The Arbitrage Theory of Capital Asset Pricing

by

Stephen A. Ross*

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RODNEY L. WHITE CENTER FOR FINANCIAL RESEARCH

University of Pennsylvania

The Wharton School

Philadelphia, Pennsylvania 19174

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The purpose of this paper is to rigorously examine the arbitrage model of capital asset pricing developed in Ross [1971, 1972]. The arbitrage model was proposed as an alternative to the mean variance capital asset pricing model, introduced by Treynor, Sharpe and Lintner, that has become the major analytic tool for explaining phenomena observed in capital markets for risky assets. The principal relation that emerges from the mean variance model holds that for any asset, i, its (ex ante) expected return

$$E_i = \rho + \lambda b_i, \tag{1}$$

where ρ is the riskless rate of interest (and even if a riskless asset does not exist, ρ is the zero-beta return on all portfolios uncorrelated with the market portfolio)[1], λ is the expected return on the market, E_m, minus ρ, and

$$b_i \equiv \frac{\sigma_{im}^2}{\sigma_m^2},$$

is the beta coefficient on the market, where σ_m^2 is the variance of the market portfolio and σ_{im}^2 is the covariance between the i^{th} asset and the market portfolio.

The linear relation in (1) basically arises from the mean variance efficiency of the market portfolio, but on theoretical grounds it is difficult to justify either the assumption of Normality in returns or of quadratic preferences to guarantee such efficiency, and on empirical grounds not only the assumptions but the conclusions of the theory have also come under attack.[2] The restrictiveness of the assumptions that underly the mean variance model have, however, long been recognized, but its tractibility and the evident appeal of the linear relation between return, E_i, and risk, b_i, embodied in (1) have ensured its popularity. An alternative and in many ways more satisfactory theory of the pricing of risky assets that retains many of the intuitive results of the original theory was developed in Ross [1971] and [1972].

In its barest essentials a simple case of the argument presented there was as follows. Suppose that the random returns on a subset of assets can be expressed by a simple factor model

$$\tilde{x}_i = E_i + \beta_i\tilde{\delta} + \tilde{\varepsilon}_i \ , \tag{2}$$

where $\tilde{\delta}$ is a mean zero common factor, and $\tilde{\varepsilon}_i$ is mean zero with the vector $\langle\tilde{\varepsilon}\rangle$ sufficiently independent to permit the law of large numbers to hold. Neglecting the noise term, $\tilde{\varepsilon}_i$, as discussed in Ross [1972] (2) is a statement that the state space tableau of asset returns lies in a two dimensional space that can be spanned by a vector with elements δ_θ, (where θ denotes the world-state) and the constant vector, $e \equiv \langle 1, \ldots, 1\rangle$. Consider forming an arbitrage portfolio, η, of all the n assets where $\eta e = o$, i.e., the arbitrage portolio uses no wealth. If n is large and η is well diversified, with obvious vector notation, we have by the law of large numbers that

$$\begin{aligned} \eta\tilde{x} &= \eta E + (\eta\beta)\tilde{\delta} + \eta\varepsilon \\ &\approx \eta E + (\eta\beta)\tilde{\delta}, \end{aligned} \tag{3}$$

(where approximation is in quadratic mean) and setting $\eta\beta = o$ yields

$$\eta\tilde{x} = \eta E \ .$$

Using no wealth, the random return $\eta\tilde{x}$ has now been engineered to be equivalent to a certain return, ηE, and to prevent arbitarily large disequilibrium positions we must have $\eta E = o$. This can only occur for all η such that $\eta e = \eta\beta = o$ if E is spanned by e and β or

$$E_i = \rho + \lambda\beta_i \tag{4}$$

for constants ρ and λ. Clearly if there is a riskless asset ρ is its rate of return and, even if there isn't such an asset, ρ is the rate of return on all zero-beta portfolios, α, i.e. portfolios with $\alpha e = 1$ and $\alpha\beta = 0$. Letting α be a portfolio of particular interest, e.g. the market portfolio, α_m, with $E_m = \alpha_m E$, (4) becomes

$$E_i = \rho + (E_m - \rho)\beta_i \ . \tag{5}$$

Condition (5) is the arbitrage theory equivalent of (1) and if $\tilde{\delta}$ is a market factor return then β_i will approximate b_i. The above approach, however, is substantially distinct from the usual mean-variance analysis and constitutes a related but separate theory. For one thing, the argument suggests that (5) holds not only in equilibrium situations, but in all but the most profound sort of disequilibria. For another, the market portfolio plays no special role.

There are, however, a number of weak points in the heuristic argument. It is not clear, for example, that for any given n, the negation of (5) will be inappropriate since the increase in risk aversion may offset the decline in risk of $\eta\tilde{\varepsilon}$. In addition, it is not a priori certain that the disequilibrium position of one agent will not be offset by the disequilibrium position of another.[3]

In Ross [1971], however, it was shown that if (5) holds then it represents an ε or quasi-equilibrium. The intent of this paper is to supply the rigorous analysis underlying the stronger arguments above. We will begin, in Section I, by using a special example to elucidate the impact of expectational mechanisms and the number of assets on wealth in a case where the arbitrage argument fails to hold. In Section II we will present some weak sufficient conditions to rule out such pathological examples and will prove a general version of the arbitrage result. A mathematical appendix contains some supportive results of a somewhat technical and tangential nature. Section III will briefly summarize the paper and suggest further generalizations.

I. A Counterexample

Surprisingly, it is not always the case that adding assets increases wealth. The impact upon wealth hinges on both the expectational schemes that agents are assumed to use and on their preferences (although it is possible to state some preference free results). However, if wealth changes then so may risk aversion and the law of large numbers argument outlined above may fail. Consider the following example.

Suppose that there is a riskless asset and that risky assets are independently and Normally distributed as

$$\tilde{x}_i = E_i + \tilde{\varepsilon}_i \ , \tag{6}$$

where

$$E\{\tilde{\varepsilon}_i\} = o$$

and

$$E\{\tilde{\varepsilon}_i^{\,2}\} = \sigma^2$$

The arbitrage argument would imply that in equilibrium all of the independent risk would disappear and, therefore,

$$E_i \approx \rho \ . \tag{7}$$

Assume, however, that the market consists of a single agent with a constant absolute risk aversion utility function of the form

$$U(z) = -\,e^{-Az} \ . \tag{8}$$

Letting α denote the portfolio and taking expectations we have

$$E\{u[w(\rho + \alpha[\tilde{x} - \rho\cdot e])]$$

$$= - e^{-Aw\rho} E\{e^{-Aw\alpha[\tilde{x} - \rho\cdot e]}\}$$

$$= - e^{-Aw\rho}\{e^{-(Aw)\alpha(E - \rho\cdot e) + \frac{\sigma^2}{2}(Aw)^2(\alpha'\alpha)}\} \tag{9}$$

and the first-order conditions at a maximum are given by

$$\sigma^2(Aw)\alpha_i = E_i - \rho. \tag{10}$$

If there is no net supply of the riskless asset the budget constraint (Walras' Law for the market) becomes

$$\sum_i \alpha_i = \frac{1}{Aw\sigma^2}\sum_{i=1}^{n}(E_i - \rho) = 1 , \tag{11}$$

and solving for wealth yields

$$w = \frac{1}{A\sigma^2}\sum_{i=1}^{n}(E_i - \rho). \tag{12}$$

Since α is the market portfolio, in equilibrium we must also have $\alpha_i > o$ for all i and, therefore, $E_i - \rho > o$ for all i.

Notice that knowledge of the stocks of assets would enable us to determine prices in a proportionate manner, but would give us no additional information on expectations in this framework. Suppose, now, that E_i alternates from $\rho + 1$ to $\rho + 2$ as $i \to \infty$. Clearly the arbitrage condition (7) is violated (for any choice of ρ). From (12), wealth diverges with the increase in n and the increase in wealth increases the agent's relative risk aversion, Aw, as rapidly as the law of large numbers diminishes the variance of the noise terms. The two effects just cancel out and there is no need for (7) to hold to prevent arbitrage.

In the next section we will develop assumptions sufficient to rule out this pathology.

II. The Arbitrage Theory

The difficulty with the constant absolute risk aversion example arises because the coefficient of relative risk aversion increases with wealth. This suggests considering risk averse agents for whom the coefficient of relative risk aversion

$$\sup_x \{- \frac{U''(x)x}{U'(x)}\} \leq R < \infty \quad , \tag{13}$$

i.e. agents with a uniformly bounded coefficient of relative risk aversion. We will refer to such agents as being of Type B (for bounded).

Pratt has shown that given a Type B utility function there exists a monotone increasing convex function, $G(\cdot)$, such that

$$U(x) = G[U(x;\, R)] \quad , \tag{14}$$

where $U(x;\, R)$ is the utility function with constant relative risk aversion, R. It is well known that

$$U\,(x;\, R) = \begin{cases} \dfrac{x^{1-R}}{1-R} \text{ if } R \neq 1 \\ \\ \log x \text{ if } R = 1 \,. \end{cases} \tag{15}$$

Essentially, then, Type B agents are uniformly less risk averse than some constant relative risk averse agent.

Assume that the returns on the particular subset of assets under consideration are subjectively viewed by agents in the market as being generated by a model of the form

$$\begin{aligned} \tilde{x}_i &= E_i + \beta_{i1}\,\tilde{\delta}_1 + \dots + \beta_{ik}\tilde{\delta}_k + \tilde{\varepsilon}_i \;, \\ &\equiv E_i + \beta_i\tilde{\delta} + \tilde{\varepsilon}_i \;, \end{aligned} \tag{16}$$

where

$$E\{\tilde{\delta}_\ell\} = E\{\tilde{\varepsilon}_i\} = o$$

and, where the $\tilde{\varepsilon}_i$'s are mutually stochastically uncorrelated. We will impose no further restrictions on the form of the multivariate distribution of $(\tilde{\delta}, \tilde{\varepsilon})$ beyond the requirement that $(\exists \ \sigma < \infty)$

$$\sigma_i^2 \equiv E\{\tilde{\varepsilon}_i^{\,2}\} \leq \sigma^2 . \tag{17}$$

In particular, then, the $\tilde{\delta}_i$ need not be jointly independent or even independent of the $\tilde{\varepsilon}_i$'s, they need not possess variances, and none of the random variables need be normally distributed.

A point on notation is also needed. In what follows α^o will denote an n-element optimal portfolio for the agent under consideration and statements of the form $\alpha^o E$ are to be interpreted as statements applicable uniformly in n. The vector β^ℓ will be the column vector $\langle \beta_{1\ell}, \ldots, \beta_{n\ell}\rangle'$ and β_i, as above, denotes the row vector $\langle \beta_{i1}, \ldots, \beta_{ik}\rangle$. The single letter β will denote the matrix $[\beta^1 \vdots \cdots \vdots \beta^k]$.

<u>Assumption 1</u> (Liability limitations) There exists at least one asset with limited liability in the sense that there is some bound (per unit invested) to the losses for which an agent is liable.

Assumption 1 is satisfied in the real world by a wide variety of assets. We can now prove a key result about Type B agents.

<u>Theorem I</u> Consider a Type B agent who lives in a world that satisfies Assumption 1 and who believes that returns are generated by a model of the form of (16). If $(\exists m < \infty)$ such that

$$\alpha^o E \leq m, \tag{18}$$

then $(\exists \rho, \gamma)$ such that

$$\sum_{i=1}^{\infty} [E_i - \rho - \beta_i\gamma]^2 < \infty . \tag{19}$$

Proof The result is independent of the particular wealth sequence $<w_n>$ and we must prove it for arbitrary sequences. Assume that $R \neq 1$.

From (18), concavity and monotonicity

$$E\{U[w_n\alpha^o\tilde{x}]\}$$

$$\leq U[w_n\alpha^o E]$$

$$\leq U[w_n m]$$

$$= G[(w^n)^{1-R}\, U(m;\, R)].$$

Now consider forming an arbitrage portfolio sequence that solves the associated quadratic problem

$$\min\ \eta V \eta\ ,$$

subject to

$$\eta e = o$$

$$\eta \beta^{\ell} = o\ ;\ \ell = 1, \ldots, k\ , \tag{20}$$

and

$$\eta E = c > m + t$$

where V is the covariance matrix of $<\tilde{\varepsilon}>$ and where t is the maximum liability loss associated with a unit investment in a limited liability asset. Assumption 1 guarantees that t is bounded.

If the constraints are unsolvable for all n, then E must be linearly dependent on e and the columns of β and we are done. (This follows since for $n > k + 1$, the system

$$\eta' [e \vdots \beta] = o$$

must have non-trivial solutions.) Suppose then, that the constraints are solvable for all n sufficiently large and, without loss of generality, let

$$x \equiv [E \vdots \beta \vdots e]$$

be of full rank. We will assume that if a sequence of random variables converges to a degenerate law (a constant) in quadratic mean, then the expected utility also converges. An examination of this point is deferred to an appendix. It follows that there must not be any subsequence on which

$$\eta' V \eta \to o \ ,$$

for then

$$E\{U(\eta \tilde{x};\ R)\} \to U(c;\ R) > U(m;\ R) \ ,$$

and, therefore, by convexity of $G(\cdot)$ there exists n such that

$$\begin{aligned} E\{U[w_n \eta \tilde{x}]\} &\equiv E\{G[(w^n)^{1-R} U(\eta \tilde{x};\ R)]\} \\ &\geq G[(w^n)^{1-R} E\{U(\eta \tilde{x};\ R)\}] \\ &> G[(w^n)^{1-R} U(m;\ R)] \ , \end{aligned}$$

violating optimality. Hence $(\exists a > o)$ such that $(\forall n)$

$$\eta' V \eta \geq a > o \ .$$

Solving (19) we have

$$V\eta = X\lambda$$

where λ is a $(k+2)$ - vector of multipliers, and applying the constraints of (20) yields

$$[X'V^{-1}X]\lambda = \begin{bmatrix} c \\ 0 \end{bmatrix} .$$

It now follows that

$$\begin{aligned} \eta' V \eta &= \lambda' \begin{bmatrix} c \\ 0 \end{bmatrix} \\ &= [c,\ 0]\,[X'V^{-1}X]^{-1} \begin{bmatrix} c \\ 0 \end{bmatrix} \\ &\geq a > o . \end{aligned}$$

Defining $b' \equiv (c, 0)$ we can apply Lemma I in the appendix to obtain the existence of a^* and $A < \infty$ such that for all n

$$(Xa^*)'(Xa^*) \leq A < \infty ,$$

where

$$a^{*\prime} b = c a^*_1 = 1$$

or

$$a^*_1 = 1/c .$$

Multiplying a^* by c proves the result.

If $R = 1$, wealth can be factored out of the utility function additively and the proof is nearly identical.

Q.E.D.

Theorem I asserts that for a Type B individual, if the optimal expected return is uniformly bounded, then it must be the case that the arbitrage condition

$$
\begin{aligned}
E_i &\approx \rho + \beta_i\gamma \\
&= \rho + \gamma_1\beta_{i1} + \ldots + \gamma_k\beta_{ik} ,
\end{aligned}
\tag{21}
$$

holds in the approximate sense that the sum of squared deviations is uniformly bounded. This implies, among other things, that as n increases

$$
|E_n - \rho - \beta_n\gamma| \to o . \tag{22}
$$

A number of simple corollaries of Theorem I are available. It is easy to see, for example, that if wealth is confined to a compact interval on which the utility function is bounded, then Theorem I will hold for any risk averse agent. More importantly, we have the following corollary.

Corollary I Under the conditions of Theorem I if there is a riskless asset then ρ may be taken to be its rate of return.

Proof The return per unit of wealth in the presence of a riskless asset is given by

$$
\rho + \alpha\,(\tilde{x} - \rho) ,
$$

where α is now the portfolio of risky assets. Deleting the constraint that $\eta e = o$ we can simply repeat the proof of Theorem I with $(E - \rho e)$ in the place of the E vector.

Q.E.D.

To turn these results into a capital market theory we will assume that there is at least one Type B individual who doesn't become negligable as the number of assets, n, is increased. The following definition is helpful.

Definition The agent, a^ν, will be said to be asymptotically negligable if, as the number of assets increases,

$$\omega^\nu \equiv \frac{w^\nu}{w} \to 0 ,$$

where w^ν is the agent's wealth and w is total wealth, i.e.,

$$w \equiv \sum_\nu w^\nu .$$

For example, an agent will not be asymptotically negligable if the sequence of proportionate quantities of assets the agent is endowed with is bounded away from zero.

Assumption 2 (Non negligability of Type B agents) There exists at least one Type B agent who believes that returns are generated by a model of the form of (16) and who is not asymptotically negligable.

To permit us to aggregate to a market relation we will make three more assumptions; essentially we must insure that Theorem 2 will not be "undone" by the rest of the economy. First we specify the generating model, (16), a bit more.

Assumption 3 (Boundedness of expectations) The sequence, $\langle E_i \rangle$ is uniformly bounded, i.e.

$$||E|| \equiv \sup_i |E_i| < \infty . \tag{23}$$

Assumption 3 will be discussed below.

Assumption 4 (Extent of disequilibria) Let ξ_i denote the aggregate demand for the ith asset as a fraction of total wealth. We will assume that only situations with $\xi_i \geq o$ are to be considered.

Notice that Assumption 4 does not rule out the possibility that an asset can be in excess supply; it only implies that the economy as a whole will wish to hold some of it. Assumptions 3 and 4 can be weakened considerably as will be shown below, but for purposes of demonstration we have chosen to leave them in a stronger than necessary form.

Lastly, we need to assume that agents hold compatible subjective beliefs.

Assumption 5 (Homogeneity of expectations) All agents hold the same expectations, E.

We can now prove our central result.

Theorem II Given Assumptions 1 through 5, $(\exists \rho, \gamma)$

$$\sum_{i=1}^{\infty} \{E_i - \rho - \beta_i \gamma\}^2 < \infty \quad . \tag{19}$$

Furthermore, if there is a riskless asset, then ρ is its rate of return.[4]

Proof From Theorem I we know that if the conclusion is false then for the Type B agent (on a subsequence)

$$\sum_i \alpha_i^o E_i \to \infty . \tag{24}$$

Let the total fraction of wealth held by the Type B agent be given by ω^o and by the rest of the economy by ω^1. If α_i^1 denotes the fraction of ω^1 held in asset i by the rest of the economy then $(\forall i)$ we must have

$$\sum_i \xi_i = 1 \ ,$$

and by Assumption 4

$$\xi_i \equiv \omega^o \alpha^o{}_i + \omega^1 \alpha^1{}_i \geq o .$$

Thus,

$$||E|| \geq \xi_i E_i$$

$$= \sum_i (\omega^o \alpha^o{}_i + \omega^1 \alpha^1{}_i) E_i$$

$$= \omega^o \sum_i \alpha^o{}_i E_i + \omega^1 \sum_i \alpha^1{}_i E_i .$$

From (23) and Assumption 2 the first sum is divergent which together with Assumption 3 implies that

$$\omega^1 \sum_i \alpha_i^1 E_i \to -\infty .$$

Since

$$\omega^1 \alpha^1{}_i \equiv \sum_{\nu \neq o} \omega^\nu \alpha^\nu{}_i ,$$

it follows that

$$\omega^1 \sum_i \alpha^1{}_i E_i = \sum_i \sum_{\nu \neq o} w^\nu \alpha_i{}^\nu E_i$$

$$= \sum_{\nu \neq o} \{\sum_i \omega^\nu \alpha^\nu{}_i E_i\} ,$$

and for some agent, a^ν,

$$\sum_i \omega^\nu \alpha^\nu{}_i E_i \to -\infty ,$$

on a subsequence. By Assumptions 1 and 5 this contradicts optimality.

The identification of ρ with the riskless return follows from Corollary 1.

Q.E.D.

As was shown in Ross [1972] the basic result of Theorem 2 can be written in a number of empirically interesting and intuitively appealing formats. For example, by appropriate normalization it can be shown that

$$E_i - \rho \approx \beta_{i1}(E^1 - \rho) + \dots + \beta_{ik}(E^k - \rho) \ , \tag{25}$$

where E^ℓ is the return on all portfolios with $\alpha\beta^s = o$ for $s \neq \ell$ and $\alpha\beta^\ell = 1$. The constant ρ is now the return on all $\alpha\beta = o$, i.e., zero-beta portfolios. Thus, the risk premium on an asset is the β-weighted sum of the factor risk premiums.

III. Generalizations and Conclusions

One of the strengths of Theorem 2 is that it does not require the stringent homogeneity of anticipations of the mean-variance theory. We are distinguishing, now, between expectations, i.e. the E vector, and anticipations, the whole model (16). If other agents have the same ex ante expectations, but believe returns are generated in a different fashion, then (25) must still hold where β is the β of the return generating model believed to hold by the Type B agent. Of course, this is a bit gratuitous since in this model as in all others it is necessary to translate the results into observable quantities and the usual ex ante-ex post identity becomes ambiguous with disparate anticipations. Even if all agents agree on (16), however, there is still considerable scope for disagreement on the underlying probability distributions. For example, if $\tilde{\delta}$, represents a market or "GNP" factor then as long as all agents agree on the impact of this factor on returns, through β_{i1}, they can hold a variety of views on the distribution of $\tilde{\delta}$ without violating the basic arbitrage condition, (25). Similarly, agents can also disagree on the distribution of the idiosyncratic noise terms, $\tilde{\varepsilon}_i$, without altering (25). In an important sense, if we are willing to accept the factor model of (16) then much of the controversy over the exact distribution of returns is irrelevant for the derivation of the capital market arbitrage condition.

The primary difficulty with the analysis arises when agents differ in their expectations, E^{ν}. Now the proof of Theorem 2 must be modified since, unless all E^{ν} vectors are positive multiples of the same vector, we cannot be assured that the divergence of $\alpha^{\nu}E^{\tau}$ to $-\infty$ for $\tau \neq \nu$, implies that $\alpha^{\nu}E^{\nu} \to -\infty$. This is a fruitful area for generalizations.

A number of other generalizations are also worth exploring. For example, it can be shown that the assumption of independence among the factors and between the factors and the noise terms has further empirical implications (see Ross [1972].

An example of a condition that should be weakened is the boundedness restriction on variances. It seems likely that this can be accomplished without altering the results. This would reinforce the distribution-free nature of the arguments.

Finally, it should be emphasized that (25) is indeed much more of an arbitrage relation than an equilibrium condition and may be expected to be quite robust. Assumptions 3 and 4 served only to guarantee that the market return,

$$E_m \equiv \sum_i \xi_i E_i \ , \tag{26}$$

would be uniformly bounded and this will hold in a wide class of disequilibrium situations. Rather then simply assuming that E_m was bounded, we chose to make Assumptions 3 and 4 directly to see how sufficient conditions for a bounded E_m would appear in alternative economic situations. For example, Assumption 4 can be weakened if instead of requiring all $\xi_i \geq o$, we assumed that $\sum_i |\xi_i|$ was bounded, i.e., we bounded the sum of the absolute proportions of wealth placed (or shorted) in all assets. This would also be sufficient to bound the market return. In practice, these are very weak conditions and easily satisfied.[5]

In conclusion, we have set forth a rigorous basis for the arbitrage relation and arguments discussed in Ross [1972] (and [1971]), and in doing so we have explored their precise underpinnings. The conditions which are sufficient to support the theory are both intuitive and reasonable. On a less optimistic note, though, while significantly weakening the assumption that investors have identical (or homogeneous) anticipations, the arbitrage theory still requires essentially identical expectations and agreement on the β coefficients if the identification of ex ante beliefs with ex post realizations is to provide empirically fruitful results. If this assumption is to be fundamentally weakened, this theory (and all others) will require a closer examination of the dynamics by which ex ante beliefs are transformed into ex post observations. Such a study properly lies in the domain of general disequilibrium dynamics and the study of the impact of information on markets. It is one of the most difficult, important and exciting areas of future research.

Appendix 1

In this appendix we prove the lemma referred to in the proofs of the paper. Define a monotone k-sequence of matrices, $\langle X^n \rangle$, to be the sequence of matrices formed by taking the first row, the first two rows, and so on of an infinite matrix with k columns.

Lemma I: Let $\langle X^n \rangle$ be a monotone k-sequence of matrices and let $\langle H^n \rangle$ be a sequence of diagonal matrices with diagonal elements $\langle h_1 \rangle$, $\langle h_1, h_2 \rangle$, and so on where, for some $h, h_i \geq h > o$ for all i. Assume $(\exists b, m)$ ($\forall X^n$ of full rank)

$$b' [X^{n'} H^n X^n]^{-1} b \geq m > o \quad . \tag{A1}$$

It follows that $(\exists a^*$ and m) ($\forall n$ such that X^n is of full rank)

$$(X^n a^*)' (X^n a^*) \leq m < \infty$$

and

$$a^{*'} b = 1 \; .$$

Proof: The result is trivial if X^n is of less than full rank for all n. In addition, if X^n is of full rank for some n ($\geq$ k) then $X^{\hat{n}}$ is of full rank, $\hat{n} > n$, and we may assume that the sequence $\langle X^n \rangle$ ($n \geq k$) is of full rank for all n. By positive definiteness $X^{n'} H^n X^n$ is of full rank and (A1) holds.

Consider the problem:

$$\min \; (X^n Z^n)' \; H^n (X^n Z^n) \; ,$$

subject to

$$Z^{n'} b = 1 \; .$$

The solution is given by

$$Z^n = \gamma[X^{n'}H^nX^n]^{-1}b \ ,$$

where

$$\gamma = (X^nZ^n)' \, H^n(X^nZ^n)$$

$$= (b'[X^{n'}H^nX^n]^{-1}b)^{-1}$$

$$\leq \frac{1}{m} < \infty \ ,$$

by (A1). Consequently, from the lower bound on $\langle h_i \rangle$ we now obtain

$$(X^nZ^n)' \, (X^nZ^n) \leq m \equiv \frac{1}{hm} < \infty \ .$$

Letting $y^n \equiv X^nZ^n$ implies that $y^{n'}y^n \leq m$. If X is a full rank submatrix of X^n then

$$XZ^n = y^n|X \ ,$$

where $y^n|X$ is the corresponding subvector of y^n, and since $y^n|X$ is bounded in the norm it has a convergent subsequence. Letting y^* be its limit we must have $Z^n \to Z^* \equiv X^{-1}y^*$ on the subsequence. It remains to show that $(\forall n)\ (X^nZ^*)'(X^nZ^*) \leq m$. Assume to the contrary that for some $\hat{n}$ (and, therefore, all $n > \hat{n}$)

$$(X^{\hat{n}}Z^*)' \, (X^{\hat{n}}Z^*) > m \ .$$

However, since $Z^n \to Z^*$ on a subsequence this would imply

$$(X^nZ^n)' \, (X^nZ^n) \geq (X^{\hat{n}}Z^n)' \, (X^{\hat{n}}Z^n) > m \text{ for some } n.$$

It follows that $(\forall n)\ (X^nZ^*)' \, (X^nZ^*) \leq m$. In addition, since $Z^{n'}b = 1$ for all n we must also have $Z^{*'}b = 1$.

Q.E.D.

Appendix 2

In this appendix we discuss the relationship between convergence in quadratic mean (q.m.) and expected utility. The technical results can be found in Loeve and Billingsley.

We can begin with a simple but powerfull result. Let $<\tilde{X}_n>$ be a sequence of random variables with $E\{\tilde{X}_n\} = o$, and $\tilde{X}_n \to o$ (q.m.), i.e. $\sigma^2(\tilde{X}_n) \to o$.

<u>Proposition</u>: If $U(\cdot)$ is concave and bounded below (which implies that the domain of $U(\cdot)$ is left bounded), then

$$E\{U[\rho + \tilde{X}_n]\} \to U(\rho) \ .$$

<u>Proof</u>: By Fatou's lemma

$$\lim\inf E\{U[\rho + \tilde{X}_n]\} \geq U(\rho) \ ,$$

but by concavity

$$E\{U[\rho + \tilde{X}_n]\} \leq U(\rho) \ .$$

$$\therefore \quad \lim\sup E\{U[\rho + \tilde{X}_n]\} \leq U(\rho)$$

$$\leq \lim\inf E\{U[\rho + \tilde{X}_n]\}$$

$$\Rightarrow \quad \lim E\{U[\rho + \tilde{X}_n]\} = U(\rho) \ .$$

<u>Q.E.D.</u>

The problem arises when $U(\cdot)$ is unbounded from below. About the weakest condition which assures convergence is uniform integrability (U.I.):

$$\lim_{\alpha\to\infty} \sup_n \int_{\Omega_\alpha} |U(\rho + X_n)| d\eta_n = 0$$

$$\Omega_\alpha \equiv \{|U(\rho + X_n)| \geq \alpha\}$$

$$\Rightarrow \quad E\{U(\rho + X_n)\} \to U(\rho) \ .$$

A number of familiar conditions imply U.I.. If the sequence $U(\rho + \tilde{X}_n)$ is bounded below by an integrable function the Lebesque convergence theorem can be invoked or if $(\exists\delta > 0)$

$$\sup_n E\{|U(\rho + X_n)|^{1+\delta}\} < \infty \ ,$$

then the sequence is U.I..

In general, then, the convergence criterion will depend on both the utility function and the random variables. It is possible, however, to find weak sufficient conditions on the random variables alone, by taking advantage of the structure of $\tilde{X}_n$, but the condition that $\tilde{X}_n = \frac{1}{n}\sum_i \tilde{\varepsilon}_i$; σ^2_i uniformly bounded and $\tilde{\varepsilon}_i$, $\tilde{\varepsilon}_j$ independent is not sufficient.[6]

In the text, it is assumed that all sequences satisfy the U.I. condition, and therefore

$$\tilde{X}_n \to a \quad (\text{q.m.})$$

will imply that

$$E\{U(\tilde{X}_n)\} \to U(a) \ .$$

Footnotes

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[1] See Black for an analysis of the mean variance model in the absence of a riskless asset.

[2] See Blume and Friend for a recent example of some of the empirical difficulties faced by the mean variance model. For a good review of the theoretical and empirical literature on the mean variance model see Jensen.

[3] Green has considered this point in a temporary equilibrium model. Essentially he argues that if subjective anticipations differ too much, then arbitrage possibilities will threaten the existence of equilibrium.

[4] Theorems I and II and Corollary I can be extended to the case where (16) holds for only a subset of the assets by generalizing the utility function to be a Lebesque dominated sequence of functions conditional on the other assets.

[5] A strong form of Theorem 2 can be obtained by assuming that the weighted sum of subjectively viewed expected portfolio returns

$$\sum_{\nu} \omega^{\nu} \sum_{i} \alpha_i^{\nu} E_i^{\nu} \qquad \text{(F1)}$$

is uniformly bounded. This would permit us to delete Assumptions 3, 4 and even 5 and, formally at least, would allow heterogeneous expectations. Alternatively, we could replace Assumption 3 with $||E^{\nu}|| < \infty$, retain Assumption 4 (or the weaker form described in Section III) and drop Assumption 5.

Furthermore, if agents agree on factors, if the actual ex post model generating returns is some convex combination (say wealth weighted, or, for that matter, any uniformly sup norm bounded linear operator) of the individual market ex ante models then the basic arbitrage condition will be expressible in ex post observables and, as such, will be directly testable. See Ross (1972) for a fuller discussion of these issues. None of this, however, is very satisfactory. For one thing, it is not clear what is the force of these boundedness conditions, particularly when the number of agents is typically much larger than the number of marketed assets. As an example, if we have two Type B agents with exactly divergent beliefs (in a sense which can be made precise in special examples) then they can exactly offset each other. There is now no reason to expect (F1), unlike (26), to be bounded simply because observed ex post return is bounded. For another, we must translate the theory into a statement about observables and this requires relating divergent subjective ex ante expectations to ex post ones via the "right" generating mechanism in a less ad hoc fashion. This is the problem posed in Section III and makes the "strong" version of Theorem II inadequate to stand alone.

[6] It is not difficult to construct counterexamples by having $U(\cdot)$ go to $-\infty$ rapidly enough as x approaches its lower bound (or $-\infty$).

Bibliography

Billingsley, Patrick. Convergence of Probability Measures. John Wiley, New York, 1968.

Black, Fischer. "Capital Market Equilibrium with Restricted Borrowing." Journal of Business (forthcoming).

Blume, Marshall and Friend, Irwin. "A New Look at the Capital Asset Pricing Model." Journal of Finance, forthcoming, March 1973.

Green, Jerry. "Temporary Equilibrium in a Sequential Trading Model with Spot and Future Transactions." Forthcoming in Proceedings of the Third NSF-NBER Conference on Decision Rules and Uncertainty, North-Holland, 1973.

Jensen, Michael. "Capital Markets: Theory and Evidence." Bell Journal of Economics and Management Science, 3 (1972), 357-398.

Lintner, John. "The Valuation of Risk Assets and the Selection of Risky Investments in Stock Portfolios and Capital Budgets." Review of Economics and Statistics, February, 1965, pp. 30-55.

Loeve, Michael. Probability Theory. Princeton, New Jersey: D. Van Nostrand Company, Inc., 1963.

Pratt, John. "Risk Aversion in the Small and in the Large." Econometrica, 32 (1964), 122-137.

Ross, Stephen. "Portfolio and Capital Market Theory with Arbitrary Preferences and Distributions - The General Validity of the Mean-Variance Approach in Large Markets." Working Paper No. 12-72, Rodney L. White Center for Financial Research, 1971.

Ross, Stephen. "Return, Risk and Arbitrage." Unpublished manuscript, 1972.

Sharpe, William. "Capital Asset Prices: A Theory of Market Equilibrium under Conditions of Risk." Journal of Finance, September, 425-442.

Treynor, Jack. "Toward a Theory of Market Value of Risky Assets." Unpublished manuscript, 1961.