An Application of the Decomposition Principle to Financial Decision Models

by

James R. Morris*

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RODNEY L. WHITE CENTER FOR FINANCIAL RESEARCH

University of Pennsylvania

The Wharton School

Philadelphia, Pennsylvania 19174

SECTION I. INTRODUCTION

Linear programming models of specialized financial decision problems such as working capital management [20], short term financing [21], or capital budgeting [23] are deficient in that they may lead to decisions which are suboptimal with respect to the firm as a whole. Each model attacks a single decision problem and neglects its interaction with the other activities of the firm. On the other hand, a model which reflects these interdependencies and interactions by including the various financing, investment and operating decisions in a single model tends to become excessively large and inefficient to use. What is needed is a model which incorporates the efficiencies inherent in smaller, more specialized models which can be utilized on a decentralized basis, and simultaneously lead to decisions which are optimal for the firm as a whole.

It has been shown ([1], [13], [14], [16], [24], [25]) that if a set of transfer prices are used which reflect the relative demand and supply of the organization's resources, then global optimal decisions can be made by decentralized decision makers who maximize their own "divisional" objective functions. This paper will utilize the theory of transfer pricing and decentralization, in the context of a linear programming model, in order to develop a scheme of transfer pricing for financial resources.

The approach to be used will be to apply the Dantzig-Wolfe [9] decomposition principle to an integrated linear programming model of the firm.[1] This large model will be decomposed according to functional areas (investment, financing, and production) into component models which make decentralized decisions. Any interaction between decisions made in different decision centers will be reflected in a set of prices for resources which are shared by the various decision centers.

The focus of the paper, and its primary contribution, is to explore the implications of this mechanism for internal pricing of financial resources. The pricing scheme will yield considerable insight into the interaction between dividends, equity financing, and capital budgeting. It is shown, for example, that under conditions of capital rationing capital budgeting decisions should be made independently of financing and production decisions only if the "conventional' cost of capital discount factor is modified by an internally generated price for funds which reflects the internal supply and demand for funds. In addition, an explicit decision rule is developed which states the conditions under which it is appropriate to utilize new equity financing.

Section II will briefly review the decomposition principle. Section III will develop a simple model of the firm which involves production, financing, and investment decisions, with an emphasis upon the development of the objective function. Section IV applies the decomposition principle to this integrated model so that we can explore the implications for internal pricing and decentralization of decisions.

SECTION II. A REVIEW OF THE DECOMPOSITION PRINCIPLE

Dantzig and Wolfe [9] have shown that certain linear programming problems can be separated, or decomposed, into a number of smaller subproblems, each of which is solved independently.[2] This is advantageous for particularly large problems which, due to their size, are difficult and inefficient to solve as one problem. Whether or not a problem can be decomposed is dependent upon its particular mathematical structure.

Consider a linear programming problem of the form

$$\max_{x,y} V = p_1x + p_2y \tag{1}$$

$$\text{subject to} \quad A_1x + A_2y \leq d_o \tag{2}$$

$$M_1x \leq d_1 \tag{3}$$

$$M_2y \leq d_2 \tag{4}$$

where, for j=1,2

p_j is a vector of dimension n_j:

A_j is an m_o x n_j matrix of coefficients:

d_o is an m_o vector:

M_j is an m_j x n_j matrix of coefficients:

d_j is an m_j vector:

x and y are vectors of decision variables.

Expression (2) represents the set of constraints which involve both sets of decision variables x and y, whereas, expressions (3) and (4) are constraints which involve just one set of decision variables, x or y, but not both. If we think of this problem as (say) a firm with two divisions[3] maximizing profits, subject to resource and technological constraints, where x is the vector of decision variables associated with division 1 of the firm, and y, the decision variables associated with division 2, then constraint (2) represents the interactions and interdependencies of the two divisions; vector d_o could represent the available amounts of resources which are shared by the two divisions. Constraints (3) and (4) represent technological relations which are unique to divisions 1 and 2, respectively.

Problem (1-4) can be solved as one centralized decision problem, or, it can be decentralized so that each division makes decisions so as to maximize a divisional objective function. This decentralization can be accomplished by formulating the problem as a single "master program" and a subprogram for each division. For our two division problem these are expressed as follows.

The Master Program:

$$\max_{\lambda} V = \sum_{i=1}^{n} \theta_1^i \lambda_1^i + \sum_{i=1}^{n} \theta_2^i \lambda_2^i \quad (5)$$

subject to

$$\sum_{i=1}^{n} E_1^i \lambda_1^i + \sum_{i=1}^{n} E_2^i \lambda_2^i \leq d_o \quad (6)$$

$$\sum_{i=1}^{n} \lambda_1^i = 1 \quad (7)$$

$$\sum_{i=1}^{n} \lambda_2^i = 1 \quad (8)$$

$$\lambda_1, \lambda_2 \geq 0$$

The Divisional Subprograms:

Division 1

$$\max_x \quad q_1^n x$$

$$M_1 x \leq d_1$$

Division 2

$$\max_y \quad q_2^n y \qquad (9)$$

$$M_2 y \leq d_2 \qquad (10)$$

The terms are defined as follows.

λ_j^i is a weight ($0 \leq \lambda_j^i \leq 1$) attached to the proposed solution of division j on the i^{th} iteration of the decomposition algorithm. Throughout this section, the subscripts denote the division, and the superscript denotes the iteration.

θ_j^i $= p_j x^i$, the contribution of division j to the value of the firm's objective function on the i^{th} iteration, where x^i denotes the optimal solution to (9), (10) for the division on the i^{th} iteration. θ_j^i is a scalar value.

E_j^i $= A_j x^i$, the amount of common resources used by division j in the i^{th} proposed solution. E_j^i is a vector of the same dimension as d_o.

q_j^i $= p_j - \gamma^{i-1} A_j$, the vector of coefficients for the division j objective function on the ith iteration. γ^{i-1} is the vector of provisional dual variables (simplex multipliers) computed on the $(i-1)^{th}$ iteration. These are the dual variables associated with constraint (6) of the master program, and $\gamma^{i-1} A_j$ is the opportunity cost of using the resources which are shared by both divisions.

As is obvious from our definitions, the decomposition algorithm is is an iterative process, where each iteration is a single step (pivot operation) of the simplex algorithm applied to the master program. This iterative process would proceed as follows.

<u>Iteration 1</u> (i=1): Each division presents to the master program, a proposed basic feasible solution to its divisional subprogram (9-10), The only details of the j^{th} division's proposed basic feasible solution disclosed to the master are (a) the resources required to implement this program: $E_j^1 = A_j\, x^1$, where the superscript denotes the 1^{st} iteration; and (b) the divisional contribution to the value of the firm's objective function, $\theta_j^1 = p_j x^1$. The values E_j^i, θ_j^1, j = 1,2 are used in the master program which tests for feasibility (i.e., demand for resources does not exceed supply) and optimality. The provisional values of the dual variables, γ^1, are computed and communicated to the division. These provisional dual variables are to act as prices of the shared resources, and are, therefore, used to modify the divisional objective functions on the next iteration.

<u>Iteration 2</u> (i=2): Using the provisional dual variables, γ^1, computed on the first iteration, each division revises its divisional objective function to take account of the value of the common resources it uses. With the revised objective function, it computes a new solution to the divisional problem (9-10). For example, assume division j has two decision variables, x_1 and x_2, and the coefficients in the original objective function (1) are p_{1j} and p_{2j}. There are two resources shared with other divisions, thus

$$A_j = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix},$$

and the provisional dual variables associated with the shared resources are $\gamma^1 = (\gamma_1^1, \gamma_2^1)$. The division's objective function coefficients would be revised to $q_j^2 = (q_{1j}^2, q_{2j}^2) = p_j - \gamma^1 Aj$:

$$q_{1j}^2 = p_{1j} - \gamma_1^1 a_{11} - \gamma_2^1 a_{21} \qquad (11)$$

$$q_{2j}^2 = p_{2j} - \gamma_1^1 a_{21} - \gamma_2^1 a_{22}$$

The new division j problem is

$$\max_x \quad q_{1j}^2 x_1 + q_{2j}^2 x_2 \qquad (9')$$

subject to, for example, the independent divisional constraints:

$$m_{11}x_1 + m_{12}x_2 \leq d_{1j}$$

$$m_{21}x_1 + m_{22}x_2 \leq d_{2j} \qquad (10')$$

$$m_{31}x_1 + m_{22}x_2 \leq d_{3j}$$

The divisional constraints (10') are the same as (3) in the original problem, but the division need not directly consider how its decisions interact with other divisions (it ignores constraint (2)). Rather, it revises its objective function, so as to take account of its use of resources shared with other divisions, by using the provisional dual variable γ as a price for shared resources. Thus, for example, in (11), $- \gamma_2^1 a_{21}$ represents the charge (per unit of activity #1) for use of resource #2, since γ_2^1 is a shadow price of resource #2, and a_{21} is the number of units of resource #2 used per unit level of activity #1.

Letting x_1^2, x_2^2 denote the optimal solution to (9'-10') for the 2nd iteration, division j tells the master program its requirements for resources 1 and 2:

$$E_j^2 = \begin{bmatrix} e_{1j}^2 \\ \\ e_{2j}^2 \end{bmatrix} = \begin{bmatrix} (a_{11}x_1^2 + a_{12}x_2^2) \\ \\ (a_{21}x_1^2 + a_{22}x_2^2) \end{bmatrix} \qquad (12)$$

and its divisional contribution to the value of the firm's objective function:

$$\theta_j^2 = p_{1j}x_1^2 + p_{2j}x_2^2. \qquad (13)$$

The values θ_j^2, E_j^2 , for j = 1,2 are included in the master program (5-8) which computes new weights λ and new values for the provisional dual variables, including γ^2 associated with constraint. (6). This solution is tested for optimality; if an optimal solution is indicated, the divisions are told to implement a weighted average of their previously proposed solutions, where the weights are the λ values. If the present solution is not optimal, the divisions are to use the new price for shared resources, γ^2, to revise their divisional objective functions for the third iteration. This process continues until an optimal solution is found. On each iteration, the divisions use new provisional dual variables as prices in their revised objective function. Thus, the master program is acting as an auctioneer, setting prices for internally shared resources so that supply equals demand.

The decomposition algorithm creates an internal market for resources which are shared by different divisions, and the dual variables are used as internal prices. So long as transfer prices are properly formulated, individual decision making units can make decentralized decisions which result in a global optimization for the firm. In this case, the transfer prices are set so that internal supply and demand are equalized, and the resources are allocated to the most efficient decision making units. Each division formulates its objective function (9) so that it is penalized according to the amount of a scarce resource it uses, and it is rewarded if it supplies a resource which can be used in other activities.

SECTION III. AN INTEGRATED MODEL OF THE FIRM

In order to explore the application of the decomposition principle to financial decisions, we will first develop a model which combines the operating (or production), investment, and financing decision problems into one integrated, although much simplified, model of the firm. The purpose of this integration is not to attempt to solve all these decision problems, rather, it is to show how these decision problems interrelate, and subsequently, how they can be separated or decentralized.

The Objective Function

The objective of the firm's decisions should be to maximize the current value per share of common equity. The model of valuation, and thus the maximand to be used here is of the form

$$p_o = \sum_{t=1}^{T} \frac{D_t}{N_t} \frac{1}{(1+\delta)^t} + \frac{V_T}{N_T} \frac{1}{(1+\delta)^T} \qquad (14)$$

where p_o = market price per share of common stock at time 0;

D_t = aggregate dividends paid at time t;

V_t = aggregate market value of equity at the horizon date, time T;

N_t = the number of common shares outstanding at time t;

δ = the rate of return on equity required by the market for a firm of its risk class, which for simplicity, we will assume is independent the amount of debt.

If our decision problem involves financing decisions, then the number of shares, N_t, is a decision variable, and our objective function is non-linear. It has been shown [4] that (14) can be written in linear form as

$$P_o = \sum_{t=1}^{T} \frac{D_t - S_t}{N_o} \frac{1}{(1+\delta)^t} + \frac{V_T}{N_o} \frac{1}{(1+\delta)^T} \qquad (15)$$

or $V_o = \sum_{t=1}^{T} (D_t - S_t)\, \rho^t + V_T \rho T$,

where S_t = the amount of new equity issued in period t, before deduction of flotation costs, and

$$\rho^t = \frac{1}{(1+\delta)^t} .$$

The reason that new equity, S_t, is subtracted out of the objective function is, as Miller and Modigliani [18] point out, that the benefit derived from the inflow of new equity funds in period t is exactly offset by the effects of dilution of ownership in future periods.

Since dividends and new equity are not the only relevant decision variables, we want to develop (15) in order to express our objective function in terms of more basic decision variables. First, we express (15) in terms of the relevant cash flows to the firm. Let

O_t = operating income, before deduction of depreciation or interest, received during period t;

Q_t = depreciation charges during period t;

I_t = interest paid at time t;

τ = corporate tax rate

C_t = the flow of funds from operations, after interest and taxes,

where,

$$C_t = O_t(1-\tau)+\tau Q_t - I_t(1-\tau). \tag{16}$$

The firm's sources of funds are: (a) from operations (C_t), (b) funds provided by the flotation of new equity, net of flotation costs ($S_t(1-f)$, where f denotes the proportional flotation costs), and (c) from net increases in debt (ΔB_t). The firm's uses for funds are dividends (D_t), and new investments (Y_t), where Y_t denotes gross investment, including replacement of depreciated assets and increases in working capital. ΔB_t is the net increase in debt, so that $\Delta B_t<0$ indicates debt repayment, a use of funds. Letting total sources of funds equal total uses, we have

$$C_t + \Delta B_t(1-f) = D_t + Y_t$$

which yields

$$D_t = C_t + \Delta B_t + S_t\ (1-f) - Y_t. \tag{17}$$

substitution of (17) for D_t in (15) yields our valuation model:[5]

$$V_o = \sum_{t=1}^{T} [C_t + \Delta B_t - Y_t + S_t\ (1-f) - S_t]\rho^t + V_T\rho^T$$

$$= \sum_{t=1}^{T} [C_t + \Delta B_t - Y_t - fS_t]\ \rho^t + V_T\rho^T \tag{18}$$

substituting for C_t from (16) we obtain

$$V_o = \sum_{t=1}^{T} [O_t(1-\tau)+\tau Q_t - I(1-\tau)+\Delta B_t - Y_t - fS_t]\rho^t + V_T\rho^T \qquad (19)$$

We must now define the relevant decision variables and show how they relate to (19).[6]

For our model we define five general types of decision variables: investment, borrowing, equity financing, dividends, and production.[7]

Investment: The firm is presented with a set of investment opportunities. Although we will explicitly consider only capital investment projects, these opportunities can include short term investment and lending.[8] Let y_k be the decision variable associated with project k, where y_k = 0, or 1, denote rejection or acceptance of the project. Let c_{tk} be the cash inflow (before interest and after taxes[9]) from project k in period t. The coefficient (in the objective function) of decision variable y_k is

$$\sum_{t=1}^{T} c_{tk}\rho^t,$$

which is the net present value[10] of project k, where the discount rate is the required rate of return on equity. The capital investment portion of the objective function is

$$\max_{y} \sum_{k} \sum_{t} c_{tk}\rho^t y_k. \qquad (20)$$

Debt Financing: The firm obtains part of its financing by issuing debt securities. For simplicity we will not distinguish between long and short term debt, we will assume all debt is issued in perpetuity, and can be repaid at any time. We will also maintain the simplifying assumption that the required return on equity, δ, is independent of the amount of debt.

Let B_t denote the amount borrowed, and R_t the amount repaid at time t. The debt financing portion of our objective function will be

$$\max_{B,R} \sum_t b_t B_t - \sum_t b_t R_t =$$

$$= \max_{B,R} \sum_t [\rho^t - \sum_{n=t=1}^{T} r\rho^n] B_t - \sum_t [\rho^t - \sum_{n=t+1}^{T} r\rho^n] R_t, \tag{21}$$

where r is the after-tax rate of interest, and we define

$$b_t = \rho^t - \sum_{n=t+1}^{T} r\rho^n. \tag{22}$$

The first term, ρ^t is the present value of the \$1 inflow when debt is issued, and the second term is the present value of all subsequent interest payments until the horizon date.

Dividends and New Equity Financing: Dividends, as a decision variable, do not appear in our objective function, even though our model will determine the dividends to be paid. Since our valuation model (19) is expressed in terms of all sources and uses except dividends, dividends are determined as a residual, when all other variables are determined. Dividends will be shown in some of the constraints.

As is shown in (19), the coefficient of the new equity variable, S_t, is $-f\rho^t$, the present value of the flotation costs.

The dividend and new equity portion of our "firm-wide" objective function is,

$$\max_{S} \quad \sum_{t} (-f\rho^{t}) S_{t}. \tag{23}$$

Production:[11] The firm has certain existing production and marketing facilities. The current decision involves the activity level at which these facilities are to be operated. Let the decision variable x_{ti} denote the activity level at which process i is to be operated in period t, and let the coefficient p_{ti} represent the present value of the after tax cash flows associated with the operation of process i at unit level in period t.[12] The portion of the firm's objective function associated with the production decisions is

$$\max_{x} \quad \sum_{t} \sum_{i} p_{ti} x_{ti}. \tag{24}$$

The Objective Function: Combining expressions 20,21,23, and 24, we write the complete, firm-wide objection function as

$$\max_{\substack{y,B,R \\ S,x}} \quad V = \sum_{k} \sum_{t} c_{tk}\rho^{t} y_{k} + \sum_{t} b_{t} B_{t} - \sum_{t} b_{t} R_{t} - \sum_{t} f\rho^{t} S_{t} + \sum_{t} \sum_{i} p_{ti} x_{ti} . \tag{25}$$

The Constraint Equations: As previously stated, the primary purpose of this paper is to explore the implications of the structure of our L. P. model, rather than to develop a detailed model of the firm. Thus, only those constraint equations regarded as basic to the structure of a model of the firm will be developed, while other, less basic, constraints will be ignored or referenced in only the most general terms.

Every financial planning model must include one set of constraint equations which are of overriding importance: a set of equations expressing accounting identities which tie the system of variables together. We will consider as typical the equation: net cash inflows = net change in cash balances.

Let K_o = cash balance at t=o, the firm's initial balance;

K_t = cash balance at time t, a decision variable;

B_o = debt outstanding at the beginning of period 1.

a_{ti} = net cash inflow in period t per unit operating level of production process i.

F_t = net cash inflows in period t which are exogenously determined.

Letting net cash receipts equal the net change in cash balances from the start to the end of the period, for period t=1 we have

$$\sum_k c_{1k}y_k - rB_o + B_1 - R_1 + (1-f)S_1 - D_1 + \sum_i a_{1i}x_{1i} + F_1 = K_1 - K_o.$$

Rewriting with the constant terms on the right, and reversing the signs so as to show cash outflows as positive and inflows as negative, we have

$$-\sum_k c_{1k}y_k - B_1 + R_1 - (1-f)S_1 + D_1 - \sum_i a_{1i}x_{1i} - K_1 = K_o + F_1 - rB_o. \quad (26)$$

Similarly, for any period t, we have

$$-\sum_k c_{tk}y_k + \sum_{n=1}^{t-1} rB_n - B_t - \sum_{n=1}^{t-1} rR_b + R_t - (1-f)S_t - \sum_i a_{ti}x_{ti} + D_t - K_{t-1} + K_t = F_t - rB_o,$$

$$t=2,\ldots,T \quad (27)$$

The cash flow equations, (26) and (27), are particularly important since they show all the relevant decision variables, including net new equity, $(1-f)S_t$, and dividends, D_t. In addition, they tie all the variables together in a "definitionally" consistent manner, and link various account balances together from period to another. Since the cash flow equations reflect the linkages between different decision centers, and they involve all decision variables, these equations will be part of the inter-divisional equations represented by expression (2), and cash will be regarded as a resource used jointly by all divisions. Thus, in our decomposition algorithm, the coefficients in (26) and (27) correspond to those represented by the A_j matrices of (2).

Debt Capacity Constraints: In order to limit the use of debt financing, we impose the policy constraint, Total Debt $\leq \alpha$ {Total Assets}, $0<\alpha<1$. We write this constraint, for period t, as

$$\sum_{n=o}^{t} B_n - \sum_{n=1}^{t} R_n \leq \alpha\{\sum_k \sum_{n=1}^{t} h_{nk} y_k + \sum_i g_i x_{ti} + K_t + H_t\},\ t=1,\ldots,T,$$

where

h_{nk} = assets purchased in period n for investment project k;

g_i = investment in working capital associated with production process i;

H_t = assets at time t which are determined exogenously to the model.

To show the constant terms on the right, we rewrite as

$$-\sum_k \sum_{n=1}^{t} \alpha h_{nk} y_k - \sum_i \alpha g_i x_{ti} + \sum_{n=1}^{t} B_n - \sum_{n=1}^{t} R_n - \alpha K_t \leq \alpha H_t - B_o,\ t=1,\ldots,T. \quad (28)$$

We will refer to this set of T constraints as debt capacity constraints, since, at any point t, the availability of additional debt financing is determined by our past borrowing and the total assets available to support this borrowing. Note that the debt capacity constraints (28) involve all our decision variables, and thus, are another part of our model which interrelates, or links together, the different divisions, or decision centers of the firm.

The set of "firm-wide" constraints represented by expression (2) would now consist of all the cash flow equations (26-27) and the debt capacity constraints (28). We will assume that these are the only constraints that involve all decision variables, and thus, reflect the interactions of the various decision centers. To the extent that we can say there are resources shared by all division, these resources are cash and debt capacity.

In order not to get lost in a forest of constraint equations, we will assume that all remaining constraints are specific to individual decision centers. We do not need to develop the complete set of constraints in order to analyze our model, although, we will develop additional detail as we proceed in order to emphasize a specific point. We will represent the constraints specific to a particular decision center j, in matrix form as

$$M_j x_j \leq d_j,$$

where x_j is the vector of decision variables controlled by division j, M_j is the matrix of coefficients, and d_j, the vector of constants.

A representative list of the types of constraints envisioned in a more detailed model is in Table I.

<u>The Integrated Model</u>; Our integrated model of the firm would have the objective function (25), and the constraints reflecting interdependence of decisions between divisions would be the cash flow equations (26-27), and debt capacity constraints (28). In addition, there are constraints $M_j x_j \leq d_j$, for all decision centers j, which are independent of the operations in other divisions; these are typified by the types shown in Table I. In the next section we will consider how the firm will decentralize their decisions, and we will explore the implications for internal pricing of the shared "resources": cash and debt capacity.

TABLE I

Type of Constraint	Production	Debt	Cash	Stock	Dividends
A. Productive Division					
1. Technological Constraints (Production Functions)	x x	B R	K	S	D
2. Physical Resource Constraints	x				
B. Long Term Capital Investment					
1. Mutual Exclusion of Projects	y				
2. Interdependence of Projects	y				
3. Integer Variables, y=0,1	y				
C. Working Capital Management					
1. Short Term Borrowing $\leq$ Line of Credit		B R			
2. Cash $\geq$ required Compensating Balances		B R	K		
3. Cash $\geq$ Minimum Balance			K		
D. Long Term Financing					
1. Long Term Borrowing $\leq$ maximum, $\geq$ minimum		B R			
2. Debt Repayment $\leq$ Debt Outstanding		B R			
3. New Equity Financing $\geq$ minimum				S	
4. Dividend Growth $\geq$ minimum					D

SECTION IV.

THE DECOMPOSITION OF THE INTEGRATED MODEL OF THE FIRM

Having developed a simple model of the firm which includes the major operating, investment, and financing decisions, we will now apply the Dantzig-Wolfe decomposition principle to the model in order to analyze it in terms of opportunities for decentralization of decision, and to explain the objective functions for the decentralized decision centers.

The Master Program: In our model, the cash flow and debt capacity constraints reflect the interaction of the various decision centers in consuming or supplying "resources" that are common to all divisions: in this case, the resources are cash and debt capacity. The master program must set internal transfer prices on these resources in order to allocate the resources among competing users. The prices used will be the values of the provisional dual variables associated with the cash and debt constraints in the master program.

On the n^{th} iteration of the decomposition algorithm, the master program, for our four division model, is written as

$$\max_{\lambda} \sum_{j=1}^{4} \sum_{i=1}^{n} \Theta_j^i \lambda_j^i \qquad (29)$$

$$\sum_j \sum_i E_j^i \lambda_j^i \le d_o \qquad (30)$$

$$\sum_j \sum_i L_j^i \lambda_j^i \le d_o' \qquad (31)$$

$$\sum_i \lambda_j^i = 1, \quad j=1,\ldots,4 \qquad (32)$$

$$\lambda_j^i \ge 0;$$

where,

Θ_j is division j's contribution to the current market value of the firm on the i^{th} iteration;

E_j^i $=(e_{1j}^i, e_{2j}^i,\ldots,e_{Tj}^i)$, is the T component vector of net cash outflows from division j on the i^{th} iteration. Element e_{tj}^i is the net cash outflow during period t of division j.

d_o is the T component vector of the firm's cash flows which are exogenously determined;

L_j^i $= (\ell_{1j}^i, \ell_{2j}^i,\ldots,\ell_{Tj}^i)$, is the T component vector where element ℓ_{tj}^i is division j's utilization of debt capacity in period t in the i^{th} proposed solution;

d_o' is the T vector of elements representing the firm's exogenously determined debt capacity, for each period t=1,...,T.

Note that constraints (30 and (31) correspond to the cash flow and debt capacity constraints (26-27) and (28) respectively.

On each iteration i of the decomposition procedure, each division j tells the master program (a) the net cash outflows for each period, E_j^i, (b) the divisional utilization of debt capacity in each period, L_j^i, and (c) the divisional contribution to the market value of equity, Θ_j^i. The master program (29-32) is solved for weights λ_j^i, and the provisional dual variables associated with (a) the cash flow constraints (30), denoted by γ_t^i, t=1,...,T, and (b) the debt capacity constraints (31), denoted by π_t^i, t=1,...,T.

Capital Budgeting: The decentralized capital budgeting problem can be formulated as

$$\max_{y} \quad \sum_{k} q_{k}^{i} \; y_{k} \tag{33}$$

$$M_1 y \quad \leq d_1, \tag{34}$$

where M_1 is the matrix of coefficients from the independent constraint equations concerned with (for example) mutual exclusion and interdependencies of projects, and constraints regarding $y_k = 0,1$; and where, on the i^{th} iteration, the objective function coefficient is q_k^i; and the subscript 1 is to denote "division #1", the capital budgeting division.

According to expression (11), the coefficient q_k^i is calculated as

$$q_k \;=\; p_k \; - \sum_{t=1}^{T} \gamma_t \, a_{tk} - \sum_{t=1}^{T} \pi_t b_{tk} \tag{35}$$

where p_k is the coefficient from the original objective function (25), a_{tk} is the coefficient from the cash flow constraints (27) for period t, and b_{tk} the coefficient from the debt capacity constraint (28) for period t. Applying (35) to the capital budgeting problem, we have

$p_k \;=\; \sum_{t=1}^{T} c_{tk} \;\; \rho^{t},$ from (20) or (25);

$a_{tk} \;=\; -c_{tk},$ from (27); and

$b_{tk} \;=\; -\alpha h_{tk}$ from (28); where

c_{tk} is the cash inflow (c_{tk}<0, an outflow) from project k in period t,

h_{tk} is the capital expenditure for assets for project k in period t

and α is the proportion in the constraint, Total Debt $\leq$ α(Total Assets).

Thus, the coefficient q_k is

$$q_k^i = \sum_t c_{tk}\rho^t - \sum_t \gamma_t^i(-c_{tk}) - \sum_t \pi_t^i(-\alpha h_{tk}) \tag{36}$$

$$= \sum_t \left[c_{tk}(\rho^t + \gamma_t^i + \pi_t \alpha h_{tk}) \right] . \tag{37}$$

The decentralized capital budgeting objective function function, for the i^{th} iteration becomes

$$\max_y \sum_k \sum_t \left[c_{tk} \ (\rho^t + \gamma_t^i) + \pi_t^i \alpha h_{tk} \right] y_k \ . \tag{38}$$

This objective function will make sense after we interpret the various terms used.

The discount factor $\rho^t + 1/(1+\delta)^t$ measures the external opportunity cost to the stockholders (evaluated at time o) of funds in period t, in the sense that the required rate of return, δ, is the highest rate of return, at a given level of risk, available to stockholders from alternative external investments. Thus, ρ^t measures the stockholders valuation of cash-flows relative to external investment opportunities.

The provisional dual variable γ_t measures the change in the current market value of equity resulting from an additional dollar of cash available at time t, and used in its most productive internal alternative use. That is, γ_t is a discount factor which measures the increment above ρ^t, in the current value of equity, derived from an additional dollar utilized by the corporation in an optimal manner, at time t. Thus, ρ^t is an <u>external</u> opportunity cost, and γ^t is an <u>internal</u> opportunity cost. If productive internal uses for funds have been exhausted so that there are no investment opportunities with a rate of return above the required rate of return δ, then $\gamma_t=0$, and the relevant discount factor applied to funds at time t is ρ^t, an external opportunity cost. If there are investment opportunities within the firm, with rates of return in excess of δ, which are foregone due to a shortage of available funds, then γ_t measures the opportunity cost, <u>in excess of ρ^t</u>, of foregoing these opportunities. Under these circumstances, the full opportunity cost of cash in period t, to the stockholder, in terms of current market value, is $\rho^t+\gamma_t$.

Given this interpretation, the term $-\sum_t \gamma_t(-c_{tk}) = \sum_t \gamma_t c_{tk}$ in (36) represents the internal opportunity cost of cash flows associated with project k. For $c_{tk}>0$, $\gamma_t c_{tk}$ represents an internal payment, or "subsidy" to project k for providing funds in period t. If $c_{tk}<0$, then $\gamma_t c_{tk}<0$ represents a penalty to project k for absorbing funds which could be utilized in alternative productive opportunities. Thus, $\sum_t \gamma_t c_{tk}$ represents a charge for the shared resource, cash, where γ_t is the internal transfer price for cash at time t. The full opportunity cost to the stockholder would be $\sum_t (\rho^t + \gamma_t)c_{tk}$.

The term $\sum_t \pi_t \alpha h_{tk}$ in (36) represents the opportunity costs, associated with project k, of "absorbing debt capacity." The debt capacity constraints (28) allow the firm to add α dollars of debt for each \$1 increase in assets, and the dual variable π_t measures the increase in the market value of equity which would result if the firm could get an additional dollar of debt financing. Since h_{tk} is the increase in assets in period t associated with project k, then adoption of project k adds αh_{tk} dollars of debt capacity to the firm in period t. Thus, $\sum_t \alpha h_{tk} \pi_t$ is the change in the market value of equity which results if project k is adopted, and αh_{tk} dollars are borrowed and put into their most productive uses. The dual variable π_t is being used as an internal transfer price for debt capacity, and since acquiring assets adds to debt capacity, the project k is subsidized, or rewarded, to the extent that it enables the firm to obtain additional debt financing.

Thus, the capital budgeting division's modified objective function recognizes a project's effect on the value of equity as coming from three sources. First, there is the direct effect of cash flows from the project upon the firm's ability to pay current dividends, as measured by $\sum_t c_{tk} \rho^t$, where ρ^t is the discount factor applied by stockholders in evaluating dividends in period t. Second, the cash flows generated by the project can also be employed in other productive uses which, in turn, enable the firm to pay dividends in future periods, as measured by $\sum_t c_{tk} \gamma_t$. Third, the increase in the asset base of the firm increases the firm's ability to borrow and thus undertake other projects. This effect is measured by $\sum_t \alpha h_{tk} \pi_t$.

On the i^{th} iteration of the decomposition, the capital budgeting manager determines the solution to (33-34). Based on this solution, he reports to the master program (a) his demand for cash in each period

$$E_1^i = (e_1^i,\ldots,e_{T1}^i),$$

where period t cash demand is $e_{t1}^i = -\sum_j c_{tk} y_k^i$, $t=1,\ldots,T$; (b) his utilization of debt capacity

$$L_1^i = (\ell_1^i,\ldots,\ell_{T1}^i),$$

where $\ell_{t1}^i = -\sum_k \alpha h_{tk} y_k^i$; and (c) the divisional contribution to the market value of equity.

$$\Theta_1^i = \sum_k \sum_t c_{tk} \rho^t y_k^i.$$

If the master program has reached the optimal solution, the process will terminate with the capital budgeting manager being told to implement a weighted average of the past proposed solutions:

$$y_k^* = \sum_{i=1}^n \lambda_1^i y_k^i.$$

If optimality is not indicated, then the master computes a new set of prices $\gamma_t^{i+1}, \pi_t^{i+1}$, $t=1,\ldots,T$, which will be used by the capital budgeting manager to compute a new investment program according to (33-34). This continues until an optimal solution is found for the master program.

Note how the capital budgeting problem (33-34) differs from the standard linear programming formulation of the capital budgeting problem as presented by Weingartner [23]. The Weingartner formulation can be stated as

$$\max_y \sum_k \sum_t c_{tk} \rho^t y_k \tag{39}$$

subject to

$$\sum_k c_{tk} y_k \leq F_t \qquad t = 1,\ldots,T \tag{40}$$

$$M_1 y \leq d_1, \tag{41}$$

where F_t represents a fixed amount of funds available for capital investment in period t, and $M_1 y \leq d_1$ represents the other constraints regarding interdependencies, mutual exclusion and $y_k = 0,1$.

The Weingartner formulation, by regarding the capital budget as fixed at F_t, doesn"t allow the L.P. to make the trade-off between competing sources and used of funds. On the other hand, our model, unlike the Weingartner model, is not necessarily a model of capital budgeting under conditions of capital rationing, it is somewhat more general. The decomposition approach explicitly reflects the interaction of capital budgeting decisions with all other competing sources and uses of funds, including current production and current dividends. With the master program acting as an internal auctioneer of funds, there is an optimal allocation of funds within the firm, where funds are allocated to the most productive uses. In addition, the internal pricing scheme provides a "subsidy" to those activities which are a source of funds in periods when funds are in great demand. Thus, the capital budgeting manager need not regard the capital budget as fixed in any given period. Rather, he can obtain additional investment capital from corporate headquarters by bidding funds away from other uses if his returns are sufficiently high, as indicated by θ_2^i, his divisional contribution to the current market value of equity.

<u>The Dividend and Equity Financing Decisions</u>: The decentralized dividend decision model would be written as

$$\max_{D} \sum_{t=1}^{T} -\gamma_t D_t = \min_{D} \Sigma\gamma_t D_t \tag{42}$$

subject to, for example,

$$\begin{aligned} D_1 \qquad\qquad & \geq D_o\ (1+\varepsilon) \\ -(1+\varepsilon)D_{t-1}+D_t & \geq 0,\ t = 2,\ldots,T \end{aligned} \tag{43}$$

where ε is the minimum growth rate of dividends, a value set as a matter of corporate policy. In this context, the dividend decision is made so as to maintain growth of dividends in order to satisfy investors. The model does not really determine an optimal dividend policy in a global sense. Rather, the policy is specified in the constraints.

The objective function (42) is formulated according to (11). Since dividends appear in the cash flow constraints (27) as an outflow, they have an opportunity cost of γ_t, where γ_t measures the opportunity cost to the stockholder of the company foregoing internal investment opportunities. If there are attractive internal investment opportunities, with $\gamma_t>0$, then dividends will be minimized subject to the growth constraints set by corporate policy. If productive investment opportunities have been exhausted and there are still funds available, then $\gamma_t=0$, and the objective function (42) will have value of zero for all D. Under these circumstances all available funds will be paid out as dividends once all internal investments with rates of return in excess of δ have been undertaken. This is consistent with the idea that the corporation (under conditions of decreasing marginal returns to scale) would, at first, face very profitable internal investment opportunities which provide a return in excess of that available to external investors. The corporation would retain earnings and expand its investments. As these very productive opportunities were exhausted, γ_t would be driven to zero, and unutilized funds would be paid out to the investor in the form of dividends. But, so long as $\gamma_t>0$, the stockholder would be better off if funds are internally invested than if he receives them as dividends.

We consider now, the decision to issue new equity. The new equity decision variable S_t appears in the integrated objective function (25) with coefficient $-f\rho^t$, and in the cash flow constraints (27) with coefficient $-(1-f)$. Thus, according to (11) we write the decentralized equity decision model as (44)

$$\max_{S} \sum_{t} (-f\rho^t)+\gamma_t(1-f)\ S_t \quad = \quad \max_{S} \sum_{t} \gamma_t(1-f)-f\rho^t \quad S_t$$

subject to independent policy constraints: $M_2S \leq d_2$, (45)

where subscript 2 denotes the division.

If productive investment opportunities have been exhausted and thus $\gamma_t=0$, the objective function (44) becomes $\min_{S} \sum_{t} \rho^t fS_t$, and new equity financing is minimized. That is, due to flotation costs fS_t, when $\gamma_t=0$, the stockholders would always be less well off if new equity is issued.

Stockholders would be indifferent to the corporation issuing new shares if $\gamma_t>0$ and $\gamma_t(1-f)=\rho^t f$, that is when the coefficients of S_t in (44) is zero. Under these circumstances, the benefits of issuing shares and using the proceeds to engage in internal investments is exactly offset by the resulting dilution of ownership. The only time the stockholders would be better off with new shares being issued would be if internal investment opportunities are sufficiently attractive so that $\gamma_t(1-f)>\rho^t f$.

In this case, the benefits derived from utilizing new equity funds for internal investments exceed the dilution of ownership imposed by flotation costs. Thus, equity financing would be maximized, subject to the various constraints represented by $M_2 S \leq d_2$.

Consider now, the interaction between dividends, new equity financing, and internal investment opportunities. When investment opportunities are particularly attractive, and funds are in short supply, and thus $\gamma_t(1-f)>\rho^t f$, then new equity would be utilized and only the minimum dividends sufficient to satisfy policy constraints (43) would be paid. So long as γ_t remains high, investment will be expanded. But, assuming diminishing marginal returns to investment, as investment expands, γ_t is forced toward zero. When $\gamma_t(1-f)=\rho^t f$, internal investments are no longer attractive enough to justify new equity financing, but dividends are still minimized so that investments can be undertaken with the funds available from other sources. When these investments are exhausted, and $\gamma_t=0$, then internal investment is stopped, and all available funds are paid out as dividends.

Debt Financing: For the i^{th} iteration of the decomposition algorithm, the debt financing decision can be made on a decentralized basis by solving the problem

$$\begin{aligned} \max_{B,R} \quad & \sum_t \left[\rho^t+\gamma_t^i - \sum_{n=t+1}^{T} r(\rho^n+\gamma_n) - \sum_{n=t}^{T} \pi_n^i \right] B_t \\ & - \sum_t \left[\rho^t+\gamma_t^i - \sum_{n=t+1}^{T} r(\rho^n+\gamma_n) - \sum_{n=t}^{T} \pi_n^i \right] R_t \end{aligned} \tag{46}$$

subject to

$$M_3 \begin{bmatrix} B \\ R \end{bmatrix} \leq d_3 \quad , \tag{47}$$

where the constraints are those independent of the other divisions' decisions, such as (total repyaments) $\leq$ (total debt). In this context, the debt manager doesn't explicitly consider constraints on debt capacity, or the cash flow constraints, since these constraints are taken into account by the internal prices π and γ.

The objective function maximizes the present value of the proceeds from borrowing over the planning horizon. This present value calculation considers not only the direct effect of borrowing upon the value of equity: $[(\rho^t - \sum_{n=t+1}^{T} r\rho^n)B_t]$, but it also considers the indirect effects of borrowing upon the value of equity. The indirect effects are reflected in the objective function (46) in two separate ways. First, an additional $1 of borrowed funds adds to the current cash flow which can be utilized to undertake productive opportunities, and of course, subsequent interest payments decrease the availability of cash. This effect is measured by the cash flow constraint evaluator γ_t. If $\gamma_t > 0$, borrowing is encouraged by the "subsidy" in the objective function as measured by $+\gamma_t B_t$. Secondly, if the firm has already borrowed up to its capacity, then there is a "penalty" associated with this borrowing, which reflects the opportunity cost of not being able to borrow additional funds to be utilized in other, profitable investment opportunities which must be foregone due to lack of financing. This penalty is measured by the dual variables π_t.

The dual variable π_t associated with the debt capacity constraint measures the increment in the current market value of the equity which will result if the firm has an additional $1 of debt capacity, or potential borrowing, in period t. If the firm is able to borrow an additional \$1 for that period, then it can potentially undertake additional investments or increase its rate of production, which it could otherwise not do. Since, in our particular model, we have assumed that debt is issued in perpetuity, the issuance of debt in period t absorbs debt capacity in the current period and all future periods. Thus, in the objective function (46), we show the penalty for issuing debt in period t as $-\sum_{n=t}^{T} \pi_k^i B_t$, and the reward for repayment of debt as $+\sum_{n=t}^{T} \pi_k^i R_t$.

As with any dual variable, if the corresponding constraint in the primal is not binding, then the value of the dual variable will be zero. Thus, if the firm doesn't borrow up to its capacity, then $\pi_t=0$, and there is no penalty for issuing additional debt in that period. One reason why the firm would not borrow up to its capacity would be that it has exhausted the opportunities for productive employment of funds. For example, if all remaining (unadopted) investment opportunities are such that their net present values are nonpositive, then there is no point in borrowing additional funds. Thus, the penalty associated with having already borrowed up to the limit would be zero. On the other hand, if there were many productive investment opportunities remaining, and the firm has already borrowed up to its limit, then $\pi_t>0$ would measure increase in the market value of equity which would result if the firm could borrow an additional \$1 in period t.

Consider now, the sequence of financing as the decomposition algorithm proceeds. At first, very profitable investment opportunities will tend to make γ_t relatively high. Both debt and equity financing would be utilized, with debt being preferred since it appears as a cheaper source of funds. If the debt capacity limit es encountered before $\gamma_t(1-f)\leq\rho^t f$, then the firm will still expand its investments, using equity financing. Further expansion of investment will drive γ_t lower so that $\gamma_t(1-f)\leq\rho^t f$, and equity financing will cease. At the same time, the extra investment, financed with equity will have pushed the firm away from its debt capacity limit ($\pi_t = 0$), and additional debt financing will occur in order to finance further investments as long as $\gamma_t > 0$.

Other decentralized decision centers such as the production division would formulate their divisional objective functions in a similar manner as the investment and financing decision centers. Each decision center would utilize the transfer prices generated by the master program, γ_t and π_t. Cash flows would be discounted with the modified discount factor $(\rho^t+\gamma_t)$, and changes in assets or liabilities which affect debt capacity will be priced with the debt capacity evaluator π_t.

SECTION V. CONCLUSION

The optimal solution to the decomposed model of the firm will result in a global optimization for the firm as a whole. From an organizational and computational point of view, there is a great advantage to having the managers most familiar with their own divisional problems and technology make their own "value maximizing" decisions, where "value" takes into account the interaction of their decisions with the rest of the firm, as reflected in the transfer prices for shared resources, generated by the master program.

The application of the decomposition principle to our financial decision model yields a transfer pricing scheme for cash flows within the firm. The appropriate transfer price for internal cash is $\rho^t+\gamma_t$, where $\rho^t \equiv 1/(1+\delta)^t$ is the discount factor applied to dividends by the market, δ is the required rate of return on equity, and γ_t is the provisional dual variable associated with the cash flow constraints. The provisional dual variable γ_t measures the increment (beyond ρ^t) to the current market value of equity if there is an additional $1 of cash available in period t. Thus, γ_t is the opportunity cost of cash.

When there are investment opportunities available with rates of of return in excess of the required rate of return on equity, and the firm has insufficient funds to exploit these opportunities, then $\gamma >0$. When the firm has exploited all such opportunities, then $\gamma_t =0$. Under conditions of decreasing marginal returns to investment, as the firm expands its investments, γ_t will decrease toward zero. Thus, investment in internal investment opportunities should be expanded until $\gamma_t=0$. At that point there are no investment opportunities remaining with rate of return greater than δ. So long as $\gamma_t>0$, dividends should be minimized, subject to various possible policy constraints. Once γ_t has been driven to zero, all funds generated by the firm should be paid out as dividends.

Only when every division has made decisions which are optimal for the firm as a whole, and thus, a global optimum has been attained, will the appropriate discount factor applied to cash flows be ρ^t. At positions which are less than the global optimum, the discount factor applied to cash flows, for internal evaluation, will be $\rho^t+\gamma_t$. For example, in order for capital budgeting decisions to be made independently of financing and production decisions, the conventional cost of capital discount factor must be modified to be $\rho^t+\gamma_t$. In this way, the capital budgeting decision implicitly considers alternative uses for funds in other divisions, since γ_t represents the opportunity cost of not adopting alternative uses for funds.

Our model provides guidance as to when the firm should and should not use equity financing. This decision rule states that new equity should be issued only if $\gamma_t(1-f)>\rho^t f$, where f is the proportional cost of flotation of new equity. When $\gamma_t(1-f)<\rho^t f$, then the returns on alternative investment opportunities are not sufficient to cover the flotation costs, and thus new equity financing should not be used. Under these conditions, so long as $\gamma >0$, additional financing should come from other sources, such as debt, providing that there is still some debt capacity remaining.

FOOTNOTES

*Assistant Professor of Finance, the Wharton School, University of Pennsylvania. Financial support was provided by the Rodney L. White Center for Financial Research of the University of Pennsylvania.

[1] Since completing this paper I have found that Carleton [6] recently wrote a similar paper, applying the decomposition principle to the capital budgeting problem. His treatment of the capital budgeting problem, while quite similar, focuses more upon the decentralization of capital budgeting decisions between corporate divisions, whereas, the present emphasis is upon decentralization according to functional areas: investment, financing, and production. The present work explores the interaction of investment and financing, whereas Carleton discusses the implementation of decentralization.

[2] For proofs and detailed development of the decomposition principle, see: Baumol and Fabian [2], Dantzig [8, Chapter 23], Dantzig and Wolfe [9], and Hadley [10, Chapter 11].

[3] The decomposition algorithm can handle any number of separate "divisions." We consider only two divisions for ease of exposition.

[4] See Miller and Modigliani [18]. Carleton [5] utilizes this valuation model in the context of a linear programming model for long range financial planning.

[5] Our expression (18) differs from Carleton's expression (16) [5, p.299], in that his expression does not subtract fS_t. Carleton's (16) is not incorrect, rather it is not developed in terms of the net sources of funds available for dividends, $F_t+S_t(1-f)$. Substitution of $F_t+S_t(1-f)$ for D_t in Carleton's (1b) shows both to be identical. Our expression (18) are to be preferred since they properly reflect the effects of flctation costs upon the sources of funds.

[6] In the objective function of the linear programming model to be developed subsequently, the terminal value of equity V_T is suppressed. This is done to simplify the notation, and does not affect our conclusions. Of course, any application of the model would require us to consider terminal value V_T in our model. Possible methods of deriving V_T would include (a) applying an anticipated price-earnings ratio to terminal period earnings, or (b) assuming a constant growth rate, g, in earnings and dividends after the terminal date, and letting $V_T = D_{T+1}/(k-g)$.

[7] Vickers [22] provides a rigorous analysis of the firm's solution to the interrelated problems of production, investment, and financing. The present work was, in part, inspired by an effort to apply the Vickers model in a linear programming context.

[8]A more detailed development of the short term investment (and financing) problem is found in Robichek, Teichroew, and Jones [21], Orgler [20] and Cohen and Hammer [7].

[9]Define $c_{tk} = O_{tk}(1-\tau)+\tau q_{tk} - h_{tk}$, in direct correspondence to $O_t(1-\tau)+\tau Q_t - Y$ in (19), where $O_{tk}(1-\tau)$ is the (after tax) operating cash inflow, before depreciation (q_{tk}), or interest, from project k in period t, and h_{tk} is the capital outlay for project k in period t.

[10]Baumol and Quandt [3] have noted the difficulties inherent in using NPV in the objective function of an LP model under conditions of capital rationing as done by Weingartner [23]. Carleton [4] and Myers [18] have both subsequently justified the use of NPV in the context of a model where the objective function involves the cash flows to the stockholder. Since our model is basically a dividend model, the use of NPV in the objective function is fully justified. That is, NPV in (20) enters the objective function only in so far as it influences dividends, since (19) is derived directly from the discounted dividend expression (14).

[11]No attempt will be made to completely model the production decision. We must emphasize, however, that the complete model of the firm must include the operating decisions involving production and marketing, as well as the investment and financing decisions.

[12]The cash flows represented by p_{ti} consist of (a) the after tax operating profit per unit operating level (corresponding to $O_t(1-\tau)$ in (19), less: (b) the incremental investment in working capital necessary to support this activity level (which corresponds to $-Y_t$ in (19). Obviously we are assuming linearity of our objective function with respect to the activity levels x_{ti}. A concomitant assumption is that the products are sold in a purely competitive market so that revenue is a linear function of the number of units sold. If these functions are non linear, but can be expressed as quadratic functions, then, as Hass [12] points out, the decomposition principle will still apply.

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