Investment for the Long Run[1]

by

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1. Summary

Kelly (1956) and others, e.g., Latané (1957) (1959), Markowitz (1959, chapter 6) and Brieman (1960) (1961), have asserted that in selecting among probability distributions of return this period, the investor who continually reinvests for the long run should maximize the expected value of the logarithm of increase in wealth. Mossin (1968) and Samuelson (1963) (1969), on the other hand, have presented examples of games in which the investor reinvests continually for the long run, has any of a wide range of apparently plausible utility functions, yet definitely should not follow the aforementioned "expected log" rule.

The argument of Kelly et al. is that, under the conditions considered, the investor who follows the expected log rule is

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[1] This article owes much to Paul A. Samuelson and to my wife Barbara. After several discussions with Professor Samuelson on the subject of this paper, I found myself neither convinced by his position nor happy with the rigor of my own. This article was written so that I could see the argument myself, show it to Professor Samuelson, and share it with whomever else was interested. Dr. Barbara Markowitz read, typed and critiqued several drafts.

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almost sure to have a greater wealth in the long run than an investor who follows a distinctly different policy. We illustrate this argument in section 3. On the other hand, Mossin and Samuelson argue that, for a wide range of plausible utility functions, the expected utility of the game as a whole provided by the expected log rule does not approach the expected utility provided by the optimum strategy, no matter how long the game is played. This argument is also illustrated in section 3.

Given my beliefs concerning expected utility, which remain as in Markowitz (1959) chapters 10 through 13, if it were in fact the case that the expected log rule did not provide asymptotically almost optimal expected utility for a wide range of utility functions, then I would have to reject the expected log rule as a general solution to the problem of reinvesting for the long run. The conclusion of the present paper, however, is that utility analysis does not refute the expected log rule. Rather it confirms the rule, and provides a more satisfactory theoretical justification for it than has been available heretofore.

As discussed in section 4, for a given game with a fixed length T, we can define the outcome of the game in terms of either the ending wealth, W_T, or the rate of return, g, achieved during the game. It makes no difference, for fixed T, whether we express utility as a function $U(W_T)$ or as $V(g)$. But as T increases it is not equivalent to assume that $U(W_T)$

remains the same for all T as to assume that V(g) remains the same for all T. Mossin and Samuelson in effect assumed that $U(W_T)$ remained the same for all T. We argue that it is a more plausible interpretation of "investment for the long run" to assume that V(g) remains constant.

Theorem 1 of this paper shows that, under very general conditions, if V(g) is continuous then the expected log rule is asymptoticly optimal; whereas if V(g) is discontinuous then the expected log rule asymptotically provides an expected utility that is within γ^{max} of the optimal expected utility, where γ^{max} is the largest jump in V(g).

Comparing theorem 1 with the Mossin-Samuelson argument, one might be tempted to conclude that the expected log rule is asymptotically desirable if constant V(g) is assumed, but generally not asymptotically desirable if constant $U(W_T)$ is assumed. But theorems 2 and 3 show, under fairly general conditions, that the expected log rule is asymptotically optimal even when constant $U(W_T)$ is assumed, provided that $U(W_T)$ is bounded from above and below. I.e., the expected log rule is asymptotically optimal, under the conditions of theorems 2 and 3, provided that U does not approach $+\infty$ as W_T approaches ∞, and does not approach $-\infty$ as W_T approaches 0. Essential to the Mossin-Samuelson result, then, was the fact that every utility function, in the class considered by them, was unbounded either from above or below.

In section 14 we argue, along the lines of Menger (1934), that given any $U(W_T)$ which is unbounded (either above or below) a St. Petersburg type of game can be constructed which shows that the particular unbounded $U(W_T)$ is absurd. From this it is argued that $U(W_T)$ should be assumed to be bounded, hence theorems 2 and 3 are applicable.

The argument in section 14 even rules out

$$U(W_T) = \log(W_T)$$

as a reasonable utility function of final wealth. Thus a word seems needed (section 15) as to how the expected log rule can be asymptotically optimal for every bounded $U(W_T)$.

In short, this paper argues that:

> it is much more plausible to analyze behavior for the long run by assuming constant $V(g)$ than constant $U(W_T)$ as T increases. In this case theorem 1 applies; but
>
> even if constant $U(W_T)$ were assumed, it must be bounded to avoid the absurd behavior shown in section 14. In this case theorems 2 and 3 apply.

Theorems 1, 2 and 3 show general conditions under which the expected log rule provides asymptotically optimum, or almost asymptotically optimum expected utility.

It is not the position of Markowitz (1959), nor is it the position of the present paper, that all investors should invest for the long run. The risk averting investor may prefer to sacrifice some return in the long run for some additional stability in the short run. One value of the results for the long run, nevertheless, is to help us narrow the range of E,V efficient portfolios which need be considered for final portfolio selection. This view of the usefulness of the long run analysis is discussed in section 6.

2. Prerequisites

Since portfolio theory has become of interest to theorists and practitioners with mathematical backgrounds ranging from none to much, we should specify the mathematical background assumed of the reader.

Sections 3 through 6 assume that the reader has had an introduction to probability and portfolio theory equivalent to the first six chapters of Markowitz (1959). The principal prerequisite assumed here that is not discussed in these six chapters is the use of the summation sign (Σ). The latter is discussed on pages 155-6 of Markowitz (1959) if the reader is not already familiar with it.

Section 7 onward in this paper requires a greater familiarity with college mathematics, particularly analysis and probability. The earlier sections, 3 through 6, are intended to illustrate the discussion. The later sections are intended to provide rigor and generality.

3. A Paradox

Rather than relating who said what when, we shall describe the controversy by analyzing a simple special case. We first analyze the case in a manner which makes the expected log rule appear desirable; and then we analyze it in a manner which makes the expected log rule appear undesirable. Having thus illustrated the problem we show, for the simplified case, how the theorems presented later in the paper resolve the apparent paradox.

The present section makes assumptions, such as unchanging probability distributions of returns over time, not made in later sections. The purpose of the present section is to illustrate the problem rather than seek generality.

Imagine a player who bets on a wheel such as that in figure 1. The wheel is marked with two or more concentric rings referred to as ring 1, ring 2, ..., ring N. The wheel is also marked into M stopping points. Numbers, r_{ij}, written on the wheel for each combination of ring i and stopping point j, indicate the return per dollar bet on the i-th ring if the wheel stops at the j-th stopping point. Thus if the wheel stops as in figure 1 the return per dollar invested in ring 1 is .05, the return per dollar invested in ring 2 is .00, and the return per dollar invested in ring 3 is .10.

We will sometimes refer to the rings as securities, and the r_{ij} as returns on securities. Cash, or a security with a fixed return, is represented by a ring with the same r_{ij} for all j.

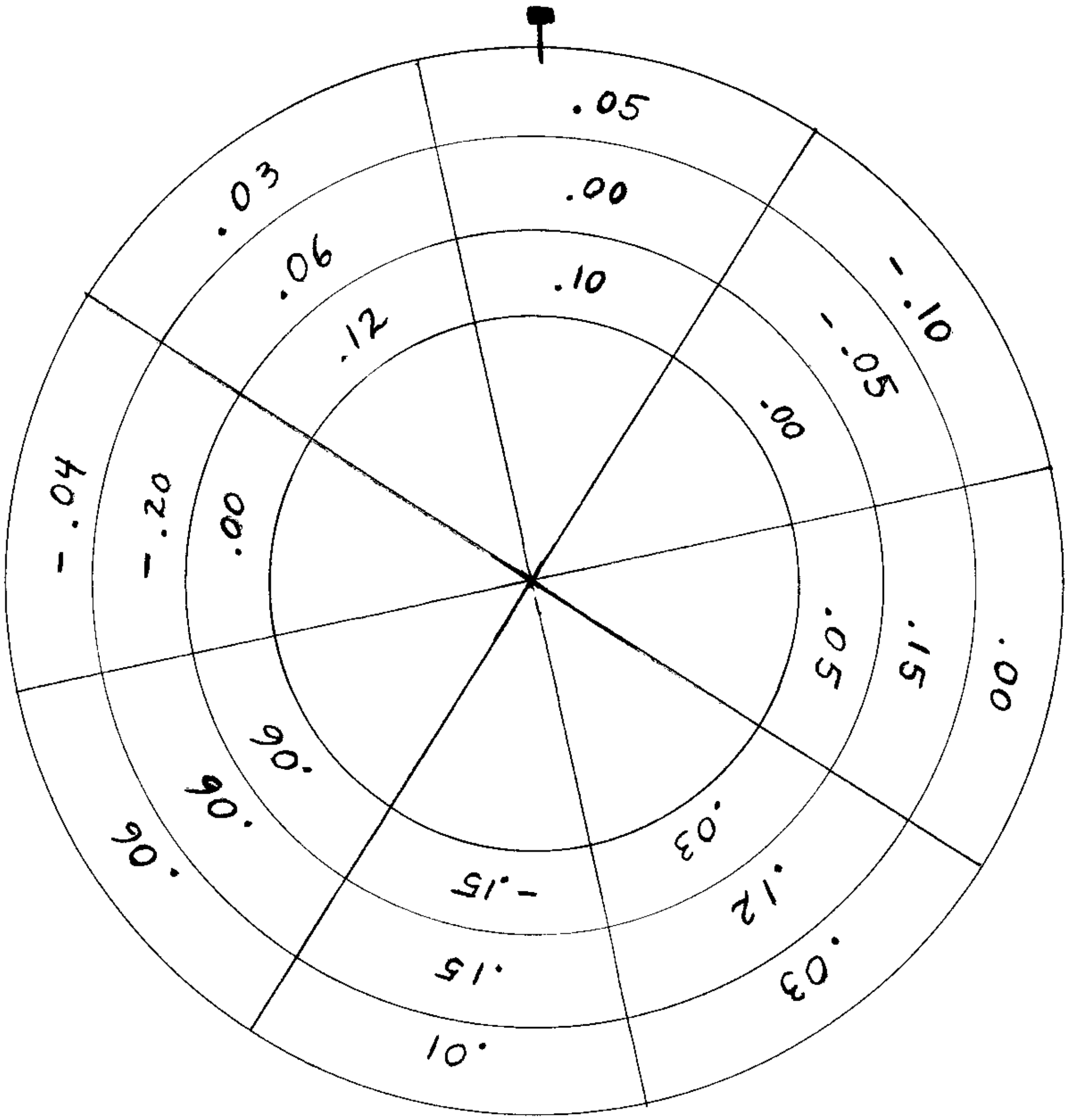

Figure 1

Wheel of Fortune Illustrating Simplified Game

The investor begins with an initial wealth $W_0 > 0$. He chooses an allocation of resources $X_1, X_2, \ldots, X_N$ such that $\sum_{i=1}^{N} X_i = 1$. The wheel is spun, stops at j_1 and his wealth then equals

$$W_1 = W_0 \cdot \sum_{i=1}^{N} X_i \cdot (1 + r_{ij_1})$$

$$= W_0 \cdot (1 + r_1)$$

where

$$r_1 = \sum_{i=1}^{N} X_i r_{ij_1} .$$

Here r_1 is the return in the first period on the portfolio as a whole.

Throughout this paper we assume that the player bets his entire accumulated wealth on each spin of the wheel. Thus for the second spin the player bets $W_1 = W_0(1 + r_1)$ in total.

In the present simplified case we will also assume that both the wheel and the investor's proportions X_i remain the same throughout the game. Thus the wheel is spun a second time, stops at j_2, and the investor's wealth becomes

$$W_2 = W_1 \cdot \sum_{i=1}^{N} X_i \cdot (1 + r_{ij_2})$$

$$= W_1 \cdot (1 + r_2)$$

$$= W_0 \cdot (1 + r_1) \cdot (1 + r_2) .$$

After T spins the player's wealth equals the product

(3.1) $$W_T = W_0 \cdot (1 + r_1) \cdot (1 + r_2) \cdot (1 + r_3) \ldots (1 + r_T)$$

where

$$r_t = \sum_{i=1}^{N} X_i r_{ij_t}, \quad \text{for } t = 1 \text{ to } T,$$

j_t being the stopping point of the wheel on the jth spin. Or, using the product sign π --which is to multiplication as Σ is to summation--we may write (3.1) as

(3.1a) $$W_T = W_0 \cdot \prod_{t=1}^{T} (1 + r_t).$$

In the present simplified case we will assume that the portfolio choosen must have

$$X_i \geqslant 0 \quad \text{for } i = 1 \text{ to } N$$

and that the wheel is such that

$$r_{ij} > -1 \quad \text{for all } i,j.$$

It follows that the investor cannot be completely wiped out in a single spin of the wheel.

Now let us consider whether the investor would be better advised to select portfolio (a) with proportions

$$x_1^a,\ x_2^a,\ x_3^a,\ \ldots,\ x_N^a$$

or portfolio (b) with proportions

$$x_1^b,\ x_2^b,\ x_3^b,\ \ldots,\ x_N^b.$$

The wealth, W_T^a, provided by portfolio (a) after T spins will be larger than the wealth, W_T^b, provided by portfolio (b), if and only if

$$W_T^a/W_0 > W_T^b/W_0$$

therefore if and only if

$$\log(W_T^a/W_0) > \log(W_T^b/W_0)$$

and therefore if and only if

$$(1/T)\log(W_T^a/W_0) > (1/T)\log(W_T^b/W_0).$$

Equation (3.1), and the basic property of logarithms that

$$\log(\prod_{t=1}^{T}(1 + r_t)) = \sum_{t=1}^{T}\log(1 + r_t),$$

imply that for any portfolio

$$(1/T)\log(W_T/W_0) = (1/T)\sum_{t=1}^{T}\log(1 + r_t);$$

hence the expected value of $(1/T)\log(W_T/W_0)$ is

$$(1/T)\cdot E \log(W_T/W_0) = (1/T)\cdot E\{\sum_{t=1}^{T} \log(1 + r_t)\}$$
$$= (1/T) \sum_{t=1}^{T} E \log(1 + r_t).$$

But since, for a given player, r_1 has the same probability distribution as r_2, which has the same probability distribution as r_3, etc., we may write

(3.2) $$(1/T)\cdot E \log(W_T/W_0) = E \log(1 + r).$$

This is unchanged as T increases.

Since the spins of the wheel are independent, the variance of $(1/T)\log(W_T/W_0)$ equals

(3.3) $$(1/T^2)\cdot \mathrm{Var}\ \{\sum_{t=1}^{T} \log(1 + r_t)\} = (1/T).\mathrm{Var}(\log(1 + r)).$$

This approaches 0 as T approaches ∞.

Therefore as T increases, the expected value of $(1/T)\log(W_T/W_0)$ remains constant while its variance approaches 0. It follows that if E log(1 + r) is larger for portfolio (a) than portfolio (b), then as $T\to\infty$ the investor who always reinvests in the former is "almost sure" to do better than the investor who always reinvests in the latter. To be precise:

> suppose that p is some probability less than 1.0 (e.g., p = .999999). Suppose that a player investing in portfolio (a) would like to be this sure that he will beat a player investing in portfolio (b). He can be this sure by choosing T large enough; since 3.2 and 3.3

and the Tchebychev inequality[1] imply that there exists a T* such that for a game of length T* or longer the probability is at least p that player (a) will beat player (b).

The player who chooses the portfolio with greater E log(1 + r) can be as sure as he pleases (short of absolute certainty) that he will do better than a player who chooses a portfolio with lesser E log(1 + r). He only has to insist that T be large enough.

It would seem then that, in this simple case at least, the way to invest for the long run is to maximize E log(1 + r), i.e., to follow the expected log rule. But let us analyze the same game from another point of view.

Suppose that the investor is to play for a fixed number of periods, T, and then "cash in" his final portfolio wealth W_T. Let us also suppose, with Mossin and Samuelson, that the investor has a utility function of the form

$$U = \alpha \; (W_T)^{\alpha}$$

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[1]Tchebychev's inequality says that the probability that a random variable will deviate from its expected value by more than k times its standard deviation is never greater than $1/k^2$. Therefore as variance and standard deviation approach 0 the probability of a given size deviation approaches 0. For example, let d be the difference between the E log (1 + r) provided by portfolio (a) and that provided by portfolio (b) in the text above. By hypothesis, d is greater than 0 and does not depend on T. The probability that the actual value of $(1/T)\log(W_T/W_0)$ provided by portfolio (a) will deviate from its expected value by as much as one-half d, or that the actual $(1/T)\log(W_t/W_0)$ provided by portfolio (b) will deviate from its expected value by as much as one-half d, approaches 0 as their two variances approach 0.

for some $\alpha \neq 0$. For example, perhaps his utility function equals $\frac{1}{2}(W_T)^{\frac{1}{2}}$ or $-\frac{1}{2}(W_T)^{-\frac{1}{2}}$. Any such function says utility increases with wealth. For functions with $\alpha < 1$ the rate of increase decreases as wealth increases. The investor's utility at the end of the T periods is

$$\begin{aligned} U &= \alpha\ (W_T)^{\alpha} \\ &= \alpha \cdot W_0^{\alpha} \cdot \{\prod_{t=1}^{T} (1 + r_T)\}^{\alpha} \\ &= \alpha W_0^{\alpha} \cdot \prod_{t=1}^{T} (1 + r_t)^{\alpha}\ . \end{aligned}$$

Since the spins of the wheel are independent, the expected utility associated with reinvesting in a given portfolio is

$$\begin{aligned} EU &= E\ \{\alpha W_0^{\alpha} \cdot \prod_{t=1}^{T} (1 + r_t)^{\alpha}\} \\ &= \alpha W_0^{\alpha} \prod_{t=1}^{T} E(1 + r_t)^{\alpha} \\ &= \alpha W_0^{\alpha}\ \{E(1 + r)^{\alpha}\}^T. \end{aligned}$$

This is maximized by choosing the portfolio with greatest $E(1 + r)^{\alpha}$. As a rule this will not be the portfolio which maximizes $E \log(1 + r)$.

Suppose portfolio (a) maximizes $E \log(1 + r)$ while portfolio (b) maximizes $E(1 + r)^{\alpha}$ for the investor's particular α . Suppose that the value of $E(1 + r)^{\alpha}$ for portfolio (b) is k times as great as that provided by portfolio (a), where $k > 1$. The ratio between expected utility from portfolio (a) and that from (b) for the game as a whole is k^T. As $T \to \infty$, $k^T \to \infty$. Hence as T increases, the superiority of portfolio (b) over portfolio (a) increases without limit.

According to our previous argument, for sufficiently large T portfolio (a) is almost sure to beat portfolio (b). In fact, for any $p < 1$ there is a T^* such that for $T \geqslant T^*$ the probability that (a) beats (b) is at least p. Yet, when $U = \alpha \, (W_T)^\alpha$ the ratio of the expected utility provided by (b) to that provided by (a) can increase without bounds as T increases.

4. The Catch

The theorems presented later in this paper address themselves to the apparent paradox illustrated in the preceding section. The theorems are proved under substantially more general assumptions than those of our previous discussion. For the time being, however, we will continue the simplified analysis, stating here without proof the implications of the theorems for the present special case, and how this reveals "the catch" in the apparent paradox. We begin by presenting some basic notions used in the theorems.

The theorems consider sequences of games $G_{T_1}, G_{T_2}, G_{T_3}, \ldots$. The first game in the sequence may have $T_1 = 100$; i.e., it is to be played for 100 periods. The next game in the sequence may have $T_2 = 101$ or $T_2 = 200$, or any other number greater than 100; and in general

$$T_{j+1} > T_j$$

for $j = 1,2,3,\ldots$. A special case of such a sequence would be, for example

$G_{100}, G_{200}, G_{300}, \ldots$

where G_{300} is a game consisting of 300 spins of the same wheel such as described in the preceding section.

As in the Mossin-Samuelson analysis we shall assume that, in some sense, the same utility function is used in each game in the sequence. We shall consider sequences of games, however, in which the utility function is assumed to stay the same in one of two different senses.

We may define the rate of growth g as

$$g = (W_T/W_0)^{1/T} - 1.$$

It follows immediately that

$$W_T = W_0 \cdot (1 + g)^T.$$

Thus if the investor had put all of his wealth in a savings account that paid (g) per period he would have ended the game with the same terminal wealth.

For a fixed T we can express utility equivalently as a function of W_T or as a function of g. E. g., if

$$U = U(W_T)$$

then

$$U = U(W_0 \cdot (1 + g)^T) = V(g)$$

by definition of V(g). Either U or V can be used equivalently to evaluate a probability distribution of W_T or the implied probability distribution of g.

Any one game, then, may be described equivalently with a $U(W_T)$ function or a $V(g)$ function. On the other hand, we have a different sequence of utility functions associated with G_{T_1}, G_{T_2}, G_{T_3}, ... if we assume $U(W_T)$ is the same for all games G_T, or assume $V(g)$ is the same for all G_T.

If we think of a period as some fixed interval of time, such as a month, then the assumption that $U(W_T)$ is constant among games assumes, for example, that the investor has the same rankings among probability distributions involving terminal wealth = \$500,000 vs. \$1,000,000 vs. \$2,000,000 whether the game is for 100 months, 200 months or 500 months. The assumption that $V(g)$ remains the same, on the other hand, asserts that the investor has the same preference rankings among probability distributions of say a ½%, 1% or 1½% rate of return per month for the game as a whole whether the game is for 100 months, 200 months or 500 months.

While results are presented below for both constant $U(W_T)$ and for constant $V(g)$, it seems to me that assuming a constant $V(g)$ is the more plausible interpretation of "investing for the long run". Suppose that the management company of a mutual fund, or the trustee organization of a large private estate, takes as its goal "return" or "increase in wealth" over the long run. Suppose indeed that they are not willing to give up anything in the long run for a second goal of reducing short run fluctuations in wealth. In this case it seems to me more plausible that their utility function is expressible in terms of a 3% vs. a 6% vs. a 9% rate of return per annum over an indefinitely long period of time, rather than in terms of a 40 fold increase in wealth vs. a 60 fold increase in wealth vs. an 80 fold increase in wealth over an indefinitely long period of time.

Let G_{T_1}, G_{T_2}, G_{T_3}, ... be a sequence of alternate possible games in which the same wheel is spun respectively T_1 times, T_2 times, T_3 times, etc. Theorem 1 implies that if $V(g)$ is the same for all games in the sequence, if utility does not decrease when g increases, and if $r_{ij} > -1$ for all i,j then:

> if $V(g)$ is continuous, the expected value of $V(g)$ provided by the expected log rule approaches the maximum obtainable expected $V(g)$ as $T \to \infty$.
>
> If $V(g)$ is not continuous, then the expected value of $V(g)$ provided by the expected log rule is within $\varepsilon + \gamma^{max}$ of the maximum obtainable expected $V(g)$, where γ^{max} is the largest jump in the $V(g)$ function, and $\varepsilon \to 0$ as $T \to \infty$.

In other words, the expected log rule provides asymptotically optimal expected utility if $V(g)$ is continuous, and at least asymptotically "nearly" optimal expected utility if $V(g)$ has only "small" jumps. This result is true even if the "maximum obtainable expected $V(g)$" is that provided by a strategy which allows the choice of portfolios to change from period to period.

Contrasting the results just quoted with those for the Mossin-Samuelson $U(W_T)$ function, one might conjecture that the expected log rule does well for constant $V(g)$ and poorly for constant $U(W_T)$. Theorems 2 and 3, on the contrary, show that under certain conditions the expected log rule is also asymptotically optimal if $U(W_T)$ is bounded both from above and from below. Specifically, in the simple case of unchanging probabilities if U does not decrease with an increase in W_T, if U is bounded from above and below, if $r_{ij} > -1$ for all i,j and if the maximum obtainable $E \log(1 + r)$ is not equal to 0, then the expected value of $U(W_T)$ provided by the expected log rule is within ε of that provided by the optimum strategy -- where ε approaches 0 as T increases. Thus the persistent difference between the expected utility provided by the expected log rule and that of the optimum strategy in the Mossin-Samuelson analysis is due to the fact that (W_T) is unbounded above if $\alpha > 0$, and unbounded below if $\alpha < 0$.

The question of bounded versus unbounded utility functions is not original to the analysis of the asymptotic optimality of the expected log rule. Section 14 argues that only a madman would act according to an unbounded $U(W_T)$ in any game G_T.

5. An Example

Before we proceed to the general discussion, let us illustrate our introductory remarks with a numerical example.

Suppose, for this example, that a "wheel" has two outcomes: heads and tails. Suppose further for this example that only two choices are allowed the game player: He can either: (a) always receive a one percent increase; or (b) have a 75 percent increase in case of heads, and a 50 percent decrease in the case of tails. Alternative (a) provides r = .01 with certainty; alternative (b) provides a 50-50 chance of r = -.5 or r = +.75. In the present example let us require that the player either always bet all of his wealth on (a), or always bet all of his wealth on (b).

Consider the choice of (a) or (b) by three investors. The first investor wants to maximize the expected value of terminal wealth W_T. This is actually the special case of maximizing the expected value of $\alpha(W_T)^\alpha$ in which $\alpha = 1$. We will refer to this player as the Mossin ($\alpha = 1$) Player. The second investor wants to maximize the expected value of the square root of W_T. He is a Mossin ($\alpha = \frac{1}{2}$) Player, since the utility functions $(W_T)^{\frac{1}{2}}$ and $\frac{1}{2}(W_T)^{\frac{1}{2}}$ are equivalent in their choices among alternative strategies. The third player follows the expected log rule. We call him our Kelly Player.

As seen in section 3, in the present game the Mossin ($\alpha = 1$) Player will pick the bet with the highest expected value of (1 + r). This will be the same bet that maximizes expected r. Since alternative (b) provides an expected return of

$$\tfrac{1}{2}(-.5) + \tfrac{1}{2}(.75) = .125$$

while alternative (a) provides an expected value of

$$\tfrac{1}{2}(.01) + \tfrac{1}{2}(.01) = .01$$

the Mossin ($\alpha = 1$) Player will prefer (b).

Section 3 also implies that the Mossin ($\alpha = \frac{1}{2}$) Player will, in the present game, choose the alternative which maximizes the expected value of $(1 + r)^{\frac{1}{2}}$ on a single spin (or flip). For alternative (a) this is

$$\tfrac{1}{2}(1.01)^{\frac{1}{2}} + \tfrac{1}{2}(1.01)^{\frac{1}{2}} = 1.005$$

For (b) the expected value of $(1 + r)^{\frac{1}{2}}$ is

$$\tfrac{1}{2}(.5)^{\frac{1}{2}} + \tfrac{1}{2}(1.75)^{\frac{1}{2}} = 1.015$$

Thus he too, in this particular example, will prefer (b).

The Kelly Player chooses the larger of

$$\tfrac{1}{2}\log_{10}(1.01) + \tfrac{1}{2}\log_{10}(1.01) = .0043$$

versus

$$\begin{aligned}\tfrac{1}{2}\log_{10}(.5) + \tfrac{1}{2}\log_{10}(1.75) &= -.1505 + .1215 \\ &= -.0190\end{aligned}$$

and selects (a) instead.

After, e.g., 2,000 flips of the coin the Kelly Player will have increased his wealth by a factor of $(1.01)^{2000}$ or over 400 million-fold. The fate of the two Mossin Players depends on the number of heads in the 2000 flips. If perchance there were exactly 1000 heads and 1000 tails, the ratio of their ending wealth to starting wealth would equal

$$\begin{aligned}(.5)^{1000} \cdot (1.75)^{1000} &= (.875)^{1000} \\ &\simeq 10^{-58}\end{aligned}$$

= .000 000 000 000 000 000 000
000 000 000 000 000 000 000
000 000 000 000 001.

There is a 50-50 chance that the Mossin Players will do worse than this. If instead of T = 2000 we choose a larger value of T the story would be the same -- only more so.

It would seem that alternative (b) is a miserable way to bet for the long run. Yet it does maximize expected W_T. If you added the probability of 2000 straight heads times $(1.75)^{2000}$ plus the probability of 1999 heads out of 2000 times $(.5)\cdot(1.75)^{1999}$ plus the sum of each other possible outcome times its probability you would find that the "expected value" of W_{2000} provided by (b) was $W_0\cdot(1.125)^{2000}$ as compared to the mere $W_0\cdot(1.01)^{2000}$ provided by (a).

This implies that if you would rather have your money ride on (a) than (b), your criteria cannot be to maximize the expected value of W_T; nor can it be to maximize the expected value of $\sqrt{W_T}$. You may still act according to the expected utility maxim. But neither $U = W_T$ nor $U = \sqrt{W_T}$ is your utility function.

We have argued that since T is of indefinite size in this discussion, utility should be expressed in terms of the rate of return g rather than terminal wealth W_T. It can be shown, as a corrollary of the discussion in section 11, that for large T alternative (a) provides a larger value of expected V(g) than does (b) in this example for any continuous everywhere increasing V(g), even for $V(g) = \alpha g^{\alpha}$ $(\alpha \neq 0)$.

The main result of this paper is theorem 1. It is not the purpose of theorem 1 to compare the expected log rule with the rule which always maximizes expected $\alpha(1 + r_t)^{\alpha}$ each period.

Rather the purpose of theorem 1 is to compare the expected log rule with whatever strategy maximizes expected V(g) for a given game G_T as a whole. The latter, precisely optimal, strategy may involve varying the portfolio from period to period even if the same wheel is spun each time. The finding of optimal strategies for realistically complex games may be beyond the optimization capabilities of our largest computers. Yet, according to theorem 1, for sufficiently large T the precisely optimum strategy can do very little better than the simple expected log rule.

6. The Moral

We conclude from theorem 1 that if you were interested only in reinvesting for the long run, in the manner assumed here, you need not bother to solve for an optimum strategy. Such an optimum solution would require you to estimate your actual V(g) function; estimate how future distributions of returns depend on time and preceding events; and perhaps may require untold calculations to determine. Instead just follow the expected log rule. For sufficiently large T there will be virtually no difference in the expected utility provided for the game as a whole.

Under conditions explored elsewhere [Markowitz (1959, pp. 121 - 125), and Young and Trent (1969)] the policy of maximizing expected logarithm for the current period may itself be approximated by a portfolio selected from the set of E,V efficient portfolios. In this case both the estimation problem and the computation problem are reduced to quite reasonable proportions, especially if some simplified model of covariance [as in Sharpe (1963), Cohen and Pogue (1967), or Markowitz (1959, pp. 96 - 101)] may be assumed.

Note that the advice to maximize E log(1 + r) applies to the portfolio as a whole rather than to some subset of the portfolio. Suppose, for example, that an analyst is asked to advise whether an investor who plans to continually reinvest for the long run should buy the common stock, the preferred stock, or some of each, of a given corporation. Unless it was in fact the investor's entire portfolio, it would not as a rule be even approximately correct to choose the combination of common and preferred which maximizes E log(1 + r).

To see the error of trying to maximize E log(1 + r) for the portfolio as a whole by maximizing it for components of the portfolio, consider again securities (a) and (b) of the preceding section. We saw that security (a), providing r = .01 with certainty, had a greater E log(1 + r) than security (b) with a 50 - 50 chance of r = -.50 or +.75. But if a sufficiently large number of securities like (b) were available, and if their returns were uncorrelated, then a portfolio consisting of many such securities would provide r = .125 with near certainty, and would have a higher E log(1 + r) than a portfolio consisting of (a) only. Many securities with a lessor E log(1 + r) thus may (or may not) combine to provide a greater E log(1 + r) for the portfolio as a whole. We should make the individual decisions for their effect on the portfolio as a whole.

Theorem 1 is relevant to the user of E,V efficient set analysis, even if he is not dedicated exclusively to investment for the long run. Suppose that a portfolio analyst has computed an E,V efficient set for an investor or investment manager, and is about to graph the probability distribution of returns for several possibly desirable efficient portfolios. In accord with Baumol's (1963) observations, the analyst would presumably not draw plots for any efficient portfolio with standard deviation σ below the point with maximum $E - k\sigma$, for k equal to about 2 or 3. Below some such point, efficient portfolios may be viewed as less variable but not safer. Similarly, the analyst would presumably not draw plots for efficient portfolios with E and σ greater than one with approximately maximum $E \log(1 + r)$; for efficient portfolios with greater E and σ are more variable in the short run without presumably[1] providing additional return in the long run.

Thus the analyst may reasonably discard from further attention efficient portfolios below a "Baumol point" and above a "Kelly-Latané point".

---

[1] We say "presumably" since (1) the asymptotic optimality of the expected log rule is shown here under certain simplified assumptions, such as no costs of transactions, as discussed in section 8; and (2) while Trent and Young show that the historical average $\log(1 + r)$ of various portfolios is closely approximated by formulae depending only on historical E and V, nevertheless E,V approximations can be quite inaccurate in the case of highly speculative portfolios. For example, if a game player were allowed to borrow and bet to such an extent that $r \leq -1.0$ could occur, then $E \log(1 + r)$ would equal $-\infty$ while the various Trent and Young approximations might even be positive. Hopefully, future research will provide broader guidelines as to when E,V approximations may be trusted than the guidelines of Markowitz (1959) pages 121-2.

7. The Game G_T

We now present our general model. We consider a game G_T played for T periods. At the beginning of each period, t, the player chooses portfolio proportions

$$X_{1t}, X_{2t}, X_{3t}, \ldots, X_{Nt}$$

such that $\sum_{i=1}^{N} X_{it} = 1$. This choice may depend on the history which precedes period t. We now may imagine that the return per dollar invested is generated by the spin of a wheel as in figure 2. The wheel in figure 2 differs from that in figure 1 in that:

it has a "wheel number", and

each stopping point on the wheel indicates the wheel number of the wheel to be spun next.

Thus if the current wheel stops as in figure 2, the returns per dollar bet on rings 1, 2 and 3 respectively are .05, .00 and .10 and the wheel to be spun for the next period is 381. In this manner both the returns this period and the opportunities of subsequent periods are generated by the spin of the wheel. The number of stopping points and securities may vary from wheel to wheel.

In our earlier simplified analysis we assumed that the wheel had a finite number of stopping points. The variety of objects (giraffes, oceans, skyscrapers) fashioned by nature and man from the finite number of atoms of the earth, suggests that

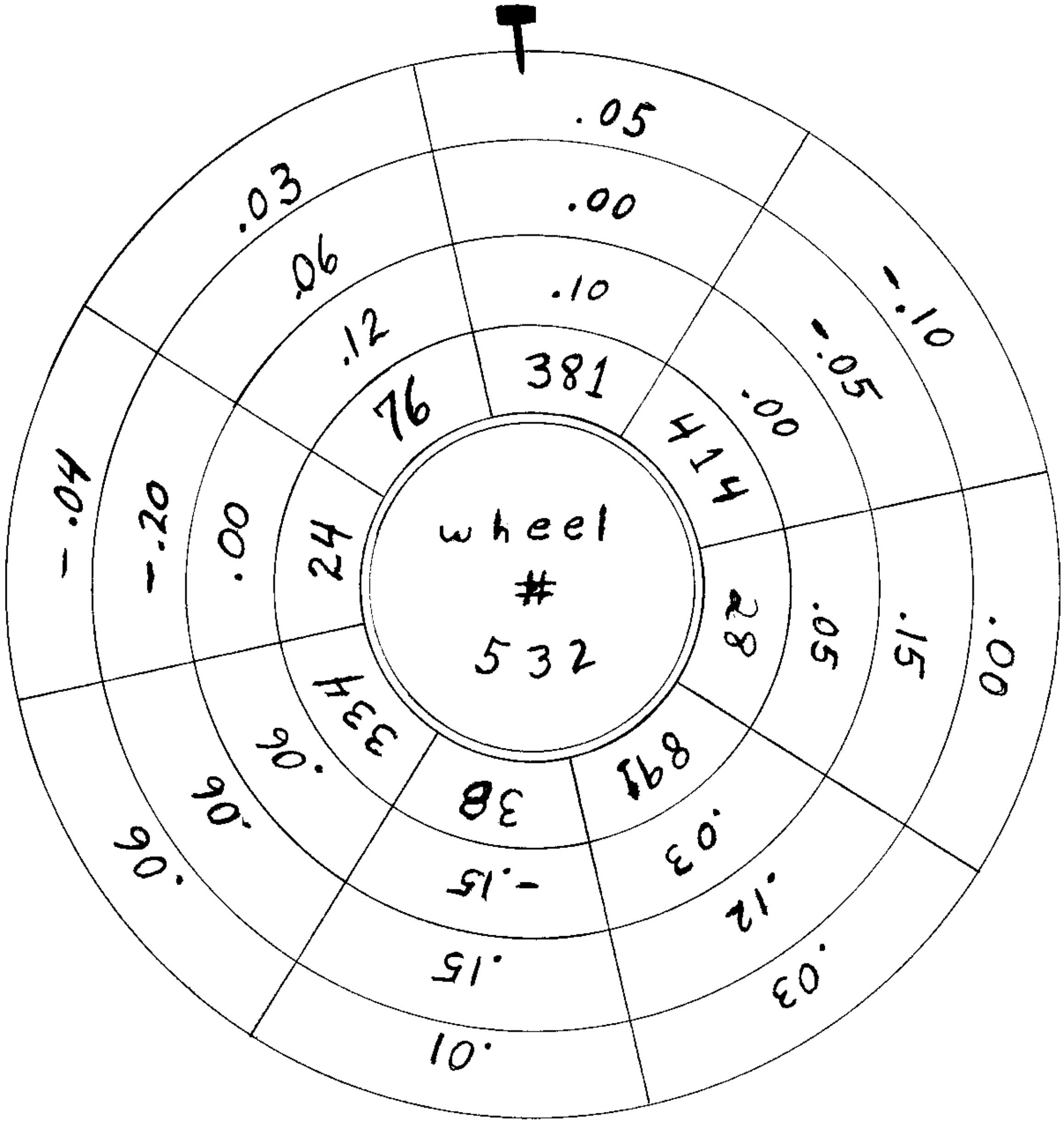

Figure 2

Wheel of Fortune Illustrating General Game the assumption is not a practical limitation. It is worth noting, however, that the theorems and their proofs apply if there are either a finite or a countably infinite number of stopping points on any wheel.

In the simplified analysis the player chose a portfolio from the constraint set described by

$X_i \geqslant 0$, for all i

(in addition to $\Sigma \ X_i = 1$). The three theorems allow the portfolio $(X_{1t}, X_{2t}, \ldots, X_{Nt})$ to be selected from a constraint set S such that:

S is not empty;

S may depend on the current wheel but not on the prior choice of portfolio;

$\Sigma \ X_{it} = 1$ for all $(X_1, X_2, \ldots, X_N) \in S$.

Later we present an additional restriction on the r_{ij} and the set S associated with any wheel in any G_T. We will not assume there, however, as we have not assumed here, that S is necessarily (for example) closed, or convex, or that E log(1 + r) necessarily achieves a maximum in S.

A strategy s is a rule specifying:

initial proportions invested, chosen from the set S of the first wheel; and

proportions to be invested at time t as a function of the history to date:

$$X_{it} = X_{it}(j_1, j_2, \ldots, j_{t-1})$$

where the portfolio $(X_{1t}, \ldots, X_{Nt})$ is contained in the set S associated with the wheel to be spun at time t.

For a given game G_T and a given strategy s, the history of a particular play is given by the sequence of stopping points. The stopping point of the first spin j_1 implies: the returns r_{ij_1}; the return r_1 on the portfolio as a whole associated with strategy s; the next wheel to be spun; and the portfolio to be selected for the next spin according to strategy s. The pair of stopping points (j_1, j_2) implies in addition: the returns r_{ij_2}; the portfolio return r_2 associated with s; the third wheel to be spun, and so on.

There are a finite or countably infinite number of possible sequences of stopping points, or "histories", $(j_1, j_2, j_3 \ldots, j_T)$. In principle we can assign numbers 1, 2, 3, ... to each possible history. This assignment may be made completely arbitrarily as long as each possible sequence is assigned a number.

Thus the history of a particular play of the game G_T may be represented by a single positive integer, h_T. The structure of a game, as described in terms of wheels, implies a probability for each h_T. The returns in each period $r_1, r_2, \ldots r_T$ obtained from following a given strategy s in a play of G_T is also implied by the integer h_T.

For each time period t there are a finite or countably infinite number of possible "partial histories" $(j_1, j_2, \ldots, j_t)$.

Each such partial history can, in principle, be assigned a number h_t. The assignment of an integer to each possible partial history at time t may be made arbitrarily -- without regard to the numbers assigned to the partial histories at time t', or to the numbers assigned to the total histories.

Any history h_T implies T-1 partial histories:

$$h_1, h_2, h_3, \ldots, h_{T-1}.$$

Any partial history has a conditional probability distribution of the total history,

$$\text{prob}(h_T \mid h_t)$$

or for any later partial history

$$\text{prob}(h_{t'} \mid h_t)$$

for $T \geqslant t' > t$. By convention we will let $h_0 = 1$ be the "partial history" before the game begins. Thus $\text{prob}(h_t \mid h_0) = \text{prob}(h_t)$.

While there are a finite or countably infinite number of possible histories, there may be a continuum of possible strategies. This will cause us no difficulty since we will be either examining the properties of one strategy or comparing two of them.

8. The Sequence of Games

We postulate a sequence of games G_T for $T = T_1, T_2, T_3, \ldots$ where $T_1 < T_2 < T_3 \ldots$. It may or may not be true that the first T_1 periods of G_{T_2} are "like" the game G_{T_1}. The sequence of games $G_{T_1}, G_{T_2}, \ldots$ can be quite loosely related.

For example, the game G_{T_1} may consist of 100 spins of a single wheel, call it wheel 1. The game G_{T_2} may consist of 200 spins of a different wheel, wheel 2. The game G_{T_3} may consist of 300 spins of wheel 1 again; G_{T_4} may consist of 400 spins of wheel 2 again; etc. We have already noted that the game $G_{T_{i+1}}$ is longer than the game G_{T_i}, and that they both have the same utility function either in the sense of $V(g)$ or $U(W_T)$. An additional major assumption concerning the sequence of games is presented later in this section.

We shall be concerned with two sequences of strategies: $s^k_{T_1}$, $s^k_{T_2}$, $s^k_{T_3}$, ... being one sequence of strategies and $s^m_{T_1}$, $s^m_{T_2}$, $s^m_{T_3}$, ... being the other. $s^k_{T_1}$ and $s^m_{T_1}$ are two out of perhaps countless ways of playing the game G_{T_1}; $s^k_{T_2}$ and $s^m_{T_2}$ are two ways of playing G_{T_2}; etc. The relationship we assume between $s^k_{T_j}$ and $s^m_{T_j}$ is that $s^k_{T_j}$ always selects an allocation with at least as high an $E \log(1 + r)$ as supplied by $s^m_{T_j}$. In other words for every game G_T, and for every partial history h_{t-1} where $1 \leqslant t \leqslant T$ we assume that the

$$E \; \{ \log(1 + r_t) \mid h_{t-1} \}$$

provided by s^k_T is as least as great as that provided by s^m_T. On the other hand s^m_T may provide greater $EV(g)$ or $EU(W_T)$. The theorems analyze the extent to which the expected utility provided by s^m_T can exceed that provided by s^k_T as $T \to \infty$.

A basic assumption used in the proofs of the theorems is the following:

There exist $r^{low} > -1$ and r^{hi} such that for all games G_T in the sequence $G_{T_1}, G_{T_2}, G_{T_3}, \ldots$, and for either strategy s_T^k or s_T^m we always have

$$r^{low} \leqslant r_t \leqslant r^{hi}.$$

For example, if $r^{low} = 10^{-8}$ and $r^{hi} = 10^9$ then our assumption says that (by law or by investment practice) the returns and constraints in all the games G_{T_1}, G_{T_2}, G_{T_3}, ... are such that

> the investor cannot be wiped out in a single spin of the wheel; in fact he must retain at least one penny per million dollars bet on any one spin of the wheel; and he cannot win more than $1,000,000,000 per dollar bet on a single spin.

While r^{low} and r^{hi} are lower and upper bounds on r_t, they are not necessarily the greatest lower bound or least upper bound. Thus for either s_T^k or s_T^m or both, and for any or all of the games G_{T_1}, G_{T_2}, ... we may have r_t always "much greater" than r^{low}, or r_t "much less" than r^{hi}.

Another way of stating this basic assumption used in proving the theorems is that $\log(1 + r)$ is bounded from above and below for both of the strategies analyzed.

In addition to the explicit assumptions of the analysis, there are the implicit assumptions suggested by asking, for example, what kind of game would have

$$W_T = W_0 \cdot \pi \ (1 + r_t)$$

or would permit

$$E\ \{\log(1 + r_t)\ |\ h_{t-1}\}$$

to always be greater for one strategy, s_T^k, than for another, s_T^m. A partial list of answers include:

> commissions and other costs of transactions are ignored (hence for semi-"realism" the period should not be thought of as "too short"; but the investment decision can only be made once per period, hence the period should not be thought of as "too long");
>
> there is no "round lot" consideration since the dollar amount invested, $W_{t-1} \cdot X_{it}$, may be very small; (the limitations in S are on the proportions X_{it} not on the amounts $W_{t-1} \cdot X_{it}$);

and so on. The world we are analyzing is clearly an abstraction; hence part of the need for some disclairmer, as in the footnote at the end of section 6, concerning the precise interpretation of the Kelly-Latané point in the EV efficient set.

9. The Theorems

The three theorems are presented here and proved in the following sections. The first theorem deals with the case in which utility is a function of g.

Theorem 1: if V(g) is a monitonically increasing function; $G_{T_1}, G_{T_2}, \ldots$ is a sequence of games as described in the last two sections; and s_T^k and s_T^m are strategies associated with game G_T such that

$$E\ \{\log(1 + r_t)\ |\ h_{t-1}\}$$

is always at least as great for s_T^k as for s_T^m then there exists a T* such that for all $T \geq T^*$ the expected value of V(g) provided by s_T^m is at most $\varepsilon + \gamma^{max}$ greater than that provided by s_T^k, where γ^{max} is the largest jump of V(g), and $\varepsilon \to 0$ as $T \to \infty$.

Our assumptions concerning the constraint sets are too general for us to conclude that for every game G_T there exists a strategy which always maximizes

$$E\ \{\log(1 + r_t)\ |\ h_{t-1}\}$$

and a strategy which maximizes EV(g) or $EU(W_T)$. In such cases as these strategies do exist, however, we may let s_T^k be the strategy which always maximizes

$$E\ \{\log(1 + r_t)\ |\ h_{t-1}\}$$

and let s_T^m be the strategy which maximizes EV(g) or $EU(W_T)$. In this case we may think of s_T^k as the Kelly-Latané strategy and s_T^m as the maximizing strategy for the game G_T. If V(g) is continuous then s_T^k is asymptotically optimal since the maximum advantage of s_T^m over s_T^k approaches 0 as T approaches ∞.

In theorems 2 and 3 we consider utility to be a bounded function $U(W_T)$. We do not derive results for this case in general, but for certain subclasses depending on

$$\bar{L}_T^k = (1/T) \sum_{t=1}^{T} E\,\{\log(1 + r_t) \mid h_{t-1}\}$$

as provided by the strategy s_T^k.

Theorem 2: If, for a sequence of games as described in the preceding section, we have

$$\text{prob}\,\{\bar{L}_t^k \geqslant \alpha > 0\} \to 1 \text{ as } T \to \infty$$

then the expected value $U(W_T)$ provided by s_T^k approaches U^{hi}, the least upper bound of $U(W_T)$.

The assumption of theorem 2 is met if there is a riskless security whose yield, while perhaps varying with time, is always at least $\beta > 0$, where $\alpha = \log(1 + \beta)$.

Theorem 3: If, for a sequence of games as described in the preceding section, we have

$$\text{prob}\,\{\bar{L}_T^k \leqslant \alpha < 0\} \to 1 \text{ as } T \to \infty$$

then the expected utility provided by either s_T^k or s_T^m approaches U^{low}, the greatest lower bound of $U(W_T)$ for $W_T > 0$.

Under either the assumption in theorem 2 or that in theorem 3 the expected value of $U(W_T)$ provided by s_T^m cannot exceed that provided by s_T^k by more than ε, where $\varepsilon \to 0$ as $T \to \infty$. This result for $U(W_T)$ is not true for assumptions as general as those for $V(g)$ in theorem 1, as shown by a counter-example in section 13.

10. Properties of V(g) and $U(W_T)$

Sections 10 and 11 establish some properties needed to prove the three theorems.

Since $r^{low} \leqslant g \leqslant r^{hi}$, the expected utility of any strategy is unchanged if we replace V(g) by a function which equals V(g) for $r^{low} \leqslant g \leqslant r^{hi}$, equals $V(r^{low})$ for $g < r^{low}$ and equals $V(r^{hi})$ for $g > r^{hi}$. Thus we may assume that V(g) in theorem 1, like $U(W_T)$ in theorems 2 and 3, is bounded. Since $W_T > 0$ in any game G_T, we may arbitrarily let $U(W_T) = U^{low}$ for $W_T \leqslant 0$ without changing the expected utility of any strategy. $U(W_T)$ thus extended, like V(g) as just defined, is a bounded, monitonicly increasing function

$$y = f(x)$$

defined for $-\infty < x < +\infty$. In the present section we review some general properties of any such function.

Texts on mathematical statistics analyze the bounded, monotonic function

$$y = P(x)$$

= the probability that a random variable is less than or equal to x.

They show that P(x) may be expressed as the sum of a continuous function and a step function, where the step function has at most a countable number of jumps. Either, but not both, functions may be identically zero. The difference between

P(x) and the general bounded, monotonically increasing function f(x), is that:

> P(x) has 0 and 1 specifically as its greatest lower bound (GLB) and its least upper bound (LUB); while f(x) may have any two numbers $y^{low} \leq y^{hi}$ as its GLB and LUB; and
>
> P(x) is continuous from the right, while f(x) may be continuous from the right, from the left or neither at any point of discontinuity.

A slight modification of the argument which establishes the character of P(x) shows that any bounded monotonically increasing f(x) has the following properties:

$$f(x) = C(x) + \sum_{\substack{d_i \varepsilon D \\ d_i < x}} \gamma(d_i) + \theta(x)$$

where C is a bounded, continuous, monotonically increasing function (perhaps identically zero); D is an empty, finite or countably infinite set of real values $x = d_i$; $\gamma(d_i) > 0$ for each $d_i \ \varepsilon\ D$; $\theta(x) = 0$ for x not in D and $0 \leq \theta(x) \leq \gamma(x)$ for $x = d_i \ \varepsilon\ D$. f(x) is continuous from the left, from the right or neither at a point $x = d_i$, depending on whether $\theta = 0$, $\theta = \gamma(x)$ or neither at the point. Since C(x) is continuous, bounded and monotonic, it is uniformly continuous[1]; i.e., for any $\varepsilon > 0$ there exists $\delta > 0$ such that if $0 \leq x_2 - x_1 \leq \delta$

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[1]Uniform continuity may be shown as follows: Let $x_a < x_b$ be such that

then $f(x_2) - f(x_1) \leqslant \varepsilon$.

Since $\gamma(d_i) > 0$ for $d_i \in D$ and $\Sigma\, \gamma(d_i) \leqslant y^{hi} - y^{low}$, the sum of any subset of the $\gamma(d_i)$ is absolutely convergent. If D is not empty, $d_1, d_2, d_3, \ldots$ is an arbitrarily chosen sequence of all of the values of x at which f is discontinuous. Given any number $\gamma > 0$ there are at most a finite number of d_i with a larger value of $\gamma(d_i)$; e.g., only a finite number of d_i have $\gamma(d_i) > \gamma(d_1)$. Otherwise the sum of the $\gamma(d_i)$ would not be finite. Hence there exists a d_i with maximum $\gamma(d_i)$. We write γ^{max} for this maximum $\gamma(d_i)$.

The following property of f(x) is particularly useful in the subsequent discussion:

Lemma 1 For every $\varepsilon > 0$ there is a $\delta > 0$ such that if $0 \leqslant x_2 - x_1 \leqslant \delta$ then $0 \leqslant f(x_2) - f(x_1) \leqslant \varepsilon + \gamma^{max}$.

Proof The uniform continuity of the monotonically increasing, continuous function C(X) implies that there is a

---

(con't)

$$f(x) - y^{low} < \varepsilon$$

for $x < x_a$, and

$$y^{hi} - f(x) < \varepsilon$$

for $x > x_b$, where y^{low} and y^{hi} are the GLB and LUB of f(x). Let $x_A = x_a - 1$; let $x_B = x_b + 1$. Within the interval $x_A \leqslant x \leqslant x_B$, f(x) is uniformly continuous because it is here continuous on a closed interval. Hence for any $\varepsilon > 0$ there is a $\delta' > 0$ such that $0 \leqslant f(x_2) - f(x_1) \leqslant \varepsilon$ provided $0 \leqslant x_2 - x_1 \leqslant \delta'$ and $x_A \leqslant x_1 \leqslant x_2 \leqslant x_B$. Let δ equal the minimum of δ' or 1. For $0 \leqslant x_2 - x_1 \leqslant \delta$, if $x_1 \leqslant x_A$ then $x_2 \leqslant x_1 + \delta \leqslant x_a$; hence $0 \leqslant f(x_2) - f(x_1) \leqslant (y^{low} + \varepsilon) - y^{low} = \varepsilon$; whereas if $x_2 \geqslant x_B$ then $x_1 \geqslant x_2 - \delta \geqslant x_b$ and $0 \leqslant f(x_2) - f(x_1) \leqslant y^{hi} - (y^{hi} - \varepsilon) = \varepsilon$. Thus the same δ "works" throughout. I.e., for every $\varepsilon > 0$ there exists $\delta > 0$, where δ depends on ε but not on x_1 or x_2, such that $0 \leqslant x_2 - x_1 \leqslant \delta$ implies

$$f(x_2) - f(x_1) \leqslant \varepsilon.$$

value $\delta_1 > 0$ such that $0 \leqslant C(x_2) - C(x_1) \leqslant \varepsilon/2$ provided $0 \leqslant x_2 - x_1 \leqslant \delta_1$. The convergence of the sum of the $\gamma(d_i)$ implies that there is an integer N such that $\sum_{i=N+1}^{\infty} \gamma(d_i) \leqslant \varepsilon/2$. Let $\delta_2 > 0$ be such that at most one $d_i \varepsilon D$ with $i \leqslant N$ appears in any interval $a \leqslant x \leqslant a + \delta_2$. Let δ equal the smaller of δ_1 and δ_2. Then for any x_1, x_2 with $0 \leqslant x_2 - x_1 \leqslant \delta$ we have

$$f(x_2) - f(x_1) = C(x_2) + \sum_{\substack{d_i < x_2 \\ d_i \varepsilon D \\ i \leqslant N}} \gamma(d_i) + \sum_{\substack{d_i < x_2 \\ d_i \varepsilon D \\ i > N}} \gamma(d_i) + \Theta(x_2)$$

$$-[C(x_1) + \sum_{\substack{d_i < x_1 \\ d_i \varepsilon D \\ i \leqslant N}} \gamma(d_i) + \sum_{\substack{d_i < x_1 \\ d_i \varepsilon D \\ i > N}} \gamma(d_i) + \Theta(x_1)]$$

$$= [C(x_2) - C(x_1)] + [\sum_{\substack{x_1 \leqslant d_i < x_2 \\ d_i \varepsilon D \\ i \leqslant N}} \gamma(d_i] + \Theta(x_2) - \Theta(x_1) + [\sum_{\substack{x_1 \leqslant d_i < x_2 \\ d_i \varepsilon D \\ i > N}} \gamma(d_i)]$$

$$\leqslant \varepsilon/2 + \gamma^{max} + \varepsilon/2$$

(since $\theta(x_2)$ may be included in the first or second sum of $\gamma(d_i)$ as appropriate). Therefore

$$f(x_2) - f(x_1) \leqslant \varepsilon + \gamma^{max}.$$

11. Properties of $(1/T)\log(W_T/W_0)$

This section discusses statistical properties of $(1/T)$ $\log(W_T/W_0)$ which are useful in establishing our theorems.

For a given game G_T and strategy s_T we define:

11.1 $$L_t = L_t(h_{t-1})$$

$$= E\,[\log(1 + r_t) \mid h_{t-1}].$$

For the game and strategy under consideration, L_t is the conditional expected value of $\log\,(1 + r_t)$ given the partial history through the (t-1)st period. L_t for s_T^k and s_T^m will be denoted by L_t^k and L_t^m. For the given game and strategy, L_t is an exact (non-stochastic) function of h_{t-1}. Generally L_t is a discrete random variable depending on the discrete random variable h_{t-1}. If the same wheel is spun each period, L_t is a constant.

For a given game and strategy we also define:

11.2 $$\lambda_t = \lambda_t(h_t)$$

$$= \log(1 + r_t) - L_t.$$

λ_t is a discrete random variable whose probability distribution may depend on G_T, s_T, and h_{t-1}. Note that λ_t is an exact (non-stochastic) function of h_t, whereas L_t is an exact function of h_{t-1}. The random variable r_t, an exact function of h_t for given G_T and s_T, always satisfies:

11.3 $$\log(1 + r_t) = L_t + \lambda_t.$$

Taking conditional expected values on both sides of 11.2 and using 11.1 we get

11.4 $$E\,[\lambda_t | h_{t-1}] = 0;$$

hence, taking the expected value of the above over all values of h_{t-1}, we have

11.5 $$E(\lambda_t) = \underset{h_{t-1}}{E}[E\,[\lambda_t | h_{t-1}]] = 0$$

For a given G_T and s_T we define

11.6 $$\bar{L} = \bar{L}(h_T) = (1/T) \sum_{t=1}^{T} L_t$$

and

11.7 $$\bar{\lambda} = \bar{\lambda}(h_T) = (1/T) \sum_{t=1}^{T} \lambda_t.$$

11.5 and 11.7 imply

11.8 $$\underset{h_T}{E}(\bar{\lambda}) = 0.$$

We have, for a given G_T, s_T and h_T

$$(1/T)\log(W_T/W_0) = (1/T)\sum_{t=1}^{T} \log(1 + r_t)$$

$$= (1/T)\sum_{t=1}^{T} L_t + (1/T)\sum_{t=1}^{T} \lambda_t$$

i.e.,

11.9 $$(1/T)\log(W_T/W_0) = \bar{L} + \bar{\lambda}.$$

For $T \geqslant t > t' \geqslant 1$, the covariance between λ_t and $\lambda_{t'}$ may be written as

11.10 $$\text{cov}(\lambda_t, \lambda_{t'}) = E\,[\lambda_t \cdot \lambda_{t'}]$$

11.11 $$= \underset{h_{t-1}}{E}[E[\lambda_{t'} \cdot \lambda_t | h_{t-1}]]$$

11.12 $$= \underset{h_{t-1}}{E}[\lambda_{t'} \cdot E[\lambda_t | h_{t-1}]]$$

$$= 0$$

In step 11.12 note that h_{t-1} implies $h_{t'}$ exactly, and hence $\lambda_{t'}$, since $t' \leq t-1$. Substitute 11.4 into 11.12 to obtain the final result -- that $\lambda_{t'}$ and λ_t are uncorrelated.

The variance of $\bar{\lambda}$ is therefore

11.13 $$\mathrm{var}(\bar{\lambda}) = (1/T)^2 \sum_{t=1}^{T} \mathrm{var}(\lambda_t) \leq$$

$$\leq (1/T)[\log(1 + r^{hi}) - \log(1 + r^{low})]^2;$$

hence, $\mathrm{var}(\bar{\lambda}) \to 0$ as $T \to \infty$.

In sum, for any sequence of games $G_{T_1}, G_{T_2}, G_{T_3}, \ldots$ and strategies $s_{T_1}, s_{T_2}, s_{T_3}, \ldots$, meeting the assumptions described in sections 7 and 8, we have $E(\bar{\lambda}) = 0$, and $\mathrm{var}(\bar{\lambda})$ approaches 0 as T approaches infinity. This plus the Tchebychev inequality implies:

<u>Lemma 2</u> For any $\varepsilon > 0$ and $p > 0$ there exists a T^* such that if $T \geq T^*$ then

$$\mathrm{prob}(|\bar{\lambda}| \geq \varepsilon) \leq p.$$

This will be used to prove

<u>Lemma 3</u> For any $\varepsilon > 0$ and $p > 0$, there is a T^* such that $T \geq T^*$ implies that

$$\mathrm{prob}[(1/T)\log(W_T^m) \geq (1/T)\log(W_T^k) + \varepsilon] \leq p$$

where W_T^k and W_T^m are the values of W_T provided by s_T^k and s_T^m respectively.

Since by definition

$$L_t^k \geqslant L_t^m$$

for all h_{t-1}, we have

11.14 $$\bar{L}^k \geqslant \bar{L}^m$$

for all h_T. This and 11.9 imply that in order to have

$$(1/T)\log(W_T^m) \geqslant (1/T)\log(W_T^k) + \varepsilon$$

we must also have

$$\bar{\lambda}^m - \bar{\lambda}^k \geqslant \varepsilon.$$

The probability of the latter relationship is less than or equal to

11.15 $$\text{prob}[|\bar{\lambda}^k| \geqslant \varepsilon/2] + \text{prob}[|\bar{\lambda}^m| \geqslant \varepsilon/2].$$

To complete the proof, note that lemma 2 implies that there is a value of T* such that each probability in 11.15 is less than p/2.

12. Proof of Theorem 1

In the present section we prove

Theorem 1 For any $\varepsilon > 0$ there is a T^* such that if $T \geqslant T^*$ then

$$EV(g^k) \geqslant EV(g^m) - \varepsilon - \gamma^{max}$$

where g^k and g^m are the values of g provided by strategies s_T^k and s_T^m, respectively, in the game G_T as defined previously.

Since

$$q = (1/T)\log(W_T/W_0) = \log(1 + g)$$

is a strictly monotonic function of g for $g > -1$, we may substitute

$$g = e^q - 1$$

in the bounded version of V(g) defined in section 10. Thus

$$U = V(g) = V(e^q - 1) = v(q).$$

U is still bounded and monotonic when expressed as a function of q for $g > -1$, i.e., for $-\infty < q < +\infty$; hence the discussion in section 10 of bounded, monotonic functions applies to v(q).

Since $E[Vg(h_T)] = E[v(q(h_T)]$ for any G_T, s_T, h_T, we may prove theorem 1 for v(q) rather than V(g). Lemma 1 implies that for any $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$0 \leqslant v(q_2) - v(q_1) \leqslant \varepsilon/2 + \gamma^{max}$$

provided

$$0 \leqslant q_2 - q_1 \leqslant \delta.$$

By lemma 3 there is a T* such that

$$\text{prob}[q^m \geqslant q^k + \delta] \leqslant \frac{\varepsilon/2}{U^{hi} - U^{low}}$$

where $U^{hi} = V(r^{hi})$ and $U^{low} = V(r^{low})$. If $U^{hi} = U^{low}$ theorem 1 is trivial. Here we assume $U^{hi} > U^{low}$. Writing U^k and U^m for $v(q^k)$ and $v(q^m)$ respectively and letting

$$p = \text{prob}[q^m \geqslant q^k + \delta]$$

we have

$$EU^m - EU^k = E[U^m - U^k]$$

$$= p \cdot [E(U^m - U^k) | q^m \geqslant q^k + \delta]$$

$$+ (1 - p) \cdot [E(U^m - U^k) | q^m < q^k + \delta].$$

Thus

$$EU^m - EU^k \leqslant p \cdot (U^{hi} - U^{low}) + (1 - p) \cdot (\varepsilon/2 + \gamma^{max})$$

$$\leqslant \frac{\varepsilon/2}{U^{hi} - U^{low}} (U^{hi} - U^{low}) + \varepsilon/2 + \gamma^{max}.$$

Thus for $T \geqslant T^*$

$$EU^m - EU^k \leqslant \varepsilon + \gamma^{max}.$$

13. Proof of Theorems 2 and 3

In the present section we prove

Theorem 2 If $U = U(W_T)$ is bounded and monotonically increasing, and if there is an $\alpha > 0$ such that

$$\text{prob}(\bar{L}^k \geqslant \alpha) \to 1 \text{ as } T \to \infty$$

then

$$EU^k \to U^{hi} \text{ as } T \to \infty$$

where

$$U^{hi} = \text{the LUB of } U(W_T).$$

As defined previously

$$\bar{L}^k = (1/T) \sum_{t=1}^{T} L_t^k.$$

The theorem includes as a special case the situation in which there is some riskless investment with positive return always equal to at least $\beta > 0$. In this case $\alpha = \log(1 + \beta)$ would meet the requirement in the theorem.

Since

$$W_T \geqslant W_0(1 + r^{low})^T > 0$$

we may define

$$w = \log W_T - \log W_0$$

and may substitute

$$U = U(W_T) = U(e^{w+\log W_0}) = u(w).$$

Since U is a bounded and monitonic function of W_T for

$$0 < W_T < \infty,$$

it is also a bounded and monitonic function of w for

$$-\infty < w < \infty.$$

Thus the previously established properties of such utility functions apply to u(w). Also, for any G_T and s_T,

$$Eu(w(h_T)) = EU(W_T(h_T))$$

since both expectations are sums of products of identical probabilities and utilities. It will thus be sufficient to show the theorem for Eu(w).

Let Δ be a number such that $\alpha > \Delta > 0$. By the hypothesis of this theorem there exists T_a such that $T \geqslant T_a$ implies

$$\text{prob}(\bar{L}^k \leqslant \alpha - \Delta/2) \leqslant \frac{\varepsilon/4}{U^{hi} - U^{low}} .$$

Lemma 2 implies that there is a T_b such that if $T \geqq T_b$, then

$$\text{prob}(\bar{\lambda}^k \leqslant -\Delta/2) \leqslant \frac{\varepsilon/4}{U^{hi} - U^{low}} .$$

Since U^{hi} = LUB of $U(W_T)$ = LUB of $u(w)$, there exists w^* such that $w \geqslant w^*$ implies $u(w) \geqslant U^{hi} - \varepsilon/2$. Let T_c be some value of T greater than $w^*/(\alpha - \Delta)$. If $T \geqslant T_c$ and

$$(1/T)w \geqslant \alpha - \Delta$$

then $w \geqslant w^*$ and thus $U \geqslant U^{hi} - \varepsilon/2$.

Note

$$(1/T)w = \bar{L}^k + \bar{\lambda}^k.$$

Let T^* be the largest of T_a, T_b and T_c. Write U^k for $u(w^k)$. Then for any $T \geqslant T^*$

$$EU^k = p \cdot E(U^k \mid (1/T)w \leqslant \alpha - \Delta) + (1 - p) \cdot E(U^k \mid (1/T)w > \alpha - \Delta)$$

where

$$p = \text{prob}((1/T)w \leqslant \alpha - \Delta)$$
$$\leqslant \text{prob}(\bar{L}^k \leqslant \alpha - \Delta/2) + \text{prob}(\bar{\lambda}^k \leqslant -\Delta/2)$$
$$\leqslant \frac{\varepsilon/2}{U^{hi} - U^{low}} .$$

Thus

$$EU^k = E(U^k \mid (1/T)w > \alpha - \Delta)$$

$$- p\{E(U^k \mid (1/T)w > \alpha - \Delta) - E(U^k \mid (1/T)w \leqslant \alpha - \Delta)\}$$

$$\geqslant U^{hi} - \varepsilon/2 - \frac{\varepsilon/2}{U^{hi} - U^{low}} \cdot (U^{hi} - U^{low}).$$

Therefore

$$EU^k \geqslant U^{hi} - \varepsilon .$$

<u>Theorem 3</u> If prob$(\bar{L}^k \leqslant \alpha < 0) \rightarrow 1$ as $T \rightarrow \infty$ then $EU(W_T) \rightarrow U^{low}$ for $s = s_T^k$ or s_T^m.

This is proved in a manner quite similar to the previous theorem. This proof is not reproduced here.

In both the case in which

$$\text{prob}(\bar{L}^k \geqslant \alpha > 0) \rightarrow 1$$

and the case in which

$$\text{prob}(\bar{L}^k \leqslant \alpha < 0) \rightarrow 1$$

we have

$$EU^k \geqslant EU^m - \varepsilon$$

where $\varepsilon \rightarrow 0$ as $T \rightarrow \infty$.

In the first case this is because $EU^k \rightarrow U^{hi}$. In the second case it is because $EU^m \rightarrow U^{low}$. It is not generally true in the constant $U(W_T)$ case, however, that

$$EU^k \geqslant EU^m - \varepsilon$$

where $\varepsilon \to 0$ as $T \to \infty$. This may be shown by a counterexample.

Suppose: that $W_0 = 1$; that in every period the investor must choose a portfolio from the set

$$X_1 + X_2 = 1$$

$$X_1 \geqslant 0,\ X_2 \geqslant 0$$

where security 1 gives $r = 0$ with certainty, and X_2 gives a 50-50 chance of $r = +.25$ and $r = -.25$. Suppose further that $U(W_T) = 0$ for $W_T \leqslant 1$; $U(W_T) = W_T - 1$ for $1 \leqslant W_T \leqslant 1.25$; and $U(W_T) = .25$ for $W_T \geqslant 1.25$. Since

$$\tfrac{1}{2}\log(1 + a) + \tfrac{1}{2}\log(1 - a) < 0$$

for any $a \neq 0$, s_T^k requires $X_1 = 1$, $X_2 = 0$ for every t. Thus

$$EU^k = U(1) = 0.$$

Consider alternatively some strategy s which is the same as s_T^k for any preassigned $T - 1$ periods, but has $X_2 = 1$, $X_1 = 0$ in one period. The expected utility for this strategy equals

$$EU = \tfrac{1}{2}U(.75) + \tfrac{1}{2}U(1.25) = .125.$$

Thus for this strategy $EU - EU^k = .125$ for all games G_T.

14. Bounded Versus Unbounded Utility Functions

Theorems 2 and 3 assume that $U(W_T)$ is bounded from above and below. In the present section we consider the plausibility of this assumption. Throughout this discussion, when we speak of a bounded function we mean one bounded both from above and below. When we speak of an unbounded function we mean one that is unbounded either above or below or both.

Dynamic programming arguments, Bellman (1957), imply that the investor who follows an optimum strategy for maximizing expected $U(W_T)$ also maximizes the expected value of single period utility functions

$$U_t = U_t(W_t, h_t).$$

Here U_t equals the conditional expected utility of terminal wealth given that the investor has wealth W_t at time t, is faced with partial history h_t, and follows an optimum strategy from time t forward. As shown in the next section, $U(W_T)$ is a bounded function if and only if the derived single period utility functions $U_t(W_t, h_t)$ are bounded. Thus the plausibility of bounded $U(W_T)$ is equivalent to the plausibility of bounded $U_t(W_t, h_t)$. We shall consider the latter function.

Dynamic investment games other than those with utility depending only on terminal wealth also reduce to a sequence of one period expected utility maximizations. Thus the plausibility of assuming a bounded single period utility function is relevant to classes of games beyond those considered here. We deal here with the one period utility function, however, not to generalize the discussion, but to make the discussion more concrete.

This section presents two variations of the "St. Petersburg game" along the lines of Bernoulli (1738) and Menger (1934). One game shows the type of absurd behavior implied by any utility function which is unbounded from above; the other shows the type of absurd behavior implied by any utility function which is unbounded from below. We shall refer to these as "Menger games" of "type A" and "type B" respectively.

The discussion in this section deals with single period utility functions, and concerns one period games that effect the player's current wealth W_t but not his opportunities for returns in the future. We may therefore refer to a one-period utility function as $U(W_t)$ rather than $U_t(W_t,h_t)$ in the present section. In the next section we will consider further the relationship between the Menger games and the game G_T.

We first consider the Menger game of type A, constructed to show the absurdity of acting according to a single period utility function which is unbounded from above. Let k be any number greater than 0. Let $w_1, w_2, w_3, \ldots$ be alternate values of W_t such that

$$U(w_i) \geqslant k \cdot 2^i .$$

Such a sequence of w_i will exist if and only if U is unbounded above. Suppose a bet gives probability $p_i = (\frac{1}{2})^i$ that wealth w_i will result. Then its expected utility equals

$$\sum_{i=1}^{\infty} U(w_i) \cdot p_i \geqslant k \cdot \sum_{i=1}^{\infty} 2^i \cdot (\tfrac{1}{2})^i = \infty .$$

Such a game would be preferred to any finite increase in wealth given with certainty.

For example, letting $W_{t-1} = 1, k = .001$ and $U = w^{\frac{1}{2}}$, a gamble with probability $(\frac{1}{2})^i$ of resulting wealth equal to $(.000001)(2^{2i})$ has infinite value. Some of the p_i and w_i for this specific k and utility function are presented in table 1. The first column contains the index i, the second contains the probability of the i-th outcome, and the third contains the ratio of ending to starting wealth $w_i = W_t/W_{t-1}$ for the i-th outcome. Thus according to the first row, for i = 1 there is a probability of $p_1 = .5$ that W_t/W_{t-1} will equal .000004. In other words, the game provides the player with a 50-50 chance of losing 99.9996% of his wealth. It similarly provides an additional one chance in four of his only losing 99.9984% of his wealth and so on. There is, however, a .000488 probability that W_t/W_{t-1} will equal 4.194304; i.e., that the player will win about 319%; .000244 that he will win 1578% and so on. The utility maximizer with $U = w^{\frac{1}{2}}$ will prefer this game to any finite increase in wealth given with certainty. I assume that the reader would not prefer to play the game in table 1 rather than have a billion-fold return on his portfolio for sure, and that he would doubt the mental competence of anyone who would.

A similar game can be constructed for any utility function which is not bounded from above. The game can be made to have an infinite value even though, with proper choice of k, the game seems hardly attractive at all to you or I.

Table 1

Payoffs for Illustrative One Period Game With $U(W_T) = w^{\frac{1}{2}}$

i	p_i	W_t/W_{t-1}
1	.500000	.000004
2	.250000	.000016
3	.125000	.000064
4	.062500	.000256
5	.031250	.001024
6	.015625	.004096
7	.007813	.016384
8	.003906	.065536
9	.001953	.262144
10	.000977	1.048576
11	.000488	4.194304
12	.000244	16.777216
13	.000122	67.108864
14	.000061	268.435456
.	.	.
.	.	.
.	.	.

Next let us consider the Menger game of type B constructed to illustrate the absurdity of utility functions which are not bounded below. The game involves values of W approaching 0, although W = 0 cannot occur. We avoid complaining about the problems of αW^{α} ($\alpha < 0$) when W = 0, since W = 0 cannot occur in the games G_T.

The Menger game of type B provides a probability $(\frac{1}{2})^i$ of having end-of-period wealth w_i with utility equal to no more than $-k \cdot 2^i$ where $k > 0$. It is possible to find such a sequence if and only if U^{low} does not exist. With w_i thus chosen expected utility equals

$$EU \leqslant -k \sum_{i=1}^{\infty} (\tfrac{1}{2})^i \cdot 2^i = -\infty.$$

Such a game would be worse than any end-of-period wealth W > 0 given with certainty.

For example, with $U = -w^{-\frac{1}{2}}$ and k = .001 we have

$$w_i = 1{,}000{,}000 \cdot (2^{-2i}).$$

This is tabulated in table 2. The game gives a 50-50 chance of a two hundred and fifty thousand-fold increase in wealth for the period, one chance in four of approximately a sixty-two thousand-fold increase in wealth, etc. It does however subject the player to a probability of .000488 that a 76% loss will occur; a probability of .000244 that a 94% loss will occur, and so on.

For the player with $U = -w^{-\frac{1}{2}}$, this game is worse than any end-of-period wealth $W_t > 0$ given with certainty. He would

Table 2

Payoffs for Illustrative One Period Game With $U(W_t) = -w^{-1/2}$

i	P_i	W_t/W_{t-1}
1	.500000	250,000.000
2	.250000	62,500.000
3	.125000	15,625.000
4	.062500	3,906.250
5	.031250	976.563
6	.015625	244.141
7	.007813	61.035
8	.003906	15.259
9	.001953	3.815
10	.000977	.954
11	.000488	.238
12	.000244	.060
13	.000122	.015
14	.000061	.004
.	.	.
.	.	.
.	.	.

rather have a 99.999999% loss with certainty, for example, than play the game. I assume that the reader agrees that it would be absurd for an investor to give up "virtually everything" with certainty rather than risk the game. But some such absurdity can always be shown if U^{low} does not exist.

15. G_T and the Menger Games

This section cares for some loose ends related to the preceding discussion. In this section, for example, we show that the single period utility function $U_t(W_t,h_t)$ in a game G_T is bounded above and below if and only if $U(W_T)$ is thus bounded. We also consider an apparent contradiction between theorems 2 and 3 and the results of the last section in that the latter rules out any unbounded single period utility function including

$$U_t(W_t,h_t) = \log(W_t).$$

To show that in any game G_T the single period function $U_t(W_t,h_t)$ is bounded above and below if and only if $U(W_T)$ is thus bounded, we may show

(a) if $U(W_T)$ is bounded above and below then so is $U_t(W_t,h_t)$; and

(b) if $U(W_T)$ is unbounded from above or below then so is $U_t(W_t,h_t)$.

(a) above follows immediately from the definition of $U_t(W_t,h_t)$ as a conditional expected value of $U(W_T)$. If $U(W_T)$ is bounded by U^{low} and U^{hi} any conditional expected value of $U(W_T)$ will also lie within these bounds. (b) above follows from the assumption that $0 < r^{low} \leqslant r_t \leqslant r^{hi}$. This implies that if $U(W_T)$ is unbounded from above so is $U_t(W_t,h_t)$, since

$$U_t(W_t,h_t) \geq U((1 + r^{low})^T \cdot W_T)$$

where the latter approaches ∞ as W_T approaches ∞; whereas if $U(W_T)$ is unbounded from below so is $U_t(W_t,h_t)$, since

$$U_t(W_t,h_t) \leqslant U((1 + r^{hi})^T \cdot W_T)$$

where the latter approaches $-\infty$ as W_T approaches 0. This completes the proof.

The following point may need clarifying. In every game G_T we have

$$0 < r^{low} \leqslant r_t \leqslant r^{hi}.$$

Both the type A and type B Menger games violate such an assumption for any $r^{low} > 0$ and any r^{hi}. Therefore no game G_T can have a Menger game for one or more of its T periods. How then can we use a Menger game to show the absurdity of unbounded utility functions in G_T?

Imagine a game H_{T+1} consisting of a one period Menger game followed by a game G_T. The initial Menger game does not affect the wheel to be spun at $t = 1$ in G_T. It only affects the initial wealth, W_0, at the start of G_T. What has essentially been shown concerning the relationship between the games G_T and H_{T+1} is this:

the single period utility functions in the game G_T, and hence in the game H_{T+1}, are bounded above and below if and only if $U(W_T)$ is thus bounded

hence

if a player is to be allowed to play a Menger game once only, and thereafter must play G_T, a properly constructed Menger game can be found to provide $+\infty$ or $-\infty$ expected single period utility if and only if $U(W_T)$ is unbounded above or below.

Thus while G_T cannot include a Menger game, the Menger games and the game H_{T+1} can be used to show the absurdity of assuming an unbounded $U(W_T)$ in a sequence of games $G_{T_1}, G_{T_2}, \ldots$.

Next, let us consider the following apparent contradiction in our results. The Menger games show that certain absurd behavior can follow from any single period utility function which is unbounded either from above or below. We have further noted that the implied single period functions for any game G_T, or any game H_{T+1}, is unbounded if and only if $U(W_T)$ is unbounded. We therefore conclude that, to avoid a utility function with such latent absurdities, we should assume that $U(W_T)$ is bounded from above and below. This in turn implies that, under the conditions of theorems 2 and 3, the s_T^k strategy is asymptoticaly optimal. But s_T^k has us use $\log(1 + r) = \log(W_t) - \log(W_{t-1})$ as a single period utility function; and this is unbounded both from above and below. Contradiction?

Not really. The Menger games show that an unbounded single period utility function, such as $U_t = \log(W_t)$, can lead to absurd results in a one period situation in which possible wealth outcomes approach either 0 or ∞. But in any game G_T the possible wealth outcomes in a single period are bounded by

$$0 < (1 + r^{low}) \leqslant W_t/W_{t-1} \leqslant (1 + r^{hi}).$$

Thus theorems 2 and 3 and the discussion of the Menger games may be reconciled as follows: The Menger games imply that the utility function $U(W_T)$ must be bounded to avoid certain absurdities. But theorems 2 and 3 imply that, under the conditions specified, the use of

$$\log(W_t/W_{t-1}) = \log(1 + r_t)$$

as the single period utility function is asymptotically optimal in the game G_T. But the game G_T, by definition, bounds r_t by

$$0 < r^{low} \leqslant r_t \leqslant r^{hi}.$$

Therefore, using the unbounded logarithmic function over this bounded range is asymptotically optimal under the conditions specified.

One last point. While using $\log(1 + r)$ provides asymptotically optimum results, it does not follow that $U_1(W_1,h_1) \to \log(1 + r)$ as $T \to \infty$. Under the conditions of theorem 2, for example, $U_1(W_1,h_1) \to U^{hi}$ as $T \to \infty$ for any $W_1 > 0$; i.e., the implied single period utility function for the first period approaches $U_1 = U^{hi}$ for $W_1 > 0$.

But if we acted according to the latter utility function we would be indifferent between, for example, a 6% gain with certainty or a 6% loss with certainty. It follows that, whereas using $\log(1 + r)$ as the single period utility function will give optimum utility asymptotically, in contrast disastrous results could follow from using throughout any G_T the limiting function approached by $U_1(W_1,h_1)$.

16. Historical Note

If the reader agrees with the arguments and conclusions of this paper it may seem that, having returned to the Kelly-Latané expected log rule, we are no further ahead than if Mossin and Samuelson had never raised their objections. But theorists since Euclid, or before, have not been satisfied to know correct answers, but have sought to know them for correct reasons. In particular they have sought to know which specific results may be deduced from which basic assumptions.

Markowitz (1959), chapter 12, postulates a simple set of axioms as its fundamental assumptions concerning rational behavior under uncertainty. From these assumptions it is fairly easy to derive the Savage (1954) conclusion that the rational economic man acts according to expected utility and personal probability. Hence, in chapters 6 and 13, the applicability and limitations of E,V analysis were explored in terms of expected utility.

Given these fundamental assumptions it was a non sequitur to justify the expected log rule for long run investment by a law of large numbers argument à la Kelly. Hopefully, the matter has now been set straight.

Bibliography

Baumol, W.J. "An Expected Gain - Confidence Limit Criteria for Portfolio Selection," Management Science, October 1963.

Bellman, Richard Ernest. Dynamic Programming, Princeton University Press, N.J., 1957.

Breiman, Leo. "Investment Policies for Expanding Businesses Optimal in a Long Run Sense," Naval Research Logistics Quarterly, 7:4, 1960, pps. 647-651.

Breiman, Leo. "Optimal Gambling Systems for Favorable Games," Fourth Berkeley Symposium on Probability and Statistics, I, 1961, pp. 65-78.

Bernoulli, Daniel. "Exposition of a New Theory on the Measurement of Risk," Econometrica, XXII, January 1954, pp. 23-63. Translated by Louise Sommer - original 1738.

Cohen, K. J. and Pogue, J. A. "An Empirical Evaluation of Alternative Portfolio-Selection Models," Journal of Business, April 1967.

Kelly, J. L., Jr. "A New Interpretation of Information Rate," Bell System Technical Journal, 917-926, 1956.

Latane, H. A. "Rational Decision Making in Portfolio Management," Ph.D. dissertation, University of North Carolina, 1957.

Latane, H. A. "Criteria for Choice Among Risky Ventures," Journal of Political Economy, April 1959.

Markowitz, Harry. "The Utility of Wealth," Journal of Political Economy, Vol. LX, No. 2, April 1952, pp. 151-158.

Markowitz, H. M. Portfolio Selection, Efficient Diversification of Investments (John Wiley and Sons, New York), 1959.

Menger, Karl. "Das Unsicherheitsmoment in der Wertlehre. Betrachtungen im Anschluss an das sogenannte Petersburger Spiel," Zeitschrift für Nationalökonomie, Vol. 5, 1934.

Mossin, J. "Optimal Multiperiod Portfolio Policies," Journal of Business, April 1968.

Samuelson, P.A. "Risk and Uncertainty: A Fallacy of Large Numbers," Scientia, 6th Series, 57th year, April-May 1963.

Samuelson, P.A. "Lifetime Portfolio Selection by Dynamic Stochastic Programming," Review of Economics and Statistics, August 1969.

Savage, Leonard J. The Foundations of Statistics, John Wiley and Sons, New York and Chapman and Holt, London, 1954.

Sharpe, W. F. "A Simplified Model for Portfolio Analysis," Management Science, January 1963.