A Generalized Theory of Velocity

by

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I. The Problem

Many studies have shown that the income velocity of circulation of the money supply is not a constant. Data relating to this behavior, however, seem to follow a common pattern. If the money supply is increasing and the economy is growing, velocity tends to increase.
Once the growth in the money supply declines, velocity tends to increase more rapidly for a time but then begins to decline. As velocity declines, the economy declines also. This decline is halted as the money supply is once again increased, but only after some time has elapsed. During this time the economy continues to decline, and velocity declines more rapidly than it had been. Although this description does not fit every business cycle or every major change in the rate of growth of the money supply, it seems to be the general case. See [3], [5], [6], and [19].

Furthermore, it has been noted that the general pattern of interest rates during this time is one of sympathy with velocity and with the economy. That is, as the economy expands and as velocity rise, interest rates rise. As the economy contracts and velocity falls, interest rates fall. Thus, it seems that any theory of velocity needs to take into account the sympathetic behavior of interest rates.

In a recent paper [12] I have attempted to show that the above described behavior can be explained with a structural macro-economic model. I concluded that the coincident movement of the velocity of circulation and interest rates is more complex than many descriptive or single-equation studies had implied and resulted from the dynamic adjustment and total response of the economic system to changes in policy.

I propose to develop the argument one step further in the present article. That is, rather than rely on one specific econometric model, as was done in [12], the current discussion will be of a more general nature that can incorporate results derived from either single-equation or structural approaches to income determination. The conclusion drawn from this effort is the same as that arrived at in [12]; that is, the behavior of velocity and interest rates results from the dynamic adjustment and total response of the economic system to changes in exogeneous policy variables.

II. A Comparison of Mathematical Models

The mathematical model of velocity behavior is not dependent on the precise approach used in analysis; it can be derived from either a single-equation, quantity theory approach or a multi-equation, structural approach. It is important, however, to develop a general concept of income determination because velocity is most frequently considered to be the ratio of the current level of income to the current money stock. Since most recent studies of income determination conclude that the current level of income is not obtained solely from contemporaneous stimuli, it becomes necessary to account for this in any theory of velocity behavior. In this section, I attempt to describe the mathematical background of the two most common methods of income determination, first, to show that the two can be combined for velocity analysis and secondly, to show that the subsequent results are truely general.

A common version of the Single-Equation approach to income determination has been presented by Friedman [3]. Assuming that

(1) $Y_t = V(r)\ M_t$

where Y is current dollar income, V is the velocity of circulation, r is "the" interest rate and M is the nominal quantity of money, and that the interest rate is a function of past values of Y and these Y's are related to past values of M, we can arrive at

(2) $Y_t = V[M(T)]\ M_t$ where $T < t$.

That is, current dollar income is functionally related to current and past values of the money supply. Since the formulation is implicit the exact functional relationship between the two variables is immaterial, as long as the relationship is non-stochastic.

This relationship has taken different explicit forms. Two such forms can be obtained from Friedman's own work ([3], [4]). Both result in somewhat clumsy and unreasonable lag structures due to the way Friedman introduces permanent income into the system to account for lags in adjustment.[1] A third form of equation (2) can be found in empirical work completed at the Federal Reserve Bank of St. Louis ([1], [2]) where the form of this equation is assumed explicitly.

At present, the exact form of the equation is not important. However, it should be kept in mind that (2) does not explicitly include an interest rate variable. This raises some question concerning the use of demand for money equations, which should include an interest rate variable, for deriving ultimate velocity equations, which should not include interest rate variables.[2] Thus, single-equation approaches imply other "single-equations" that account for the behavior of other important variables. This point, I think, will be clearer after the discussion of the Fundamental Dynamic Equations of Structural Models that is presented in the next portion of the paper.

Structural models present a somewhat different problem and cannot really be identified with any one economist. The general model common to all structural efforts, with lagged endogenous and exogenous variables, may be written as a set of N equations, the $i^{\underline{th}}$ of which is

$$(3) \qquad \sum_{n=1}^{N} \alpha_{in}^{(o)} y_{nt} = \sum_{j=1}^{r} \sum_{n=1}^{N} \alpha_{in}^{(j)} Y_{n,t-j} + \sum_{j=o}^{S} \sum_{k=1}^{K} \beta_{ik}^{(j)} X_{k,t-j}$$

where $y_{1t},\ldots, Y_{nt}$ and $X_{1t},\ldots, X_{kt}$ are respectively the values of the endogenous and exogenous variables at time t. It is assumed that the general conditions of identifiability are met by the model. Introducing the lag operator L, defined by $L^j W_t = W_{t-j}$; the NxN and NxK matricies

Γ (L) and β (L); and if $\underset{\sim}{Y}_t = (Y_t,\ldots,Y_{nt})$ and $\underset{\sim}{X}_t = (X_{1t},\ldots,X_{kt})$, then the set of N structural equations becomes

$$(4) \qquad \Gamma(L)\underset{\sim}{y}_t = \beta(L)\ \underset{\sim}{X}_t \ .$$

This is but one form a structural model can take[3] and it specifically represents the full interaction of the exogenous and endogenous variables of the system. In this paper, however, we are interested in the Final Form or the System of Fundamental Dynamic Equations (FDE's). Each FDE contains only one current dated endogenous variable and all of the exogenous variables. Thus, if Γ^* (L) is the adjoint of Γ(L) and the determinant, $|\ \Gamma\ (L)\ | = \lambda\ (L)$, (4) can be rewritten as

$$(5) \qquad \underset{\sim}{Y}_t = \frac{A(L)}{\lambda(L)} \underset{\sim}{X}_t$$

where $A(L) = \Gamma^*(L)\ \beta(L)$ a $N \times K$ matrix. Specifically, the $n^{\underline{th}}$ $(1 \leq n \leq N)$ of the set of fundamental dynamic equations

$$(6) \qquad y_{nt} = \sum_{k=1}^{K} \frac{\alpha_{nk}(L)}{\lambda\ (L)}\ X_{kt}.$$

Each of the FDE's consist of a set of rational distributed lags on the exogenous variables. While all these FDE's are quite important, we are particularly interested in the one that relates income to various exogenous variables. This is the equation most closely related to the single-equation models of income determination. The structural model generally contains more exogenous variables than just the money supply; for example, government expenditures and/or exports, to name two. For the sake of the current analysis these variables can be suppressed to make this equation comparable to equation (2). One such method is to assume changes in these other exogenous variables to be zero (see [12]). Another, would be to assume that the coefficients attached to the other exogenous variables $\sum_{k=1}^{K} \alpha_{nk}(L)/\lambda(L)$ are zero or are sufficiently close to zero to ignore. Thus, both structural and single-equation models can be shown to reduce to an equation relating current dollar income to current and past levels of the money supply (or some monetary aggregate) in the form of a rational distributed lag.[4] In general, therefore, we will assume the general form of equation (7) in the analysis presented below.

$$(7) \quad Y_t = [A(L)/\lambda(L)]M_t = \sum_{i=o}^{\infty} (\Psi_i \, M_{t-i})$$

It will be assumed that the economy described is stable

$$(8) \quad \lim_{T\to\infty} \sum_{i=o}^{T} \Psi_i = k \; .$$

Furthermore, the structural models can provide additional information that is not provided by the single-equation approach. Generally, one (or in other cases, several) of the FDE's obtained in equation (5) is an equation expressing an interest rate variable as a function of the exogenous variables. For example, we can write (6), so that r_t, the current dated interest rate, is the endogenous variable.

$$(9) \quad r_t = \sum_{i=o}^{\infty} \gamma_i M_{t-i} \; .$$

III. The Analysis of Movements in Velocity

In the previous section it was shown that movements in income can be described by rational distributed lags of the money supply on income. It was also shown that this lagged relationship can be derived from single-equation or structural approaches. This section takes the analysis one step further and attempts to develop the sufficient conditions incorporated into the lag structure that leads to the various movements in the velocity of circulation and interest rates.

This section itself is divided into four parts. The first treats movements in velocity as responses to monetary growth of one direction: the secular question. The second part analyzes velocity during periods of change in monetary growth : the cyclical behavior. The analysis of interest rates is included in the third part, and a treatment of other influences (such as government expenditures) on velocity is treated at the end of the section.

A . The Secular Movement of Velocity

It is easy to show that in situations of increasing velocity the marginal velocity is greater than the average. Conversely, when velocity is decreasing the marginal velocity is less than average.[5] These results can be combined with those obtained in Section II, equations (7) and (8), to obtain the first set of conclusions. Equation (7) can be expressed in either linear or non-linear form in the variables. In the following we will work with non-linear case and assume that the variables are in logrithmic form. The linear case is developed in an attached appendix.

In deriving the multipliers Ψ_i it is generally assumed that the independent variable, M_t, grows at a constant rate of growth; a one-percent growth rate is most often used. Thus k becomes descriptive of the system and (7) can be rewritten once time derivatives are taken:

(10)
$$\frac{1}{y}\frac{dy}{dt} = \frac{1}{m}\frac{dm}{dt}\sum_{i=o}^{\infty}\Psi_i = \left(\frac{1}{m}\frac{dm}{dt}\right)(k) .$$

To simplify, I will use dots over variables to denote percentage rate of change, (10) thus becomes $\dot{Y}_t = (\dot{M}_t)\ (k)$.

Using this information it can easily be seen that the sufficient condition for velocity to be increasing due to an increasing money supply is for k to be greater than one. Thus,

(11) $$0<\dot{V}_t = \dot{Y}_t - \dot{M}_t$$

leads to

(12) $$1 < \dot{Y}_t/\dot{M}_t = k^6$$

B. The Cyclical Movement of Velocity

Specific values of the lag structure have played no part, so far, in movements in velocity. The Ψ_i's could be geometrically declining as $i \to T$, or could form an inverted u shape. All that is necessary for the above results to hold is that the sum of the Ψ_i's be greater or less than one. This is not true of turning points. It is to this problem that we now turn. This is important because velocity generally keeps going in the direction it had been for several periods after a turning point takes place in monetary growth.

Thus, in the case of turning points, we are primarily interested, not in the fact that velocity is increasing, but that it increases more rapidly than it had in the past. Thus, we are not interested in whether $\dot{V}_t > 0$ but whether

(13) $$\dot{V}_t - \dot{V}_{t-1} > 0$$

Friedman has indicated, for example, that the rate of growth of velocity will accelerate with a reduction in the rate of growth of the money supply [3, p. 335]. The reverse will be true with an increase in the rate of growth of the money supply after a period of restricted growth. This result, however, comes from the lagged effect of monetary change and not from specific "impact" elasticities. Furthermore, the mathematical model derived above can show exactly what the conditions are under which this situation will be true. The results tend to support the work of Friedman, even though he has expressed his conclusions entirely in terms of reduced form equations.

Expanding (13) we get,

(14) $$\dot{V}_t - \dot{V}_{t-1} = \dot{Y}_t - \dot{M}_t - \dot{Y}_{t-1} + \dot{M}_{t-1}.$$

But,

(15) $$\dot{Y}_t - \dot{Y}_{t-1} = (\Psi_o \dot{M}_t + \sum_{i=1}^{\infty} \Psi_i \dot{M}_{t-1}) - (\Psi_o \dot{M}_{t-1} + \sum_{i=1}^{\infty} \Psi_i \dot{M}_{t-1})$$
$$= \Psi_o [\dot{M}_t - \dot{M}_{t-1}] .$$

Substituting into (14) we find that for condition (13) to hold when the rate of growth of money becomes less is that

(16) $$1 > \Psi_o.$$

That is, the multiplier of the money supply in the first period of the change in the rate of growth must be less than one in absolute value. It is easily shown that if $\Psi_o = 1$, then the income velocity of circulation will increase at the rate it had been increasing before the change.

For the velocity change to be positive after the initial period of restriction the condition $\Psi_i < 1$ must continue to exist. For the increase in velocity to be greater in any period after the restrictive policy was introduced than it was before the introduction of the restrictive policy, it must be true that the sum of the multipliers affected due to restriction must be less than one, i.e., $\sum_{i=0}^{n} \Psi_i < 1$. Otherwise, velocity will finally be reduced below its value at the start of the restrictive period. To show this let

$$(17) \qquad \dot{V}_{t+n} - \dot{V}_{t-1} > 0.$$

Expanding and substituting as before, we find that

$$(18) \qquad \sum_{i=o}^{n} \Psi_i < 1$$

must hold for velocity to increase at a pace in period n greater than it had in period t.

Velocity begins to decline once the rate of growth of income is less than the rate of growth of the money stock, or

$$(19) \qquad \dot{M}_t > \dot{Y} = \dot{M}_t > \dot{M}_t \sum_{i=o}^{n} \Psi_i + \dot{M}_{t-1} \sum_{i=n+1}^{\infty} \Psi_i = \dot{Y}_t$$

An interesting situation arises if $\Psi_o = 1$. Then if $\Psi_i > 0$ for some $i = 1, 2, 3, \ldots \infty$, the rate of growth of velocity will never increase at turning points in the money supply, but will increase at the rate it had been for one period and then increase at a decreasing rate. The special case is if $\Psi_o = 1 = k$. This implies, from equation (12) above, that velocity had been constant before the change in growth, and now, since $\Psi_i = o$ for all i's = 1, 2, ... ∞, it would remain constant.[7]

The remarkable thing about the results just reported is that it <u>doesn't matter</u> what the lag structure implied by equation (7) is, as long as the sum of the Ψ_i's are less than one velocity will increase more rapidly than it had been during monetary expansion once a restrictive policy sets in. This result is true even when (7) is in the form of a geometrically declining lag. Assume, for example, that $0<\lambda<1$, and $\Psi_i = \lambda^{i+1}$ for $i=0, 1, 2, 3, \ldots$. This must be the case for solving the equation and also for the assumption of increasing velocity. Then, according to (10)

$$(20) \qquad k = \sum_{i=o}^{\infty} \Psi_i = 1/(1 - \lambda) > 1.$$

Of particular importance is whether $\lambda < 1$. If this is true then $\Psi_o < 1$ and velocity will increase at an accelerated rate once a given monetary expansion is achieved.

A simple example of cyclical behavior is provided in the table. The results in terms of velocity are charted in the accompanying figure[8]. Initially, money was growing at a four percent rate of growth. Velocity as a result was increasing at an 0.8 percent rate of growth. In period t, the rate of growth of the money supply is slowed to 2 percent. Velocity growth immediately jumps to a 1.6 percent rate of growth. Continuing this rate of growth in the money supply through another quarter allows velocity to increase at a 1 percent rate. This is higher than velocity had been increasing before the restrictive policy was introduced; the reason is that the sum of Ψ_o and Ψ_1 is less than one as implied by equation (18). Velocity finally drops below the original rate in the next period as the sum of the multipliers becomes greater than one. Velocity continues to increase, however, until the fifth period after the change of policy; this is when the condition set forth in equation (19) is satisfied and monetary growth is greater than income growth. It should be added that the result is independent of the lag structure chosen for the example.

Figure 1

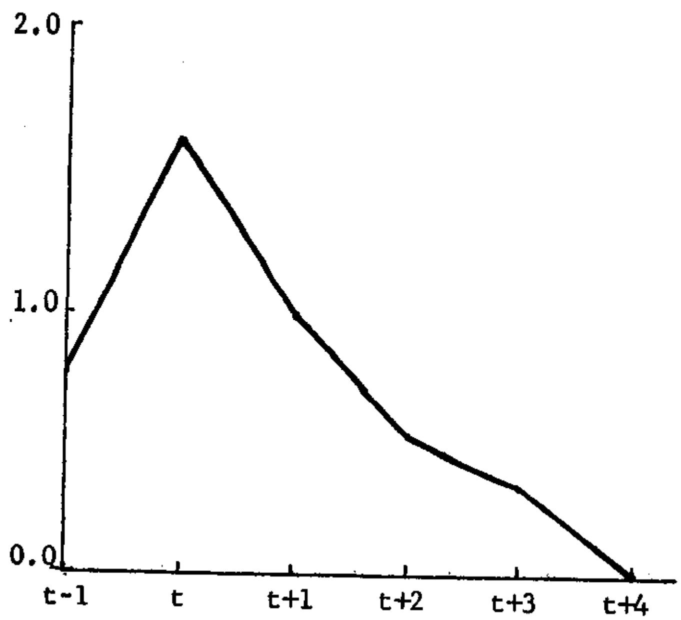

Table 1 - Non-Linear Case

$$\dot{Y}_t = \sum_{i=o}^{\infty} \psi_i \dot{M}_{t-i}$$

where $\psi_o = 0.6,\ \psi_1 = 0.3,\ \psi_2 = 0.2,\ \psi_3 = 0.1.$

$k = 1.2$

Money supply had been increasing at 4%; in period t = o, the growth of the money supply drops to 2%.

Time Period	$\frac{1}{y}\frac{dy}{dt}$	$\frac{1}{v}\frac{dy}{dt}$
t - 1	4.8%	0.8%
t	3.6	1.6
t + 1	3.0	1.0
t + 2	2.6	0.6
t + 3	2.4	0.4
t + 4	2.0	0

C. Interest Rates

If interest rates are to increase secularly with increases in the money supply it is necessary for them to ultimately be positively related to changes in the money stock; that is the sum of the multipliers attached to the money supply in the interst rate FDE must be positive. This is perhaps the hardest condition to be satisfied in econometric models without incorporating the influence of other exogenous policy variables. However, taking equation (9), if in assuming a constant growth rate in the money supply we obtain

$$(21) \qquad \dot{r}_t = \sum_{i=o}^{\infty} \gamma_i \dot{M}_{t-1} = \dot{M}_t \sum_{i=o}^{\infty} \gamma_i$$

and

$$(22) \qquad \sum_{i=o}^{\infty} \gamma_i = C > 0$$

In terms of cyclical performance, it is perhaps easier to see that interest rates will perform as we would expect during periods when monetary growth is changing, if we anticipate the results of empirical studies. These results generally indicate that at least some of the γ's (usually the first several) are negative because of the "liquidity" effects of changes in monetary growth. This result insures that interest rates will increase (decrease) more rapidly once money expansion is slowed (accelerated) than they had been under the constant growth conditions specified above. In general, in the non-linear case, for interest rates to change more rapidly than they had under constant growth in the money supply the following conditions must hold.

$$(23) \qquad 0 < \dot{r}_t - \dot{r}_{t-1} = (\gamma_o \dot{M}_t + \dot{M}_{t-1} \sum_{i=1}^{\infty} \gamma_i) - (\gamma_o \dot{M}_{t-1} + \dot{M}_{t-1} \sum_{i=1}^{\infty} \gamma_i)$$

or

(24) $0 < \gamma_o \, (\dot{M}_t - \dot{M}_{t-1})$

and since $(M_t - M_{t-1}) < 0$ due to a reduction in the monetary growth rate

(25) $0 > \gamma_o$.

Interest rates will continue to increase more rapidly under the change in monetary policy as long as the sum of the γ_i's affected by the change is negative.

(26) $0 < r_{t-n} - r_{t-1} = (\dot{M}_t \sum_{i=o}^{n} \gamma_i + \dot{M}_{t-1} \sum_{n+1}^{\infty} \gamma_i) - (\dot{M}_{t-1} \cdot \sum_{i=o}^{n} \gamma_i + \dot{M}_{t-1} \sum_{n+1}^{\infty} \gamma_i)$

implying

(27) $0 > \sum_{i=o}^{n} \gamma_i$.

The timing relationship between changes in interest rates and changes in velocity is entirely an empirical matter. However, if the conditions set down in equation (12) and (22) hold, velocity and interest rates will generally be moving in the same direction at the same time.

D. Other Influences on Velocity

The final portion of this section considers the role other variables might have on the behavior of velocity. The general conclusion presented above is that the result of changes in the money supply are unambiguous. An increase in the money supply supply at a constant rate causes velocity to increase only if the "total" multiplier of income with respect to the money supply is greater than one. Velocity accelerates at an even faster pace for several periods, once monetary growth is slowed, if the multipliers affected by the change in growth rates meet certain requirements. The influence of other exogenous variables must be added to this in order to arrive at a complete explanation of the behavior of velocity over the credit cycle. To do this, we need to alter equations (5) and (7) to obtain the following result:

$$(28) \qquad Y_t = \sum_{i=o}^{\infty} \Psi_i M_{t-i} + \sum_{i=o}^{\infty} \sigma_i E_{t-i}$$

where E_t represents other exogenous influences on Y_t. This includes Government expenditures, Taxes (or base amounts), Net Exports or any other exogenous influence on national income. Depending upon the magnitude and sign of the σ_i's and their sum, the effects of changes in money supply on velocity may be augmented or diminished depending upon the movements in these other exogenous variables. The important thing to note, however, is that the influence of changes in the money supply on velocity can be quite unambiguous even though these effects might be offset in actual experience. The inclusion of other exogenous influences in structural models becomes particularly important in the case of the secular movement of interest rates.

A final consideration needs to be discussed. This is the fact that the money supply is not generally considered to be an exogenous variable; usually other variables have been assumed to be more directly under the control of the monetary authorities. This does not negate the results obtained above, however, for two reasons. One, the money supply usually reacts more rapidly to changes in the variables controlled by the monetary authorities thus representing an important intermediate target of the policy makers.[9] Consequently, the results will hold, given a stable relationship between the two monetary variables under consideration. Secondly, even though there may be substantial feedbacks from the real sectors to the money supply, the money supply will generally react more strongly to other monetary variables at turning points in the growth of the latter thus continuing to support the results presented above.

IV. Summary

In this article we have been concerned with the concept of the velocity of circulation of the money supply. As such, we have seen that it is a much more complex relationship than that indicated by the simple demand for money approaches to it. I have tried to show that the behavior of the velocity of circulation is a result of the lagged adjustment of the economy to changes in monetary variables. In doing so, I have presented the sufficient conditions for velocity to increase with increases in the money supply and for it to accelerate its rate of increase once policy has become more restrictive. The reverse is true of the opposite case. I have also shown what is necessary for interest rates to show sympathetic behavior.

One question that arises from this work is that concerning the value of the concept of velocity itself. This is a difficult question, and requires some extended thinking. Because the subject is too large to add to this paper, I have attempted to put down my own ideas in another [16]. Regardless of a persons feeling on this particular question, however, there remains a great deal of research to be completed in this area.

In many respects it would be most fruitful if research tried to draw the structural and reduced form approaches together in the future, since so much analysis depends upon the magnitude and length of lagged relationships. David A. Pierce and I have tried to do this in [18] and I have also attempted to further the effort in [14] and [15]. However, it seems to me that one cannot be dogmatic about the proper approach, for both have shown remarkable support in empirical estimation, although less acumen in forecasting. Thus, it would seem that there is much to gain by drawing knowledge from each; this pertains to both theoretical and empirical work. One such area of vital importance concerns problems of identifying adjustment processes and empirically estimating them. It is with this effort in mind that the present work has been advanced.

Footnotes

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[1]For an analysis of the lag structure implied by [4] see John Hause [8]. Hause concludes that Friedman's formulation implies lags that are much more complicated than those usually employed in empirical studies because the result is a combination of lag structures: one rising to a peak and then declining exponentially towards zero; the other oscillating towards zero as the time lag increases.

A similar result is achieved from the formulation presented by Friedman in [3]. Of particular note is the fact that the theoretical lag coefficients alternate between positive and negative values.

[2]The basic point is that interest rates are exogenous to a velocity equation. Thus, it is underidentified in terms of determining causal behavior. For further examples of this problem see [10], [11] and [19].

[3]For a more complete development of the various forms a structural model can take see [9] and [18].

[4]This result is not inconsistent with other theoretical work relating structural models to single-equation models. For example Gramlich and Pierce [7 , p. 2] state the " ... not only is (the) typical macroeconometric model not biased against monetarists, but its very structure insures long-run results which could only be called classical."

[5]This is easily seen as follows:

$$\log V = \log Y - \log M$$

Taking time derivatives we get

$$\frac{1}{v}\frac{dv}{dt} = \frac{1}{y}\frac{dy}{dt} - \frac{1}{m}\frac{dm}{dt}$$

In the case of increasing velocity where $1/v \cdot dv/dt > 0$

$$\frac{1}{m}\frac{dm}{dt} < \frac{1}{y}\frac{dy}{dt}$$

or

$$\frac{y}{m} < \frac{dy}{dt} / \frac{dm}{dt}$$

[6]Conversely, a sufficient condition for velocity to fall is $k < 1$. This is obtained by reversing the inequality in equation (11). It can be noted at this point that k can be interpreted as 1/long-run income elasticity of demand for money (see [13]) and thus related in this way to secular increases or declines in the velocity of circulation. Of particular note is the fact that if $k = 1$ implying that the long-run income elasticity of demand for money is equal to the generally assumed short-run elasticity velocity will be secularily constant in the absence of other exogenous influences on velocity. In addition, this will be true at turning points, something that will be shown below. Furthermore, to facilitate analysis, the appendix shows that the linear result in this instance can be converted to the non-linear in order to bring the empirical studies using the former structure into long-run velocity analysis.

[7]This is equivalent to saying that the long-run income elasticity of the demand for money is equal to one.

[8]An example using a linear model is presented in the appendix.

[9]The reaction of the money supply to changes in policy variables has been estimated to be rapid so that a quarterly econometric model might even consider adjustments to take place within the quarter. Thus, the money supply for all intents and purposes might be considered to be exogenous for a quarterly model.

Appendix

A. The Secular Case

The results of the linear case are not quite as direct as those of the non-linear case. Assuming that velocity increases with an increase in the money supply we get

(A.1)
$$\frac{\Delta M}{M_{t-1}} < \frac{\Delta Y}{Y_{t-1}}$$

and

(A.2)
$$V_{t-1} = \frac{Y_{t-1}}{M_{t-1}} < \frac{\Delta Y}{\Delta n} = k'.$$

The right-hand equality can be obtained from equations (7) and (8) in the text of this article. The result implies that k'must be greater than the past period's velocity in order for velocity to increase.

The relationship between the results of the linear and non-linear examples can be determined in the following manner. Since $(M_{t-1}/Y_{t-1})/(\Delta Y/\Delta m) = k$ and $\Delta Y/\Delta m = k'$ we have

(A.3)
$$M_{t-1}/Y_{t-1} \cdot k' = k$$

and

(A.4)
$$V_{t-1} = k'/k$$

Thus estimates from linear models can be compared with the non-linear conditions derived above.

B. The Cyclical Case

Results similar to the secular case do hold for the cyclical behavior of velocity.

(B.1)
$$\Delta Y_t - \Delta Y_{t-1} = \Psi_o' \Delta M_t + \Delta M_{t-1} \sum_{i=1}^{\infty} \Psi_i' - \Psi_o' \Delta M_{t-1} - \Delta M_{t-1} \sum_{i=1}^{\infty} \Psi_i' = \Psi_o' (\Delta M_t - \Delta M_{t-1})$$

Using the average value for Y, i.e., $\overline{Y}$, and the average value for M, i.e., $\overline{M}$, we can obtain

$$(B.2) \quad (\overline{Y}/\overline{Y})\ (\Delta Y_t - \Delta Y_{t-1}) = \Psi_o' \ (\Delta M_t - \Delta M_{t-1})\ (\overline{M}/\overline{M})$$

which approximates

$$(B.3) \quad \dot{Y}_t - \dot{Y}_{t-1} = \Psi'_o \ (M_t - M_{t-1})\ (\overline{M}/\overline{M})$$

If, $\dot{V}_t - \dot{V}_{t-1} > 0$, then

$$(B.4) \quad 0 < \Psi'_o (\dot{M}_t - \dot{M}_{t-1})\ (\overline{M}/\overline{Y}) - (\dot{M}_t - \dot{M}_{t-1})$$

and

$$(.5) \quad (\overline{Y}/\overline{M}) = \overline{V} > \Psi'_o \ .$$

which is different from the result obtained in (16) due to the use of discrete data but comparable to that obtained in equation (A.2) for secular change.

C. Interest Rates

The linear interest rate case works out exactly as the non-linear case.

$$(C.1) \quad \Delta r_t = \sum_{i=o}^{\infty} \gamma'_i \Delta M_{t-1} = \Delta M_t \sum_{i=o}^{\infty} \gamma'_i$$

and

$$(C.2) \quad \sum_{i-o}^{\infty} \gamma'_i = C' > 0$$

for the secular results to hold. Some of the γ_i's must be negative for the cyclical conditions to be satisfied.

D. An Example

An example similar to the non-linear example presented in the text of the article can be derived for the linear case.

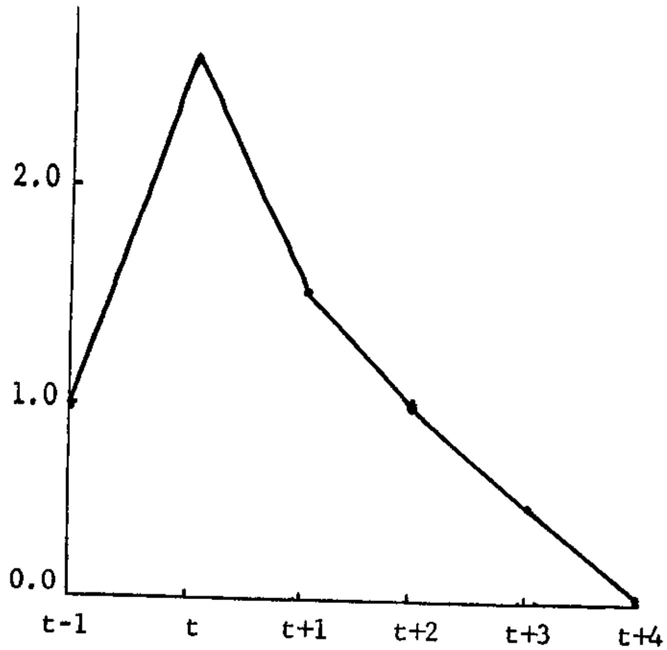

Table 2 - Linear Case

$$\Delta Y_t = \sum_{i=o}^{4} \psi_i' \Delta M_{t-i}$$

where $\psi_o' = 0.40$, $\psi_1' = 0.60$, $\psi_2' = 0.8$, $\psi_3' = 0.5$ and $\psi_4' = 0.2$

$k = 2.5$

Money supply had been increasing at \$4 billion per period; in period $t = o$, the money supply increases at \$2 billion per period. In this example the initial conditions are M = \$100 billion and Y = \$200 billion.

Time Period	ΔY	ΔV	$\frac{1}{v}\frac{dv}{dt}$
t - 1	10.0	0.02	1.0%
t	9.2	0.05	2.5
t + 1	8.0	0.03	1.5
t + 2	6.4	0.02	1.0
t + 3	5.4	0.01	0.5
t + 4	5.0	0.00	~

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