The Distribution of Common Stock Price Changes: An Application of Transactions Time and Subordinated Stochastic Models

by

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Working Paper No. 6-73b

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I. INTRODUCTION

The empirical distributions of price changes for speculative assets (e.g. common stocks, bonds, etc.) measured over calendar time yield a higher frequency of observations near the mean and at the tails than would be expected for a normal distribution. The sample kurtosis is almost always greater than 3--the value expected for a normal distribution--and the distributions are commonly characterized as fat tailed and peaked (i.e., leptokurtic).

The most widely accepted theory to explain the observed distributions of security returns received its introduction from Mandelbrot [15]. Fama [7], Roll [19], Blume [2], and Teichmoeller [22] gave additional empirical support to this theory--the so-called "stable Paretian hypothesis." The theory states that price changes measured over calendar time intervals conform most closely to symmetric stable laws with characteristic exponent α, where $1 < \alpha < 2$. For symmetric stable laws, population moments of order r are not finite beyond α. These symmetric stable distributions have unbounded population kurtosis and usually exhibit values for sample kurtosis much greater than 3. Some recent empirical evidence casts doubt upon the stable Paretian hypothesis (see e.g. Blattberg and Gonedes [1] and Hsu, Miller and Wickern [14]).

Another but less widely known hypothesis that purports to explain the high kurtosis values for common stock returns was first introduced by Mandelbrot and Taylor [16] and Granger and Morgenstern [12] and uses the theory of subordinated stochastic processes. Subordinated models of security returns can frequently be described as mixtures of normal distributions. The hetereoscedasticity associated with such mixtures of normals will cause the sample kurtosis to have large values. The subordinated stochastic model can be introduced by recalling that common stock price changes reflect an accumulation of new information during a particular calendar time period. This accumulation is the sum of many small, independent bits of information, i.e., the sum of a large number of independent random variables. If the number of these independent random variables is, itself, a random variable, then the price change of a common stock observed over a period in calendar will be from a subordinated stochastic model. The subordinated normal model states that each price change measured over calendar time can be thought of as the sum, S_n, of a random number, n, of identically distributed random variables, X_i.[1] In the Praetz model [18], and Blattberg and Gonedes model [1], n obeys a log-normal distribution. In the Mandelbrot-Taylor model [16], n obeys a non-normal stable law.

Mandelbrot and Taylor [16], Granger and Morgenstern [12], and Clark [4] have introduced the concept of transactions time and employed it to refine the subordinated model. They first examined the price changes of securities in the interval between successive transactions. Next, they defined a price change over a fixed interval in calendar time (e.g. a day) as the sum S_n from time t to t + 1 of the price changes from n transactions, where n is a random variable. Thus the fundamental unit of time in measuring return is the time it takes to affect a given number of transactions. The subordinated model with transaction time predicts that the variability of security return will be positively related to elapsed transactions time during a given calendar time interval and that the leptokurtosis in empirical distributions will become less after properly adjusting security returns for the elapsed transaction time. Several discussions of subordinated models link transaction time to the level of volume of trading. On this basis, the subordinated model has achieved some empirical support from the work of Brada, Ernst and Van Tassel [3], Granger and Morgenstern [12], and Clark [4]. However, to date, the subordinated model has not been directly tested on a large scale common stock data file.

The purpose of the present investigation is to extend the existing empirical work to a large sample of common stocks to reach clearer and more reliable conclusions about the validity of the subordinated model of security returns. The evidence in this paper tends to support the subordinated model and suggests that the pricing process of the stock market evolves at different rates on different days, and that the rate of the evolution of the pricing process is intimately connected with transactions time. Evidence is presented that shows price changes are not stationary in calendar time and the standard deviation is not a well behaved measure of dispersion (see Officer [17] and Fielitz [11]). Moreover, after properly adjusting for transactions time, the distributions of price change become much less leptokurtic and behave more like a normal.

II. PRELIMINARY EXAMINATION OF THE DISTRIBUTION OF PRICE CHANGES

2.1 The Sample

The data consist of dividend adjusted daily return relatives for 315 common stocks listed on the New York Stock Exchange (NYSE) from the period January 1968 to September 1969--a period of 412 trading days. There are 412 daily observations for each common stock. Accompanying these relatives are the number of shares traded daily for each security.[2,3]

2.2 The Sample Moments and Characteristic Exponents

Estimates of measures of the sample moments of each security provide a description of the sample distributions of these relatives. In Table 1, the average, median and quartile values for skewness indicate the distributions are slightly asymmetric.[4] The average, median and quartile values for kurtosis show that the distributions are highly leptokurtic.[5] It has been demonstrated that estimates of a common stock's characteristic exponent, $\hat{\alpha}$, will be less than 2 when taken from leptokurtic distributions.[6] Using the procedure described by Fama and Roll [9], α is estimated for each security and the results compiled in Table 1. As expected, the average estimate of α is less than 2 ($\bar{\hat{\alpha}}$ = 1.58) and is similar in magnitude to values reported by Teichmoeller [22] and Officer [17].

All of the results of this section indicate the sample frequencies conform to a leptokurtic distribution for daily price changes.

III. STATIONARITY AND SUBORDINATED MODELS

It is well known that mixtures of normal distributions (e.g. with the same mean and changing variances) can cause leptokurtosis in observed price changes (see Sims[21]).Thus, if distributions of price changes over successive time periods take the form of mixtures of normal distributions with changing variances, fatter tails than the normal would be expected. Fama [7] was among the first to articulate this type of non-stationarity but he concluded that his evidence was more consistent with non-normal symmetric stable distributions than with mixtures of normals. More recently, Officer [17] found for a sample of NYSE securities that estimates of the characteristic exponent computed using daily price changes were constant over various time periods; this is consistent with stationarity.[7] In fact, the particular stable Paretian model developed by Fama and Roll assumes price change sequences are stationary in calendar time; whereas, the subordinated model generally assumes that price changes are non-stationary in calendar time and stationary in transactions time. A special case introduced by Mandelbrot and Taylor has the property that price changes can be stationary, stable, non-normal in calendar time, yet subordinate to a normal distribution when translated into transactions time.

3.1 A Subordinated Normal Model of Stock Price Changes

A subordinated normal model follows directly from the theory of random sums. Let $\zeta(\Delta d)$ be the price change measured over an interval on a transaction time scale Δd. Further assume that $\zeta(\Delta d)$ is a Gaussian process with the following properties:

1. $E[\zeta(\Delta d)] = \mu(\Delta d)$
2. $Var[\zeta(\Delta d)] = \sigma^2(\Delta d)$
3. $E[\zeta(\Delta d), \zeta(\Delta d')] = 0; \Delta d \neq \Delta d'$
4. $\zeta(\Delta d)$ is a stationary process.

Consider another stochastic process $d(\Delta t)$ which evolves over calendar time intervals, Δt, with the properties:

5. $d(\Delta t) > 0$
6. $E[d(\Delta t), d(\Delta t')] = 0; \Delta t \neq \Delta t'$

Finally, establish the subordinated process $P(\Delta t) = \zeta[d(\Delta t)]$. Given the properties of $\zeta(\Delta d)$ and $d(\Delta t)$, the process $P(\Delta t)$ must have the following properties:

7. $E[P(\Delta t)] = \mu d(\Delta t)$
8. $Var\,[P(\Delta t)|d(\Delta t)] = \sigma^2 d(\Delta t)$
9. $E[P(\Delta t), P(\Delta t')] = 0; \Delta t \neq \Delta t'$
10. $P(\Delta t) = \mu d(\Delta t) + \sqrt{Var[P(\Delta t)|d(\Delta t)]}\ Z(\Delta t)$ where $Z(\Delta t)$ is distributed as $\zeta(\Delta d)$, and is unit normal.

The distribution of $P(\Delta t)$ will be subordinate to a normal distribution and has a conditional normal distribution function with variance $\sigma^2 d(\Delta t)$.

If we posit a transactions time scale measuring the velocity of price changes, $\zeta(\Delta d)$, changes in transactions time can correspond to the directing variable $d(\Delta t)$, where $d(\Delta t)$ tells how much transactions time has elapsed between points in calendar time. Feller [10] has shown that if $d(\Delta t)$ is a true stochastic process with properties 5 and 6, an additional process may be formed for $P(\Delta t)$. In reality we hypothesize this process generates the observed distribution of price changes and is subordinated to $\zeta(\Delta d)$ and directed by the distribution $d(\Delta t)$. $\zeta(\Delta d)$ becomes the unobserved price change in transactions time and is a single effect of the pricing process where $d(\Delta t)$ is the amount of transactions time that has elapsed over a unit in calendar time t to t+1. Clark and Granger and Morgenstern suggested that transactions time can be usefully approximated by cumulative volume of trading up to calendar time t+1. Empirically, $d(\Delta t)$ would become a function volume of trading over a unit in calendar time t to t+1.

3.2 Non-Stationary Variances

The subordinated normal model provides obvious implications for observed prices change over calendar time. For example, the variance of $P(\Delta t)$ will vary with increments in the directing process $d(\Delta t)$; hence, the larger the volume of daily trading the greater should be the absolute magnitudes of observed price changes for a security. Note that if we rank calendar time intervals (e.g., days) by the size of the increments in the directing process (e.g., daily volume) we should expect to observe different magnitudes of $Var[P(\Delta t)|d(\Delta t)]$ as a function of the rank. This would indicate shifting variance of price change as a function of volume.

To test the hypothesis that price changes are not stationary in calendar time and that the number of transactions is connected with the true velocity of price evolution, we first pursued the implications of the relationship between variance of price change and volume.

In sample data this phenomenon would be supported by a relationship between volume of trading on day Δt, (taken as a surrogate for $d(\Delta t)$, and the square of the difference between the price change and its mean on day Δt. Such an association would indicate a mixture of normal distributions and could produce the large sample values on kurtosis. (See Clark [4] for a discussion of this association).

The 411 daily price changes for each security were first ranked from high volume to low volume, and then arranged in 10 groups of 41 days each. The first group included the 41 price changes corresponding to the highest volume of trading days,etc. (one observation is lost). The variance of daily price changes was computed for each group for each security as illustrated in Table 2 for Allied Chemical.

Consider the effect of this grouping technique on daily estimates of price change variance. Daily trading volume and estimates of price change variance appear to have a positive relationship. This result is consistent with the hypothesis that the observed leptokurtosis in empirical distributions of daily price change come from a generating process that is a mixture of distributions with differing variances. The variance itself appears to be a function of trading volume.

For a closer look, the variance of daily price change in each volume group for a security has been standardized by dividing by the securities' total variance of daily price change. This procedure was repeated for all 315 securities. The ranks of average values of the standardized variance of price change are perfectly correlated with the ranks of the daily volume of trading, and the values, from the highest volume group to the lowest, are the following: 2.24, 1.35, 1.11, .94, .80, .73, .63, .58, .48, and .37. Hence, it can be concluded larger than average price changes (both positive and negative) are associated with relatively high volume of trading over the same calendar time intervals. This supports the subordinated stochastic process hypothesis and casts serious doubt on the stationary version of the symmetric stable model.

3.3 A Regression Model for Transactions Time

Using trading volume as an approximate transactions time clock, it can be hypothesized that the conditional variance of price change for a security will be a linear function of trading volume, e.g.

$Var[P(\Delta t)|V(\Delta t)] = a_o + a_1 \; V(\Delta t)$, $\Delta t = 1,2,\ldots, T$; where $V(\Delta t)$ is the number of shares traded on Δt, and $Var[P(\Delta t)|V(\Delta t)]$ is the variance of price change conditional upon the volume of trading on Δt. However, the conditional variance cannot be directly observed on Δt. Over each calendar time interval, say a day, the observed square of daily price change after adjusting for the daily mean $P^2(\Delta t)$ will be equal to the conditional variance plus a random error term.

$$P^2(\Delta t) = Var[P(\Delta t)|\; V(\Delta t)] + e(\Delta t)$$

Following these developments a linear regression model can be established, i.e.,

$$P^2(\Delta t) = a_o + a_1 \; V(\Delta t) + e(\Delta t),$$

which can be used to provide time sequencial estimates of conditional variance, by

$$\hat{Var}[P(\Delta t)|V(\Delta t)] = \hat{a}_o + \hat{a}_1 \; V(\Delta t)$$

$\Delta t = 1,2,\ldots, T$. In practice, the particular form of the transaction time predictor was determined after the linear model was compared to various non-linear models. The empirical comparisons indicated the linear model with a constant term provided the best description.[8]

The average, median and quartile values of the regression statistics, reported in Table 3, show that there is a significant relationship between magnitude of price change and level of trading volume for most securities;

therefore, trading volume can be taken for an instrumental variable in measuring transactions time. In most instances, the value of the constant term was positive.

3.4 Volume Groups and Kurtosis

Under the subordinated model, the value of the sample kurtosis from daily price change should increase with increases in the amount of transactions time that elapses within a day. Specifically, if the volume of trading approximates the number of increments of elasped transactions time and the variance of daily volume is large relative to the mean of daily volume, the values of the sample kurtosis should be large.[9] Thus we can expect the sample kurtosis within each relatively homogeneous volume group to be significantly lower than the sample kurtosis estimated over all 411 trading days, for each security.

As expected from the volume groupings, kurtosis can be greatly reduced within volume groups as compared to the overall sample.[10] The results are illustrated for Allied Chemical in Table 2; the within volume group kurtosis values were less than the overall value of 6.88852. Moreover, this type of reduction in kurtosis was observed in all 315 securities. Thus, it appears that much of the leptokurtosis in empirical distributions of price changes can be eliminated by combining price changes into similar volume of trading groups.

In Table 4, decile values of the sample kurtosis values are reported for the 315 securities. This distribution is compared to a revised distribution of sample kurtosis values from each security and each volume group (making 3150 observations). The median is reduced from 4.92 to 3.34. The results indicate distributions of price change with "thinner tails," much closer to what could be drawn from a normal distribution (with mean 3.00 and variance equal to 24/N, where N is equal to the number of observations). Hence, we conclude that kurtosis can be significantly reduced after the non-stationarity of price change variance is properly accounted for.

IV. GOODNESS OF FIT TESTS OF THE STABLE AND SUBORDINATED MODELS

One obvious method of analyzing the distribution of price changes for securities compares the theoretical standardized cumulative probability distribution functions (CDF's) with the observed standardized CDF's. Although there are many procedures for making these comparisons, the two alternative procedures commonly used involve the Kolmorogov-Smirnov (KS) statistic and the Chi Square (χ^2) statistic.[11]

Both the KS and the χ^2 statistics have been computed as summary measures of the degree of approximation of the actual CDF's to the theoretical CDF's. However, only the χ^2 values are reported, as both the χ^2 and KS give substantially the same results, and the KS statistic seems less theoretically appropriate.

The specific procedures used in this section are based upon those of Blume [2] and Hsu, Miller and Wickern [14]. The standardized variate is calculated under the stable model as

$$T_\alpha(\Delta t) = \frac{P(\Delta t) - \hat{\mu}}{\hat{c}} \;;\; \Delta t = 1,2,\ldots,411,$$ for each security, where $P(\Delta t)$

is the daily price change for a particular security, $\hat{\mu}$ is the Fama-Roll [8] truncated mean estimate of the location parameter of the stable model, and $\hat{c}$ is the Fama-Roll [9] estimate of the scale parameter of the stable model. c is defined as $\gamma^{1/\alpha}$ where γ is the scale parameter of a stable model and α is the characteristic exponent of a stable model. Fama and Roll [9] show that c can be estimated without assuming a value of α. Next, the theoretical standardized distributions and computed conditional upon values of α = 1.5, 1.6, 1.7, 1.8, 1.9, and 2.0 (see the tables in [8]). Also we set $\alpha = \hat{\alpha}$ using the Fama-Roll estimation procedure, referred to previously, for α. Hence, the theoretical standardized cumulative distribution functions corresponding to $T_\alpha(\Delta t)$ are calculated for each of 7 values of α for each security.

The unit interval for the theoretical, standardized CDF's are partitioned into 17 unequal subintervals and the corresponding number of cumulative frequencies in each subinterval are counted for each security and each value of α[12]. A χ^2 value is computed for each α, and the results are displayed in Table 5.

Under the subordinated normal model the price change generating process for each security can be assumed equal to (see equations 7 to 11)

$P(\Delta t) - \mu(\Delta t) = \sqrt{Var[P(\Delta t) | V(\Delta t)]}\ Z(\Delta t)$; $\Delta t = 1,2,\ldots, 411$, where $\sqrt{Var[P(\Delta t) | V(\Delta t)]}$ is the conditional standard deviation of $P(\Delta t)$ and is, itself, a random variable. $Z(\Delta t)$ is unit normal with mean equal to zero, variance equal one, and kurtosis equal to three. The conditional standard deviation is estimated on each day for each security via the regression models of section 3.3, as

$$\sqrt{\hat{Var}\ P(\Delta t) | V(\Delta t)} = \sqrt{\hat{a}_o + \hat{a}_1\ V(\Delta t)}\ .$$

Thus, an adjustment procedure for translating daily price changes into transaction time is used to establish an "adjusted variate"

$$\hat{Z}(\Delta t) = \frac{P(\Delta t) - \mu(\Delta t)}{\sqrt{\hat{Var}[P(\Delta t) | V(\Delta t)]}}\ ; \ \Delta t = 1,2,\ldots,411$$

If $\sqrt{\hat{a}_o + \hat{a}_1\ V(\Delta t)}$ correctly measures elapsed transaction time, $\hat{Z}(\Delta t)$ should be normally distributed under the subordinated normal model. Furthermore, it should have mean equal to zero, variance equal to one and kurtosis equal to three. However, if the correct transaction time function has not been found via the regressions of 3.3, there may be random elements of transactions time still in the adjusted series and $\hat{Z}(\Delta t)$ may not be normally distributed. To assess the distribution of $\hat{Z}(\Delta t)$, the location and scale parameters are estimated, as before, and the standardized variate, $T_{\alpha}(\Delta t)$, is recorded for $\Delta t = 1,2,\ldots, 411$. The theoretical CDF values are obtained conditional upon $\alpha = 1.5, 1.6, 1.7, 1.8, 1.9, 2.0$ and $\alpha = \hat{\alpha}$. The unit interval is partitioned into 17 unequal subintervals and the number of cumulative frequencies in each **subinterval** counted for each security. Tabulations of χ^2 at various critical values are then made.

The number of times the χ^2 value exceeded a corresponding critical value (which is a function of the degrees of freedom and confidence level) is counted for four critical values and each assumed characteristic exponent (e.g., the critical value for the χ^2 statistic at the .005 level of confidence is 31.3 for 14 degrees of freedom). The purpose of computing the χ^2 values is to summarize the degree of approximation that exists between the actual security returns and those hypothesized under the stable and subordinated models. The results are reported in Table 5.

As expected, the tabulated values of the χ^2 statistic suggest the normal model for unadjusted security returns ($\alpha = 2$) gives the poorest approximations. To illustrate, the normal model yields χ^2 values greater than the corresponding critical χ^2 value at the .005 level for 262 of 315 securities. As the characteristic exponent used in generating the theoretical distributions for unadjusted security returns is reduced from 2.0, the tendency is for the actual distributions to be closer to the theoretical distributions. The symmetric stable model with $1.5 < \alpha < 1.6$, usually gives the best approximations, and when the characteristic exponent is reduced beyond 1.5 the χ^2 values tend to increase.

Let us turn to the subordinated normal model. After price changes have been properly adjusted for volume, the χ^2 values show the subordinated normal model achieves better approximations than the normal model for unadjusted security returns. In other words, $\hat{Z}(\Delta t)$ is more normal than $P(\Delta t)$. In fact, at each assumed characteristic exponent, from $\alpha = 2.00$ to $\alpha = 1.60$, the corresponding subordinated model gives closer approximations to the actual return distributions than the corresponding symetric stable model.

Although the χ^2 tests are encouraging and reveal that price changes adjusted for transactions time are much less fat tailed than can be expected from a non-normal stable distribution with $1.5 < \alpha > 1.6$. The sample kurtosis values of $Z(\Delta t)$ are still too high for normality; the average is equal to 3.65.

The results support the subordinated normal model by showing that adjusting price changes for volume achieves closer approximations to observed price changes than otherwise. However, the subordinated normal model does not describe actual distributions of price changes better than the symmetric stable model with $1.5 < \alpha > 1.6$. Furthermore, the best approximations are achieved when the theoretical CDF's are generated assuming that adjusted price changes are from a subordinated stable model with characteristic exponent α, where $1.65 < \alpha > 1.80$. This is consistent with a general version of the subordinated model presented by Mandelbrot and Taylor. They argue that $Z(\Delta t)$ might be symmetric, stable with characteristic exponent equal to α_1, and directed by the symmetric, stable transactions time variate with characteristic exponent equal to $\alpha_2 < 1$. Accordingly, $P(\Delta t)$ could be symmetric stable with characteristic exponent α equal to the product of α_1 and α_2, i.e. $\alpha_1 \cdot \alpha_2$, and subordinate to $Z(\Delta t)$.

There are other explanations for the kurtosis that remains in the adjusted price change series. For example, it may be that $\hat{Var}[P(\Delta t)| V(\Delta t)]$ is measured with error by the linear volume regression model. In this case, the adjusted variate, $\hat{Z}(\Delta t)$, would include a random element of transaction time, and as Clark has shown, the inclusion of transaction time will almost always lead to increased kurtosis value and to results like those in table 5.[13]

From these results there is little doubt that subordination exists in security return data. However, this fact does not rule out a stable, non-normal subordinated model (e.g., see Mandelbrot and Taylor), nor does it confirm a subordinated normal model. These goodness of fit results suggest that the χ^2 statistic is inadequate to distinguish between the several competing hypotheses, since from Clark it is clear that for the subordinated model any variation in transaction time increments can result in relatively large sample kurtosis. Thus, if the true transaction time has not been found with the assumed regression model, using the regression model to adjust price changes will result in distributions which are still leptokurtic. Hence, it is not surprising that reducing the characteristic exponent from 2.0 achieves better approximations after price changes are adjusted for volume.[14]

We conclude that adjusting price changes by volume reduce the fat tails of the distribution of price change (note the average $\bar{\hat{\alpha}}$ for $Z(\Delta t)$ increase from 1.59 to 1.71 after this adjustment is made), but not enough for a subordinated normal model to be unequivocally established.

V. TESTS OF STABILITY

One issue that remains is whether empirical procedures can be developed to distinguish between the finite variance subordinated model of Clark and the general class of subordinated models proposed by Mandelbrot and Taylor. The stability test may help us here. Recall that Mandelbrot and Taylor have put forth a subordinated model in which common stock prices measured on a calendar scale are symmetric, stable with characteristic exponent $\alpha < 2$. They posit a special case for which calendar time price changes are subordinate to Gaussian transactions time price changes, directed by a transactions time variable that is a non-normal stable random function with infinite mean. The Mandelbrot and Taylor example generalizes to arbitrary stable processes where $P(\Delta t)$ has characteristic exponent $\alpha = \alpha_1 \cdot \alpha_2$ and is subordinate to $Z(\Delta t)$ with exponent $\alpha_1 \leq 2$, directed by $d(\Delta t)$ with exponent $\alpha_2 < 1$.[15] Thus the Mandelbrot and Taylor model is a case where common stock returns are both stable and subordinated.

A test for the stability of calendar time price changes, recently employed by Teichmoeller [22], Officer [17], and Blattberg and Gonedes [1] , is performed on our sample of daily stock returns. The characteristic is estimated for each security for non-overlapping sums of 1,3,5,7 and 9 daily returns, chronologically ordered. As a result, for each estimate of the characteristic exponent, there are 411 observations for sums of 1 and 45 observations for sums of 9. If calendar time price changes are stable, the estimate of the characteristic exponent will remain the same regardless of the sum size (note the converse is not true).

The results, in Table 6, show a tendency for the average, median and quartile values of the characteristic exponent to increase for larger sums. This is in agreement with Officer [17] and Blattberg and Gonedes [1] but contradicts Teichmoeller [22][16]. The tendency to increase is inconsistent with any stationery, symmetric stable model with $\alpha<2$ and consistent with a finite variance subordinated model. Thus it is inconsistent with the Mandelbrot and Taylor infinite variance model.

Recently, an alternative procedure for testing for stability has been put forth by Hsu, Miller and Wickern [14]. To incorporate their modifications, the order of the observations has been randomized (as opposed to a chronological ordering) and the stability tests performed again. As before, the results presented in Table 7 show a pronounced tendency for the average, median and quartile values of the estimates of characteristic exponent to increase with larger sum sizes (α increases from 1.58 for N = 1 to 1.91 for N = 9) and suggest that the price changes are not stable Paretian. In addition to randomizing the order of returns for each security in estimating α, Hsu, Miller and Wickern also propose a stability test that adjusts for the possibility of shifting scale parameter over time.

To demonstrate that the tendency for the characteristic exponent to increase with sum size is not due to shifting scale parameter over time the sample time period was divided into two equal parts and the scale parameters estimated for each security. The stability tests were repeated after mutliplying the observations from the second half of the time period by the ratios of the scales estimates from the first half of the sample divided by those from the second half of the sample, $\frac{\hat{c}(1)}{\hat{c}(2)}$, for each of the 315 securities. The results do not change and confirm the earlier findings that the $\bar{\hat{\alpha}}$ is equal to 1.58 for N = 1 and 1.91 for N = 9.

The same test for stability can be performed on the adjusted price change variate $\hat{Z}(\Delta t)$ where calendar time price changes have adjusted via the linear regression models of trading volume. If $\hat{Z}(\Delta t)$ is non-normal, stable, and properly adjusted for transactions time,estimates of the characteristic exponent of $Z(\hat{\Delta}t)$ should not change as a positive function of the length of the calendar time interval used to observe it. Accordingly, the characteristic exponent for $Z(\Delta t)$, $t = 1,2\ldots,411$ for each security has been estimated for non-overlapping (chronologically ordered) sums of 1 and 9. The average estimate of α for sum size of 1 is 1.71. For sum size of 9 it is 1.91. Recall that under any stable model the characteristic exponent α is invariant under addition, thus it should not systematically increase as we create sums of random price changes. This tendency to increase is consistent with a subordinated normal model and inconsistent with a non-normal stable subordinated model, and can only occur if the sample is not drawn from a stable, non-normal distribution. Similar results are obtained when the stability test is constructed with randomly ordered returns (see Table 8).

VI. CONCLUSIONS

Distributions of stock price changes were examined to determine if the stable Paretian model articulated by Fama and Roll provided as good a hypothesis as the subordinated models of Clark [4] and Mandelbrot and Taylor [16] and Praetz [18] . The results indicate that the distributions of stock price changes have many of the properties predicted by a subordinated model.

The important implications of the **findings are:**

1. The sample standard deviation is not stationary in calendar time and, in some regards, is not as well behaved as found by Officer [17] supporting the findings of Fielitz [11].

2. The absolute magnitude of daily price change appears to vary with the number of transactions during the day. Hence, the distribution of price change is probably not-stationary in calendar time. We proposed a change in the time scale to account properly for the non-stationary and to establish a subordinated model of price changes.

3. Actual price changes for common stocks appear to be better described by a subordinated probability model and not a stationery, symmetric stable model. The evidence taken in its entirety makes a good case for the finite variance of price changes when they are properly adjusted for transactions time. It tends to contradict the infinite variance subordinated model proposed by Mandelbrot and Taylor, but is not so strong that it rules out this particular version of the stable Paretian theory.

1. Estimates of Sample Properties of the

Distribution of Price Changes

Parameter or fractile	Skewness[a]	Kurtosis[b]	Characteristic exponent[d]
average	-.58	4.93	1.58
F25[c]	+.25	4.22	1.49
median	-.64	4.92	1.57
F75	-1.43	5.93	1.67

[a]Skewness is computed as a function of the percentage of observations below the mean. (see footnote 4.)

[b]Kurtosis is computed as the sample value of the fourth moment divided by the square of the variance.

[c]The .25 fractile value.

[d]The Fama-Roll [9] estimator of the characteristic exponent, using the .95 fractile value.

2. Volume Groups and Price Change Variance: Allied Chemical Corporation

Volume Group	Mean Daily Volume (000)	Variance of Daily Price Change	Standardized Variance of Daily Price Change[b]	Kurtosis[c] Daily of Price Change
1[a]	1116.67	.00106	4.07692	3.38234
2	506.57	.00021	.80769	5.41482
3	350.97	.00023	.88462	3.40513
4	264.61	.00024	.92307	4.99985
5	218.26	.00027	1.03846	4.30899
6	187.80	.00018	.69230	3.53504
7	162.13	.00015	.57697	2.10356
8	143.36	.00010	.38386	3.55952
9	122.66	.00010	.38461	2.65631
10	88.12	.00009	.34615	2.34435
Entire Sample	300.17	.00026	1.00000	6.88852

a. The volume group representing the 41 days of greatest volume.

b. Computed by dividing total variance of price change into variance of price change for each volume group.

c. Kurtosis has an expected value of 3 and variance of 24/N under a normal hypothesis (where N equals the sample size).

3. Regression Estimates for Price Change

Variance-Volume Model

Parameter or fractile	intercept (10^{-6})	slope (10^{-6})	R^2	F(1,409)	T
mean	179.42	2.71	.088[b]	40.99[c]	6.82[d]
F25	67.61[a]	.86	.045	19.78	4.74
median	180.38	2.34	.084	38.01	6.12
F75	316.86	5.81	.152	72.45	8.62

[a]The .25 fractile of the intercept from 315 OLS regressions.

[b]The average coefficient of determination adjusted for degress of freedom.

[c]The average value of the F statistic with one degree of freedom and 409 observations.

[d]The average value of the T statistic.

4. The Distribution of Kurtosis of Daily Price Change

Fractile Estimates[d]	Sample Kurtosis Values From Each Security	Within Group Kurtosis Values From Each Security and Each Volume Group
0.00	2.99368[a]	1.79037
0.10	3.82579[b]	2.46627
0.20	4.14091	2.70693
0.30	4.29446	2.92188
0.40	4.61911	3.11201
0.50 (median)	4.91955	3.34285
0.60	5.20642	3.60357
0.70	5.63602	3.89934
0.80	6.32510	4.39578
0.90	7.98264	5.26704
1.00	116.77020[c]	29.68614

a. The minimum value of sample kurtosis from 315 securities (NYSE).

b. The .10 fractile value from the sample.

c. The maximum value of sample kurtosis from 315 securities (NYSE).

d. An unbiased estimator of the .x fractile ,except for the 0 and 1 fractile for which the sample values were used.

5. Goodness of Fit Tests: The Number of Times Chi Square Values Exceed Critical Values

A. Daily Price Change

Characteristic Exponents	Levels of Confidence .05	.025	.01	.005[c]
1.58[a]	189.	155.	124.	105.
1.50	243.	209.	176.	148.
1.60	194.	160.	131.	109.
1.70	180.	159.	137.	120.
1.80	212.	196.	170.	161.
1.90	254.	240.	227.	214.
2.00	293.	279.	267.	262.

B. Daily Price Change Adjusted for Volume of Trading

Characteristic Exponents	.05	.025	.01	.005
1.71[b]	143.	122.	97.	79.
1.50	259.	241.	208.	178.
1.60	202.	170.	136.	107.
1.70	146.	122.	102.	87.
1.80	142.	118.	99.	80.
1.90	178.	159.	138.	125.
2.00	218.	207.	188.	174.

[a]The average of the estimates of α for price change of 315 securities (NYSE).

[b]The average of the estimates of α for adjusted price change of 315 securities (NYSE).

[c]$P(X^2 > 31.3 \mid df = 14) = .005$, df = degrees of freedom.

6. The Characteristic Exponent for Sums of Daily Returns: Chronological Ordering

Parameter or fractile	Sum Size 1	3	5	7	9
Mean	1.58	1.70	1.76	1.78	1.85
F25[a]		1.59	1.61	1.63	1.67
Median		1.70	1.75	1.77	1.85
F75		1.82	1.91	1.97	2.00

[a]The .25 fractile value.

7. The Characteristic Exponent for Sums of Daily Returns: Random Ordering

Parameter or fractile	Sum Size 1	3	5	7	9
Mean	1.58	1.82	1.90	1.90	1.91
F25		1.70	1.77	1.74	1.74
Median		1.81	1.91	1.93	1.94
F75		1.96	2.00	2.00	2.00

8. The Characteristic Exponent for Sums of Daily Adjusted Returns

Parameter	Sum Size 1	3	5	7	9
Chronological ordering mean	1.71	1.78	1.89	1.90	1.91
Random ordering mean	1.71	1.82	1.85	1.89	1.89

FOOTNOTES

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[1] Suppose that $X_i,,,X_n$ are random variables with common distribution F. The sum $S_n = X_i + ,,, + X_n$ has the distribution F*, namely the n fold convolution of F with itself. However, the fixed n can be replaced by a random variable N with its own distribution. Thus the number of N transactions can also be a random variable with probability distribution $\P\{N=n\} = \P_n$, the conditional distribution of S_n given N = n is F* and the unconditional distribution is given by $U = \sum_{n=1}^{\infty} \P_n F^{n*}$.

The distribution of U is not necessarily a member of the stable class and probability limits on sums of U may not exist (Feller [10], p. 258).

However, if the number of transactions (elements) $N_i,,,N_n$ is a random variable such that

$$\text{plim}\frac{N_n}{n} = 1$$

then the central limit theorem holds and S_N will tend toward normality (Clark [4], Feller [10]).

[2] The data file is a condensed version of the ISL Quarterly Historical Stock Tapes distributed by Standard and Poors Corporation. The 315 NYSE securities have continous price and volume data from January 1, 1968 to September 30, 1969. The volume data do not include third market and regional exchange volume and do include all block trades. Adjustments have

been made for all stock splits and stock dividends and new issues during the sampling period attempting to hold the number of shares outstanding constant. If, after these adjustments were made, the number of shares outstanding changed by more than 2.5% over the period the security was eliminated.

[3]A price change is measured as a return relative which is defined as $\frac{P_t + D_t}{P_{t-1}} = R_t$; where P_t is price at time t and D_t are dividends paid out from time t-1 to t. The return relative is converted to natural logarithms.

In addition, much of the analysis was repeated using the percentages. Fama and Roll have relied on the natural logarithm of return, whereas Blume and Officer have relied on the percentage return. There appears to be no difference in daily price changes. The theoretical implication of using natural logarithms or percentages is unclear.

[4]Skewness is measured as a function of the percentage of sample observations less than the .5 truncated mean, or

$$\frac{(\text{percentage below mean} - \frac{1}{2})}{\frac{1}{2}} N^{\frac{1}{2}}$$

with sample size N. This statistic has a normal distribution with mean zero and variance one. It has been used by Roll [19]. Kurtosis is measured by dividing the square of the variance into the fourth moment. The mean is estimated by employing the truncated mean estimator developed by Fama and Roll [9]. Moment estimator of skewness has also been calculated with similar results. The fractile estimator was reported for expository reasons, since Roll reported it for bond returns, and because it is more robust than the moment estimator.

[5]The variance of kurtosis from a normal population is 24/N where N is the sample size. Thus sample kurtosis that lies between 2.52 and 3.48 is within two standard deviations of the true value. As expected over 95%

of all sample values are beyond this range. Note that the sample error of sample variance σ^2 from a normal population is $2\sigma^4/N$ and correspondingly, of the sample fourth moment μ_4 from a normal parent is (see Clark (4)).

[6] Even if price change distributions do not obey stable laws, estimates of α will measure the fatness of the tails (see [1] [4]).

[7] Recently Fielitz [11] and **Hsu,** Miller and Wickern [14] have provided evidence indicating that the probability distributions of stock price changes are not stationary in calendar time.

[8] Specifically, the following forms of $f[V(\Delta t)]$ were tried:

$$f[V(\Delta t)] = V(\Delta t)^{a_1}$$

$$f[V(\Delta t)] = e^{V(\Delta t)a_1}$$

In several specific instances $P^2(\Delta t)$ was replaced with $|P(\Delta t)|$ with no clear superiority in the tests.

[9] The kurtosis of price change of a subordinated distribution can be expressed as a function of the variance of volume by

$$\text{plim}\frac{\mu_4}{(\sigma^2)^2} = 3(1 + \frac{\sigma_v^2}{\delta_v^2})$$

where σ_v^2 is the variance of the number of transactions over Δt and δ_v is the mean of the number of transactions over Δt. Clark [4] relied on a similar volume grouping to examine hetereoscedasticity and kurtosis for cotton futures.

[10] Any within volume group sample kurtosis that lies between 1.47 and 4.53 is within two standard deviations of 3, which is expected under a normal model. 85% lie within this range.

Note we do not expect the average sample value to reduce to 3 after the volume grouping since this is an imperfect procedure and within each

[11]Using the K-S test is appropriate since the critical values depend upon known parameter values for generating the observed standardized CDF's Praetz correctly relied on the χ^2, but he makes a significant error. He estimated his observed standardized CDF's by subtracting the sample means and dividing by the sample standardized deviations. However, as proved in Cramer [5], for composite hypotheses one must obtain maximum liklihood estimates for the unknown parameters. Fama and Roll's results demonstrate that sample means and sample standard deviations are not adequate estimates for the parameters of stable laws when $\alpha<2$. Blume [23] and Hsu, Miller and Wickern [14] relied on the χ^2 test.

It should be noted that the χ^2 is a robust goodness of fit test but is not the most powerful [8,p.335]

[12]The unit interval was partitioned into the following subintervals:

Subinterval

0.00-0.02	0.60-0.70
0.02-0.04	0.70-0.80
0.04-0.06	0.80-0.90
0.06-0.08	0.90-0.92
0.08-0.10	0.92-0.94
0.10-0.20	0.94-0.96
0.20-0.30	0.96-0.98
0.30-0.40	0.98-1.00
0.40-0.60	

Obviously the choice of subintervals is partially arbitrary. One requirement is that the expected frequency in each subinterval be greater (or equal to) 5. The Pearson approximation to the χ^2 is not adequate for lower subinterval sizes.

We choose finer subintervals under the tails of the distribution because this is where it is easiest to distinguish between leptokurtic distributions and non-leptokurtic distributions. Note that Blume [2] used equal subintervals in developing a χ^2 goodness of fit test.

[13]There is potential bias as a consequence of introducing error with an incorrect specification of $d(\Delta t)$ in obtaining the adjusted price series. It might be asked whether it is possible to transfrom a fat tailed variate, e.g., $P(\Delta t)$, into a normal variate, $Z(\Delta t)$, by introducing specification error via the adjustor $d(\Delta t)$. This is potentially possible if $a_o + a_1 V(\Delta t)$ overestimates the variance of $P(\Delta t)$ on very high price

changes and understates the variance of $P(\Delta t)$ on small price changes, the result would be to cut off the tails of $P(\Delta t)$. This might be true if the "true" volume clock was $a_o V(\Delta t)^{a_1}$ and, instead $a_o + a_1 V(\Delta t)$ was used. However, as reported in the text, the linear volume function gave the best results i.e. closer fits and better Durbin-Watson values. Inspection of several scatter diagrams revealed that the linear procedure always has both positive and negative errors on the high and low ends of the trading volume range.

[14] Another related issue is if $Z(\Delta t)$ is truly a non normal, stable variate, with infinite variance and $Var(P(\Delta t) | V(\Delta t))$ is used to adjust $P(\Delta t)$. Interestingly, $Z(\Delta t)$ will likely be more fat tailed than $P(\Delta t)$ if a variance estimate is used in the adjustment procedure when, in fact, its theoretical value is infinite.

One procedure for reducing measurement error and assessing this potential bias in the time series transaction time clock used in section 33 is to estimate $f(V(\Delta t))$ cross sectionally, by establishing, $c_i = a_o + a \bar{V}_i$, $i = 1,2,...315$ where, c_i is the Fama Roll estimate of the scale parameter of calendar time prices for security i over all time and $\bar{V}_i$ is the average daily volume of trading for security i (adjusted for the number of shares outstanding). This regression model is estimated with $\hat{a}_o = .00981$, $\hat{a} = .17053$ and, coefficient of determination, $R^2 = .45599$. Next the adjusted variate $Z(\Delta t)$ is computed using $\hat{c}$ where $\hat{a}_o$ and $\hat{a}_1$ are the same for all securities. The χ^2 is determined as before, over a grid of c, α values and the number of times the value of χ^2 exceeds a corresponding critical value is counted for each α, c(for each security). The results are tabulated.

Daily Price Change Adjusted
for Volume of Trading

	Levels of Confidence			
Characteristic Exponents	.05	.025	.01	.005
1.69[b]	157.	127.	102.	89.
1.50	255.	232.	200.	173.
1.60	202.	167.	135.	114.
1.70	144.	119.	102.	90.
1.80	163.	138.	115.	107.
1.90	197.	185.	169.	146.
2.00	235.	224.	213.	204.

a. The average of the estimates of α for price change of 315 securities (NYSE).

b. The average of the estimates of α for adjusted price change of 315 securities (NYSE).

c. $P(\chi^2 > 31.3 \mid \varphi = 14) = .005$, φ = degrees of freedom.

Clearly the results are not an improvement over those reported in the text.

[15] Mandelbrot and Taylor [16] discuss the properties of a stable non-normal subordinated model. From Feller [10], it can be shown that if $d(\Delta t)$ follows a positive symmetric stable distribution with bounded $0 < \alpha_1 < 1$, the unconditional distribution $d(\Delta t)$ will be symmetric stable distribution with α_2 equal to $2\alpha_1 = \alpha_2 < 2$. The importance of the Mandelbrot and Taylor analysis is to show that under one set of assumptions the non-normal stable Paretian model is deriveable from a subordinated model. Thus, the possibility exists that there is no inconsistency between the subordinated model and the stable Paretian model. However, the Clark subordinated model is logically inconsistent with the non-normal stable Paretian model.

[16] There is a potential problem in interpreting the results of these stability tests since Fama and Roll [9] presented evidence that estimates of α are biased downward and the bias is greater with small sample sizes, Hence if estimates of α are corrected for bias there could be a more pronounced upward trend in $\hat{\alpha}$ as sum size increases and sample size decreases. Fama and Roll [9] show that if α is assumed to be equal to 1.5, the downward bias in $\hat{\alpha}$, estimated in the fractile range .95 to .97 is .01 for sample size 99, and .09 for sample size 49. In the present study the sample size is equal 411 for sum size equal to 1 and 46 for sum sizes equal to 9. Thus adjusting for the downward bias could produce $\hat{}$values as much as .09 greater than otherwise. Clearly the upward drift in $\hat{\alpha}$ may be significantly understated if the bias is not properly incorporated in the analysis. Fama and Roll's estimating procedure utilizes estimates of fractile values for standardized price change variates. Also, Fama and Roll's results indicate that $\hat{\alpha}$ should be the same at each fractile; thus any systematice change from fractile to fractile would indicate wither a poor estimator (large sample error) or non-symmetric stable underlying empirical distributions. For example, it is possible that $\hat{\alpha}$ may be a function of the fractile selected. This type of systematic variation in estimates of $\hat{\alpha}$ as the feactiles change would reflect unfavorably om the hypothesis that sample data are characterized by symmetric, stable non-normal distribution. The results are tabulated in Table I'.

The pattern of these estimates casts some doubt on the adequacy of the symmetric stable model for describing the distribution of price changes of common stocks in the sample. The results may mean there is enough skewness to effect the Fama-Roll estimator of α which was developed assuming symmetry. It is also possible that the shapes of the subordinated distribution are different than the symmetric stable distributions, although they both allow for fat tails.

1' ESTIMATES OF THE CHARACTERISTIC EXPONENT AT DIFFERENT FRACTILES

Parameter or fractile	F90[a]	F94	F95	F96	F97
average	1.51	1.55	1.58	1.60	1.64
F25	1.39[b]	1.46	1.49	1.53	1.57
median	1.51	1.54	1.57	1.61	1.65
F75	1.64	1.66	1.67	1.70	1.73

[a]The .90 order statistic used to estimate α.

[b]The .25 fractile of the distribution of α for 315 securities (NYSE).

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