How Diversification Reduces Risk:
Some Empirical Evidence

by

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Introduction

Modern capital market theory has indicated that the variability of returns of portfolios can be separated into two distinct elements. The first has been identified as systematic variability and is the part of return of a portfolio that is attributable to changes in the return of the market as a whole. Another element of return variability is referred to as unsystematic variability and is the portion of portfolio return that is unrelated to the returns of any other portfolio. Obviously some individual securities and portfolios have more or less systematic variability than others because not all firms will respond to over-all economic conditions with the same sensitivity or persistence.

Many research studies have found that systematic variability accounts for approximately 30 percent and 40 percent of the total variability of individual common stock returns over time, whereas the unsystematic variability contributes the rest. However, for large, well diversified portfolios it is more likely that 80 percent and 90 percent of total variability of portfolios return can be explained by the variability of market returns. Thus Merril Lynch observes,"In a typical diversified portfolio of thirty or more common stocks, diversification eliminates so much of the specific risk that roughly 85-90% of all risk (in the portfolio) is market risk ... " [9, p. 6]. Evans and Archer conclude that "much of the unsystematic variation of return is reduced by the time the eighth security is added to a portfolio." [4, p. 767] Finally, Cohen, Zinbarg and Zeikel note, "The conclusion for practicing investment managers seems clear. While portfolio diversification can substantially reduce the nonmarket risk of common stock holdings, the law of diminishing returns takes hold very early in the process." [3, p. 773].

Thus, it is clear that diversification reduces the nonmarket element of portfolio return variability and the reduction of variability appears very quickly. Eventually, as diversification continues, the variability of portfolio return will be almost exactly proportional to the variability of the market returns.

The purpose of this paper is to re-examine the question of how many securities should be included in a portfolio. There is no doubt that increasing the number of securities in a portfolio will reduce the non-market related return variability. However, there are several areas of controversy. For example, it should be recognized that the term "variability" can have different definitions. Furthermore, the ability of diversification to eliminate unsystematic variability of return is important only if this reduction can be incorporated into predictions of future risk. In addition, the method of portfolio formation may determine the extent to which diversification enables portfolio managers to more accurately assess future risk.

In Part I, a new look is taken at the relationship between diversification and the reduction of portfolio variance. This examination tends to confirm earlier research that only moderate amounts of diversification are needed to obtain most of potential reduction in portfolio return variability. Part II examines the problem of predicting portfolio variability using only historical data. As part of the examination, the symmetric stable model and the subordinated stochastic model are compared to each other in the context of developing procedures to improve the accuracy of predictions. The results of this part of the paper indicate that the symmetric stable probability model can be better utilized to improve the accuracy of prediction of future portfolio risk than the subordinated stochastic model.

I. Diversification and the Reduction of Unsystematic Variability

In order to predict the effect of diversification on the return variability of a portfolio we must first establish a model for describing how the returns of securities are generated. A common procedure is to assume that the returns on individual securities are distributed normally and that the all interrelationships between securities stem from a common market factor that affects all securities. The model incorporating these assumptions is referred to as the market model; and it posits that the return on a security i, R_{it}, at time t is linearly related to the return on a market asset, R_{mt}, or

(1) $$R_{it} = \alpha_i + \beta_i R_{mt} + \varepsilon_{it}$$

(2) $$\sigma(R_{it}) = [|\beta_i|^2 \sigma^2(R_{mt}) + \sigma^2(\varepsilon_{it})]^{1/2}$$

Here α_i and β_1 are constants and ε_{it} is a random error term whose probability distribution is assumed to be normal with mean equal to zero. Further, it is assumed that the pair ε_{it} and R_{mt} and the pair ε_{it} and ε_{jt} are independently distributed. σ^2 is a symbol indicates the variance. The term $|\beta|_1^2 \sigma^2(R_{mt})$ is commonly referred to as the systematic variability of return for asset i and $\sigma^2(\varepsilon_{it})$ is referred to as the unsystematic portion of return variability.

If an equally weighted portfolio of n assets is formed, the portfolio return will be a simple average of the independent variables R_{mt} and ε_{it},

(3) $$R_{pt} = \frac{1}{N} \sum_{i=1}^{N} R_{it} = \frac{1}{N} \sum_{i=1}^{N} (\alpha_i + \beta_i R_{mt} + \varepsilon_{it}) = \overline{\alpha}_i + \overline{\beta}_i R_{mt} + \frac{1}{N} \sum_{i=1}^{N} \varepsilon_{it}$$

where the $\overline{\alpha_i}$ indicates an average α_i, $i = 1, 2, \ldots, N$. The standard deviation of an equally weighted portfolio of N assets is

$$(4) \quad \sigma(R_{pt}) = (|\beta_i|^2 \sigma^2(R_{mt}) + \overline{\sigma^2(\varepsilon_{it})}/N)^{1/2}$$

From the portfolio standard deviation equation of (4), it is clear that the number of assets in a portfolio can have a profound effect on portfolio standard deviation; hence diversification is likely to be a very effective tool for reducing portfolio variability.

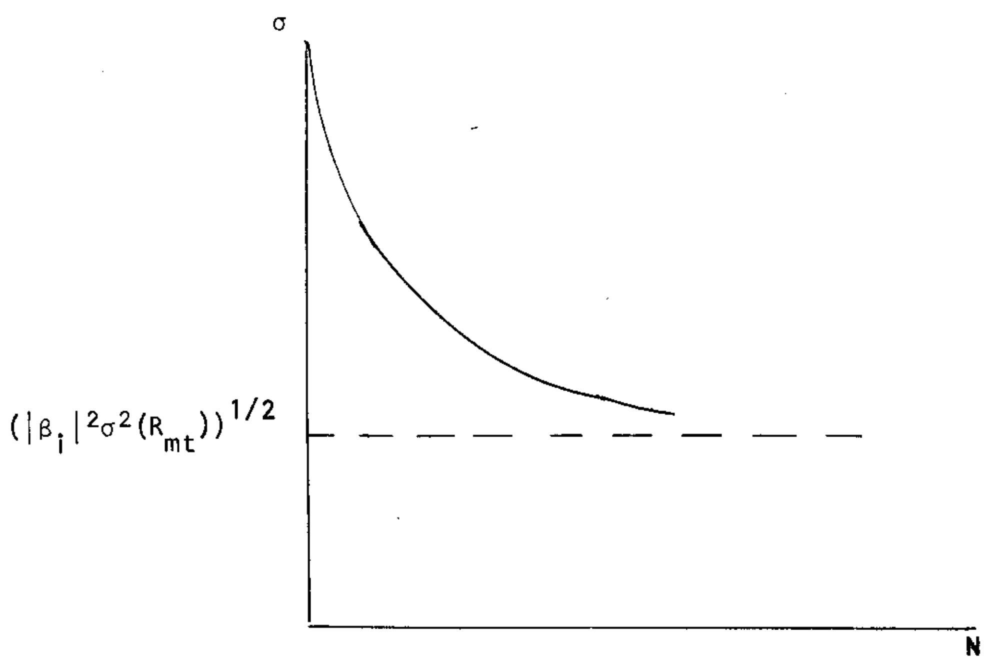
From Figure 1, it can be seen that the predicted reduction in portfolio standard deviation as the number of separate assets included in the portfolio is increased is quite pronounced at the early stages of diversification and becomes less and less as more assets are added to the portfolio. This relationship has led some writers to conclude that diversification beyond 10 or 15 assets is superfluous and should be avoided.[1] To test this assertion, the empirical association between the variability of portfolio returns and the size of the portfolio must be examined. We now proceed to assess how diversification reduces return variability under a normal probability model. Note that under the normal probability hypothesis it is sufficient to measure the variability of portfolio returns as the standard deviation (or variance) of portfolio returns.

A. The Sample

The data consist of dividend adjusted daily logrithms of return relatives for 489 common stocks listed on the New York Stock Exchange and the American Stock Exchange from the period January 2, 1968 to September 30, 1969 -- a period of 412 trading days. There are 411 observations for each security. Accompanying these relatives are the number of shares traded daily for each security.

Figure 1

A Graph of Portfolio Standard Deviation as a Function of Diversification



σ = Standard Deviation

N = The number of separate assets included in the portfolio

B. Forming Portfolios

Portfolios were formed by, first, ordering the set of individual securities randomly. After the securities were randomly ordered, portfolios of size N were formed by grouping the first N securities together and weighting each one equally to form portfolio 1; and repeating this grouping procedure for the second N securities and so on.[2] Portfolios were revised daily to maintain the equal weights. The portfolio formation procedure continued for all sizes N = 1, 5, 10, 15, 20, 489; so that, there were 489 portfolios of 1 security, 97 portfolios of 5 securities, 48 portfolios of 10 securities, 32 portfolios of 15 securities, 24 portfolios of 20 securities and 1 portfolio of 489 securities.

C. The Initial Results

To see how diversification can reduce return variability, the standard deviation, skewness and kurtosis of the return distributions have been computed for each portfolio.[3] To describe the results, the average, the .10 decile value and the .90 decile value of each of the measures of return variability are displayed in Table 1. The mean-standard deviation market model developed in (1) to (4) predicts that the standard deviation of portfolios will decrease as a function of the number of assets included, at the rate $(\frac{1}{N})^{1/2}$. Moreover, this model predicts that skewness values will be equal to zero and kurtosis values will be equal to 3.

In some respects, the results of Table 1 are consistent with the findings of others with regard to the effects of portfolio diversification on the portfolio returns variability and, in particular, consistent with the predictions of equations (1) to (4). Standard deviation of return decreases relatively quickly as the number of assets in the portfolio is increased.

Table 1

Portfolio Diversification
and Measures of Return Variability

Portfolio Size	Standard Deviation		
	F.10*	Average	F.90
1	0.0132	0.0258	0.0367
5	0.0105	0.0137	0.0162
10	0.0093	0.0113	0.0134
15	0.0089	0.0103	0.0115
20	0.0088	0.0098	0.0106
489	--	0.0082	--

Portfolio Size	Skewness		
	F.10	Average	F.90
1	-1.430	0.787	3.009
5	-0.444	0.022	0.444
10	-0.543	0.045	0.542
15	-0.641	0.089	0.444
20	-0.345	0.144	0.345
489		0.444	

Portfolio Size	Kurtosis		
	F.10	Average	F.90
1	3.915	6.697	10.001
5	3.339	4.608	5.362
10	3.239	4.077	4.814
15	3.294	4.113	4.737
20	3.390	4.200	5.048
489		4.905	

*Corresponds to the .10 fractile value.

This tendency is more closely examined in Table 2, where the reduction in standard deviation is computed as a percentage decline over the various N values and compared with the percentage reduction predicted by equation (4).

Table 2
The Incremental Percentage Reduction of Portfolio Standard Deviation

N	Actual % Decline	Predicted % Decline
1	0	0
5	69.11%	55.29%
10	13.61	13.10
15	5.38	5.80
20	2.91	3.45
489	8.98	22.36
total	100.00%	100.00%

The fit is not remarkable but close enough so that a preliminary conclusion can be put forth that diversification reduces the dispersion of the distribution of returns on portfolios as predicted by the mean-standard deviation version of the market model of equations (1) to (4). One explanation for this result is that a law of large numbers is operating on the error term in the market model of equation (1), so that the independent effects incorporated in these error terms become more and more offsetting as the portfolio is more diversified. For portfolios including at least 15 different securities approximately 75% of the total possible reduction in standard deviation has taken place (and a 93% reduction in variance).

Several unpredicted results appear in Table 1. Skewness of portfolio return decreases to approximately zero, on the average, as portfolio size increases to five assets. Thus the distribution of portfolio returns becomes approximately symmetric for moderately diversified portfolios. However, as portfolios become more diversified, asymmetry reappears and tends to persist. Moreover, the .10 and .90 decile values of the skewness values for the portfolios do not change as portfolio diversification increases. A portfolio manager is no more certain that a portfolio will exhibit symmetric returns for 20 securities than for 5 securities.

Another interesting result is that high values of the kurtosis of returns for portfolios tend to persist for well diversified portfolios. Recall that under a normal probability model, kurtosis values should equal 3, and for nearly all securities, kurtosis is much higher than 3 (only 1 security has kurtosis ≤ 3). The existence of kurtosis in the distribution of portfolio returns suggests that the relative frequency of return observations is greater in the tails and in the center than would be predicted if returns were normally distributed.

II. Predicting Portfolio Risk and Diversification

The results of the previous section reveal that the standard deviation of a portfolios' return not related to a common market factor is greatly reduced with moderate amounts of diversification. This provides empirical substance to the notion that investors should generally diversify, since as the number of securities included in a portfolio increases the contribution of the standard deviation of any individual security to the standard deviation of the portfolio becomes hardly noticeable. However, the results also support the notion that diversification beyond 10 or 15 securities achieves a non-negligable reduction in standard deviation of returns.

return variability because of the persistence of skewness and kurtosis in the distribution of returns for well diversified portfolios.

This section examines the ability to predict return variability for portfolios of varying degrees of diversification from two points of view. First, it is assumed that the basic probability model underlying the distribution of portfolio returns is from the class of symmetric stable probability distributions. Second, it is assumed that the probability model generating observed portfolio returns is a particular subordinated probability process. Consequently, the paper departs from the point of view developed in Part I where it was shown that standard deviation of returns decreased as the number of separate securities contained in the portfolio.

A. The Predictability of Portfolio Variability Under a Symmetric Stable Law.

A significant amount of empirical evidence exists indicating that security returns are symmetric, stable random variates.[4] Unfortunately, most of the evidence suggests that security returns are not normal and the standard deviation is not an adequate measure of security return variability. If it assumed that portfolio return R_{pt}, is symmetric stable with characteristic exponent α, the portfolio standard deviation equation (4) can be easily modified as

$$(5) \quad \sigma(R_{pt}) = [\,|\overline{\beta}_i|^{\alpha} \sigma^{\alpha}(R_{mt}) + \overline{\sigma^{\alpha}(\varepsilon_{it})}\, \frac{N}{N^{\alpha}}]^{1/\alpha}$$

Using (5) it is clear that portfolio dispersion, $\sigma(R_{pt})$, is reduced as N is increased as long as $\alpha>1$.[6] Thus, under a symmetric stable law we would predict that diversification generally reduces the effects of the independently distributed error term of (1) and thus it reduces the total dispersion of portfolio returns in (5). Here the exact nature of the reduction of portfolio dispersion depends upon α. For a closer look, consider more the last term of (5), or

$$(6) \quad \sigma(\varepsilon_{pt}) = [(\frac{N}{N^{\alpha}}) \overline{\sigma^{\alpha}(\varepsilon_{it})}]^{1/\alpha}$$

Portfolio dispersion $\sigma(\varepsilon_{pt})$ can be expected to decrease with increases in N, conforming to

$$(7) \quad (\frac{N}{N^{\alpha}})^{1/\alpha}$$

An important question is which characteristic exponent provides the best explanatory power in describing how $\sigma(\varepsilon_{pt})$ is reduced as N increases.

The Best Fitting Stable Law. In Section I it was found that the standard deviation of return for portfolios tended to decrease as predicted at the rate of $(1/N)^{1/2}$. This result suggests that the normal probability assumption underlying many models of common stock returns can provide useful predictions concerning the efficacy of real-world diversification. However, it is possible to challenge this assumption and to test it directly since normality is the special case of $\alpha = 2$ and σ can be estimated independent of α. That is, it is possible to predict the rate of decrease of $\hat{\sigma}$ as a function of α and N by using equation (7). To illustrate, for $N = 2$, $\alpha = 1.5$ equation (7) predicts that $\hat{\sigma}$ will decrease by 21 percent, whereas, for $N = 2$, $\alpha = 2$ (the normal case) equation (7) predicts that $\hat{\sigma}$ will decrease by 29 percent. Using the Fama-Roll estimator σ, in Table 3, the average values of $\hat{\sigma}$ have been recorded for portfolios size N.[7] Also included, are the actual percentage declines in $\hat{\sigma}$ as portfolio diversification increases. Table 4, compares these actual percentage delcines in estimates of portfolio dispersion with the percentage declines predicted by the various hypotheses concerning α. The method of comparison is to subtract

Table 3

The Effect of Diversification on
The Dispersion of Portfolio Returns

Portfolio Size	Average Portfolio Dispersion $\hat{\sigma}$	Percentage Decline
1	0.0132	
5	0.0084	57.00%
10	0.0071	15.50
15	0.0065	7.00
20	0.0062	3.80
489	0.0048	16.40

Table 4

Predictive Error Under Different
Characteristic Exponents

Characteristic Exponent	Predictive Error*
1.0	0.202
1.1	0.095
1.2	0.078
1.3	0.089
1.4	0.107
1.5	0.126
1.6	0.143
1.7	0.159
1.8	0.173
1.9	0.185
2.0	0.196

*Predictive error is calculated as the average of the sum the squared deviations of the predicted percentage decline and the actual percentage decline in $\hat{\sigma}$ as a function of N.

the actual percentage decline from the hypothesized percentage decline at each α for all N, sum the square of this difference over N and divide by N -- the statistic is commonly called the mean square error.

The best fit over all N is for $\alpha = 1.2$ casting serious doubt upon the adequacy of the normal model. In another study it was found that $\hat{\alpha}$ generally took the bounded values $1.4 \leq \hat{\alpha} \leq 1.6$ for the securities in the sample [11].

Predicting Portfolio Risk Under the Stable Model. The time interval is divided into two equal sub-intervals of 206 trading days. Using the Fama-Roll estimator of σ, portfolio dispersion has been estimated for all portfolios of N = 1, 5, 10, 15, 20, 489 during the first 206 trading days of our total sample (starting from January 1, 1968). Repeating the procedures of Part I, the first portfolio of N securities consisted of those securities with N highest estimates of σ. The second portfolio consisted of those securities with the next N highest estimates of σ, and so on until the remaining number of securities was less than N.

Finally, the Fama-Roll estimator of σ is re-computed for all portfolios in the second time sub-interval. Consequently, for each portfolio there are two estimates of σ corresponding to the two time sub-intervals.

Portfolio Size	β_i	R^2	D.W.
1	.7295	.639	1.227
5	.7531	.765	1.657
10	.8239	.890	2.438
15	.8439	.844	1.348
20	.9099	.986	1.759

β_1 = Slope coefficient estimate

R^2 = Coefficient of determination (adjusted for degrees of freedom)

D.W. = Durban Watson statistic

Table 5

Regressions Estimates Using
The Stable Probability Model

Portfolio Size	β_i	R^2	D.W.
1	.7295	.639	1.227
5	.7531	.765	1.657
10	.8239	.890	2.438
15	.8439	.844	1.348
20	.9099	.986	1.759

β_1 = Slope coefficient estimate

R^2 = Coefficient of determination (adjusted for degrees of freedom)

D.W. = Durban Watson statistic

Table 6

Regression Estimates of Return Dispersion and Volume of Trading

	Value
a	.01315
b	.03229
R^2	.07729
T	5.65638
D.W.	1.86865
N	371

a = intercept

b = slope

R^2 = coefficient of determination (adjusted in degrees of freedom)

T = student t statistic

D.W. = Durban Watson statistic

N = the number of securities

Table 7

Regression Estimates Using
The Subordinated Stochastic Model

Portfolio Size	β_i	R^2	D.W.
1	.7133	.667	1.355
5	.8108	.813	2.097
10	.8208	.744	1.870
15	.8373	.764	2.088
20	.6973	.752	1.485

β_i = The intercept estimate

R^2 = The coefficient of determination (adjusted for degrees of freedom)

D.W. = The Durban Watson statistic

Consider the coefficient of determination. For completely undiversified portfolios it has an average value equal to .667. For portfolios that include 20 distinct assets the corresponding average value is .752. Thus there is marginal improvement in one's ability to predict future portfolio dispersion with well diversified portfolios when compared to completely undiversified portfolios. In fact, all benefits of diversification occur with portfolios of no more than 5 assets. So far the grouping procedure supplied by the subordinated probability model does not appear to be standing up too well. This seems to be the case because one's ability to predict portfolio dispersion is much greater when relying solely upon the stable probability model.

Next examine the regression coefficients of each of the five regressions. The non-stationarity, referred to as regression towards the mean, remains. Remarkably, the drift in dispersion of return values over time is greatest for the better diversified portfolios. Thus, we can not be happy with the subordinated model and we reject it as a tool to enable portfolio grouping procedures to allow for increased portfolio dispersion predictive ability.

III. Conclusions

This paper has re-examined how diversification reduces risk. The traditional mean-standard deviation probability model provided inadequate measures of return variability and the symmetric, stable non-normal probability model was utilized. It was shown that diversification reduces the prediction error of assessing future risk. However, the usefulness of diversification depends upon the particular procedures used to form portfolios and how historical values of return dispersion are extrapolated into the future.

Footnotes

[1] For example Francis [8] states, "If 10 or 15 different assets are selected for the portfolio, the maximum benefits from naive diversification have most likely been attained - further spreading of the portfolio's assets is superfluous diversification and should be avoided."

[2] Recalling the purpose of the experiment, the portfolio formation method provides a completely naive, mechanistic portfolio averaging strategy and avoids using commonly used portfolio optimization procedures.

[3] Skewness is computed by the following statistic

$$\frac{\left(\frac{\%\text{ below mean}}{100} - \frac{1}{2}\right)\sqrt{N}}{1/2}$$

which has a normal distribution with mean zero and variance one. Kurtosis is the fourth movement divided by the square of the variance.

[4] For example, Fama [5] and Roll [10].

[5] For a discussion of this point see Fama and Miller [6].

[6] What is referred to as dispersion in this paper is the scale parameter describing a stable law with characteristic exponent α. The scale parameter corresponds to the standard deviation when returns are normal and $\alpha = 2$.

[7] Letting σ be the dispersion parameter and z a random variate and x its standardized variate

$$x_f - x_{1-f} = (z_f - z_{1-f}) / \sigma$$

where x_f denotes the f fractile of the x distribution. Restating the result

$$\sigma = (z_f - z_{1-f}) / (x_f - x_{1-f})$$

Now since $x_{.72} - x_{.28}$ is nearly constant and equal to 1.654 for $\alpha \geq 1$ an estimate of σ is (Fama and Roll [7], p. 822-824)

$$\hat{\sigma} = (\hat{z}_{.72} - \hat{z}_{.28}) | 1.657$$

where $\hat{z}_f$ refers to the sample estimate of the f fractile. Note $\hat{\sigma}$ is not a function of α.

[8] For example, see Westerfield [11] and Clark [2].