (2-21) becomes

$$E_{t-1}[\tilde{R}_{it}] = \rho_{it-1} + E_{t-1}[(\delta\tilde{\overline{C}}_{it})(\delta\tilde{\rho}_{it})]$$

The second term involving the cross products of $\delta\tilde{\overline{C}}_{it}$ and $\delta\tilde{\rho}_{it}$ can be assumed to be essentially equal to zero for two reasons. First the relationship between changes in mean future net cash payments and changes in capitalization rates has no naturally dominant sign. An increase in the productivity of capital (and associated positive $\delta\tilde{\overline{C}}_{it}$) could be met with an increase or a reduction in the amount of resources devoted to capital formation, and hence in the marginal productivity of capital (strongly associated with $\delta\rho$). Since neither sign is dominant, an assumption that the interrelationship is neutral should cause no harm. However, the case need not rest here because this term is the product of two fractional changes and will tend to be much smaller than the individual change term. Although it might be argued that over a relative long sample period even a small mean for this term might cause it to dominate the larger variance but zero mean components in its contribution to sample mean return, empirical estimates of the relative size of the components substantiate the validity of the negligible contribution of this cross product term in explaining sample mean returns. The negligible result for sample mean returns assures that the term can be ignored in considering individual period returns where the delta terms will play a relatively stronger role. Therefore it will be assumed henceforth that

$$E[\delta\tilde{\overline{C}}_{it}\ \delta\tilde{\rho}_{it}] = 0 \tag{2-22}$$

and moreover

$$\delta\tilde{\overline{C}}_{it}\ \delta\tilde{\rho}_{it} \ \tilde{\approx}\ 0 \tag{2-23}$$

The resultant expected return is

$$E[\tilde{R}_{it}] = \rho_{it} \tag{2-24}$$

and the deviation of realized return about its expected value contingent on the simple valuation model of (2-11) is

$$\tilde{R}_{it} - E_{t-1}\left[\tilde{R}_{it}\right] = \delta\tilde{\overline{C}}_{it} - \delta\tilde{\rho}_{it} + (\rho_{it-1})\delta\tilde{C}_{it} \tag{2-25}$$

The third term of equation (2-25) can be rewritten using the pricing equation, (2-11) and the definition of $\delta\tilde{C}_{it}$ as

$$(\tilde{C}_{it} - \overline{C}_{it-1})/\rho_{it-1}$$

which is the unanticipated portion of the current divident yield. The difference between realized returns and prior anticipated returns, therefore, can be decomposed into portions which result from the fractional change in cash flow expectations, the fractional change in the capitalization rate, and the unanticipated portion of realized dividend yield.

As will be shown below, the number of factors which explain the deviation of individual security returns about their prior means bears no relationship to the number of factors required to explain the deviation of diversified portfolio returns around their prior means, therefore this question must be treated specifically.

One of the principle results of the capital asset pricing model is that variation which can be reduced through diversification is not worth considering as seriously as variation which persists in influencing the returns of a diversified portfolio. While the capital asset pricing model is concerned primarily with diversifying errors from an ex ante standpoint, a study of realized returns may find instances where variables which were expected to behave independently do not in fact do so in the observed series. Therefore, in looking backward there may be errors which were diversified against in the ex ante sense but cannot be ignored in the observed series. For this reason the only errors or factors which can be ignored are those which tend to produce no net effect on the return of a well diversified portfolio (which may be subject to restrictions such as the inclusion of certain types of securities, for example "low Beta"). Using this as a guideline, it is possible to decompose the changes in individual security cash flow mean and capitalization rate used in (2-26) into the components which tend to carry over into portfolio returns, and the components which are specific to the individual security, and may be diversified away. One simple way of doing this is to define

$$\begin{aligned} PIDOT_i &= \delta\tilde{\bar{C}}_{it} + \text{realized dividend yield} \\ &= \delta\tilde{\bar{C}}_{it} + (\rho_{it-1})(1+\delta\tilde{C}_{it}) \qquad (2\text{-}29) \end{aligned}$$

= component of returns arising from cash rather than capitalization sources

Expected return to the security is entirely represented in PIDOT because:

E_{it} [$PIDOT_{it}$] = expected dividend yield = ρ_{it-1}

Then using (2-25)

$$\tilde{R}_{it} = PIDOT_{it} - \delta\tilde{\rho}_{it} \qquad (2\text{-}30)$$

Equation (2-30) may be used as a basis for introducing a market model which relates variations in the observed values of $PIDOT_{it}$ and $\delta\tilde{\rho}_{it}$ to variations in market wide variables. To write $\delta\tilde{\rho}_{it}$ in terms of market wide influences, the capital asset pricing model can be used to substitute changes in market-wide variables for changes in ρ_i. The capital asset pricing model can be written as

$$E[\tilde{R}_{it}] = R_{ft} + \beta_i \, [E[\tilde{R}_m] - R_{ft}] \qquad (2\text{-}31)$$

where R_{ft} is the risk free rate at time t and $E[\tilde{R}_{mt}]$ is the expected return on the market portfolio at time t.

or using (2-24)

$$\rho_i = R_{ft} + \beta_i [\rho_{mt} - R_{ft}] \qquad (2\text{-}32)$$

to find $\delta\tilde{\rho}_{it}$ substitute (2-32) into the definition of $\delta\tilde{\rho}_{it}$, (2-15)

$$\rho_{it} - \rho_{it} = R_{ft} - R_{ft-1} + \beta_i [\rho_{mt} - R_{ft}] - \beta_i \, [\rho_{mt-1} - R_{ft-1}]$$

denoting $\rho_{mt} - R_{ft}$ as χ_{mt}, the risk premium on the market portfolio at time t, and using Δ to indicate changes:

$$\delta\rho_{it} \equiv \Delta\rho_{it}/\rho_{it-1} = \frac{\Delta R_{ft}}{\rho_m}\frac{\delta_m}{\rho_i} + \beta_i \frac{\Delta\chi_{mt}}{\rho_m}\frac{\rho_m}{\rho_i}$$

by defining $RDOT_t = \dfrac{\Delta R_{ft}}{\rho_m}$ (2-33)

$$XDOT_t = \frac{\Delta\chi_{mt}}{\rho_m} \qquad (2\text{-}34)$$

$$\tilde{\delta\rho}_i = \rho_m/\rho_i \text{ RDOT}_t + \beta_i(\rho_m/\rho_i)\text{XDOT}_t \tag{2-35}$$

or be defining

$$NU_{it} = \rho_{mt}/\rho_{it} \tag{2-36}$$

and

$$MU_{it} = NU_{it}\ (\beta_i) = (\rho_{mt}/\rho_{it})\ \beta_i \tag{2-37}$$

the $\tilde{\delta\rho}_{it}$ term can be written as

$$\tilde{\delta\rho}_{it} = NU_{it}\ (\text{RDOT}_t) + MU_{it}\ (\text{XDOT}_t)$$

It should be noted that the component of security returns which results from changing capitalization rate is related to the marketwide variables by an exact rather than a stochastic relationship which depends only on the value of β_i and market wide variables.

It will be assumed that the systematic aspects of **realized values** of PIDOT_i may be represented as in equation (2-38):

$$\text{PIDOT}_{it} = \gamma_i + \lambda_i\ \text{PIDOT}_{mt} + \tilde{\varepsilon}_{it} \tag{2-38}$$

so that realized security returns may be written

$$\tilde{R}_{it} = \text{PIDOT}_i - NU_{it}\ (\text{RDOT}_t) - MU_{it}\ (\text{XDOT}_t)$$

$$= \gamma_i + \lambda_i\ (\text{PIDOT}_{mt}) - NU_{it}\ (\text{RDOT}_i) - MU_{it}\ (\text{XDOT}_t) + \tilde{\varepsilon}_{it} \tag{2-39}$$

The value of γ_i can be found by taking the expected value of both sides of (2-39) and using the capital asset pricing model to relate expected returns on the security to expected returns on the market.

$$E[\tilde{R}_{it}] = \gamma_i + \lambda_i\rho_m$$

and substituting for expected returns using (2-24) and (2-32)

$$\gamma_i = R_f + \beta_i(\rho_m - R_f) - \lambda_i\rho_m$$

Substituting this into equation (2-39) yields the market model for realized security returns given in equation (2-40).

$$\tilde{R}_{it} = R_f + \beta_i(\rho_m - R_f) - \lambda_i\rho_m + \lambda_i \, PIDOT_{mt} - NU_{it}(RDOT_t) - MU_{it}(XDOT_t) + \tilde{\varepsilon}_{it} \tag{2-40}$$

In equation (2-40) the non-systematic or firm changes in $\overline{C}_i$ and ρ_i have been relegated to the error term. The impact of the error term on portfolio returns will presumably be reduced to zero as the number of assets in the portfolio is increased. The assumption that (2-35) and (2-38) adequately represent the systematic components of $PIDOT_{it}$ and $\delta\tilde{\rho}_{it}$ allow (2-28) to be written in terms of the three marketwide factors $PIDOT_m$, $XDOT_t$, and $RDOT_t$. A representation such as (2-40) which describes security returns in terms of a limited number of marketwide factors and specific asset coefficients will be called a return generating process. The choice of a return generating process is a central issue in the process of calculating a conditional expectation, and converting the realized return to an <u>ex ante</u> expectation, since any error in the specification of the return generating process will result in an error in the computed unconditional expectation. An examination of (2-38) may cast some doubt on the existance of any adequate return generating process. In (2-38), it is assumed that any change in net cash flow expectations for the economy will on average be reflected by a proportional change in expectations for the individual security regardless of the source of the change in economic expectations. This must be an oversimplification in view of the fact that the individual asset response would probably vary depending on whether the change in economic expectations was due to changing technology, changing consumer behavior, or changing financial conditions. The number of factors which might be included in (2-38) is equal to the number of possible sources of changing income which have an impact on the entire economy. Unless the firm responds in a similar manner to each of these factors so that several factors can be represented by a single composite factor, they will all have to be represented in a model which contains the undiversifiable components of realized returns. Since this number could exceed both the number of firms and the number of observations available for study, the return generating function may not exist in a useful form. However, in the context of current research on security prices,(2-35) and (2-38) offer a convenient starting point for loosening the restrictions which have been imposed on the return generating process in earlier studies.

The return generating function of equation (2-40) decomposes security returns into three components which result from the three components of returns to the market portfolio. This can be seen by using equations (2-25), (2-29), (2-33) and (2-34) to write returns to the market portfolio

$$\tilde{R}_{mt} = PIDOT_m - XDOT_m - RDOT_m \qquad (2\text{-}41)$$

$$\bar{R}_m = \rho_m$$

This fact can be used to rewrite (2-40) in terms of that part of security returns which comes from beta times the market return, and the part of returns generated by the separate individual factors in equation (2-40) which is left unexplained by that source. By adding beta times the left side of (2-41) and subtracting beta times the right side to equation (2-40) and using the definition of PIDOT given in (2-29), (2-40) becomes

$$\tilde{R}_{it} = R_f(1-\beta_i) + \beta\tilde{R}_{mt} + (\lambda_i-\beta_i)(PIDOT_{mt}-\rho_{mt}) - (MU_{it}-\beta_i)XDOT_{mt} - (NU_{it}-\beta_{it})RDOT_{mt} + \tilde{\varepsilon}_{it} \qquad (2-42)$$

The first two terms on the right hand side of (2-42) are the type of single market factor model which has traditionally been used to represent security returns. The last four terms, therefore are components of the error term of such a single factor model. To assert that such single market factor model errors will average out to zero over a particular sample of observed data is equivalent to asserting that there has been no important unanticipated (and secular) changes in expectations, capitalization rates, or the risk free rate. Since the coefficients NU_{it}, MU_{it} and by implication λ_i depend only on β_i, if returns are generated by a process such as (2-40) and a single factor model is used to compute conditional expectations, the actual returns, even for diversified portfolios, will differ systematically from the computed conditional expectation.

If μ_{it} is used to represent the error which results from using a single factor model based on returns to the market portfolio so that

$$\tilde{\mu}_{it} = \tilde{R}_{it} - R_f - \beta_i(\tilde{R}_{mt} - R_f)$$

Then from equation (2-42)

$$\tilde{\mu}_{it} = (\lambda_i-\beta_i)(PIDOT_{mt}-\rho_m) - (MU_{it}-\beta_i)XDOT_{mt} - (NU_{it}-\beta_i)RDOT_{nt} + \tilde{\varepsilon}_{it} \qquad (2-43)$$

For given values of $PIDOT_{mt}$, $XDOT_{mt}$ and $RDOT_{mt}$, the observed values of $\tilde{\mu}_{it}$ will be a systematic function of beta. This results from the fact that MU_i and NU_i depend only on ρ_i which is a function of beta and on beta itself. Since returns are determined by MU_i, NU_i, and λ_i along with the three components of market returns, the value of beta will depend on these three coefficients and the variability of the three components of market return. Therefore λ_i is implicitly a function of beta. An alternative interpretation of this relationship may prove more appealing. If the value of λ_i which measures the responsiveness of cash flow expectations and dividend yield of the firm to the corresponding variables for the market is known, then the value of beta can be determined. This is true since the overall covariance of returns results jointly from the fact that the payment stream is tied to the economic market in its particular way and from the fact that variation in the discount rate for this payment stream will tend to be associated with changes in discount rates applied to other assets. The systematic change in the discount rate will be determined by the nature of the cash flow sensitivity, λ_i, through its influence on β. Thus the sensitivity of earnings expectations to changes in the market expectations can be viewed as the sole determinant of beta, ρ_i, NU_i and MU_i.

If this systematic variation in $\tilde{\mu}_{it}$ is projected back to the return axis (where β_i=0), the observed result will be a number which results from the sample data and which can be called a zero beta factor. There are several alternative explanations available for the difference between expected returns to the zero beta portfolio of securities and the risk free rate.[4] Equation (2-42) adds one more possible explanation of the difference between observed returns to the zero beta portfolio and the risk free rate which does not require a difference in the prior mean values. What is of particular interest about equation (2-42) is that it provides a theory of residual errors of diversified portfolios over time as a function of the overall portfolio β and the sources of market returns. It is therefore possible to test the validity of the form of the model of equation (2-40) and to examine the extent to which it explains the zero beta portfolio returns which have been studied by previous authors. This will be done next.

3. Empirical Tests

Equations (2-40) and (2-42) describe returns and the systematic component of residual errors in terms of several marketwide variables, capitalization rate applied to income from the security, ρ_i, the coefficient which relates income changes in the market portfolio to income changes for the security, λ_i, and the overall value of β_i for the security. Of the three firm specific variables, ρ_i, λ_i and β_i, only β_i need be considered as exogeneous. This is true because for given values of the market wide variables R_f and ρ_m along with a specific value of β_i, the capital asset pricing model can be used to yield a value of ρ_i (and by extension MU_i and NU_i). The value of λ_i can also be found given β_i and the variances and covariances of the sources of market returns in a manner which will be described below. Therefore, in order to test the return generating function represented in equation (2-40) and the degree to which it can explain the residual errors of a single factor market model as characterized in equation (2-42) the following items of data are required:

Series of returns on assets or portfolios with identifiable values of β_i

A series of fractional changes in cash flow expectations plus realized dividends yields (PIDOT) for the market portfolio

A series of capitalization rates for the market portfolio (ρ_m)

A series of observations for the risk free rate (R_f)

A series for changes in the risk free rate divided by the capitalization rate for the market portfolio (RDOT)

A series for changes in the risk premium ($\rho_m - R_f$) divided by the capitalization rate for the market portfolio (XDOT)

To construct this data numerous assumptions were required. Only those assumptions which appear to have had an identifiable impact on the results will be discussed in detail. The data used to test the model was derived from the dividend price ratio equation of the Penn-Brookings econometric model. Cash flow expectations were taken as a weighted average of corporate profits and dividend payments on the Standard and Poor's index stocks amplified to reflect the underlining earnings base. The capitalization rate for the market portfolio was derived by assuming that the price of the market was equal to the expected earnings found above discounted at a rate ρ_m. Therefore, ρ_m was calculated by dividing the dividend price ratio for the Standard and Poor's stocks by their dividend payments and multiplying by expected earnings. The dividend price ratio equation of the Penn-Brookings model explains the dividend price ratio in terms of expectations, interest rate, and uncertainty variables. Since this study has separated market return into three corresponding components, expectations, the risk free rate, and the risk premium, the risk free rate was calculated from the terms representing the interest rate effect in the dividend-price equation of the Penn-Brookings equation. This made the risk free rate depend on a four period lag of the corporate bond rate and a three year lag on the rate of inflation. These lags would correspond to a measured interest rate of considerably longer duration than the short term rate usually used (a short term rate was used to compute portfolio beta values). A long term rate was used rather than a short term risk free rate because the discounting model implicit in equation (2-11) and (2-12) involves an average of future short term rates and a long term rate implicitly supplies such an average. The change variables PIDOT and RDOT were computed directly using their respective definitions from the level variables as defined above.

For the change in the risk premium divided by the capitalization rate for the market portfolio (XDOT),a different approach was used. Equation (2-40) provides a return generating function in terms of three unobservable independent variables. The three unobservable variables are related, however, to the observable return on the market portfolio by equation (2-41). In order to compare the results of estimating the parameters of equation (2-40) to the use of a single market factor model, it is desirable to have equation (2-41) hold as an identity. Therefore XDOT was taken to be equal to the part of market return which did not result from PIDOT or RDOT in equation (2-41). While this procedure insures that equation (2-40) can be no less effective than a single factor market model, it does bias tests seriously against acceptance of either the model of (2-40) or (2-42). This happens because any error in the proxy variable used for changing expectations will cause an equal and opposite error in the value used for XDOT. Similarly, a change in the market's expectations which is not picked up in the value of PIDOT used will be counted as a part of XDOT. Since the coefficients of PIDOT and XDOT in (2-42) are of opposite signs for any value of beta, this effect will cause a severe antennuation of the measured coefficients. The effect any type of error will have on equation (2-40) is to drive the coefficient of all the variables toward β_i since even if any or all of the variables in equation (2-40) had no impact on security returns, or were measured only with large errors, (2-40) would work as well as a single factor model if all the coefficients were equal to β_i. The test of the relevance of (2-40) will in fact be whether the estimated coefficients conform more nearly to the hypothesized coefficients given by equation (2-40) or to the value of β_i which would result from a single factor model.

The returns used for the test were returns to portfolios chosen on the basis of the estimated beta value of the securities in the preceding 5 years. Quarterly data were used beginning in 1951-I and ending in 1969-IV. At the beginning of each quarter all listed New York Stock Exchange stocks were checked for five years of return history. Those without five years of prior data were eliminated. Beta values for the remaining securities were estimated on the basis of the five most recent years of data and the security was assigned to one of fifteen possible portfolios on the basis of the estimated beta value. This was done for each of the 76 quarters in the sample, then beta values were estimated for each of the fifteen "managed" portfolios. The returns to these portfolios were used to test (2-40) and (2-42). Portfolios were used to reduce the impact of the residual error term. The beta value which was used was the <u>ex post</u> measured beta to eliminate the impact of measurement or estimation error in the beta term. The result of estimating (2-40) for each of the 15 portfolios is given in Table 1, where the estimation of the first three terms was approximated by a constant term.

The results are not overwhelming in their confirmation of the model. The estimation of the return generating process fits the data better, but not at a high level of significance, than a single market factor model. The dominance of a single factor model can be rejected by the fact that the factor coefficients are not all equal except for the sign for each portfolio. Looking at the changes in the estimated coefficients for changing values of beta, the estimated values of MU_i and λ_i have the right form, having elasticities with respect to beta of less than one and more than one respectively, but both being closer to the actual value of beta than theory would predict. The estimated values of NU_i violate prior expectations, however, since they should decline in magnitude with increasing beta whereas the estimated coefficients increase in magnitude. This may have resulted from the manner in which the risk free rate was constructed, or it may be due to the bias towards beta which results from measurement error in this and the other variables. Such errors would tend to impose a particularly strong association between NU_i and MU_i since these two combine to form the capitalization rate portion of market return. The question of the overall value

Table 1

Regressions Explaining Returns

$$Return_i = a_i + \lambda_i PIDOT - NU_i(RDOT_t) - MU_i(XDOT_t) + \varepsilon_{it}$$

Portfolio Number	a_i t-stat.	a_i t-stat.	λ_i	λ_i t-stat.	$-NU_i$	$-NU_i$ t-stat.	$-MU_i$	$-MU_i$ t-stat.	R^2	Portfolio β
1	.00898	2.4	0.390	4.6	-0.316	-1.3	-0.518	-12.2	.72	0.494
2	.01086	2.7	0.400	4.5	-0.340	-1.3	-0.612	-13.5	.75	0.577
3	.00666	1.8	0.587	7.1	-0.335	-1.4	-0.733	-17.6	.84	0.693
4	.00344	0.8	0.673	7.2	-0.505	-1.9	-0.765	-16.3	.82	0.739
5	.00795	2.3	0.678	8.9	-1.231	-5.6	-0.748	-19.4	.88	0.780
6	.00641	2.1	0.779	11.4	-0.903	-4.6	-0.810	-23.5	.91	0.815
7	-.00193	-0.7	0.950	15.3	-0.951	-5.3	-0.865	-27.5	.93	0.878
8	-.00183	-0.6	1.036	15.3	-0.985	-5.1	-0.940	-27.4	.93	0.950
9	-.00428	-1.4	0.967	14.4	-0.936	-4.9	-0.960	-28.2	.94	0.959
10	-.00056	-0.2	1.084	16.0	-1.332	-6.9	-1.055	-30.9	.95	1.078
11	-.00926	-2.9	1.193	16.7	-1.208	-5.9	-1.173	-32.5	.95	1.177
12	-.01312	-4.4	1.292	19.4	-1.426	-7.5	-1.219	-36.2	.96	1.239
13	-.01612	-2.8	1.500	11.6	-1.294	-3.5	-1.329	-20.3	.88	1.338
14	-.01991	-3.9	1.524	13.3	-1.403	-4.3	-1.406	-24.2	.92	1.413
15	-.03317	-5.8	2.113	16.4	-1.296	-3.5	-1.629	-25.0	.92	1.636

of using an equation such as (2-40) instead of a simpler single market factor model can best be explored by examining the ability of this model to explain the errors made by such a simple model. The structure for such a test is set up in equation (2-42).

To test equation (2-42) each of the right hand side components was generated from the market wide variables ρ_m and R_f, and the beta value for the portfolio. MU_i and NU_i were computed according to equations (2-36) and (2-37). The value of λ_i was found by observing that beta must be a weighted average of λ_i, MU_i and NU_i with the weights being the fractions of variance of R_m due to PIDOT, XDOT, and RDOT respectively.[5] By assuming that these weights remained constant over the entire sample period and could be approximated from their sample values, λ_i could be found for given values of β_i and the market variables. Residual errors of the market model were computed by subtracting the risk free rate and the estimated beta times the realized market risk premium from the portfolio risk premia. This yielded residual errors for each of the portfolios in each quarter. These residual errors do not have zero sample mean values because the estimated constant term from the regression used to estimate the values of beta was not included. The generated right hand side variables did not have a zero mean since the macroeconomic data showed positive drift in expectations and negative drift in capitalization rates. In order to compensate for any tendency of low beta portfolios to outperform the predicted results and high beta portfolios to underperform and also to compensate for any antennuation in the coefficients of the constructed right hand side variables which might otherwise explain this deviation in performance, a dummy variable equal to $(1-\beta_i)$ was added so that each portfolio could be individually compensated for the mean of its corresponding right hand side observations. The resulting equations were estimated over 1,140 observations consisting of 76 quarters of data for each of 15 portfolios.

A direct test of the ability of equation (2-42) to explain the observed residual errors of a single factor model using the generated data is shown in panel 1 of Table 2. If the independent variables were computed without error, each of variable coefficients would be equal to one or minus one. The extreme antennuation of the coefficients of earnings changes and risk premia changes from this ideal value may be due to the manner in which the variables were defined, which causes the risk premium term to reflect any errors in the measured value of the earnings term. The manner in which the data were constructed is clearly subject to a considerable amount of error. The coefficient of the change in the risk free rate term is more of an anomoly. It is significant and has the wrong sign. One reason for this may be that this term is adjusting for the non zero mean performance of high and low beta securities. Even if equation (2-42) could account entirely for this performance, the antennuated coefficients of the other two terms would leave some of it unexplained. To adjust for this a new variable was inserted to allow each portfolio to have a different mean return. The result is shown in panel 2 of Table 2. The addition of this variable approximately doubles the explained variance of the residual errors and removes entirely the significance

Table 2

Regressions Explaining Residual Errors

Type	Regression
(1)	$RESIDUAL_{it} = R_{it} - R_{ft} - \beta_i(R_{mt} - R_{ft}) = \underset{(-4.28)}{-.00327} + \underset{(4.01)}{.1323}(\lambda_{it} - \beta_i)(PIDOT_t - \rho_{mt}) + \underset{(2.29)}{.3207}(NU_{it} - \beta_i)(RDOT_t) + \underset{(-2.59)}{-.1926}(MU_{it} - \beta_i)(XDOT_t) + \varepsilon_{it}$ $R^2 = .041$, d.f.=1,136 $F(3,1137) = 16.23$
(2)	$RESIDUAL_{it} = R_{it} - R_{ft} - \beta_i(R_{mt} - R_{ft}) = \underset{(-4.76)}{-.00356} + \underset{(7.38)}{.01882}(1 - \beta_i) + \underset{(5.16)}{.1683}(\lambda_{it} - \beta_i)(PIDOT_t - \rho_{mt}) + \underset{(0.01)}{.01315}(NU_{it} - \beta_i)(RDOT_t) - \underset{(-0.40)}{.0300}(MU_{it} - \beta_i)(XDOT_t) + \varepsilon_{it}$ $R^2 = .085$, d.f.=1,135 $F(4,1136) = 26.37$
(3)	$ZERO\ BETA\ RESIDUAL_{it} = \hat{R}_{zt}(1 - \beta_i) = \underset{(0.0)}{0.0} - \underset{(-0.20)}{.00032}(1 - \beta_i) + \underset{(8.16)}{.1652}(\lambda_{it} - \beta_i)(PIDOT_t - \rho_{mt}) + \underset{(0.11)}{.0099}(NU_{it} - \beta_i)(RDOT_t) + \underset{(-0.42)}{-.1993}(MU_{it} - \beta_i)(XDOT_t) + \varepsilon_{it}$ $R^2 = .079$, d.f.=1,135 $F(4,1136) = 24.22$
(4)	$RESIDUAL_{it} = R_{it} - R_{ft} - \beta_i(R_{mt} - R_{ft}) = \underset{(-5.99)}{-.0035} + \underset{(9.23)}{.0195}(1 - \beta_i) + \underset{(0.22)}{.0059}(\lambda_{it} - \beta_i)(PIDOT_t - \rho_{mt}) + \underset{(-0.12)}{-.0142}(NU_{it} - \beta_i)(RDOT_t) + \underset{(0.06)}{.0040}(MU_{it} - \beta_i)(XDOT_t) + \underset{(26.54)}{.9976}(\hat{R}_{zt})(1 - \beta_i) + \varepsilon_{it}$ $R^2 = .439$, d.f.=984

of the change in the risk free rate term and to a lesser extent the significance of the risk premium term. The change in earnings variable, however, does contribute signficantly to explaining the residual errors.

At this point a comparison can be made between the explanation of residual errors of a single factor model which has been developed above, and the explanation of residual errors which results from the "zero beta factor" model which has currently replaced the single factor model in most performance studies. In order to generate the explained residual which results from the use of that model, residual errors were regressed on a constant and $(1-\beta_i)$ for each of the 76 cross sections in the sample. This process could be done in a single regression on all 1,140 observations by using 76 dummy variables for the constant terms and 76 dummy variables for the beta variation in each quarter. In practice the estimates were made from 76 cross section regressions, but either method absorbs 152 degrees of freedom. The first test which was performed was to examine to what extent the proxy used and equation (2-42) is able to explain the estimated residuals produced by the zero beta factor model. This was done by regressing the zero beta factor explained residuals on the same independent variables used in panel 2 of Table 2. The results of this estimation are shown in panel 3 of Table 2, and in many respects are quite similar. The $(1-\beta_i)$ term which might have been necessary to compensate for the non zero mean of the change variables does not contribute much. The change in earnings variable is quite significant and has the right sign; however, it again has a coefficient much less than one. The risk premium term is large and has the right sign, but again is significant at only the 35 percent level if a one-tailed test is used. As before the change in the risk free rate term is completely insignificant. Overall the variables account for about eight percent of the variance in the residual error explanation which results from the use of an <u>ex post</u> estimated zero beta factor.

The final comparison which has been made is to compare panel 2 with panel 4 which shows the amount of variance of the residual errors which can be accounted for by both explanations. The inclusion of the terms from equation (2-42) might be considered redundant here, since in any time period their overall impact could be well approximated by $(1-\beta_i)$ times a weighted sum of the change factors. The weights would be the regression coefficients of λ_i, NU_i and MU_i on $(1-\beta_i)$. As expected the significance of all the variables from equation (2-42) is eliminated in panel 4. The closeness of the regression coefficient of the zero beta residuals to zero is also as expected and the $(1-\beta_i)$ term is significant in explaining the non zero mean of the computed left hand side residuals. An F-test can be used to measure the contribution of the zero beta factors after compensation is made for the degrees of freedom used to estimate the factors. From the data in panels 2 and 4 the appropriate computation is

$$F(152,984) = \frac{(.915 - .561)/152}{(.561)/984}$$

$$= 4.08 .$$

The level of F required for significance at the one percent level is

1.30 so the zero beta factor clearly pays for itself in terms of an improved explanation of residual errors relative to the set of proxy variables used. However, being an ex post measure the zero beta factor is equivalent to having perfect knowledge as to the change variables in (2-42) and approximating their coefficients by a linear function of $(1-\beta_i)$ if (2-42) is valid. The true significance of an unexplained extra zero beta factor could only be measured if the change variables were also measured accurately.

Footnotes

*Lecturer in Finance, University of Pennsylvania. The contents of this paper represent the product of a joing effort of myself and Professor Franco Modigliani. Although we shared in the development of the ideas, I alone take responsibility for this exposition.

1 The Capital Asset Pricing Model or CAPM refers to the models of market equilibrium for risky assets developed primarily by Treynor [14], Sharpe [13], and Lintner [6]. These models can be written either in terms of the distribution of returns or the distribution of terminal prices as was done by Mossin [10]. In either case equilibrium must be established by setting the current market price since this is the only variable which a disequilibrium condition can cause to change.

2 The term "expectation" refers to the collective expectation of the market participants and requires that those expectations be consistent with the mathematical expectation.

3 The only attempt which has been made to examine the problem of what type of generating function would be adequate without restricting the dimensionality of the problem was performed by Brennan [4], who performed a factor analysis of the residual errors of diversified portfolios differentiated by β. He found that two additional factors were required to explain the residual errors.

4 There are numerous explanations in the literature of why the mean return on a zero beta portfolio might differ from the risk free rate. See, for example, Miller and Scholes [9], Ross [12], and Dieffenbach [5], and Merton [7].

5 For a rigorous proof of this relationship, see Appendix III of Rie [11].

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