Return, Risk and Arbitrage

by

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Return, Risk and Arbitrage

One of the strongest statements that can be made in positive economics is the assertion that if two riskless assets offer rates of return of ρ and ρ', then (in the absence of transactions costs):

$$\rho = \rho'. \tag{1}$$

As an arbitrage condition, this equality of rates of return may be expected to hold in all but the most profound disequilibrium. If economic agents can both borrow and lend at the rates ρ and ρ', then infinite profits are envisioned if rates diverge and (1) must obtain to prevent such arbitrage possibilities. Furthermore, even when the rates are both only lending rates, for example, if (1) did not hold then there would be no (gross) demand for the asset with the lower rate of return. The introduction of risk, however, alters these strong conclusions. Under neoclassical theory if an asset is risky its expected return will equal the riskless rate plus a premium to compensate the holder of the asset for bearing risk. The explanations that have been advanced in an effort to understand this premium have focused on somewhat special forms of equilibrium theories and have essentially abandoned the robust sort of argument that supports (1). The intent of this paper is to develop an arbitrage theory for risky assets analogous to that for riskless assets, and, in so doing, to analyze the nature of risk premiums.

There are at present two major theoretical frameworks for the analysis of markets for risky assets; the state space preference approach and the mean variance model and its variants. The arbitrage model which we will develop is a third approach to capital market theory; empirically distinguishable from the mean variance theory and more directly related to the state space approach. While formally all models may be viewed as special cases of the state space preference framework, it is in the restrictions imposed either on preferences or distributions that the empirical

Section I

The mean variance model of capital market equilibrium represents a very strong restriction on the structure of asset returns across states. To develop the model we assume that all agents subjectively, and ex ante, view the n assets as being jointly normally distributed with a vector of means, E, and covariance matrix V.[2] It is well known (see, e.g., Sharpe [1970]) that the feasible set, F, of means, m, and standard deviations, σ, attainable on portfolios formed from the n assets has the convex shape illustrated in Figure I. If a riskless asset with a sure return, ρ, is introduced, the efficient frontier including portfolios formed with this asset is the line tangent to F in Figure I.

Figure I also illustrates the familiar separation property (see Markowitz or Tobin) in the presence of a riskless asset; risk averse investors will choose their portfolios as combinations of investment (or borrowing) in the riskless asset and a single portfolio of risky assets. In equilibrium, then, this efficient portfolio must consist of risky assets held in proportion to their total dollar value, i.e., it must be the market portfolio obtained by purchasing all available risky assets.[3] Conversely, the market portfolio must be efficient; it is the minimum variance portfolio of risky assets that attains the market return, E_m.

Letting $\tilde{x} = (\tilde{x}_1, \ldots, \tilde{x}_n)$ denote the vector of random returns on the n risky assets (represented as the columns of a state space tableau), the first order conditions for the mean variance efficiency of the market portfolio are given by

$$E_i = \rho + \lambda b_i \; ; \; i = 1, \ldots, n, \qquad (2)$$

where

$$b_i \equiv \frac{\text{cov}\,\{\tilde{x}_i, \tilde{x}_m\}}{\sigma_m^2} = (V\alpha_m)_i \qquad (3)$$

and

$$\lambda \equiv E_m - \rho \, , \qquad (4)$$

with α_m denoting the market portfolio, $E_m = \alpha_m' E$ and $\sigma_m^2 = \alpha_m' V\alpha_m$ being, respectively, the mean and variance of the random market return, $\tilde{x}_m$. Equation (2) is the famous

Figure I

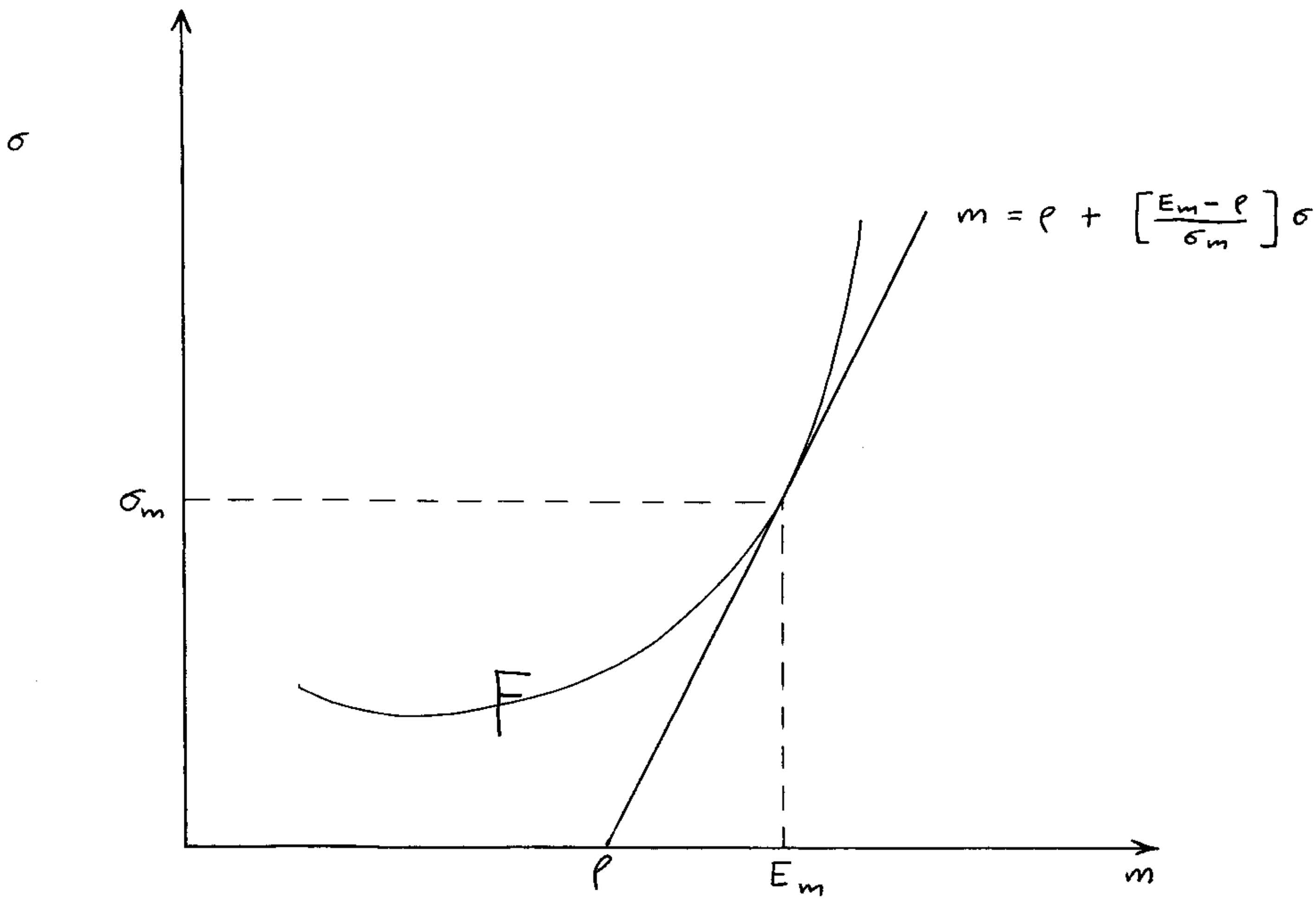
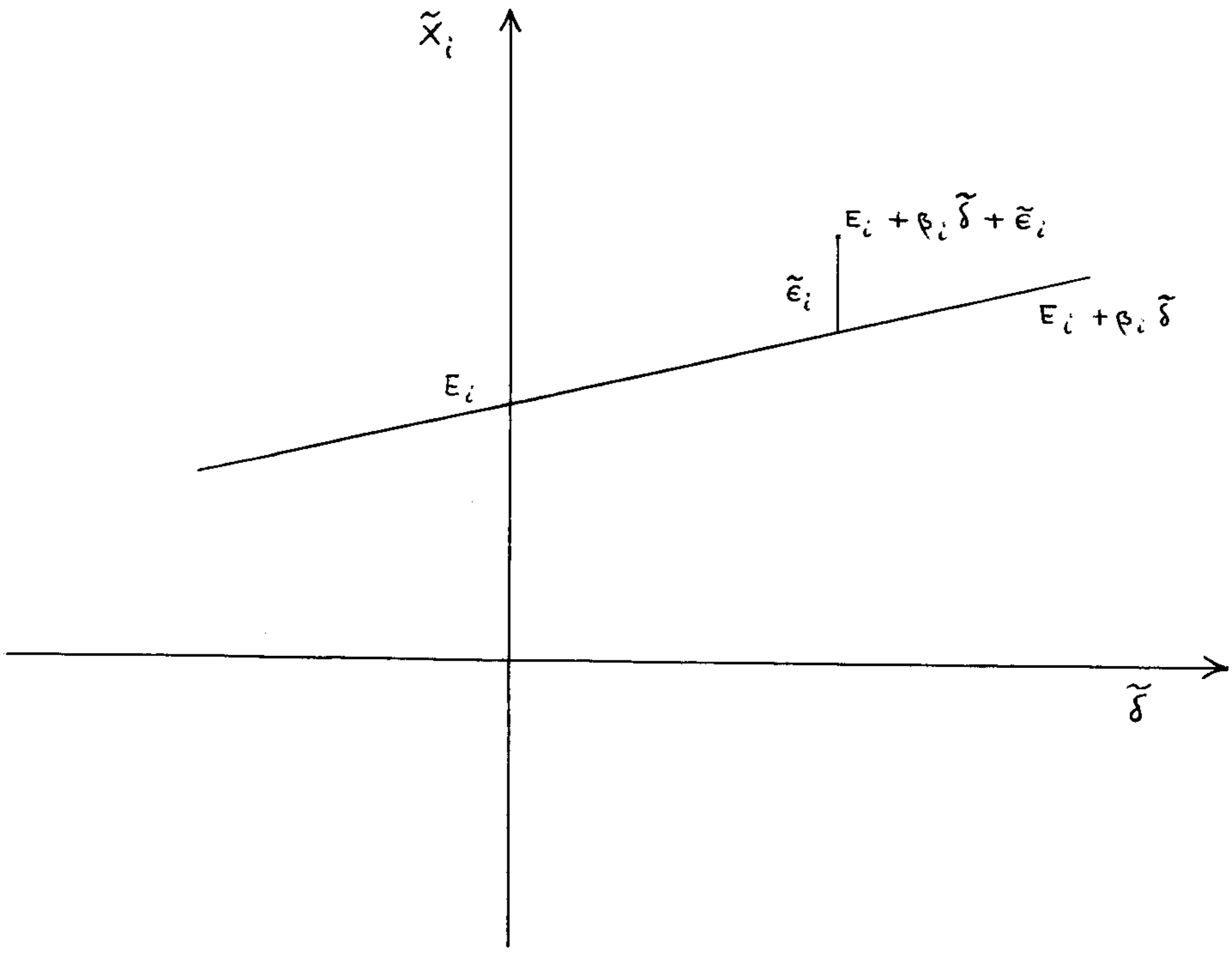


Figure II



$\tilde{X}_i$
$E_i + \beta_i \tilde{\delta} + \tilde{\epsilon}_i$
$\tilde{\epsilon}_i$
$E_i + \beta_i \tilde{\delta}$
E_i
$\tilde{\delta}$

Alternatively, it is possible to explain this phenomenon by arguing that the relation (5) of and by itself constitutes a satisfactory basis for a capital market theory without the additional assumptions of mean variance theory. We will develop such a theory in this section and for reasons that will become apparent refer to it as the arbitrage theory.[6] We begin with an algebraic argument and then illustrate the argument by an example. Throughout we assume that the number of assets, n, is sufficiently large to permit our arguments to hold and that the noise vector, $\tilde{\varepsilon}$, is sufficiently independent to permit the law of large numbers to work. (Notice that in the generating model of (5) this would allow for independent industry effects if there were enough industries.) Under these assumptions we can show that, if an appropriate pricing relation does not hold, then arbitrage opportunities will exist.

Suppose, first, that we form an arbitrage portfolio, η, of the n assets, where η_i is the proportion of wealth placed in the i^{th} asset. An arbitrage portfolio, η, is a portfolio that uses no wealth, hence

$$\eta' e = 0 \quad , \tag{6}$$

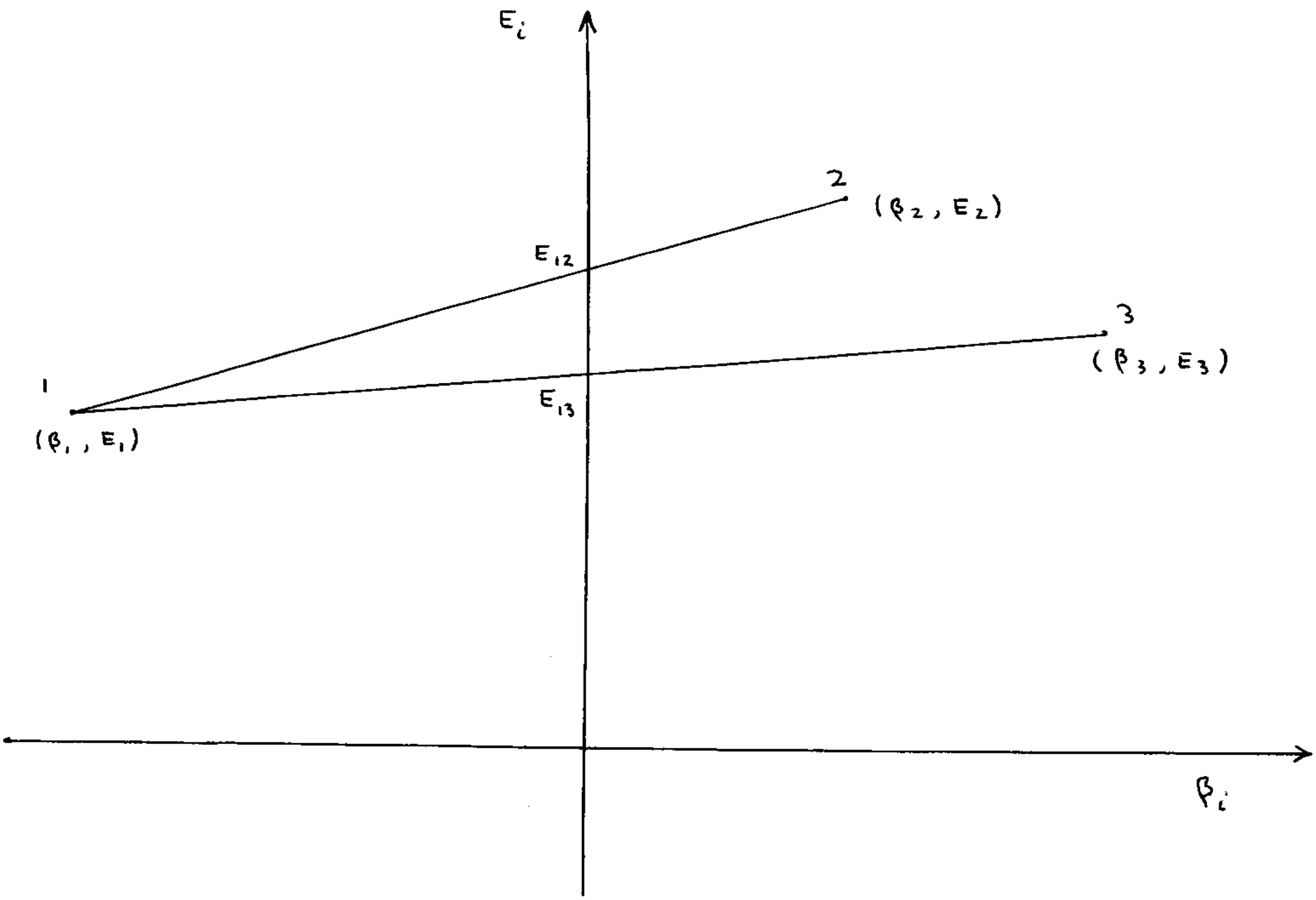
where e is a vector of ones. In other words, the wealth invested long in assets is exactly balanced by the amount borrowed from short sales and, net, the portfolio uses no wealth.

Denoting the vector of mean returns by E and the vector of beta coefficients by β, and using (5), the return on the arbitrage portfolio will be given by

$$\begin{aligned} \tilde{R} \equiv \eta'\tilde{x} &= (\eta'E) + (\eta'\beta)\,\tilde{\delta} + \eta'\tilde{\varepsilon} \\ &\approx (\eta'E) + (\eta'\beta)\,\tilde{\delta} \ , \end{aligned} \tag{7}$$

where we have, secondly, assumed that the arbitrage portfolio is sufficiently well diversified to permit us to use the law of large numbers to approximately eliminate

Figure IV



$$\beta_1 + \beta_3 = 0$$

$$\beta_1 + 2\beta_2 = 0$$

where E_m is the expected return on the market portfolio.[10] Equation (12) is the arbitrage equivalent of the mean variance security line equation.[11]

Notice that we did not have to assume that the market was in equilibrium to derive our basic arbitrage condition (12). It is a stronger result and depends essentially on the absence of arbitrage possibilities rather than on the much more restrictive condition that the market be in equilibrium as is required in the mean variance theory.

The basic arbitrage condition generalizes easily to the ℓ-factor case, as well, provided only that the number of common factors is significantly less than the number of assets.[12] By considering alternative arbitrage portfolios it is possible to show that when the generating model has the form

$$\tilde{x}_i = E_i + \beta_{i1} \tilde{\delta}_1 + \dots + \beta_{ik} \tilde{\delta}_k + \tilde{\varepsilon}_i \ , \tag{13}$$

the basic arbitrage condition takes the form

$$E_i = E_0 + (E_m - E_0) [\gamma_1 \beta_{i1} + \dots + \gamma_k \beta_{ik}] \ , \tag{14}$$

where the γ_ℓ are non-negative constants normalized so that

$$\sum_\ell \gamma_\ell = 1 \ .^{13} \tag{15}$$

In other words, the risk premium on the i^{th} asset, $E_i - E_0$, is a convex combination of its beta weights times the risk premium on the market portfolio.

There is no need, however, for the market portfolio to play any special role in this theory. The basic arbitrage condition (14) can also be written as

$$E_i - E_0 = (E^1 - E_0) \beta_{i1} + \dots + (E^k - E_0) \beta_{ik} \ , \tag{16}$$

where $E^k - E_0$ is the risk premium on a portfolio which has only the risk of the

Section III

The simplest state space framework describes a world with n assets and m discrete, exclusive states, $<\theta_1, \ldots, \theta_m>$, that the world could be in. (Implicitly we will deal with a two-period world -- where uncertainty is resolved in the second period -- to avoid the problem of intertemporal allocation, but many of the results extend to the multi-period context in an obvious fashion.) The matrix

$$X \equiv \text{States} \begin{matrix} \\ \theta_1 \\ \vdots \\ \theta_m \end{matrix} \overset{\begin{matrix} \text{Assets} \\ X^1 \; . \; . \; . \; X^n \end{matrix}}{\begin{bmatrix} & & \\ & x_{ij} & \\ & & \end{bmatrix}} , \qquad (19)$$

will denote the commonly agreed upon array of per-dollar (accounting unit) returns, where we define x_{ij} to be the gross return per dollar invested in asset j if state i occurs.[15]

Individuals choose their portfolios as combinations of the n assets so as to maximize the utility of their wealth and the only assumption on preferences is that all utility functions are monotone functions of the form $v(w_1, \ldots, w_m)$, where w_θ is wealth in state θ and for each θ there is some individual who exhibits no satiation in w_θ.[16]

Suppose, now, that the capital market is in equilibrium. It follows that there cannot exist any successful arbitrage strategies. To be specific, it must not be possible to insure a non-negative return in all states and a positive return in at least one state with no net investment, i.e., without taking a position. Mathematically, if we consider the return vector on an arbitrage portfolio, η,

$$y = X \eta \qquad (20)$$

equilibrium is not efficient relative to a complete Arrow-Debreu equilibrium. As a consequence not all individual marginal rates of substitution across wealth in different states will be equal.

In the special case explored by Cass and Stiglitz where X is of full row rank a number of interesting results can be obtained. Now, $p' = e'X^{-1}$ is uniquely defined and $p'e = e'X^{-1}e$ is the aggregate cost of obtaining a sure dollar return in all states, or $(e'X^{-1}e)^{-1}$ is the riskless interest factor. In general, though, the existence of a non-negative state price vector is all that can be ascertained from the assertion that the market is in equilibrium.

To verify this we must show that for any tableau of return, X, with an associated positive price vector, p, for which $p'X = e'$, there exists some structure of preferences such that the market is in equilibrium. To construct such a market consider a single individual who seeks to maximize his expected utility of wealth,

$$E\{u(\tilde{w})\} = \sum_{\theta} \pi_{\theta}\; u(wx_{\theta}'\alpha) \tag{23}$$

over portfolios α subject to $e'\alpha = 1$, where π_{θ} is the subjective probability assigned to state θ and is assumed positive, and where X_{θ} denotes the θ^{th} row of X. The first-order conditions take the form

$$\Sigma_{\theta}\; \pi_{\theta}\; u'(wx_{\theta}'\;\alpha)\; x_{\theta i} = \lambda;\; i = 1, \ldots, n\;, \tag{24}$$

where λ is a Langrange multiplier. Since $p'X = e'$, set

$$\pi_{\theta}\; \frac{1}{\lambda}\; u'(wx_{\theta}'\alpha) = p_{\theta} \tag{25}$$

or

$$u'(wx_{\theta}'\alpha) = \lambda\; p_{\theta}/\pi_{\theta} > 0. \tag{26}$$

Letting u be a concave utility function with everywhere positive marginal utility and α any portfolio, it is clear that λ and a vector of probabilities , Section IV

In evaluating the two theories on theoretical grounds the argument centers on the a priori appeal of their respective major assumptions. The assumptions that underly mean variance theory have been exhaustively studied in the literature and are outlined above and we will not repeat them. The primary assumption underlying the arbitrage theory is that only a small number of factors is significant.[21] Without presenting a formal argument for why this should be so we can, at least, argue that it is quite plausible.

The question turns on whether substitution or income effects in both consumption and production are more important in the very short run. Abstracting from capital gains, for the moment, suppose that the returns on investment are the returns on productive activities. In the short run, ideally in a differential time, productive activities will be fixed (as in a putty-clay model) and the returns to capital will be composed of ordinary returns plus (disequilibrium) profits. When the level of aggregate demand changes, sectors will accomodate in a Leontief-like fashion, but the output and profit response of each will be a linear function of the change. If capital receives the residual, after payments to other productive agents, then there will be a specific aggregate demand factor. Of course, if the change in aggregate demand is accompanied by important systematic shifts in demand across sectors and, consequently, in prices then this will not be the case.[22] To the extent to which such secular demand shifts take place and are significant they must be represented by factors in the generating model. In the short run, though, models of habit formation would suggest that price shifts would be of a second order and could be ignored. Finally, to the extent that anticipations of capital gains are extrapolative and based on learning from past experience the number of factors would be further limited if the slow shift responses postulated on both the consumption and production sides were valid. These should prove to be important areas of future research. The precise determination of what the relevant factors are

Given a one or a dependent two-factor model, (17), both mean variance theory and arbitrage theory lend to the simple pricing relation of (12). It is only with the assumption that the generating model is more complex that the arbitrage theory is distinguishable from mean variance theory on the basis of the pricing relation. The verification of a second factor in the cross section studies of return on the beta coefficient, for example, rather than destroying all of our capital market theory would instead constitute a significant piece of evidence in support of the arbitrage theory as against mean variance theory.

This raises the question of whether in the absence of a second significant independent factor it is possible to test the arbitrage theory as against mean variance theory. In fact, if the number of assets is large such a test will be very difficult. As we have shown in Ross [1971:2], with a one factor generating model and a large number of assets, mean variance theory will be approximately correct, to the same order of approximation as arbitrage theory, irrespective of the underlying distributions (provided that second moments exist) or preference structure. In effect, then, there will be no empirical basis on which to distinguish the conclusions of the two theories in such a world.

With an independent two factor model, however, a test based on the pricing relation is possible. Suppose, that one of the factors is the market factor with beta coefficients $<\beta_i>$ and the other factor has coefficients $<\gamma_i>$. (Both $\tilde{\varepsilon}$ and the second factor are uncorrelated with the market factor.) The basic pricing relation given by arbitrage theory asserts that

$$E_i - E_0 = \gamma_i \, [E^1 - E_0] + \beta_i \, [E_m - E_0] \, . \tag{27}$$

If the factor term, $\gamma_i \, [E^1 - E_0]$ proves significant with γ_i different from $1 - \beta_i$ this will constitute a rejection of the mean variance theory in favor of the arbitrage theory.

Section V

In the above sections we have developed a new theory of the pricing of risky assets. The arbitrage theory was built on the foundation of the state space framework and both the theoretical and empirical implications of the theory in contrast to mean variance capital market theory were discussed. It was argued, in particular, that the arbitrage theory follows directly from the generating models that are commonly used to test the mean variance theory and that, as a consequence, in this context the additional assumptions of mean variance theory are not necessary. Furthermore, the arbitrage theory permitted a significant weakening of the assumption that markets were in equilibrium.

Much work, however, remains. The arbitrage theory is constructed in the tradition of Popper and a number of empirical tests have been suggested. On the theoretical side, as footnotes have indicated, it is not difficult to extend the arbitrage theory to an intertemporal context. More pressing is the need to expand, both on the theoretical and empirical fronts, the argument outlined in Section IV supporting the limited dependence of asset returns. For example, if the degree of interdependence in returns is high, it will be very difficult to develop a meaningful theory of competitive (or "small") firm behavior short of assuming that firms can fully assess the complex market valuations of risky assets in the absence of complete price signals. As a final point, neither the arbitrage theory nor any other theory has made a serious attempt to describe the disequilibrium dynamic adjustment of ex post observations to ex ante assumptions. Understanding the impact of information on market adjustment will be a prerequisite for such an analysis and of great interest in its own right.

[4] The exact meaning of the security relation is not always made clear in the literature. It is not important that (2) is linear, what is important is that the market portfolio is an observable economic variable, the totality of wealth held at risk, and that it plays a pivotal role in the theory. The linearity of (2) is only of interest when it is understood that correlation with the market portfolio is the proper measure of risk in equilibrium. In fact, it is easy to show that unless arbitrage is possible there always exists some portfolio α such that

$$E_i = \rho + \lambda(V\alpha)_i \quad , \tag{f1}$$

where λ is a constant so that $\alpha' e = 1$.

However appealing the linear relation (f1) may appear, then, it is nearly devoid of economic content. In fact, given any collection of random variables with a nonsingular V, (f1) will be satisfied for some portfolio, α, whether we are in equilibrium or not. As such, (f1) is empty of any empirical content as well. The result (f1) only becomes meaningful when something can be said about the nature of α. (This is in contrast to the work of Beja who derives (f1) assuming complete contingent markets and somehow feels that it is a deep result.)

When we add the assumption that returns are jointly normal, or that utility functions are quadratic, the portfolio α can be shown to be the market portfolio, and the security line equation (2) constitutes an empirically significant restriction on asset returns. The fact that a surrogate for the market portfolio is nearly always used in empirical work might suggest that the exact choice of α is unimportant. Nothing could be further from the truth and the theoretical justification for the use of surrogates is found in Section II and not in the mean variance model. To put the matter somewhat differently, with given expectations, E, and a given covariance matrix, V (or given dependence of E and V on prices) then (f1) can be

[8] We've eliminated the condition that the portfolio be well diversified since the argument is algebraic and if the implication follows for all portfolios in some open neighborhood of a point of the order of $(\pm \frac{1}{n}, \ldots, \pm \frac{1}{n})$ it will hold on the whole space.

[9] Of course, the result holds as an approximation with a finite number of assets, but we will not worry about this. See Ross [1971:2] and [1972] for a fuller discussion of the exact nature of the approximation.

[10] This normalization is possible whenever $\alpha_m'\beta \neq 0$, and this will always be true for the market portfolio if agents are risk averse. From (5) the return on the risky portfolio held by the ν^{th} agent is given by

$$\begin{aligned}\tilde{R}^\nu &= \alpha^\nu \tilde{x} \\ &= \alpha^\nu E + (\alpha^\nu \beta)\tilde{\delta} + \alpha^\nu \tilde{\varepsilon} \\ &= E_0 + (\alpha^\nu \beta)\,[a + \tilde{\delta}] + \alpha^\nu \tilde{\varepsilon} \\ &\approx E_0 + (\alpha^\nu \beta)\,[a + \tilde{\delta}]\ ,\end{aligned}$$

and if the ν^{th} agent is risk averse he will insist on a compensation for bearing risk (more formally it follows from the concavity of his utility function) and, therefore,

$$a(\alpha^\nu \beta) \geq 0\ ,$$

with strict inequality in general (except where the utility function is improper at the certain wealth level of wE_0). If ω^ν denotes the proportion of wealth held by the ν^{th} agent, then

$$\begin{aligned}E_m - \rho &= a(\alpha_m \beta) \\ &= a \sum_\nu \omega^\nu (\alpha^\nu \beta) \\ &> 0\ ,\end{aligned}$$

must hold with $E_0 = \rho$ in the presence of a riskless asset. The difference between this derivation and the original arbitrage argument comes at the stage where we concluded that because $\eta'\tilde{\varepsilon}$ was uncorrelated with $\tilde{R}_m$ the expected return, $\eta'E = 0$. In the absence of assumptions which justify focusing only on means and (co)variances this conclusion would be unwarranted.

[12] In an intertemporal model if E_i or β_i are stochastic it may be necessary to make limited dependence factor assumptions about their movement as well to obtain a useful theory. More generally we can assume that all of the stochastic parameters are governed by a generating model of the form of (13).

[13] There is also no need to impose any restrictions on the multivariate distribution of the factors.

[14] Consider a portfolio, α, which is zero beta on all factors except the ℓ^{th}. Formally,

$$\alpha'\beta j = 0 \text{ if } j \neq \ell \text{ ,} \quad \text{(f3)}$$

$$\alpha'\beta_\ell = 1$$

and (f4)

$$\alpha'e = 1 \text{ ,}$$

where β_i denotes the vector of i^{th} factor weights, $\langle\beta_{1i}, \ldots, \beta_{ni}\rangle$. (If such portfolios cannot be formed, then factors can be linearly combined simplifying the generating model.) From (13) the risk premium on any such portfolio, $E^\ell - E_0$, is simply $(E_m - E_0)\,\gamma_\ell$.

and without loss of generality required that $(\alpha_1, \alpha_2) \geqq 0$. In addition, $e'y = 1$ insures that y is semipositive if it is non-negative.)

By Farkas' Lemma, see, e.g., Gale, the above system will not have a solution only if its dual

$$q'X + ve' = 0 ,$$

$$-q' + ze' \geqq 0 ,$$

$$z < 0 ,$$

possesses a solution (q, v, z) .

To obtain (21) and (22) we define $p' \equiv -\frac{1}{v} q'$. Equation (21) follows directly from the definition of p' and the dual equations, and since

$$-q' \geqq -ze' > 0 ,$$

it only remains to show that $v > 0$. Noting that X is a semi-positive matrix, $-q'X$ is also semipositive and this implies that $v > 0$.

Notice that even if the subjective probability of a state θ occuring is zero, we still have $p_\theta > 0$. If the expected utility hypothesis governs preferences, though, we would be indifferent to wealth in state θ, violating non-satiation. In this case we can modify the analysis to eliminate states for which all agree there is no probability of occurence. However, even if the probability is zero, the possibility, as with picking a rational with the uniform measure on the unit interval, remains. In the absence of transactions costs even the most ardent believer in expected utility would demand infinite wealth in state θ if it was free. The lack of upper semi-continuity for the demand correspondence can pose difficulties for the existence of equilibrium in more complex models.

[19] Of course, this leaves open the possibility that restrictions on investor risk aversion may have further implications. These results are similar to the more general work of Sonnenschein on the implications of agent optimization for aggregate excess demand functions.

risk premiums and if the ex post generating model is a linear combination of the individual ex ante models (e.g., a weighted average), then the basic arbitrage relation will hold ex post and will be testable as it stands. It is only necessary that agents agree on what the factors are, not on their impact on asset returns.

To make this point algebraically, let agents be indexed by "ν" and suppose the ν^{th} agent believes ex ante returns are generated by the model

$$\tilde{x}_i^{\nu} = E_i^{\nu} + \beta_i^{\nu} \tilde{\delta}^{\nu} + \tilde{\varepsilon}_i^{\nu} , \qquad \text{(f5)}$$

where the assumptions (13) are assumed to hold. Under the conditions described in the text (see Ross [1972]) the boundedness of a market return surrogate will imply that for each ν, the arbitrage condition will hold,

$$E_i^{\nu} = \rho + \lambda^{\nu} \beta_i^{\nu} \; ; \; \lambda^{\nu} = E_m^{\nu} - \rho , \qquad \text{(f6)}$$

where we have assumed a riskless asset and ignored the approximation. If the true ex post model can be obtained from (f5) by aggregating with weights γ_{ν}, then the ex post

$$\begin{aligned} \tilde{x}_i &= \sum_{\nu} \gamma_{\nu} \tilde{x}_i^{\nu} \\ &= \sum_{\nu} \gamma_{\nu} E_i^{\nu} + \sum_{\nu} \gamma_{\nu} \beta_i^{\nu} \tilde{\delta}^{\nu} + \Sigma_{\nu} \gamma_{\nu} \tilde{\varepsilon}_i^{\nu} \\ &\equiv E_i + \beta_i \tilde{\delta} + \tilde{\varepsilon}_i , \end{aligned}$$

assuming that $\tilde{\delta}_{\theta}^{\nu} = \tilde{\delta}_{\theta}$. If the risk premiums, $\lambda^{\nu} = \lambda$ agree then (f6) aggregates to

$$E_i = \rho + \lambda\beta_i$$

which is now directly testable. To generalize the result, we can scale the $\tilde{\delta}^{\nu}$ factors, rescaling the β_i^{ν} variates accordingly, so that $\lambda^{\nu} = \lambda$ trivially. The

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